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A Remark on Weak Tracial Approximation

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Abstract: In this paper, we extend the notion of generalized tracial approximation to the non-unital case. An example of this approximation, which comes from dynamical systems, is also provided. We use the machinery of Cuntz subequivalence to work in this non-unital setting.

Keywords: tracial approximation; large subalgebras; C*-algebras; functional analysis

MSC: 46L05; 46L35; 46L55

1. Introduction

Before Elliott and Niu in [1] introduced the notion of tracial approximation by abstract classes of unital C^*-algebras, tracially approximately finite-dimensional (TAF) algebras and tracially approximately interval (TAI) algebras were considered by Lin [2,3]. The idea here is that if A can be well-approximated by well-behaved algebras in trace, then we can expect that A is well-behaved too. Many properties can be inherited from the given class to tracially approximated C^*-algebras. In [1], finiteness (stably finiteness), being stable rank one, having nonempty tracial state space, the property that the strict order on projections is determined by traces, and the property that any state on the K_0 group comes from a tracial state are considered. Elliott, Fan, and Fang in [4] investigated the inheritance of several comparison properties and divisibility. Elliott and Fan [5] also proved that tracial approximation is stable for some classes of unital C^*-algebras. Fu and Lin developed the notion of asymptotical tracial approximation in [6], trying to find a tracial version of Toms–Winter conjecture. Approximation is widely used in other mathematical areas. The reader is referred to [7,8] for new trend in fractional calculus and functional equations.

Putnam introduced the orbit-breaking subalgebra of the crossed product of the Cantor set by a minimal homeomorphism in [9]. His technique is widely used and has many applications; for examples, see [10–13]. Phillips introduced large subalgebras in [14], which is an abstraction of Putnam's orbit-breaking subalgebra. In [14], Phillips studied properties such as stably finite, pure infinite, the restriction map between trace state spaces, and the relationship between their radius comparison. Later, Archey and Phillips in [15] introduced centrally large subalgebras and proved that the property of stable rank one can be inherited. In [16], Archey, Buck, and Phillips studied the property of tracial $\mathcal{Z}$-absorbing when A is stably finite. Fan, Fang, and Zhao investigated several comparison properties in [17], and they also studied the inheritance of almost divisibility [18] and the inheritance of the tracial nuclear dimension for centrally large subalgebras [19].

Following Lin's tracial approximation and Phillips' large subalgebras, Elliott, Fan, and Fang introduced the notion of weakly tracially approximated C*-algebras, which generalizes both tracial approximation and centrally large subalgebras. Tracially $\mathcal{Z}$-absorbing, tracial nuclear dimension, property (SP), and m-almost divisibility have been studied in their work [20] thus far.

In this paper, we point out that the definition of weak tracial approximation can be reformulated to adapt to non-unital cases. The symmetry of C*-algebras allows us to use the machinery of Cuntz subequivalence to work in this non-unital setting. A known example of weak tracial approximation is proved in Section 3.

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Citation: Fang, X.; Yang, J. A Remark on Weak Tracial Approximation. *Axioms* **2023**, *12*, 242. https://doi.org/10.3390/axioms12030242

Academic Editors: Silvestru Sever Dragomir, Emanuel Guariglia and Gheorghita Zbaganu

Received: 4 January 2023
Revised: 9 February 2023
Accepted: 21 February 2023
Published: 27 February 2023

2. Preliminaries

Let A be a C*-algebra. $a, b \in A_+$. We say that a is Cuntz subequivalent to b, denoted by $a \precsim_A b$, if there exists a sequence $(v_n) \subset A$ such that $\|a - v_n b v_n^*\| \to 0$. We say that a is Cuntz equivalent to b if $a \precsim_A b$ and $b \precsim_A a$ (written as $a \sim_A b$).

Let $\mathrm{M}_\infty(A)$ denote the algebraic direct limit of the system $(\mathrm{M}_n(A))_{n=1}^\infty$ with upper-left embedding. The Cuntz subequivalence can be defined similarly for the positive elements in $\mathrm{M}_\infty(A)$; see [14] (Rek. 1.2).

Sometimes we write $a \approx_\epsilon b$ to represent $\|a - b\| < \epsilon$.

The following lemma about Cuntz subequivalence is well known and frequently used. We list them below for the reader's convenience.

Lemma 1. *Let A be a C*-algebra.*

(1) Let $a, b \in A_+$. If $a \leq b$, then $a \precsim_A b$.
(2) Let $a, b \in A_+$. If $a \in \overline{bAb}$, then $a \precsim_A b$.
(3) Let $a, b \in A_+$. If a, b are mutually orthogonal, then $a \oplus b \sim_A a + b$.
(4) Let $x \in A$ be a nonzero element, and $b \in A_+$. Then, for any $\epsilon > 0$,

$$(xbx^* - \epsilon)_+ \precsim_A x(b - \epsilon/\|x\|^2)x^*.$$

In particular, if $\|x\| \leq 1$, then $(xbx^* - \epsilon)_+ \precsim_A x(b-\epsilon)_+ x^* \precsim_A (b - \epsilon)_+$.

Proof. Part (1) is [21] (Lem. 2.3(b)). Part (2) is [22] (Lem. 2.7(i)). Part (3) is [22] (Lem. 2.8 (ii)). Part (4) is from [23] (Lem. 2.3). □

The following lemma is also well known. See [14] (Lem. 2.4) or [24] (Lem. 4.7) for an elementary proof.

Lemma 2. *Let A be a simple non-elementary C*-algebra. Then, for any $n \in \mathbb{N}$, there are the nonzero positive elements $e_1, \cdots, e_n \in A$ which are mutually orthogonal and $e_1 \sim_A \cdots \sim_A e_n$.*

Elliott, Fan, and Fang introduced a class of generalized tracial approximation C^*-algebras in [20] (Def. 1.1). They defined as follows the class of C^*-algebras that can be weakly tracially approximated by C^*-algebras in Ω and denote this class by WTAΩ.

Definition 1 ([20])**.** *Let Ω be a class of unital C^*-algebras. A simple unital C^*-algebra A is said to belong to the class* WTAΩ *if, for any $\epsilon > 0$, any finite subset $F \subset A$ and any nonzero element $a \in A_+$, there exists a C^*-subalgebra $B \subset A$ with $B \in \Omega$, a projection $p \in A$ with $p = 1_B$, and an element $g \in B_+$ with $0 \leq g \leq 1$, such that*

(1) $(p-g)x \in_\epsilon B$, $x(p-g) \in_\epsilon B$ for all $x \in F$;
(2) $\|(p-g)x - x(p-g)\| < \epsilon$ for all $x \in F$;
(3) $1_A - (p-g) \precsim_A a$;
(4) $\|(p-g)a(p-g)\| \geq \|a\| - \epsilon$.

Note that $p - g$ appears as a whole in the definition above. We use a new variable $r = p - g$ and a different notation GTAΩ to represent a possible non-unital Ω or a non-unital A. The idea of condition (3′) below comes from [23] (Def. 1.1).

Definition 2. *Let Ω be a class of C^*-algebras. A simple C^*-algebra A is said to belong to the class* GTAΩ *if, for any $\epsilon > 0$, any finite subset $F \subset A$, any positive elements $a \in A_+$ with $\|a\| = 1$, and any $b \in A_+$ with $\|b\| \leq 1$, there exist a C^*-subalgebra $B \subset A$ with $B \in \Omega$ and an element $r \in B_+$ with $\|r\| \leq 1$ such that*

(1′) $rx \in_\epsilon B$, $xr \in_\epsilon B$ for all $x \in F$;
(2′) $\|rx - xr\| < \epsilon$ for all $x \in F$;
(3′) $(b^2 - brb - \epsilon)_+ \precsim_A a$;
(4′) $\|rar\| \geq 1 - \epsilon$.

Proposition 1. *If A and all algebras in Ω are unital, then $A \in \mathrm{WTA}\Omega$ is equivalent to $A \in \mathrm{GTA}\Omega$.*

Proof. If $A \in \mathrm{WTA}\Omega$, apply Definition 1 with (ϵ, F, a) for any given (ϵ, F, a, b), obtaining the corresponding B, p, and g. Then, $r = p - g$ satisfies

$$(b^2 - brb - \epsilon)_+ \precsim_A b(1 - r - \epsilon/\|b\|^2)_+ b \precsim_A (1 - r - \epsilon/\|b\|^2)_+ \precsim_A 1 - r \precsim_A a\,,$$

where the first $\precsim_A$ is from Lemma 1(4), the second $\precsim_A$ is by the definition of Cuntz subequivalence, the third $\precsim_A$ is from Lemma 1(1).

Now, suppose that $A \in \mathrm{GTA}\Omega$. For any given (ϵ, F, a), set $\epsilon_1 = \epsilon/m$, where $m = 4M$ and $M = \max\{\|x\| : x \in F\} \vee 1$. Let

$$f(t) = \begin{cases} 0, & t \le 0; \\ \text{linear}, & 0 \le t \le 1 - \epsilon_1; \\ 1, & t \ge 1 - \epsilon_1\,. \end{cases}$$

Then, $|f(t) - t| \le \epsilon_1$ for $t \in [0, 1]$. By [25] (Lem. 2.5.11(1)), find $\delta > 0$ such that if $\|hc - ch\| < \delta$ for any C^*-algebra C with $c, h \in C$, $\|c\| \le M$, and $0 \le h \le 1$, then $\|f(h)c - cf(h)\| < \epsilon$.

Applying Definition 2 with $\epsilon' = \epsilon_1 \wedge \delta$, $b = 1_A$ and the same F and a, we obtain a C^*-subalgebra $B \subset A$ in Ω and $r \in B$ with $0 \le r \le 1$ such that $rx \in_{\epsilon_1} B$ for all $x \in F$. By combining with $rx \approx_{\epsilon_1} f(r)x$, we have $f(r)x \in_\epsilon B$ and, similarly, $xf(r) \in_\epsilon B$ for all $x \in F$. Note that $\|f(r)x - xf(r)\| < \epsilon$ is also true by our choice of δ and $\epsilon' \le \delta$.

We next prove condition (3) with $p = 1_B$ and $g = p - f(r)$. Using Lemma 1(1) at the third step and condition (3′) at the last step, we have

$$1 - f(r) = \frac{1}{1 - \epsilon_1}(1 - r - \epsilon_1)_+ \sim_A (1 - r - \epsilon_1)_+ \precsim_A (1 - r - \epsilon')_+ \precsim_A a.$$

Finally, $f(r)af(r) \approx_{\epsilon_1} raf(r) \approx_{2\epsilon_1} rar$. Therefore, we have $\|f(r)af(r)\| \ge \|rar\| - 2\epsilon_1 \ge 1 - 3\epsilon_1 > 1 - \epsilon$. □

Remark 1. *From the proof above, we know that if A is unital, then condition (3′) in Definition 2 can be replaced by $1_A - r \precsim_A a$.*

The following two lemmas can be seen as a positive element analog of [26] (Prop. 9.5) which modify condition (4′) of Definition 2. The first lemma allows that condition (4′) can be replaced by another positive element of norm one.

Lemma 3. *Let Ω be a class of C^*-algebras. Let A be a simple C^*-algebra that belongs to the class $\mathrm{GTA}\Omega$. Then, for any $\epsilon > 0$, any finite subset $F \subset A$, and any positive elements $a, b, c \in A$ with $\|a\| = \|c\| = 1$ and $\|b\| \le 1$, there exist a C^*-algebra $B \subset A$ with $B \in \Omega$ and an element $r \in B_+$ with $\|r\| \le 1$ such that*

(1) $rx \in_\epsilon B$, $xr \in_\epsilon B$ for all $x \in F$;
(2) $\|rx - xr\| < \epsilon$ for all $x \in F$;
(3) $(b^2 - brb - \epsilon)_+ \precsim_A a$;
(4) $\|rcr\| \ge 1 - \epsilon$.

Proof. A similar method and technique from [23] (Lem. 3.9). □

When A is finite, condition (4′) is implied by condition (3′) in Definition 2.

Lemma 4. *Let Ω be a class of C^*-algebras. Let A be a finite-/infinite-dimensional simple unital C^*-algebra. Suppose that, for any $\epsilon > 0$, any finite set $F \subset A$, and any nonzero element $a \in A_+$, there exist a C^*-subalgebra $B \subset A$ with $B \in \Omega$ and an element $r \in B_+$ with $\|r\| \leq 1$ such that*

(1) $rx \in_\epsilon B$, $xr \in_\epsilon B$ *for all* $x \in F$;
(2) $\|rx - xr\| < \epsilon$ *for all* $x \in F$;
(3) $1_A - r \precsim_A a$.

Then, A belongs to the class GTAΩ.

Proof. Since A is finite and unital, apply [14] (Lem. 2.9) to the given A, a, and ϵ, obtaining a nonzero positive element $y \in \overline{aAa}$ such that whenever $g \precsim_A y$ for $g \in A$ with $0 \leq g \leq 1$, then $\|(1_A - g)a(1_A - g)\| > 1 - \epsilon$.

Apply the hypothesis with y in place of a and everything else as given, obtaining C^*-subalgebra $B \subset A$ and $r \in B_+$ with $\|r\| \leq 1$. We have $1_A - r \precsim_A y \precsim_A a$ and $\|rar\| = \|(1_A - (1_A - r))a(1_A - (1_A - r))\| > 1 - \epsilon$ from the choice of y. □

The following proposition slightly strengthens condition (1′) in Definition 2. It is similar to [14] (Prop. 4.4), but we could not follow the proof there. We give another proof for the sake of completeness.

Proposition 2. *Let Ω be a class of C^*-algebras and A be a unital C^*-algebra in* GTAΩ. *Then, for any $m \in \mathbb{Z}_{>0}$, any $x_1, \cdots, x_m \in A$, and any positive element $a \in A_+$ with $\|a\| = 1$, there exists $y_1, \cdots, y_m \in A$, a C^*-subalgebra $B \subset A$ with $B \in \Omega$, and an element $r \in B_+$ with $\|r\| \leq 1$, such that*

(1) $\|y_j - x_j\| < \epsilon$ *for all* $j = 1, \cdots, m$;
(2) $ry_j \in B$, $y_jr \in B$ *for all* $j = 1, \cdots, m$;
(3) $\|rx_j - x_jr\| < \epsilon$ *for all* $j = 1, \cdots, m$;
(4) $1_A - r \precsim_A a$;
(5) $\|rar\| \geq 1 - \epsilon$.

Proof. Let $1 > \epsilon > 0$. Let

$$f'(t) = \begin{cases} 0, & t \leq \frac{\epsilon}{3}; \\ \text{linear}, & \frac{\epsilon}{3} \leq t \leq 1 - \frac{\epsilon}{3}; \\ 1, & t \geq 1 - \frac{\epsilon}{3}. \end{cases}$$

Therefore, $f' \in C_0(0,1]$, $f'(t) \approx_{\epsilon/3} t$, and $1 - f'(t) = f'(1-t)$ for $0 \leq t \leq 1$. Let $f_1(t)$ be the unit of $f'(t)$ such that $f_1 f' = f' f_1 = f'$. For example, one may take

$$f_1(t) = \begin{cases} 0, & t \leq \frac{\epsilon}{6} \\ \text{linear}, & \frac{\epsilon}{6} \leq t \leq \frac{\epsilon}{3} \\ 1, & t \geq \frac{\epsilon}{3}. \end{cases}$$

Let $h \in C_0(0,1]$ such that $th(t) = f_1(t)$. Applying [25] (Lem. 2.5.11(1)) to $\epsilon/3$ and $f(t) = (1 - f_1(t))^{1/2}$, there exists $\delta > 0$ such that for any elements $u \in A$ and $v \in A_+$ with $\|v\| \leq 1$, if $\|uv - vu\| < \delta$, then $\|uf(v) - f(v)u\| < \epsilon$.

Apply the definition with $F = \{x_1, \cdots, x_m\}$ and with $\min\{\delta, \epsilon/3\}$ in place of ϵ, obtaining $r_0 \in B_+$ with $\|r_0\| \leq 1$ such that $1_A - r_0 \precsim_A a$, and $\|r_0\|^2 \geq \|r_0 a r_0\| / \|a\| \geq 1 - \epsilon/3$. Let $r = f'(r_0) \in C^*(r_0) \subset B_+$. Then, $\|r\| \leq 1$, $r = f'(r_0) \precsim_A r_0$ and $r = 1_A - f'(1_A - r_0)$. Therefore, $1_A - r = f'(1_A - r_0) \precsim_A 1_A - r_0 \precsim_A a$.

Since $r_0 x_j \in_{\epsilon/3} B$ and $r_0 x_j \approx_{\epsilon/3} x_j r_0$, there exists $b_j \in B$ such that $b_j \approx_{\epsilon/3} r_0 x_j$ and $b_j \approx_{2\epsilon/3} x_j r_0$. Set

$$y_j = h(r_0) b_j + (1_A - f_1(r_0))^{1/2} x_j (1_A - f_1(r_0))^{1/2}.$$

To verify $ry_j \in B$, note that $h(r_0)b_j \in B$ and $f'(r_0)(1_A - f_1(r_0)) = 0$. Similarly, we have $y_j r \in B$. Condition (2) is verified.

To check $\|y_j - x_j\| < \epsilon$, note that

$$\begin{aligned} y_j &= h(r_0)b_j + (1_A - f_1(r_0))^{1/2} x_j (1_A - f_1(r_0))^{1/2} \\ &\approx_{\epsilon/3} f_1(r_0)x_j + (1_A - f_1(r_0))^{1/2} x_j (1_A - f_1(r_0))^{1/2} \\ &\approx_{2\epsilon/3} f_1(r_0)x_j + (1_A - f_1(r_0))x_j \\ &\approx_{2\epsilon/3} x_j . \end{aligned}$$

Finally, since $\|r - r_0\| < \epsilon/3$, we have

$$\begin{aligned} \|rar - r_0 a r_0\| &\le \|rar - r_0 a r\| + \|r_0 a r - r_0 a r_0\| \\ &\le \|r - r_0\| \|a\| \|r\| + \|r_0\| \|a\| \|r - r_0\| \\ &\le \frac{2\epsilon}{3} \|a\| . \end{aligned}$$

Therefore, $\|rar\| \ge \|r_0 a r_0\| - 2\epsilon/3 \ge 1 - \epsilon$. □

3. Main Results

Niu's tracial approximation theorem (see Theorem 1) provides us with an example of generalized tracial approximation. Ref. [20] mentioned it after Definition 1.1 without an explicit explanation. For the reader's convenience, we state Niu's theorem below.

Let (X, Γ) be a topological dynamical system, where Γ is a discrete amenable group. Recall that a tower (B, S) is a subset $B \subset X$ and a finite subset $S \subset \Gamma$ such that the sets $B\gamma$ for $\gamma \in S$ are pairwise disjoint. A castle is a finite collection of towers $\{(B_i, S_i)\}_{i=1}^N$, such that the sets $B_i S_i$ for $i = 1, \cdots, N$ are pairwise disjoint. The castle $\{(B_i, S_i)\}_{i=1}^N$ is closed if each base B_i is closed in X.

Recall that (X, Γ) is said to have the Uniform Rokhlin Property (URP), if for any $\epsilon > 0$ and finite subset $K \subset \Gamma$, there exists a closed castle $\{(B_i, S_i)\}_{i=1}^N$ such that each shape S_i is (K, ϵ)-invariant and the orbit capacity of the remainder $X \setminus \bigsqcup_{i=1}^N B_i S_i$ is less than ϵ (see [24]) (Def. 3.1). Denote by $\mathcal{M}_{\mathrm{inv}}(X, \Gamma)$ the Γ-invariant Borel probability measures on X. Then, the second condition of the URP ensures that $\mu(X \setminus \bigsqcup_{i=1}^N B_i S_i) < \epsilon$ for all $\mu \in \mathcal{M}_{\mathrm{inv}}(X, \Gamma)$.

Theorem 1 ([24] (Thm. 3.9))**.** *Let (X, Γ) be a free minimal topological dynamical system, where X is a compact Hausdorff space and Γ is an infinite countable discrete amenable group. If (X, Γ) has the URP, then the C*-algebra $A := C(X) \rtimes \Gamma$ has the following property: for any finite set $\{f_1, \cdots, f_n\} \subset \mathrm{M}_m(A)$, $h \in C(X)_+$ with $h(x) \ge \frac{3}{4}$, $x \in F$ for a closed set $F \subset X$, and any $\delta > 0$, there exists a positive element $p \in C(X)$ with $\|p\| \le 1$, a C*-subalgebra $C \subset A$ with $C \cong \bigoplus_{i=1}^N \mathrm{M}_{K_i}(C_0(Z_i))$ for $K_1, \cdots K_N \in \mathbb{N}$, and some locally compact Hausdorff spaces $Z_1, \cdots, Z_N$ together with compact subsets $[Z_i] \subset Z_i$, $i = 1, \cdots, N$, $\{f_1', \cdots, f_n'\} \subset \mathrm{M}_m(A)$, and $h' \in C(X)_+$ such that if $p := p \otimes 1_m$, then*

(i) $\|h' - h\| < \delta$, $\|f_j - f_j'\| < \delta$, $j = 1, \cdots, n$;

(ii) $\|p_m f_j' - f_j' p_m\| < \delta$, $j = 1, \cdots, n$;

(iii) $p \in C$, $ph'p \in C$, $pf_j'p \in \mathrm{M}_m(C)$, $j = 1, \cdots, n$;

(iv) $\dim([Z_i])/K_i < \mathrm{mdim}(X, \Gamma) + \delta$, $i = 1, \cdots, N$;

(v) $\mu(X \setminus p^{-1}(1)) < \delta$ *for all* $\mu \in \mathcal{M}_{inv}(X, \Gamma)$;

(vi) *under the isomorphism* $C \cong \bigoplus_{i=1}^N \mathrm{M}_{K_i}(C_0(Z_i))$, *one has*

$$\mathrm{rank}((ph'p - \frac{1}{4})(z)) \ge K_i(\min_{\mu} \mu(F) - \delta), \quad z \in [Z_i],$$

where μ runs through $\mathcal{M}_{inv}(X, \Gamma)$;

(vii) *under the isomorphism* $C \cong \bigoplus_{i=1}^{N} \mathrm{M}_{K_i}(C_0(Z_i))$, *the element* p *has the form* $p = \bigoplus_{i=1}^{N} \operatorname{diag}(p_{i,1}, \cdots, p_{i,K_i})$, *where* $p_{i,\ell} : Z_i \to [0,1]$, *and*

$$\left|\{1 \le \ell \le K_i : p_{i,\ell}(z) = 1\}\right| > K_i(1-\delta), \quad z \in [Z_i],\ i = 1, \cdots, N;$$

(viii) *under the isomorphism* $C \cong \bigoplus_{i=1}^{N} \mathrm{M}_{K_i}(C_0(Z_i))$, *any diagonal element of* C *is in* $C(X)$, *and if* $f \in C_+$ *is a diagonal element satisfying* $f|_{[Z_i]} = 1_{K_i}$, $i = 1, \cdots, N$, *then, as an element of* $C(X)$,

$$\mu(X \setminus f^{-1}(1)) < \delta, \quad \forall \mu \in \mathcal{M}_{inv}(X, \Gamma).$$

In the proof of Theorem 1, Niu constructs each Z_i from an open cover of the base B_i with a bounded order. In his proof, each Z_i is actually a locally compact subset of $[0,1]^{d_i}$ with $d_i < \infty$. We need only conditions (i), (ii), (iii), and (v) in Theorem 1 to show that under another condition named (COS), such A is in GTAΩ for some class Ω.

Recall that (X, Γ) is said to have a Cuntz comparison of open sets (COS), if there exists $\lambda \in (0,1]$ and $m \in \mathbb{Z}_{>0}$ such that for any open subsets $U, V \subset X$ with $\mu(U) \le \lambda\mu(V)$ for all $\mu \in \mathcal{M}_{\mathrm{inv}}(X, \Gamma)$, one has $\varphi_U \precsim \varphi_V \otimes 1_m$ in $\mathrm{M}_\infty(C(X) \rtimes \Gamma)$, where φ_E can be any non-negative function on X such that its open support is E; see [24] (Def. 4.1).

Proposition 3. *Let* (X, Γ) *be a free minimal topological dynamical system, where* X *is a compact Hausdorff space and* Γ *is an infinite countable discrete amenable group. If* (X, Γ) *has the URP and (COS), then the C*-algebra* $A := C(X) \rtimes \Gamma$ *belongs to the class* GTAΩ, *where* Ω *is the class of C*-algebras of the form* $\bigoplus_{i=1}^{N} \mathrm{M}_{K_i}(C_0(Z_i))$, *where* $N \in \mathbb{N}$, *each* $K_i < \infty$, *and each* Z_i *is a locally compact Hausdorff space.*

Proof. Suppose that (X, Γ) satisfies (λ, m')-(COS) for $\lambda > 0$ and $m' \in \mathbb{Z}_{>0}$. Given any finite subset $F = \{f_1, f_2, \cdots, f_n\}$ of A, any $\epsilon > 0$, and any nonzero $a \in A_+$, we are supposed to find a C*-algebra B in Ω such that A can be generally tracially approximated by B. By Lemma 2, there exists mutually orthogonal nonzero elements $b_1, \cdots, b_{m'} \in (\overline{aAa})_+$ such that $b_1 \sim_A \cdots \sim_A b_{m'}$. Apply [24] (Lem. 4.2), obtaining a nonzero function $b \in C(X)_+$ such that $b \precsim_A b_1$.

Since X is compact, $\inf_\mu \mu(b^{-1}(0,\infty))$ is a positive number, where μ runs over all $\mathcal{M}_{\mathrm{inv}}(X, \Gamma)$. Applying Theorem 1 with $m = 1$, $F = \{f_1, \cdots, f_n\}$, $h = 1_A$,

$$\delta = \min\left\{\frac{\epsilon}{3}, \lambda \inf_\mu \mu(b^{-1}(0,\infty))\right\},$$

one gets $p \in C(X)_+$ (not necessarily a projection) with $\|p\| \le 1$, a C^*-subalgebra $B \subset A$ with $B \cong \bigoplus_{s=1}^{S} \mathrm{M}_{K_i}(C_0(Z_i))$, and $f_1', \cdots, f_n'$ in A such that

(i) $\|f_j - f_j'\| < \delta$ for $j = 1, \cdots, n$;
(ii) $\|pf_j - f_jp\| < \delta$ for $j = 1, \cdots, n$;
(iii) $p \in B$, $pf_j'p \in B$ for $j = 1, \cdots, n$;
(iv) $\mu(X \setminus p^{-1}(1)) < \delta$ for all $\mu \in \mathcal{M}_{\mathrm{inv}}(X, \Gamma)$.

Let $r = p^2$. Then, $rf_j = p^2 f_j \approx_\delta p^2 f_j' \approx_{2\delta} pf_j'p \in B$ and, similarly, $f_jr \in_{2\delta} B$. So, condition (1′) in Definition 1 is verified. Additionally, we have $rf_j = p^2 f_j \approx_\delta pf_j'p$ and, similarly, $f_jr \approx_\delta pf_j'p$. Thus, $rf_j \approx_{2\delta} f_jr$ for all $j = 1, \cdots, n$. This verifies condition (2′). Condition (iv) implies that $\mu((1-r)^{-1}(0,\infty)) < \lambda\mu(b^{-1}(0,\infty))$ for all $\mu \in \mathcal{M}_{\mathrm{inv}}(X, \Gamma)$. Therefore, $1_A - r \precsim_A b \otimes 1_{m'}$ by (λ, m')-(COS). Using Lemma 1(3) during the second step, Lemma 1(2) during the last step, we have $1_A - r \precsim_A b_1 \oplus \cdots \oplus b_{m'} \sim_A b_1 + \cdots + b_{m'} \precsim_A a$. This verifies condition (3′) by Remark 1. Condition (4′) can be omitted when A is finite by Proposition 4. In fact, under these conditions A has stable rank one by [27] (Thm. 7.8). □

One might hope that Ω can be restricted to a smaller class, such as C^*-algebras of the form $\bigoplus_{i\in I} \mathrm{M}_{K_i}(C_0(Z_i))$ with a uniformly bounded dimension ratio, i.e.,

$$\frac{\dim(Z_i)}{K_i} \leq D \qquad \text{for all } i \in I.$$

Unfortunately, this is not the case in Niu's approach. In fact, in Theorem 1 only a compact subset of each Z_i satisfies this inequality, and the subset is large (under the isomorphism) in a dynamical sense.

According to [24] (Lem. 3.5) and [28] (Thm. 5.5), if X is a compact metric space with an infinite covering dimension and $h : X \to X$ is a minimal homeomorphism, then $(X, h, \mathbb{Z})$ has the URP and (COS). Proposition 3 holds for the crossed product algebra of such a dynamical system.

Let $\widetilde{\Omega}$ be the class of unitization of algebras in Ω.

Lemma 5. *For a unital C^*-algebra A and an arbitary class Ω of C^*-algebras, if $A \in \mathrm{GTA}\Omega$, then $A \in \mathrm{WTA}\widetilde{\Omega}$.*

Proof. Suppose that A is unital and $A \in \mathrm{GTA}\Omega$. By Remark 1, for any $\epsilon > 0$, any finite subset $F \subset A$, and any nonzero positive element $a \in A$, applying Definition 2 with $\epsilon/(1 \vee \|a\|)$ in place of ϵ, F in place of F, and $a/\|a\|$ in place of a, there exists $B \in \Omega$ and $r \in B_+$ with $\|r\| \leq 1$ satisfying

(1) $rx \in_\epsilon B$, $xr \in_\epsilon B$ for all $x \in F$;
(2) $\|rx - xr\| < \epsilon$ for all $x \in F$;
(3) $1_A - r \precsim_A \frac{a}{\|a\|} \sim_A a$;
(4) $\|rar\| \geq \|a\| - \epsilon$.

Then, $\widetilde{B} \cong \mathrm{C}^*(B, 1_A) \in \widetilde{\Omega}$ with $p = 1_A$ and $g = p - r$ satisfy the definition of $A \in \mathrm{WTA}\widetilde{\Omega}$. $\square$

Corollary 1. *Let $A = C(X) \rtimes \Gamma$ and Ω be as in Proposition 3. Then, A belongs to the class $\mathrm{WTA}\widetilde{\Omega}$.*

As we can see, the notion of $\mathrm{GTA}\Omega$ allows us to approximate an algebra in trace without performing a unitization. Therefore, it may help us to prove directly that a non-unital algebra is well-behaved.

Author Contributions: Conceptualization, X.F. and J.Y.; methodology, X.F. and J.Y.; writing—original draft preparation, X.F. and J.Y.; writing—review and editing, X.F. and J.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Sciences Foundation of China (Grant Number 11871375).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors would like to thank the referees for their helpful comments.

Conflicts of Interest: The authors declare no conflict of interest.

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