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Boundary Value Problems for Fractional Differential Equations of Caputo Type and Ulam Type Stability: Basic Concepts and Study

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Abstract: Boundary value problems are very applicable problems for different types of differential equations and stability of solutions, which are an important qualitative question in the theory of differential equations. There are various types of stability, one of which is the so called Ulam-type stability, and it is a special type of data dependence of solutions of differential equations. For boundary value problems, this type of stability requires some additional understanding, and, in connection with this, we discuss the Ulam-Hyers stability for different types of differential equations, such as ordinary differential equations and generalized proportional Caputo fractional differential equations. To propose an appropriate idea of Ulam-type stability, we consider a boundary condition with a parameter, and the value of the parameter depends on the chosen arbitrary solution of the corresponding differential inequality. Several examples are given to illustrate the theoretical considerations.

Keywords: boundary value problems; Ulam-type stability; fractional differential equations; generalized proportional Caputo fractional derivative

MSC: : 34A08; 34D99; 34B99

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Citation: Agarwal, R.; Hristova, S.; O'Regan, D. Boundary Value Problems for Fractional Differential Equations of Caputo Type and Ulam Type Stability: Basic Concepts and Study. *Axioms* **2023**, *12*, 226. https://doi.org/10.3390/axioms12030226

Academic Editor: Nicolae Lupa

Received: 17 January 2023
Revised: 18 February 2023
Accepted: 20 February 2023
Published: 21 February 2023

1. Introduction

Ulam stability is an important problem investigated in differential equations, which include fractional derivatives. It has applications in optimization, biology, economics, etc., and it is a special type of data dependence of solutions (see, for example, [1–5]). Usually, one first proves the existence and uniqueness of the problem and then one considers a corresponding differential inequality so, in the definition of Ulam-type stability, one assumes that, for any solution of the corresponding differential inequality, there exists a solution of the given problem, such that both solutions are close enough. In the case of an initial value problem (IVP), usually one chooses the initial value of the differential equation, which will depend on an arbitrary chosen solution of the corresponding differential inequality. For example, Ulam-type stability for IVPs was studied in [6], the Caputo fractional differential equation (FDE) was studied in [7], the Darboux problem for a partial FDE was studied in [8], generalized fractional derivatives were studied in [9], fractional Volterra-type integral equations with delay were studied in [10], non-linear delay differential equations with fractional integrable impulses were studied in [11], delay differential equations were studied in [12,13], Caputo FDEs were studied in [14], Caputo FDEs with impulses were studied in [15], Riemann-Liouville FDEs were studied in [16], Riemann-Liouville FDEs with delay were studied in [17], Caputo FDEs with delays were studied in [18], Hadamard FDEs were studied in [19], Hilfer-Katugampola FDEs with impulses were studied in [20], and sequential FDEs were studied in [21] for first-order impulsive fuzzy differential equations.

In the case of boundary value problems (BVP), the situation is more complicated. There are mainly two different types of boundary conditions. One type is with a parameter, and the other type is without a parameter (see, for example [22,23]), and, in this paper, we propose an appropriate idea for Ulam-type stability. To motivate the idea, we study linear boundary value conditions. First, we consider well known ordinary differential equations, and we will illustrate the above ideas on a simple example, and we will set up the boundary condition and the definition of Ulam-Hyers stability. Then, we study, in detail, a linear BVP for nonlinear differential equations with generalized proportional Caputo fractional derivatives(GPFDE). The generalized proportional Caputo fractional derivative was recently introduced (see, [24,25]), and it provides wider possibilities for modeling more complexity of real world problems, and this type of derivative is a generalization of the Caputo fractional derivative. We consider the case when a parameter is involved in the boundary condition and an integral representation of the solution of the studied linear BVP for GPFDE is presented. The existence and uniqueness of the solution of the BVP for GPFDE for any value of the parameter is also studied. In an appropriate way, the Ulam-Hyers stability is defined, and some sufficient conditions are obtained. The main idea is to choose an arbitrary solution of the corresponding differential inequality with a generalized proportional Caputo fractional derivative and then to use a parameter depending on this solution in the given boundary condition. As a special case, we provide some results for linear BVPs for Caputo fractional differential equations. Our theoretical results are illustrated with examples.

2. Ordinary Differential Equations and Ulam-Hyers Stability

Without loss of generality, we assume the initial time point is 0. Let $0T \leq \infty$.
Consider the scalar ordinary differential equation (ODE):

$$y'(t) = f(t, y(t)), \ \ t \in (0, T] \tag{1}$$

where $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$.

Let $\varepsilon > 0$. Consider the differential inequality (ODI):

$$|\nu'(t) - f(t, \nu(t))| \leq \varepsilon, \quad t \in [0, T]. \tag{2}$$

If $\nu \in C^1[0, T]$ is a solution of (2) then there exists a function $g \in C[0, T] : \ |g(t)| \leq \varepsilon, \ t \in [0, T]$ such that $u(t)$ is a solution of the ODE

$$\nu'(t) = f(t, \nu(t)) + g(t), \ \ t \in [0, T]. \tag{3}$$

Thus,

$$\nu(t) = \nu(0) + \int_0^t [f(s, \nu(s)) + g(s)]ds, \quad t \in [0, T]. \tag{4}$$

Case 1. *Initial value problem (IVP) for ODE* (1). Consider the initial condition:

$$y(0) = y_0, \tag{5}$$

where $y_0 \in \mathbb{R}$.

Assume that, for any initial value y_0, the IVP for the ODE (1), (5) has a solution on $[0, T]$.

Let us recall the definition for Ulam-Hyers stability.

Definition 1 ([3,4])**.** *The IVP* (1), (5) *is Ulam-Hyers stable (UHS) if there exists a real number $C_f > 0$, such that, for each $\varepsilon > 0$, and for each solution $\nu \in C^1[0, T]$ of the ODI* (2)*, there exists a solution $y \in C^1[0, T]$ of* (1), (5) *with $|\nu(t) - y(t)| \leq C_f \varepsilon$ for $t \in [0, T]$.*

Remark 1. *In Definition 1, for the solution $\nu \in C^1[0,T]$ of the ODI (2), we consider the solution $y \in C^1[0,T]$ of (1), (5) with $y_0 = \nu(0)$. Then, for UHS, it is necessary to obtain conditions, such that $|\nu(t) - y(t)| \le C_f \varepsilon$ for $t \in [0,T]$, where the constant C_f does not depend on ε, and it is the same for all solutions of the ODI (2).*

In the case of the initial value condition, we introduce the following definition:

Definition 2. *The solution $y \in C^1[0,T]$ of (1), (5) with an initial given value x_0 is called* modified Ulam-Hyers stable *(MUHS) if there exists a constant $C_f > 0$, such that, for each $\varepsilon > 0$ and for each solution $\nu \in C^1[0,T],\ \nu(0) = y_0$ of the ODI (2), the inequality $|\nu(t) - y(t)| \le C_f \varepsilon$ holds for $t \in [0,T]$.*

Remark 2. *If we consider Definition 2 for all initial values x_0, then one gives a description of Definition 1.*

We will use the following integral operator $\Omega : C[0,T] \to C[0,T]$ by

$$(\Omega y)(\tau) = y_0 + \int_0^\tau f(s, y(s))ds, \quad \tau \in [0,T], \tag{6}$$

and the integral operator $\Omega_1 : C[0,T] \to C[0,T]$ by

$$(\Omega_1 \nu)(\tau) = \nu(0) + \int_0^\tau [f(s, \nu(s)) + g(s)]ds, \quad \tau \in [0,T]. \tag{7}$$

The solution $y \in C^1[0,T]$ of (1), (5) is a fixed point of the operator Ω and vice versa. The solution $\nu \in C^1[0,T]$ of the ODE (3) is a fixed point of Ω_1 and vice versa.

Case 2. *Boundary value problem (BVP) for ODE* (1). Consider the boundary condition

$$ay(0) + by(T) = \mu, \tag{8}$$

where $a, b \in \mathbb{R} :\ a + b \neq 0, \mu \in \mathbb{R}$.

Assume that, for any $\mu \in \mathbb{R}$, the BVP (1), (8) has a solution on $[0,T]$. Note: if $b = 0$ in (8), we obtain the IVP for (1).

Following the ideas of Definition 1, we define UHS by:

Definition 3. *The BVP* (1), (8) *is* Ulam-Hyers stable *(UHS) if there exists a constant $K > 0$, such that, for any number $\varepsilon > 0$ and any solution $\nu \in C^1[0,T]$ of the ODI (2), there exists a solution $y \in C^1[0,T]$ of BVP (1), (8), such that $|y(t) - \nu(t)| \le K\varepsilon,\ \ t \in [0,T]$.*

Remark 3. *In Definition 3, for the solution $\nu \in C^1[0,T]$ of ODI (2), we consider the solution $y \in C^1[0,T]$ of (1), (8) with $\mu = a\nu(0) + b\nu(T)$. Then, for UHS, it is necessary to obtain conditions, such that $|\nu(t) - y(t)| \le K\varepsilon$ for $t \in [0,T]$, where the constant K does not depend on ε, and it is the same for all solutions of the ODI (2).*

In the case of an initial given value of $\mu \in \mathbb{R}$, following the ideas of Definition 2, we introduce the following definition.

Definition 4. *The solution $y \in C^1[0,T]$ of BVP (1), (8), with an initial given value of μ, is* modified Ulam-Hyers stable *(MUHS) if there exists a constant $K > 0$, such that, for any number $\varepsilon > 0$ and any solution $\nu \in C^1[0,T],\ a\nu(0) + b\nu(T) = \mu$, of the ODI (2), the inequality $|y(t) - \nu(t)| \le K\varepsilon$ holds for $t \in [0,T]$.*

Define the operator $\Omega : C[0,T] \to C[0,T]$ by

$$(\Omega y)(t) = \frac{\mu}{a+b} - \frac{b}{a+b}\int_0^T f(s, y(s))ds + \int_0^t f(s, y(s))ds, \quad t \in [0,T], \tag{9}$$

and the operator $\Omega_1 : C[0,T] \to C[0,T]$ by

$$(\Omega_1 \nu)(t) = \frac{\mu}{a+b} - \frac{b}{a+b}\int_0^T (f(s,\nu(s)) + g(s))ds + \int_0^t (f(s,\nu(s)) + g(s))ds, \quad t \in [0,T]. \tag{10}$$

For any value of $\mu \in \mathbb{R}$ the solution $y \in C^1[0,T]$ of BVP (1), (8) is a fixed point of the operator Ω and vice versa. The solution $\nu \in C^1[0,T]$ of the ODE (3) is a fixed point of the operator Ω_1 iff $\mu = a\nu(0) + b\nu(T)$ and vice versa.

Example 1. *Consider the following scalar ordinary differential equation:*

$$x'(t) = 1.5x(t), \quad t \in [0,1]. \tag{11}$$

Let $\varepsilon > 0$ be a given arbitrary number and consider the differential inequality:

$$|\nu'(t) - 1.5\nu(t)| \le \varepsilon, \quad t \in [0,1]. \tag{12}$$

Case 1. *(IVP) Consider the initial condition*

$$x(0) = x_0, \tag{13}$$

where $x_0 \in \mathbb{R}$. The IVP (11), (13) has a unique solution $x(t) = x_0 e^{1.5t}$, $t \in [0,1]$.

Case 1.1. *We will look at Definition 1.*

Choose an arbitrary solution of the ODI (12), for example $\nu(t) = \varepsilon t^2 : \ |\nu'(t) - 1.5\nu(t)| = |2t - 1.5t^2|\varepsilon \le \varepsilon, \ t \in [0,1]$. Then, choose $x_0 = \nu(0) = 0$ in (13) and the solution of the IVP (11), (13) is $x(t) = 0$. Thus, the inequality $|x(t) - \nu(t)| = |0 - \varepsilon t^2| \le \varepsilon$ holds for $t \in [0,1]$ and all $\varepsilon > 0$.

Choose another solution of the ODI (12), for example $\nu(t) = \varepsilon(t^2 + 0.5) : \ |\nu'(t) - 1.5\nu(t)| = |2t - 1.5(t^2 + 0.5)|\varepsilon \le \varepsilon, \ t \in [0,1]$. Then, choose $x_0 = \nu(0) = 0.5\varepsilon$ and the solution of the IVP (11), (13) is $x(t) = 0.5\varepsilon e^{1.5t}$, and we have the inequality $|x(t) - \nu(t)| = |0.5e^{1.5t} - (t^2 + 0.5)|\varepsilon \le \varepsilon$ for $t \in [0,1]$ and all $\varepsilon > 0$.

Case 1.2. *We will look at Definition 2.*

Let x_0 be given, for example, $x_0 = 2$. Then, the IVP (11), (13) has a unique solution $x(t) = 2e^{1.5t}$, $t \in [0,1]$.

Let ε be an arbitrary number, for example, $\varepsilon = 0.01$. Choose the solution $\nu(t) = 2.004e^{-t}(-0.00199601 + e^{2.5t})$ of the ODI (12) and note $|x(t) - \nu(t)| \le 1.65\varepsilon$ for $t \in [0,1]$, i.e., $C_f = 1.65$.

Let $\varepsilon = 1$. Choose the solution $\nu(t) = 2e^{-t}$ of (12) satisfying the initial condition $\nu(0) = 2$. Then, $|x(t) - \nu(t)| \le 1.65\varepsilon$ for $t \in [0,1]$.

Now, let $x_0 = -5$. Then, the IVP (11), (13) has a unique solution $x(t) = -5e^{1.5t}$, $t \in [0,1]$.

Let $\varepsilon = 0.01$. Choose the solution $\nu(t) = -0.004e^{-t} - 4.996e^{1.5t}$ of (12), satisfying the initial condition $\nu(0) = -5$. Then, $|x(t) - \nu(t)| \le 1.65\varepsilon$ for $t \in [0,1]$.

Let $\varepsilon = 1$. Choose the solution $\nu(t) = -0.4e^{-t} - 4.6e^{1.5t}$ of (12) satisfying the initial condition $\nu(0) = -5$. Then, $|x(t) - \nu(t)| \le 1.65\varepsilon$ for $t \in [0,1]$.

Summarizing, if we have a particular initial value, then we consider a particular solution of the corresponding IVP, and we study MUHS. If we have an arbitrary initial value, then we study UHS.

Case 2. *(BVP) Consider the boundary condition:*

$$0.5x(0) + 0.5e^{-1.5}x(1) = \mu. \tag{14}$$

Case 2.1. *We will look at Definition 3.*

Consider $\mu \in \mathbb{R}$ as a parameter. Then, the BVP (11), (14) has a unique solution $x(t) = \mu e^{1.5t}$ for any given value of the parameter μ.

Let $\varepsilon > 0$ be an arbitrary number. Choose an arbitrary solution of the differential inequality (12), for example $\nu(t) = \varepsilon t^2$. Then, choose the parameter $\mu = 0.5\nu(0) + 0.5e^{-1.5}\nu(1) = 0.5e^{-1.5}\varepsilon$.

There exists a solution of the BVP (11), (14) for the chosen μ, *namely,* $x(t) = 0.5e^{-1.5}\varepsilon e^{1.5t}$, *and* $|x(t) - \nu(t)| = |0.5e^{-1.5}e^{1.5t} - t^2|\varepsilon \le 0.5\varepsilon$ *for* $t \in [0,1]$.

Additionally, in this case, the function $g(t) = (2t - 1.5t^2)\varepsilon$ *and the solution* $\nu(t) = 0.5e^{-1.5}\varepsilon$ *are fixed points of the operator* Ω_1 *with the chosen value of* μ.

Case 2.2. *We will look at Definition 4.*

Let μ *have a particular initial given value, for example* $\mu = 0.5e^{-1.5}$. *Then, the BVP (11), (14) has a solution* $x(t) = 0.5e^{1.5(t-1)}$.

Let $\varepsilon > 0$ *be an arbitrary number, for example,* $\varepsilon = 1$.

Choose an arbitrary solution of the differential inequality (12) with $\varepsilon = 1$, *such that* $0.5\nu(0) + 0.5e^{-1.5}\nu(1) = 0.5e^{-1.5}$. *For example,* $\nu(t) = t^2$. *Then, the inequality* $|x(t) - \nu(t)| = |0.5e^{1.5(t-1)} - t^2| \le 0.5\varepsilon$ *holds for* $t \in [0,1]$.

Summarizing, if we have a particular boundary condition (initially given the value μ*), then we consider a particular solution of the corresponding BVP, and we study MUHS. If we have a boundary condition with a parameter (arbitrary value of* μ*), then we study UHS.*

Remark 4. *In connection with the above example and the above discussion to study UHS of any type of differential equation, we need a boundary condition with a parameter. Otherwise, we could only study MUHS of a particular solution.*

3. Generalized Proportional Caputo Fractional Differential Equations and Ulam-Hyers Stability

3.1. Preliminary Results from Fractional Calculus

Fractional differential equations arise in many investigations in recent years, and they are widely used in dynamical models with chaotic dynamical behavior, quasi-chaotic dynamical systems, the dynamics of complex material or porous media, and random walks with memory (for some physical background and numerical algorithms of fractional partial differential equations, see Chapter 1 [26] and the recent book [27], for example). Recently, there are several types of fractional derivatives defined and studied in the literature. In our paper, we will use the generalized proportional Caputo fractional derivative, which is a generalization of the Caputo fractional derivative (see, [24,25]).

Let $\nu : [0,T] \to \mathbb{R}$, $(T \le \infty)$. The generalized proportional fractional integral is defined by (as long as all integrals are well defined, see [24,25])

$$({}_0\mathcal{I}^{\alpha,\rho}\nu)(\tau) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)}(\tau-s)^{\alpha-1}\nu(s)\,ds, \quad \tau \in (0,T], \ \alpha > 0, \ \rho \in (0,1],$$

and the generalized Caputo proportional fractional derivative is defined by (see, [24,25])

$$\begin{aligned}({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(\tau) &= \frac{1-\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)}(\tau-s)^{-\alpha}\nu(s)\,ds \\ &+ \frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)}(\tau-s)^{-\alpha}\nu'(s)\,ds, \ \ \tau \in (0,T], \ \alpha \in (0,1), \ \rho \in (0,1].\end{aligned} \tag{15}$$

Remark 5. *The Caputo fractional derivative* ${}_0^C D_\tau^\alpha \nu(\tau)$ *is a partial case of the generalized Caputo proportional fractional derivative* $({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(\tau)$ *with* $\rho = 1$.

We consider the following classes of functions:

$$\begin{aligned}C^{\alpha,\rho}[0,T] &= \{\nu : [0,T] \to \mathbb{R} : \ ({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(\tau) \ \text{exists for all} \ \tau \in (0,T]\}, \\ I^{\alpha,\rho}[0,T] &= \{\nu : [0,T] \to \mathbb{R} : \ ({}_0\mathcal{I}^{\alpha,\rho}\nu)(\tau) \ \text{exists for all} \ \tau \in (0,T]\}.\end{aligned}$$

Remark 6. *Note, in this paper, the space* $C^{\alpha,\rho}[0,T]$ *is not the Hölder space in the literature.*

For $\nu \in C^{\alpha,\rho}[0,T]$, ${}_0^C\mathcal{D}^{\alpha,\rho}\nu(.) \in I^{\alpha,\rho}[0,T]$ we have the following result:

Lemma 1 (Theorem 5.3 [24])**.** *Let $\rho \in (0,1]$ and $\alpha \in (0,1)$. Then, we have:*

$$({}_0\mathcal{I}^{\alpha,\rho}({}_0^C\mathcal{D}^{\alpha,\rho}\nu))(t) = \nu(t) - \nu(0)e^{\frac{\rho-1}{\rho}t}, \ \ t \in (0,T].$$

For $\nu \in I^{\alpha,\rho}[0,T]$, ${}_0\mathcal{I}^{\alpha,\rho}\nu(.) \in C^{\alpha,\rho}[0,T]$, $\rho \in (0,1]$ we have:

Corollary 1 ([24])**.** *Let $\alpha \in (0,1)$. Then:*

$$({}_0^C\mathcal{D}^{\alpha,\rho}({}_0\mathcal{I}^{\alpha,\rho}\nu))(\tau) = \nu(\tau), \quad \tau \in (0,T].$$

Lemma 2 (Theorem 5.2 [24])**.** *For $\rho \in (0,1]$ and $\alpha \in (0,1)$, we have:*

$$({}_0\mathcal{I}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t^{\beta-1})(\xi) = \frac{\Gamma(\beta)}{\rho^\alpha\Gamma(\beta+\alpha)})e^{\frac{\rho-1}{\rho}\xi}\xi^{\beta-1+\alpha}, \quad \beta > 0.$$

Corollary 2 ([24], Remarks 3.2 and 5.4)**.** *The equality $({}_0^C\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}(.)})(\tau) = 0$ for $\tau \in (0,T]$, $\rho \in (0,1]$, $\alpha \in (0,1)$, holds.*

Lemma 3 (Proposition 5.2 [24])**.** *For $\rho \in (0,1]$ and $\alpha \in (0,1)$, we have:*

$$({}_0\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t^{\beta-1})(\xi) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta+\alpha)})e^{\frac{\rho-1}{\rho}\xi}\xi^{\beta-1-\alpha}, \quad \beta > 0.$$

Later, we will use the following result (for the proof see Example 5.7 [24] with a slight modification):

Lemma 4. *The solution of the scalar linear generalized proportional Caputo fractional initial value problem*

$$({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(t) = \rho^\alpha\lambda\nu(t) + F(t), \ \ \nu(0) = \nu_0, \ \ \alpha \in (0,1), \ \rho \in (0,1]$$

where $F \in C([0,T],\mathbb{R})$, has a solution

$$\nu(t) = \nu_0 e^{\frac{\rho-1}{\rho}t}E_\alpha(\lambda t^\alpha) + \rho^{-\alpha}\int_0^t E_{\alpha,\alpha}(\lambda(t-s)^\alpha)e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}F(s)ds, \ t \in [0,T]. \quad (16)$$

Applying $E_\alpha(0) = 1$, $E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$, we obtain the following result:

Corollary 3. *The solution of the scalar linear generalized proportional Caputo fractional initial value problem*

$$({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(\tau) = F(\tau), \ \ \nu(0) = \nu_0, \ \ \alpha \in (0,1), \ \rho \in (0,1]$$

where $F \in C([0,T],\mathbb{R})$, has a solution

$$\nu(\tau) = u_0 e^{\frac{\rho-1}{\rho}\tau} + \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)}(\tau-s)^{\alpha-1}F(s)ds, \ \tau \in [0,T]. \quad (17)$$

3.2. Ulam Type Stability for Initial Value Problems

Consider the nonlinear generalized proportional Caputo fractional differential equation (GPFDE)

$$({}_0^C\mathcal{D}^{\alpha,\rho}x)(t) = f(t, x(t)), \quad t \in (0,T], \quad (18)$$

with an initial condition

$$x(0) = x_0, \quad (19)$$

where $x_0 \in \mathbb{R}$, $\alpha \in (0,1)$ and $\rho \in (0,1]$.

Assume the following holds:

(**A**). The function $f \in C([0,T]\times\mathbb{R}):\ f(.,\nu(.)) \in I^{\alpha,\rho}[0,T]$ for any function $\nu \in C[0,T]$. Consider the fractional integral operator $\Omega : C[0,T] \to C[0,T]$ (see, Corollary 3)

$$(\Omega\nu)(\tau) = x_0 e^{\frac{\rho-1}{\rho}\tau} + \frac{1}{\rho^q\Gamma(\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)}(\tau-s)^{\alpha-1}f(s,\nu(s))ds,\ \ \tau\in[0,T]. \tag{20}$$

For any type of fractional differential equation, an appropriate integral operator is defined, and its fixed point is called a mild solution (see, for example, Definition 3.1 [28] for fractional neutral evolution equations, [29] for fractional evolution equation, and [30] for Caputo-Hadamard fractional differential equations). We use the ideas in the above mentioned papers to define a mild solution of (18), (19).

Definition 5. *The fixed point $x \in C([0,T])$ of the fractional integral operator Ω, defined by (20) (if any), is called a mild solution of the IVP for GPFDE (18), (19).*

Theorem 1. *If condition (A) is satisfied, then any solution $x \in C^{\alpha,\rho}[0,T]$ of (18), (19) is a mild solution and vice versa.*

Proof. Let $x \in C^{\alpha,\rho}[0,T]$ be a mild solution of (18), (19), i.e.,

$$x(t) = (\Omega x)(t) = x_0 e^{\frac{\rho-1}{\rho}t} + ({}_0\mathcal{I}^{\alpha,\rho}f(.,x(.)))(t),\ t\in[0,T] \tag{21}$$

Take the fractional derivative ${}_0\mathcal{D}^{\alpha,\rho}$ on both sides of (21), use Corollaries 1 and 2, and obtain (18).

Let $x \in C^{\alpha,\rho}[0,T]$ be a solution of (18), (19). Take the fractional integral ${}_0\mathcal{I}^{\alpha,\rho}$ on both sides of (18), use Lemma 1, and obtain $x(t) = (\Omega x)(t),\ t\in[0,T]$. □

Let $\varepsilon > 0$ and consider the generalized proportional Caputo fractional differential inequality (PFDI):

$$\left|({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(t) - f(t,\nu(t))\right| \le \varepsilon,\ \ t\in[0,T]. \tag{22}$$

Note, if $\nu \in C^{\alpha,\rho}[0,T]$ is a solution of (22), then there exists a function $g \in I^{\alpha,\rho}[0,T]$: $|g(t)| \le \varepsilon,\ t\in[0,T]$, such that $\nu(t)$ is a solution of the GPFDE

$$({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(t) = f(t,\nu(t)) + g(t),\ \ t\in[0,T]. \tag{23}$$

Define the fractional integral operator $\Omega_1 : C[0,T] \to C[0,T]$ by

$$(\Omega_1\nu)(\tau) = \nu(0)e^{\frac{\rho-1}{\rho}\tau} + \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)}(\tau-s)^{\alpha-1}(f(s,\nu(s))+g(t))ds,\ \ \tau\in[0,T]. \tag{24}$$

The fixed point of the operator Ω_1 (if any) is called a mild solution of (23) with the initial value $\nu(0)$.

Similar reasoning as in Theorem 1 guarantees:

Theorem 2. *If condition (A) is satisfied, then, for solution $\nu \in C^{\alpha,\rho}[0,T]$ of PFDI (22), there exists a function $g \in I^{\alpha,\rho}[0,T]$: $|g(t)| \le \varepsilon,\ t\in[0,T]$, such that $\nu(t)$ is a mild solution of (23) and vice versa.*

Definition 6. *The IVP (18), (19) is Ulam-Hyers stable (UHS) if there exists a real number $C_f > 0$, such that, for each $\varepsilon > 0$, and for each solution $\nu \in C^{\alpha,\rho}[0,T]$ of the PFDI (22), there exists a solution $x \in C^{\alpha,\rho}[0,T]$ of (18), (19) with $|\nu(t) - x(t)| \le C_f\varepsilon$ for $t\in[0,T]$.*

Remark 7. *In Definition 6, for the solution $\nu \in C^{\alpha,\rho}[0,T]$ of the PFDI (22), we consider the solution $x \in C^{\alpha,\rho}[0,T]$ of (18), (19) with $x_0 = \nu(0)$. Then, for UHS, it is necessary to obtain* *conditions, such that* $|\nu(t) - x(t)| \leq C_f \varepsilon$ *for* $t \in [0, T]$*, where the constant* C_f *does not depend on* ε*, and it is the same for all solutions of the PFDI* (22)*.*

In the case of an initial condition (19), we introduce the following definition:

Definition 7. *The solution* $x \in C^{\alpha,\rho}[0, T]$ *of* (18), (19)*, with an initial given value* x_0*, is called* modified Ulam-Hyers stable *(MUHS) if there exists a constant* $C_f > 0$*, such that, for each* $\varepsilon > 0$*, and for each solution* $\nu \in C^{\alpha,\rho}[0, T]$*,* $\nu(0) = x_0$*, of the PFDI* (22)*, the inequality* $|\nu(t) - x(t)| \leq C_f \varepsilon$ *holds for* $t \in [0, T]$*.*

Remark 8. *If we consider Definition 7 for all initial values* x_0*, then one gives a description of Definition 6.*

3.3. Existence and Uniqueness of the Solution of Boundary Value Problem

Consider the following linear generalized proportional Caputo fractional differential equation

$$({}_0^C\mathcal{D}^{\alpha,\rho}\nu)(t) = F(t), \quad t \in (0, T], \tag{25}$$

with the boundary condition

$$a\nu(0) + b\nu(T) = \mu, \tag{26}$$

where $a, b \in \mathbb{R}: \ a + be^{\frac{\rho-1}{\rho}T} \neq 0$, $\mu \in \mathbb{R}$ is a parameter, $\alpha \in (0, 1)$ and $\rho \in (0, 1]$.

Note that, in the case $b = 0$ in (26), we obtain an initial condition.

We introduce the assumption:

(**B**). The condition $a + be^{\frac{\rho-1}{\rho}T} \neq 0$ holds with $\alpha \in (0, 1)$, $\rho \in (0, 1]$.

Lemma 5. *Let* $F \in I^{\alpha,\rho}[0, T]$ *and condition (B) be satisfied.*

Then, for any value of the parameter $\mu \in \mathbb{R}$*, the BVP* (18), (25) *has a unique solution:*

$$\begin{aligned} \nu(t) = & \frac{\mu}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}t} - \frac{b}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}t} \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^T e^{\frac{\rho-1}{\rho}(T-s)} \frac{F(s)}{(T-s)^{1-\alpha}} ds \\ & + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{F(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0, T]. \end{aligned} \tag{27}$$

The claim of Lemma 5 follows directly from Corollary 3, and we omit the proof.

We use the supremum norm in $C[0, T]$, namely, $||x|| = \sup_{t \in [0,T]} |x(t)|$, where $x \in C[0, 1]$.

Consider the fractional integral operator $\Omega : C[0, T] \to C[0, T]$ (see Lemma 5):

$$\begin{aligned} (\Omega\nu)(\tau) = & \frac{\mu}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}\tau} - \frac{b}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}\tau} \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^T e^{\frac{\rho-1}{\rho}(T-s)} \frac{f(s, \nu(s))}{(T-s)^{1-\alpha}} ds \\ & + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)} \frac{f(s, \nu(s))}{(\tau-s)^{1-\alpha}} ds, \quad \tau \in [0, T]. \end{aligned} \tag{28}$$

Definition 8. *The solution (if any)* $u \in C([0, T])$ *of the fractional integral operator* Ω*, defined by* (28)*, is called a mild solution of the BVP for GPFDE*(18), (26)*.*

Remark 9. *Both the mild solution and the solution of BVP for GPFDE* (18), (26) *depend on the parameter* μ*.*

Theorem 3. *Let condition (B) be satisfied and the function* $x \in C([0, T])$ *be a mild solution of the BVP for GPFDE* (18), (26) *for a given value of the parameter* μ*. If* $x \in C^{\alpha,\rho}[0, T]$*, then the function* $x(t)$ *is a solution of the same problem.*

Proof. From (28), it follows that $x(t)$ satisfies boundary condition (26). Take the derivative ${}_0^C\mathcal{D}^{\alpha,\rho}$ on both sides of (28), use Corollaries 1 and 2, and obtain

$$({}_0^C\mathcal{D}^{\alpha,\rho}x)(t) = ({}_0^C\mathcal{D}^{\alpha,\rho}\Omega(x))(t) = \Big({}_0^C\mathcal{D}^{\alpha,\rho}({}_0\mathcal{I}^{\alpha,\rho}f(.,x(.)))\Big)(t) = f(t,x(t)). \tag{29}$$

□

Theorem 4. *Let condition (B) be satisfied and let the function $x \in C^{\alpha,\rho}[0,T]$ be a solution of BVP for GPFDE (18), (26) for a given value of the parameter μ. If the function $F \in I^{\alpha,\rho}[0,T]$ where $F(t) = f(t,x(t))$, then the function $x(t)$ is a mild solution of the same problem.*

Proof. Take the fractional integral ${}_0^C\mathcal{I}^{\alpha,\rho}$ on both sides of (18), use Lemma 1, and obtain

$${}_0^C\mathcal{I}^{\alpha,\rho}\left({}_0^C\mathcal{D}^{\alpha,\rho}x\right)(t) = \left({}_0^C\mathcal{I}^{\alpha,\rho}f(.,x(.))\right)t. \tag{30}$$

Thus,

$$x(t) = x(0)e^{\frac{\rho-1}{\rho}t} + {}_0^C\mathcal{I}^{\alpha,\rho}f(.,x(.)))t, \tag{31}$$

apply the boundary condition (26), and we obtain $ax(0) + bx(0)e^{\frac{\rho-1}{\rho}T} + ({}_0^C\mathcal{I}^{\alpha,\rho}f(.,x(.)))(T) = \mu$ or

$$x(0) = \frac{1}{a + be^{\frac{\rho-1}{\rho}T}}\left[\mu - \left({}_0^C\mathcal{I}^{\alpha,\rho}f(.,x(.))\right)(T)\right]. \tag{32}$$

Substitute (32) in (31) and obtain $x(t) = (\Omega x)(t)$. □

Corollary 4. *If conditions (A) and (B) are satisfied, then any solution $x \in C^{\alpha,\rho}[0,T]$ of (18), (26) for a given value of the parameter μ is a mild solution and vice versa.*

We now consider the existence and uniqueness of the solution of the BVP for GPFDE (18), (26) for any value of the parameter μ.

Theorem 5. *Let the following conditions be fulfilled:*

1. *The condition (B) is satisfied.*
2. *The condition (A) is satisfied and there exists a constant $L > 0$, such that, for $t \in [0,T]$, $x_j \in \mathbb{R}$, $j = 1,2$, the inequality $|f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|$ holds.*
3. *The inequality $LT^\alpha\left(1 + \frac{|b|}{\left|a+be^{\frac{\rho-1}{\rho}T}\right|}\right) < \rho^\alpha\Gamma(1+\alpha)$ holds.*

Then, for any given value of the parameter μ, the BVP for GPFDE (18), (26) has a unique mild solution.

Proof. Let $u,v \in C([0,T])$. From condition (A), the operators $(\Omega u)(\tau)$ and $(\Omega v)(\tau)$ are well defined for any given value of the parameter μ, and for $\tau \in [0,T]$ we have:

$$\begin{aligned} |(\Omega u)(\tau) - (\Omega v)(\tau)| &\le \left|\frac{b}{a+be^{\frac{\rho-1}{\rho}T}}\right| e^{\frac{\rho-1}{\rho}\tau}\frac{L}{\rho^\alpha\Gamma(\alpha)}\int_0^T e^{\frac{\rho-1}{\rho}(T-s)}\frac{|u(s)-v(s)|}{(T-s)^{1-\alpha}}ds \\ &+ \frac{L}{\rho^\alpha\Gamma(\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)}\frac{|u(s)-v(s)|}{(\tau-s)^{1-\alpha}}ds \\ &\le \frac{L}{\rho^\alpha\Gamma(\alpha)}\frac{T^\alpha}{\alpha}\left(1 + \left|\frac{b}{a+be^{\frac{\rho-1}{\rho}T}}\right|\right)||u-v||. \end{aligned} \tag{33}$$

From inequality (33) and condition 3, it follows that the operator Ω is a contraction. □

Remark 10. *According to Corollary 4, if the conditions of Theorem 5 are satisfied, then the BVP for GPFDE (18), (26) for a given value of the parameter μ has a unique solution $x(t)$, such that $x(t) = (\Omega x)(t), t \in [0,T]$, where Ω is defined by (28).*

For an arbitrary $\varepsilon > 0$, we consider PFDI (22). As above, if $\nu \in C^{\alpha,\rho}[0,T]$ is a solution of (22), there exists a function $g \in I^{\alpha,\rho}[0,T] : |g(t)| \leq \varepsilon, \ t \in [0,T]$, such that $\nu(t)$ is a solution of the GPFDE (23). Then, as above, we have the following results:

Lemma 6. *Let conditions (A) and (B) be satisfied, and the function $v \in C^{\alpha,\rho}[0,T]$ is a solution of the PFDI (22). Then, there exists a function $g \in I^{\alpha,\rho}[0,T] : |g(t)| \leq \varepsilon$ for $t \in [0,T]$ such that v is a fixed point of the fractional integral operator Ω_1, where*

$$\begin{aligned}(\Omega_1 v)(t) =& \frac{av(0) + bv(T)}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}t} \\ &- \frac{b}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}t} \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^T e^{\frac{\rho-1}{\rho}(T-s)} \frac{f(s,v(s)) + g(s)}{(T-s)^{1-\alpha}} ds \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{f(s,v(s)) + g(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0,T].\end{aligned} \tag{34}$$

Remark 11. *Note that, if we modify operator Ω_1 by*

$$\begin{aligned}(\bar{\Omega}_1 v)(t) =& \frac{\mu}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}t} \\ &- \frac{b}{a + be^{\frac{\rho-1}{\rho}T}} e^{\frac{\rho-1}{\rho}t} \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^T e^{\frac{\rho-1}{\rho}(T-s)} \frac{f(s,v(s)) + g(s)}{(T-s)^{1-\alpha}} ds \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{f(s,v(s)) + g(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0,T],\end{aligned} \tag{35}$$

then the solution ν of the PFDI (22) is a fixed point of the modified $\bar{\Omega}_1$ iff $\mu = a\nu(0) + b\nu(T)$.

3.4. Ulam Type Stability for Boundary Value Problems

Consider the boundary condition (26). This condition depends on μ, which could be a given value, or it could be a parameter. According to Remark 10, the BVP (18), (26) has a solution $x(t)$ in both cases.

Following the ideas of Definition 6, we define UHS by:

Definition 9. *The BVP (18), (26) is* Ulam-Hyers stable *(UHS) if there exists a constant $K > 0$, such that, for any number $\varepsilon > 0$ and any solution $\nu \in C^{\alpha,\rho}[0,T]$ of the PFDI (22), there exists a solution $x \in C^{\alpha,\rho}[0,T]$ of BVP (18), (26), such that $|x(\tau) - \nu(\tau)| \leq K\varepsilon$ for $\tau \in [0,T]$.*

Remark 12. *In Definition 9, for the solution $\nu \in C^{\alpha,\rho}[0,T]$ of PFDI (22), we consider the solution $x \in C^{\alpha,\rho}[0,T]$ of (18), (26) with $\mu = a\nu(0) + b\nu(T)$. Then, for UHS, it is necessary to obtain conditions, such that $|\nu(\tau) - x(\tau)| \leq K\varepsilon$ for $\tau \in [0,T]$, where the constant K does not depend on ε, and it is the same for all solutions of the PFDI (22).*

In the case of an initial given value of $\mu \in \mathbb{R}$ in the boundary condition (26), following the ideas of Definition 7, we could introduce the following definition.

Definition 10. *The solution $x \in C^{\alpha,\rho}[0,T]$ of BVP (18), (26) with an initial given value of μ is* modified Ulam-Hyers stable *(MUHS) if there exists a constant $K > 0$, such that, for any number $\varepsilon > 0$ and any solution $\nu \in C^{\alpha,\rho}[0,T], \ a\nu(0) + b\nu(T) = \mu$, of the PFDI (22), the inequality $|x(\tau) - \nu(\tau)| \leq K\varepsilon$ holds for $\tau \in [0,T]$.*

Theorem 6. (UHS). *Let the following conditions be satisfied:*

1. *The conditions of Theorem 5 are satisfied.*
2. *For any $\varepsilon > 0$, the inequality (22) has at least one solution from $C^{\alpha,\rho}[0,T]$.*

Then, the BVP for GPFDE (18), (26) is Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$ be an arbitrary number and let the function $\nu \in C^{\alpha,\rho}[0,T]$ be a solution of inequality (22). According to Lemma 6, there exists a function $g \in I^{\alpha,\rho}[0,T]$, such that the function ν is a fixed point of the operator Ω_1 defined by (35).

Let $\mu = a\nu(0) + b\nu(T)$ in the boundary conditions (26). According to Theorem 5 and Remark 10, the BVP for GPFDE (18), (26) has a solution $x \in C^{\alpha,\rho}[0,T]$ for this particular value of the parameter μ, and it is a fixed point of the operator Ω, defined by (28).

Let $\tau \in [0,T]$ be fixed. From condition 2 of Theorem 5, we obtain:

$$\begin{aligned}
&|u(\tau) - x(\tau)| = |(\Omega_1 u)(\tau) - (\Omega x)(\tau)| \\
&\leq \frac{|b|}{\left|a + be^{\frac{\rho-1}{\rho}T}\right|} e^{\frac{\rho-1}{\rho}t} \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^T e^{\frac{\rho-1}{\rho}(T-s)} \frac{L|u(s)-x(s)|+C}{(T-s)^{1-\alpha}} ds \\
&\quad + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-s)} \frac{L|u(s)-x(s)|+C}{(\tau-s)^{1-\alpha}} ds \\
&\leq ||u-x||L\left(1 + \frac{|b|}{\left|a + be^{\frac{\rho-1}{\rho}T}\right|}\right) \frac{1}{\rho^\alpha \Gamma(\alpha)} \frac{T^\alpha}{\alpha} + C\left(1 + \frac{|b|}{\left|a + be^{\frac{\rho-1}{\rho}T}\right|}\right) \frac{1}{\rho^\alpha \Gamma(\alpha)} \frac{T^\alpha}{\alpha},
\end{aligned} \tag{36}$$

and therefore

$$||x-u|| \leq \mathcal{G}\epsilon + \mathcal{G}L||x-u||,$$

where $\mathcal{G} = \left(1 + \frac{|b|}{\left|a+be^{\frac{\rho-1}{\rho}T}\right|}\right) \frac{T^\alpha}{\rho^\alpha \Gamma(1+\alpha)}$. According to condition 3 of Theorem 5, the inequality $L\mathcal{G} < 1$ holds.

According to Definition 9, the BVP (18), (26) is UHS with $K = \frac{\mathcal{G}}{1-L\mathcal{G}}$. □

Theorem 7. (MUHS). *Let the conditions be satisfied:*

1. *The condition (B) is satisfied, and the value of μ is given.*
2. *The function $x \in C^{\alpha,\rho}[0,T]$ is a solution of the BVP for GPFDE (18), (26) with the given μ.*
3. *The function $f \in C([0,T] \times \mathbb{R})$ is such that $f(.,x(.)) \in I^{\alpha,\rho}[0,T]$, where $x(t)$ is the solution from condition 2, and there exists a constant $L > 0$, such that, for $t \in [0,T]$, $x_j \in \mathbb{R}$, $j = 1,2$, the inequality $|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|$ holds.*
4. *The inequality $LT^\alpha \left(1 + \frac{|b|}{\left|a+be^{\frac{\rho-1}{\rho}T}\right|}\right) < \rho^\alpha \Gamma(1+\alpha)$ holds.*
5. *For any $\varepsilon > 0$, the inequality (22) has at least one solution from $C^{\alpha,\rho}[0,T]$.*

   Then, the solution $x(t)$ of the BVP for GPFDE (18), (26) is modified Ulam-Hyers stable.

The proof of Theorem 7 is similar to the one in Theorem 6, so we omit it.

3.5. Ulam Type Stability for BVP for Caputo Fractional Differential Equations

Since the Caputo fractional derivative is a special case of the generalized proportional Caputo fractional derivative ($\rho = 1$), we will give only the sufficient condition for UHS and MUHS for the Caputo fractional differential equation (CFDE)

$$ {}_0^C D^\alpha x(t) = f(t, x(t)), \quad t \in (0, T], \quad \alpha \in (0, 1). \tag{37} $$

with a boundary condition (26).

Let $\varepsilon > 0$. We consider the following Caputo fractional differential inequalities:

$$ \left| ({}_0^C D^\alpha u)(t) - f(t, u(t)) \right| \le \varepsilon, \quad t \in [0, T]. \tag{38} $$

Theorem 8. *(UHS). Let the following conditions be satisfied:*

1. *The condition (A) is satisfied with $\rho = 1$, and there exists a constant $L > 0$, such that, for $t \in [0, T], x_j \in \mathbb{R}, \ j = 1, 2$, the inequality $|f(t, x_1) - f(t, x_2)| \le L|x_1 - x_2|$ holds.*
2. *The inequalities $a + b \neq 0$ and $LT^\alpha \left(1 + \frac{|b|}{|a+b|}\right) < \Gamma(1 + \alpha)$ hold.*
3. *For any $\varepsilon > 0$, the inequality (38) has at least one solution.*

Then, the BVP for CFDE (37), (26) is Ulam-Hyers stable.

Theorem 9. *(MUHS). Let the conditions be satisfied:*

1. *The inequality $a + b \neq 0$ holds, and $\alpha \in (0, 1)$ and μ are given.*
2. *The function $x(t), \ t \in [0, T]$, is a solution of the BVP for GPFDE (37), (26) with the given μ.*
3. *The function $f \in C([0, T] \times \mathbb{R})$ is such that $f(., x(.)) \in I^{\alpha,1}[0, T]$, where $x(t)$ is the solution from condition 2, and there exists a constant $L > 0$, such that, for $t \in [0, T], x_j \in \mathbb{R}, \ j = 1, 2,$, the inequality $|f(t, x_1) - f(t, x_2)| \le L|x_1 - x_2|$ holds.*
4. *The inequality $LT^\alpha \left(1 + \frac{|b|}{|a+b|}\right) < \Gamma(1 + \alpha)$ holds.*
5. *For any $\varepsilon > 0$, the inequality (38) has at least one solution.*

   Then, the solution $x(t)$ of the BVP for GPFDE (37), (26) is modified Ulam-Hyers stable.

3.6. Examples

Since Caputo fractional differential equations are studied by many authors, we will start with a simple example with the Caputo fractional derivative ($\rho = 1$).

Example 2. *Consider the linear Caputo fractional differential equation ($\alpha \in (0, 1)$)*

$$ {}_0^C D^\alpha x(t) = -0.5x(t), \quad t \in (0, 1]. \tag{39} $$

Let $\varepsilon > 0$, and consider the fractional differential inequality

$$ \left| {}_0^C D^\alpha \nu(\tau) + 0.5\nu(\tau) \right| \le \varepsilon, \quad \tau \in (0, 1]. \tag{40} $$

The functions $\nu(t) = \varepsilon E_\alpha(-0.5t^\alpha)$ and $\nu_1(t) = \varepsilon$ are solutions of (40) for a constant ε.
Case 1. *(IVP). Consider the initial condition*

$$ x(0) = x_0. \tag{41} $$

Case 1.1. *We will look at Definition 6 with $\rho = 1$.*

The IVP (39), (41) has a unique solution $x(t) = x_0 E_\alpha(-0.5t^\alpha)$, where $E_\alpha(.)$ is the Mittag-Leffler function with one parameter (see Lemma 4 with $\rho = 1, \lambda = -0.5, \ F(t) \equiv 0$).

Consider the solution $\nu(t) = \varepsilon E_\alpha(-0.5t^\alpha)$ of the fractional inequality (40). Take $x_0 = \nu(0) = \varepsilon$, then the inequality $|x(t) - \nu(t)| = 0 \le \varepsilon, \ t \in [0, 1]$, holds.

Consider the solution $\nu_1(t) = \varepsilon$ of (40). Take $x_0 = \nu_1(0) = \varepsilon$. Then, the inequality $|x(t) - \nu_1(t)| = \varepsilon |E_\alpha(-0.5t^\alpha) - 1| \le \varepsilon, \ t \in [0, 1]$, holds.

Case 1.2. *We will look at Definition 7 with* $\rho = 1$.

Let the value of x_0 *be given, for example,* $x_0 = 0$. *The IVP (39), (41) has a unique solution* $x(t) = 0$.

For an arbitrary $\varepsilon > 0$, *we choose a solution of (40), which satisfies the initial condition (41) with* $x_0 = 0$, *for example* $\nu(t) = \varepsilon t^2$. *It satisfies (40) because* ${}_0^C D^\alpha t^2 = \frac{2t^{2-\alpha}}{\Gamma(1-\alpha)(2-3\alpha+\alpha^2)}$ *and also note* $|\nu(t) - x(t)| = |\varepsilon t^2 - 0| \leq \varepsilon, t \in [0,1]$.

Summarizing, if we have a particular initial value, then we consider a particular solution of the corresponding IVP, and we study MUHS. If we have an arbitrary initial value, then we study UHS.

Case 2. *We will consider the linear Caputo fractional differential equation* ($\alpha \in (0,1)$) *(39) with a boundary condition.*

Case 2.1. *We will look at Definition 9 with* $\rho = 1$.

Consider the boundary condition

$$\nu(0) + \frac{\nu(1)}{E_\alpha(-0.5)} = \mu \tag{42}$$

where μ *is an arbitrary constant.*

The BVP (39), (42) has a unique solution $x(t) = 0.5\mu E_\alpha(-0.5t^\alpha)$ *for any value of the parameter* μ.

In this case, $L = 0.5$, $T = 1$, $a = 1$, $b = \frac{1}{E_\alpha(-0.5)}$ *and the inequality* $LT^\alpha\left(1 + \frac{|b|}{|a+b|}\right) = 0.5(1 + \frac{1}{1+E_\alpha(0.51)}) < \Gamma(1+\alpha)$ *is satisfied for all* $\alpha \in (0, 10)$. *Therefore, the conditions of Theorem 8 are satisfied, and the BVP for CFDE (39), (42) is Ulam-Hyers stable.*

In this case, the operator Ω *defined by (28) with* $\rho = 1$ *is*

$$\begin{aligned}(\Omega\nu)(\tau) = &\frac{\mu E_\alpha(-0.5)}{E_\alpha(-0.5)+1} + \frac{1}{E_\alpha(-0.5)+1}\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{\nu(s)}{(1-s)^{1-\alpha}}ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{\nu(s)}{(\tau-s)^{1-\alpha}}ds, \quad \tau \in [0,1].\end{aligned} \tag{43}$$

Consider the solutions $\nu(t) = \varepsilon E_\alpha(-0.5t^\alpha)$ *and* $\nu_1(t) = \varepsilon$ *of the fractional Equation (40).*

The operator Ω_1 *is defined by (35), and it reduces to:*

$$\begin{aligned}(\Omega_1\nu)(\tau) = &\frac{\mu E_\alpha(-0.5)}{E_\alpha(-0.5)+1} + \frac{1}{E_\alpha(-0.5)+1}\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{\nu(s)-\varepsilon}{(1-s)^{1-\alpha}}ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{\nu(s)-\varepsilon}{(\tau-s)^{1-\alpha}}ds, \quad \tau \in [0,1].\end{aligned} \tag{44}$$

If $\mu = \nu(0) + \frac{1}{E_\alpha(-0.5)}\nu(1) = 2\varepsilon$, *then* $\nu(t)$ *is a fixed point of the operator* Ω_1, *and* $|x(t) - \nu(t)| = |0.5(2\varepsilon)E_\alpha(-0.5t^\alpha) - \varepsilon E_\alpha(-0.5t^\alpha)| = 0 < \varepsilon$ *for* $t \in [0,1]$.

If $\mu = \nu_1(0) + \frac{1}{E_\alpha(-0.5)}\nu_1(1) = \varepsilon(1 + \frac{1}{E_\alpha(-0.5)})$, *then* $\nu_1(t)$ *is a fixed point of operator* Ω_1 *and* $|x(t) - \nu(t)| = \varepsilon|0.5(1 + \frac{1}{E_\alpha(-0.5)})E_\alpha(-0.5t^\alpha) - 1| < \varepsilon$ *for* $t \in [0,1]$ *(note* $\alpha \in (0,1)$*).*

Case 2.2. *We will look at Definition 10 with* $\rho = 1$.

Consider the boundary condition without any parameter, i.e.,

$$\nu(0) = \frac{1}{E_\alpha(-0.5)}\nu(1). \tag{45}$$

The BVP (39), (45) has a unique solution $x(t) = E_\alpha(-0.5t^\alpha)$.

In this case, $L = 0.5, T = 1, b = -\frac{1}{E_\alpha(-0.5)}, a = 1$, *and* $LT^\alpha\left(1 + \frac{|b|}{|a+b|}\right) = 0.5(1 + \frac{1}{1-E_\alpha(-0.5)}) \geq \Gamma(1+\alpha)$ *for* $\alpha \in (0,1)$. *Then, condition 3 of Theorem 9 is not satisfied, and we cannot conclude that the solution* $x(t) = E_\alpha(-0.5t^\alpha)$ *of BVP for CFDE (39), (45) is modified Ulam-Hyers stable.*

Note that, in this case for MUHS, we need to take only solutions of PFDI (40), which satisfy the boundary condition (45). For example, the solution $u(t) = \varepsilon$ of (40) could not be applied because it does not satisfy the boundary condition (45). The solution $\nu(t) = \varepsilon E_\alpha(-0.5t^\alpha)$ of (40) satisfies the boundary condition (45), and the inequality $|x(t) - \nu(t)| = |1 - \varepsilon|E_\alpha(-0.5t^\alpha) \leq C\varepsilon$ is not satisfied for ε close to 0.

Summarizing, the type of boundary condition is very important for Ulam-type stability. If it has a parameter, one might be able to select this parameter, so that PFDI (40) is satisfied, and the BVP has some sort of Ulam-type stability.

In connection with the above example and the above discussion to study UHS of any type of differential equations, we need a boundary condition with a parameter.

In the case when the initially given boundary value problem does not have a parameter, we could only study MUHS of a particular solution. In this case, we have to use only solutions of the corresponding differential inequality, which are satisfying the given boundary condition (see, for example, the given BVPs without any parameter, the definitions of Ulam type stability and the study of the stability of the unique solution in [31] for Riemann-Liouville FDEs, Lemma 4.1 in [32] for nonlinear coupled systems of Riemann-Liouville fractional differential equations, for implicit Caputo FDEs, and [33,34] for Caputo FDEs).

Example 3. *Consider the following GPFDE:*

$$ {}_0^C\mathcal{D}^{0.3,0.7} y(t) = 0.6\frac{t}{t+1}e^{-|y(t)|}, \quad t \in [0,1] \tag{46}$$

with boundary condition

$$3y(0) + 2y(1) = \mu. \tag{47}$$

In this case, $f(t,y) = 0.6\frac{t}{t+1}e^{-|y|}$ and $L = 0.6$, $\alpha = 0.3, \rho = 0.7$, $T = 1$, $a = 3$, $b = 2$.

Additionally,

$$\mathcal{G} = \left(1 + \frac{|b|}{\left|a + be^{\frac{\rho-1}{\rho}T}\right|}\right)\frac{T^\alpha}{\rho^\alpha\Gamma(1+\alpha)} = \frac{1}{0.7^{0.3}\Gamma(1.3)}\Big(1 + \frac{2}{3+2e^{\frac{-0.3}{0.7}}}\Big)\Big) = 1.81648, \tag{48}$$

and $L\mathcal{G} = 0.908239 < 1$, $C = \frac{1.63215}{0.020711} = 19.7958$.

According to Theorem 6, the BVP for GPFDE (46), (47) is Ulam-Hyers stable.

For example, let $\varepsilon = 0.35$. Consider the function $\nu(s) = e^{-\frac{3}{7}s}s$, $s \in [0,1]$. According to Lemma 3 with $\rho = 0.7$, $\beta = 2$, we have $({}_0\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t)(\tau) = \frac{0.7^{0.3}\Gamma(2)}{\Gamma(2.3)})e^{\frac{-0.3}{0.7}\tau}\tau^{1-0.3}$ and

$$\left|({}_0\mathcal{D}^{\alpha,\rho}e^{-\frac{3}{7}(.)}(.))(t) - 0.6\frac{t}{t+1}e^{-e^{\frac{\rho-1}{\rho}t}t}\right| < 0.35, \quad t \in [0,1].$$

Consider the solution $y(t)$ of (46), (47) with $\mu = 3\nu(0) + 2\nu(1) = 0 + 2e^{\frac{-3}{7}} = 1.30288$. Then, the inequality $\max_{t\in[0,1]}|x(t) - e^{\frac{-3}{7}t}t| \leq 0.35C = 6.92853$ holds.

Remark 13. *The above definition and ideas for Ulam-Hyers stability could be easily used as a basis for definitions of other types of Ulam stability, such as Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability, and for other types of boundary conditions for differential equations with different types of derivatives.*

4. Conclusions

A linear boundary value problem for a generalized proportional Caputo fractional differential equation is considered. The boundary condition depends on a parameter. The existence and uniqueness of the solution depending on this parameter are discussed, and the Ulam-Hyers stability is defined and discussed. Several examples are given to illustrate the theory. The obtained results and the ideas in this paper could be applied to study other types of Ulam stability for various types of fractional differential equations with boundary conditions, i.e., the proposed ideas could be applied to study Ulam-type stability for various types of boundary value problems for other types of fractional equations, such as fractional differential equations of order $\alpha \in (1, 2)$, integro-differential fractional equations, and partial differential equations with time fractional derivatives.

Author Contributions: Conceptualization, R.P.A., S.H. and D.O.; methodology, R.P.A., S.H. and D.O.; validation, R.P.A., S.H. and D.O.; formal analysis, R.P.A., S.H. and D.O.; investigation, R.P.A., S.H. and D.O.; writing—original draft preparation, R.P.A., S.H. and D.O.; writing—review and editing, R.P.A., S.H. and D.O. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by the Bulgarian National Science Fund under Project KP-06-PN62.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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