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On Symbol-Pair Distance of a Class of Constacyclic Codes of Length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

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Abstract: Let $p \neq 3$ be any prime. In this paper, we compute symbol-pair distance of all γ-constacyclic codes of length $3p^s$ over the finite commutative chain ring $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where γ is a unit of $\mathcal{R}$ which is not a cube in $\mathbb{F}_{p^m}$. We give the necessary and sufficient condition for a symbol-pair γ-constacyclic code to be an MDS symbol-pair code. Using that, we provide all MDS symbol-pair γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$. Some examples of the symbol-pair distance of γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ are provided.

Keywords: constacyclic codes; dual codes; chain rings; MDS codes; symbol-pair codes

MSC: 94B15; 94B05; 11T71

1. Introduction

In coding theory, constacyclic codes are important, since many optimal linear codes are derived from constacyclic codes. The class of constacyclic codes has practical applications as constacyclic codes are effective for encoding and decoding with shift registers.

A λ-constacyclic code of length n over $\mathbb{F}$ is an ideal $\langle g(x) \rangle$ of the ambient ring $\frac{\mathbb{F}[x]}{\langle x^n - \lambda \rangle}$ where $g(x)$ is a divisor of $x^n - \lambda$ and λ is a unit in the finite field $\mathbb{F}_{p^m}$. If $(n, p) = 1$, the code is called a *simple-root code*. Otherwise, it is called a *repeated-root code* . Repeated-root codes were studied earlier, from the 1960s, in some papers (for example, refs. [1–8]).

After the celebrated results in the 1990s [9–11] by Nechaev and Hammons et al., that many important yet seemingly non-linear codes over finite fields are actually closely related to linear codes over the ring of integers modulo four, codes over $\mathbb{Z}_4$ in particular, and codes over finite rings in general, have received a great deal of attention. The class of finite rings of the form $\mathcal{R} = \frac{\mathbb{F}_{p^m}[u]}{\langle u^2 \rangle} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ have been used widely as alphabets of certain constacyclic codes. For example, the structure of $\frac{\mathbb{F}_2[u]}{\langle u^2 \rangle}$ is interesting because this ring lies between $\mathbb{F}_4$ and $\mathbb{Z}_4$, in the sense that it is additively analogous to $\mathbb{F}_4$ and multiplicatively analogous to $\mathbb{Z}_4$. Codes over $\frac{\mathbb{F}_2[u]}{\langle u^2 \rangle}$ have been extensively studied by many researchers, whose work includes cyclic and self-dual codes [12], decoding of cyclic codes [13], Type II codes [14], and duadic codes [15]. The most general form of these rings, $\frac{\mathbb{F}_{p^m}[u]}{\langle u^a \rangle} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{a-1}\mathbb{F}_{p^m}$, has been used as code alphabet as well. For instance, Ozen and Siap [16] addressed linear codes over this ring with respect to the Rosenbloom–Tsfasman metric, and Alfaro et al. obtained a construction for self-dual codes over it [17].

Let σ be the code alphabet consisting of q elements. Then, each element $v \in \sigma$ is called a *symbol*. In symbol-pair read channels, a codeword $(v_0, v_1, \ldots, v_{n-1})$ is read as $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_0))$. A q-ary code of length n is a nonempty subset $C \subseteq \sigma^n$.

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Assume that $v = (v_0, v_1, \ldots, v_{n-1})$ is a vector in σ^n. Then, $\pi(v) = ((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_0))$ is said to be a symbol-pair vector of v. Hence, for each v, we have a unique symbol-pair vector $\pi(v) \in (\sigma, \sigma)^n$. In 2010, Cassuto and Blaum [18] introduced the symbol-pair distance as the Hamming distance over the alphabet (σ, σ). Given $v = (v_0, v_1, \ldots, v_{n-1})$, $t = (t_0, t_1, \ldots, t_{n-1})$, the symbol-pair distance between v and t is defined as

$$\mathrm{d_{sp}}(v, t) = \mathrm{d_H}(\pi(v), \pi(t)) = |\{i : (v_i, v_{i+1}) \neq (t_i, t_{i+1})\}|.$$

The symbol-pair distance of C is defined as $\mathrm{d_{sp}}(C) = \min_{v,t \in C, v \neq t}\{\mathrm{d_{sp}}(v, t)\}$. The symbol-pair weight of a vector v, denoted by $\mathrm{wt_{sp}}(v)$, is defined as

$$\mathrm{wt_{sp}}(v) = \mathrm{wt}(\pi(v)) = \left|\{i \mid (v_i, v_{i+1}) \neq (0,0), 0 \leq i \leq n-1, v_n = v_0\}\right|.$$

If C is linear, $\mathrm{d_{sp}}(C)$ is equal to the minimum symbol-pair weight of nonzero codewords of C:

$$\mathrm{d_{sp}}(C) = \min\{\mathrm{wt_{sp}}(v) \mid v \neq 0,\ v \in C\}.$$

With the development of high-density data storage technologies, symbol-pair codes are proposed to protect efficiently against a certain number of pair-errors. In 2010, Cassuto and Blaum [18] studied the model of symbol-pair read channels. However, the problem of determining symbol-pair distance of a code C is very difficult in general. In 2012, lower bound on the symbol-pair distances for binary cyclic codes are considered by Yaakobi et al. [19]. They proved that the symbol-pair distance of linear code C is at least $\mathrm{d_H} + \left\lceil \frac{\mathrm{d_H}}{2} \right\rceil$ Theorem 4 of [19]. This result is better than the previous result provided by Cassuto and Litsyn. However, the algorithms in [19,20] must improve because these algorithms can not be used to decode all symbol-pair codes. Motivated by this, a new algorithm is given by Hirotomo et al. [21], using the parity-check matrix for decoding symbol-pair codes. By extending Theorem 10 of [20], Kai et al. [22] provided a new lower bound on simple-root constacyclic codes. Recently, Dinh et al. [23,24] succesfully established the symbol-pair distances for all constacyclic codes of length p^s and cyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$. In addition, refs. [25,26] investigated Hamming and symbol-pair distances of repeated-root constacyclic codes of length p^s over $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

Motivated by these, in this paper, we determine symbol-pair distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is not a cube in $\mathcal{R}$. As an application, we identify all the MDS symbol-pair codes among such codes.

The rest of this paper is organized as follows. Section 2 gives some preliminaries. Section 3 obtains the symbol-pair distance of all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is not a cube in $\mathcal{R}$. In Section 4, we give the necessary and sufficient condition for a symbol-pair λ-constacyclic code to be an MDS symbol-pair code, and we identify all such codes. The conclusion of this paper is given in Section 5.

2. Preliminaries

For a unit λ of R (R is a finite chain ring size p^m), the λ-constacyclic (λ-twisted) shift ρ_λ on R^n is the shift

$$\rho_\lambda(x_0, x_1, \ldots, x_{n-1}) = (\lambda x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and a code C is called λ-constacyclic if $\rho_\lambda(C) = C$. If $\lambda = \{1, -1\}$, then C is a cyclic and negacyclic code, respectively.

Proposition 1 ([27,28]). *Let C be a linear code. Then C is a λ-constacyclic code of length n over R if C is an ideal of the ring $\frac{R[x]}{\langle x^n - \lambda \rangle}$.*

Proposition 2 ([29]). *The dual of a λ-constacyclic code is a λ^{-1}-constacyclic code.*

Let p be a prime and R be a finite chain ring of size p^m.

Proposition 3 ([27,30]). *Let C be a linear code C of length n over R. Then $|C| = p^k$, for some integer $k \in \{0,1,...,mn\}$. In addition, $|C| \cdot |C^{\perp}| = |R|^n$, where $C^{\perp}$ is the dual code of C.*

Assume that α and β are elements in $\mathbb{F}_{p^m}$. It is simple to check that $\alpha + u\beta$ is invertible over $\mathcal{R}$ if $\alpha \neq 0$. Thus, we divide all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ into the following cases: λ is a cube and $p^m \equiv 1 \pmod 3$, λ is a cube and $p^m \equiv 2 \pmod 3$, $\lambda = \alpha + u\beta$ is not a cube and $0 \neq \alpha, \beta \in \mathbb{F}_{p^m}$, λ is not a cube and $0 \neq \lambda \in \mathbb{F}_{p^m}$. We give all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ in the following theorem.

Theorem 1 ([31]). *Let $p \neq 3$ be any prime. Let C be a λ-constacyclic code of length $3p^s$ over $\mathcal{R}$.*

(1) Assume that λ is a cube in $\mathcal{R}$ and $p^m \equiv 1 \pmod 3$. Let $\lambda_0 \in \mathcal{R}$ such that $\lambda_0^3 = \lambda$ and $\delta, \theta \in \mathbb{F}_{p^m}$ such that $\delta\theta = 1$ and $\delta + \theta = -1$. Then $C = C_1 \oplus C_2 \oplus C_3$ where C_1 is a λ_0-constacyclic code of length p^s over $\mathcal{R}$, C_2 is a $\delta\lambda_0$-constacyclic code of length p^s over $\mathcal{R}$ and C_3 is a $\theta\lambda_0$-constacyclic code of length p^s over $\mathcal{R}$. In particular, $|C| = |C_1||C_2||C_3|$.

(2) Assume that λ is a cube in $\mathcal{R}$ and $p^m \equiv 2 \pmod 3$. Let $\lambda_1 \in \mathcal{R}$ such that $\lambda = \lambda_1^3$. Then

(a) $C = C_1 \oplus C_2$ where C_1 is a λ_1-constacyclic code of length p^s over $\mathcal{R}$, and C_2 is an ideal of $\frac{\mathcal{R}[x]}{\langle x^{2p^s} + \lambda_1 x^{p^s} + \lambda_1^2 \rangle}$.

(b) $|C| = |C_1||C_2|$, where C_1 is determined as in Theorem 2.2 and all ideals of $\frac{\mathcal{R}[x]}{\langle (x^2+\lambda_1 x+\lambda_1^2)^{p^s} \rangle}$ are determined as follows:

- *Type 1:*

$$\langle 0 \rangle \text{ and } \langle 1 \rangle.$$

Then $n_{C_2} = 1$ and $n_{C_2} = p^{4mp^s}$, respectively.

- *Type 2:*

$$\langle u(x^2 + \lambda_1 x + \lambda_1^2)^j \rangle,$$

where $0 \leq j \leq p^s - 1$. Then $n_{C_2} = p^{2m(p^s - j)}$

- *Type 3:*

$$\langle (x^2 + \lambda_1 x + \lambda_1^2)^j + u(x^2 + \lambda_1 x + \lambda_1^2)^t v(x) \rangle,$$

where $1 \leq j \leq p^s - 1, 0 \leq t < j$, and either $v(x)$ is 0 or a unit which can be represented as $v(x) = \sum_{i=0}^{j-t-1} (v_{1i}x + v_{0i})(x^2 + \lambda_1 x + \lambda_1^2)^i$ with $v_{0i}, v_{1i} \in \mathbb{F}_{p^m}$ and $v_{10}x + v_{00} \neq 0$. In this case,

$$n_{C_2} = \begin{cases} \bullet\ p^{4m(p^s-j)}, \text{ if } v(x) \text{ is } 0,\ 1 \leq j \leq p^s - 1 \text{ or } v(x) \text{ is a unit }, 1 \leq j \leq \frac{p^s+t}{2}, \\ \bullet\ p^{2m(p^s-t)}, \text{ if } v(x) \text{ is a unit, and } \frac{p^s+t}{2} < j \leq p^s - 1. \end{cases}$$

- *Type 4: $\langle (x^2 + \lambda_1 x + \lambda_1^2)^j + u(x^2 + \lambda_1 x + \lambda_1^2)^t v(x), u(x^2 + \lambda_1 x + \lambda_1^2)^{\omega} \rangle$, with $v(x)$ as in Type 3, $\deg v(x) \leq \omega - r - 1$ and $\omega < R$ and R is the smallest integer satisfying $u(x^2 + \lambda_1 x + \lambda_1^2)^R \in \langle (x^2 + \lambda_1 x + \lambda_1^2)^j + u(x^2 + \lambda_1 x + \lambda_1^2)^t v(x) \rangle$. In this case, $n_{C_2} = p^{2m(2p^s - j - \omega)}$*

(3) Assume that $\lambda = \alpha + u\beta$ is not a cube in $\mathcal{R}$. There is an $\alpha_1 \in \mathbb{F}_{p^m}$ satisfying $\alpha = \alpha_1^{p^s}$. Then $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$ are the ideals $\langle (x^3 - \alpha_1)^i \rangle \subseteq \mathcal{R}_{\alpha,\beta}$, where $0 \leq i \leq 2p^s$ and each $(\alpha + u\beta)$-constacyclic code $\langle (x^3 - \alpha_1)^i \rangle$ has $p^{3m(2p^s-1)}$ codewords.

(4) Assume that $\lambda \in \mathbb{F}_{p^m} \setminus \{0\}$ is not a cube in $\mathbb{F}_{p^m}$. Let $\lambda_0 \in \mathbb{F}_{p^m}$ such that $\lambda_0^{p^s} = \lambda$. Then λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ are

- *Type 1:*

$$\langle 0 \rangle \text{ and } \langle 1 \rangle.$$

- *Type 2:*

$$\langle u(x^3 - \lambda_0)^i \rangle,$$

where $0 \leq i \leq p^s - 1$.

- *Type 3:*

$$\langle ((x^3 - \lambda_0)^i + u(x^3 - \lambda_0)^t)v(x) \rangle,$$

where $1 \leq i \leq p^s - 1, 0 \leq t < i$, and $v(x)$ is 0 or a unit where it can be written as $v(x) = \sum_{j=0}^{i-t-1}(h_{2j}x^2 + h_{1j}x + h_{0j}j)(x^3 - \lambda_0)^j$ where $h_{0j}, h_{1j}, h_{2j} \in \mathbb{F}_{p^m}$ and $h_{00} \neq 0$.

- *Type 4:*

$$\langle (g(x))^i + u(\sum_{j=0}^{\omega-1}(t_{0j}(x))(g(x))^j), u(g(x))^\omega \rangle$$

where $g(x) = x^3 - \lambda_0$, $1 \leq i \leq p^s - 1, a_{0j}, b_{0j}, c_{0j} \in \mathbb{F}_{p^m}$, $t_{0j}(x) = a_{0j}x^2 + b_{0j}x + c_{0j}$, and $\omega < T$, where T is the smallest integer satisfying

$$u(g(x))^T \in \langle (g(x))^i + u\sum_{j=0}^{w-1}(t_{0j}(x))(g(x))^j \rangle$$

or equivalently,

$$\langle (g(x))^i + u(g(x))^t h(x), u(g(x))^\omega \rangle$$

with $h(x)$ as in Type 3 and $\deg h(x) \leq \omega - t - 1$.

In addition, the number of codewords of C, denoted by n_C, is determined as follows:

- *If $C = \langle 0 \rangle$ and $C = \langle 1 \rangle$, then $n_C = 1$ and $n_C = p^{6mp^s}$, respectively.*
- *If $C = \langle u(x^3 - \lambda_0)^i \rangle$, where $0 \leq i \leq p^s - 1$, then $n_C = p^{3m(p^s-i)}$.*
- *If $C = \langle (x^3 - \lambda_0)^i + u(x^3 - \lambda_0)^t h(x) \rangle$ where $1 \leq i \leq p^s - 1, 0 \leq t < i$, and $h(x)$ is 0 or a unit, then*

$$n_C = \begin{cases} \bullet\, p^{6m(p^s-i)}, \text{ if } h(x) \text{ is } 0,\ 1 \leq i \leq p^s - 1 \text{ or } h(x) \text{ is a unit },1 \leq i \leq p^{s-1} + \frac{t}{2}, \\ \bullet\, p^{3m(p^s-t)}, \text{ if } h(x) \text{ is a unit},\ p^{s-1} + \frac{t}{2} < i \leq p^s - 1. \end{cases}$$

- *If $C = \langle (x^3 - \lambda_0)^i + u(x^3 - \lambda_0)^t h(x), u(x^3 - \lambda_0)^\kappa \rangle$, where $1 \leq i \leq p^s - 1, 0 \leq t \leq i$, either $h(x)$ is 0 or a unit, and*

$$\kappa < T = \begin{cases} i, & \text{if } h(x) = 0, \\ \min\{i, p^s - i + t\}, & \text{if } h(x) \neq 0, \end{cases}$$

then $n_C = p^{3m(2p^s-i-\kappa)}$.

Let b be an integer and $b \geq 1$. For a codeword $v = (v_0, v_1, \cdots, v_{n-1}) \in \sigma^n$, the b-symbol read codeword of v is defined as

$$\pi_b(x) = ((v_0, \cdots, v_{b-1}), (v_1, \cdots, v_b), \cdots, (v_{n-1}, v_0, \cdots, v_{b-2})) \in (\sigma^b)^n.$$

Then the b-symbol distance between two codewords v and t in σ^n is denoted by $\mathrm{d_b}(v, t)$ and defined as

$$\mathrm{d_b}(v, t) = \mathrm{d_H}(\pi_b(v), \pi_b(t)).$$

In 2016, [32] generalized the coding framework for symbol-pair read channels to that for b-symbol read channels, where the read operation is performed as a consecutive sequence of $b > 2$ symbols. The authors of [32] also generalized some of the known results for symbol-pair read channels to those for b-symbol read channels. In [33], Dinh et al. computed the b-symbol distance for $C = \langle (x^n - \lambda_0)^j \rangle$ for $0 \leq j \leq p^s$ and $b \leq \eta$, where $(x^n - \lambda_0)$ is irreducible. For symbol-pair distance, we have the following theorem.

Theorem 2. *Let $C = \langle (x^3 - \lambda_0)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \lambda \rangle}$ for $0 \leq j \leq p^s$, the symbol-pair distance $\mathrm{d_{sp}}(C)$ is completely given by*

$$\mathrm{d_{sp}}(C) = \begin{cases} 2, & \text{if } j = 0 \\ 2(\delta + 1)p^{\xi}, & \text{if } p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq j \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1} \end{cases}$$

where $1 \leq \delta \leq p - 1$, $0 \leq \xi \leq s - 1$.

3. Symbol-Pair Distance

The authors of [23] obtained the symbol-pair distance of all constacyclic codes of length p^s over $\mathbb{F}_{p^m}$. After that, Dinh et al. [25,26] provided the symbol-pair distance of all constacyclic codes of length p^s over $\mathcal{R}$. In this section, we compute the symbol-pair distance of all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is not a cube in $\mathcal{R}$. First, we determine the symbol-pair distance of all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is not a cube in $\mathbb{F}_{p^m}$. Obviously, if $C = \langle 0 \rangle$, then $\mathrm{d_{sp}}(C) = 0$. If $C = \langle 1 \rangle$, then $\mathrm{d_{sp}}(C) = 1$. Now, we determine the symbol-pair distance for all λ-constacyclic code of Types 2,3,4 of length $3p^s$, where λ is not a cube in $\mathbb{F}_{p^m}$. Note that when λ is not a cube in $\mathbb{F}_{p^m}$, the structure of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ is given in part 4 of Theorem 1. Denote $\mathrm{d_{sp}}(C_F)$ as the symbol-pair distance of $C|_{\mathbb{F}_{p^m}}$. The symbol-pair distance of λ-constacyclic code of Type 2 can be determined as follows.

Theorem 3. *Let $C_2 = \langle u(x^3 - \lambda_0)^j \rangle$ be a λ-constacyclic code of Type 2 of length $3p^s$ over $\mathcal{R}$, where $0 \leq j \leq p^s - 1$. Then we have $\mathrm{d_{sp}}(C_2) = \mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^j \rangle_F)$, and*

$$\mathrm{d_{sp}}(C_2) = \begin{cases} 2, & \text{if } j = 0 \\ 2(\delta + 1)p^{\xi}, & \text{if } p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq j \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1} \end{cases}$$

where $1 \leq \delta \leq p - 1$, $0 \leq \xi \leq s - 1$.

Proof. We consider the case $j = 0$ and $p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq j \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$.

Case 1: If $j = 0$, then $\mathrm{d_{sp}}(C_2) = 1$.

Case 2: If $p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq j \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then n_{C_2} is exactly same as $n_{\langle (x^3 - \lambda_0)^j \rangle}$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \lambda \rangle}$ multiplied by u. Hence, we see that $\mathrm{d_{sp}}(C_2) = \mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^j \rangle_F)$ and

$$\mathrm{d_{sp}}(C_2) = \begin{cases} 2, & if\ j = 0 \\ 2(\delta+1)p^{\xi}, & if\ p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le j \le p^s - p^{s-\xi} + \delta p^{s-\xi-1} \end{cases}$$

where $1 \le \delta \le p-1, 0 \le \xi \le s-1$. □

In the following theorem, we discuss the symbol-pair distance of λ-constacyclic codes of Type 3 of length $3p^s$ over $\mathcal{R}$.

Theorem 4. *Let* $C_3 = \langle (x^3-\lambda_0)^j + u(x^3-\lambda_0)^r v(x) \rangle$ *be a* λ*-constacyclic code of Type 3 of length* $3p^s$ *over* $\mathcal{R}$*, where* $1 \le j \le p^s - 1$*,* $0 \le r < j$ *and either* $v(x)$ *is a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \lambda \rangle}$ *or* 0*. Then, we have* $\mathrm{d_{sp}}(C_3) = \mathrm{d_{sp}}(\langle (x^3-\lambda_0)^T \rangle_F)$*, where* T *is the smallest integer satisfying* $u(x^3-\lambda_0)^T \in \langle (x^3-\lambda_0)^j + u(x^3-\lambda_0)^r v(x) \rangle$*, and*

$$T = \begin{cases} j, & if\ v(x) = 0 \\ \min\{j, p^s - j + r\}, & if\ v(x) \neq 0. \end{cases}$$

Then

$$\mathrm{d_{sp}}(C_3) = 2(\delta+1)p^{\xi},$$

where $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le T \le p^s - p^{s-\xi} + \delta p^{s-\xi-1}$*,* $1 \le \delta \le p-1$ *and* $0 \le \xi \le s-1$*.*

Proof. Since T is the smallest integer satisfying $u(x^3-\lambda_0)^T \in \langle (x^3-\lambda_0)^j + u(x^3-\lambda_0)^r v(x) \rangle$, we have

$$\mathrm{d_{sp}}(C_3) \le \mathrm{d_{sp}}(\langle u(x^3-\lambda_0)^T \rangle) = \mathrm{d_{sp}}(\langle (x^3-\lambda_0)^T \rangle_F).$$

Let $c(x) \in C_3$ be an arbitrary polynomial. Then we see that there are two polynomials $f_0(x)$ and $f_u(x)$ over $\mathbb{F}_{p^m}$ satisfying

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(x^3-\lambda_0)^j + u(x^3-\lambda_0)^r v(x)] \\ &= f_0(x)(x^3-\lambda_0)^j + u[f_0(x)(x^3-\lambda_0)^r v(x) \\ &\quad + f_u(x)(x^3-\lambda_0)^j]. \end{aligned}$$

Now, we consider two cases as follows:
Case 1: Assume that $v(x) = 0$. Then, we have

$$\begin{aligned} \mathrm{wt_{sp}}(c(x)) &\ge \max\Big\{\mathrm{wt_{sp}}(f_0(x)(x^3-\lambda_0)^j), \mathrm{wt_{sp}}(f_u(x)(x^3-\lambda_0)^j)\Big\} \\ &\ge \max\Big\{\mathrm{wt_{sp}}(f_0(x)(x^3-\lambda_0)^j), \mathrm{wt_{sp}}(f_0(x)(x^3-\lambda_0)^j)\Big\} \\ &\ge \mathrm{d_{sp}}(\langle (x^3-\lambda_0)^j \rangle_F), \\ &= \mathrm{d_{sp}}(\langle (x^3-\lambda_0)^T \rangle_F), \end{aligned}$$

Case 2: Assume that $v(x) \neq 0$. Then we see that

$$\begin{aligned} \mathrm{wt_{sp}}(c(x)) &\ge \max\Big\{\mathrm{wt_{sp}}(f_0(x)(x^3-\lambda_0)^j), \mathrm{wt_{sp}}(h(x))\Big\} \\ &\ge \max\Big\{\mathrm{wt_{sp}}(f_0(x)(x^3-\lambda_0)^j), \mathrm{wt_{sp}}(f_0(x)(x^3-\lambda_0)^{p^s-j+r})\Big\} \\ &\ge \mathrm{d_{sp}}(\langle (x^3-\lambda_0)^{\min\{j,\, p^s-j+r\}} \rangle_F), \\ &= \mathrm{d_{sp}}(\langle (x^3-\lambda_0)^T \rangle_F), \end{aligned}$$

where $h(x) = f_0(x)(x^3 - \lambda_0)^r v(x) + f_u(x)(x^3 - \lambda_0)^j$. Hence, by combining both the cases, we get $\mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^T \rangle_F) \leq \mathrm{d_{sp}}(C_3)$, which implies that, $\mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^T \rangle_F) = \mathrm{d_{sp}}(C_3)$. □

The symbol-pair distance of λ-constacyclic codes of Type 4 is computed in the following result.

Theorem 5. *Let $C_4 = \langle (x^3 - \lambda_0)^j + u(x^3 - \lambda_0)^r v(x), u(x^3 - \lambda_0)^\omega \rangle$ be a λ-constacyclic code of Type 4 of length $3p^s$ over $\mathcal{R}$, where $v(x)$ is same as given in Type 3, $\deg(v) \leq \omega - r - 1$, $\omega < T$, and T is the smallest integer satisfying $u(x^3 - \lambda_0)^T \in \langle (x^3 - \lambda_0)^j + u(x^3 - \lambda_0)^r v(x) \rangle$; i.e., $T = j$, if $v(x) = 0$ and otherwise $T = \min\{j, p^s - j + t\}$. Then, we have $\mathrm{d_{sp}}(C_4) = \mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^\omega \rangle_F)$, and is given by*

$$\mathrm{d_{sp}}(C_4) = 2(\delta + 1)p^\xi,$$

where $p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq \omega \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, $1 \leq \delta \leq p - 1$ and $0 \leq \xi \leq s - 1$.

Proof. Since $\omega < T \leq j$, we see that $C_4 = \langle (x^3 - \lambda_0)^j + u(x^3 - \lambda_0)^r v(x), u(x^3 - \lambda_0)^\omega \rangle \supseteq \langle u(x^3 - \lambda_0)^\omega \rangle \supseteq \langle u(x^3 - \lambda_0)^j \rangle$. Hence, $\mathrm{d_{sp}}(C_4) \leq \mathrm{d_{sp}}(\langle u(x - \lambda_0)^\omega \rangle) = \mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^\omega \rangle_F)$. We will prove that $\mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^\omega \rangle_F) \leq \mathrm{d_{sp}}(C_4)$. Now, taking an arbitrary polynomial $c(x) \in C_4$, we need to prove that $\mathrm{wt_{sp}}(c(x)) \geq \mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^\omega \rangle_F)$. Now, the following polynomials exist $f_0(x), f_u(x), g_0(x)$ and $g_u(x)$ over $\mathbb{F}_{p^m}$ satisfying

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(x^3 - \lambda_0)^j + u(x^3 - \lambda_0)^r v(x)] + u(x^3 - \lambda_0)^\omega [g_0(x) + ug_u(x)] \\ &= f_0(x)(x^3 - \lambda_0)^j + u[f_0(x)(x^3 - \lambda_0)^r v(x) + f_u(x)(x^3 - \lambda_0)^j + g_0(x)(x^3 - \lambda_0)^\omega] \\ &= f_0'(x)(x^3 - \lambda_0)^\omega + u[f_0(x)(x^3 - \lambda_0)^r v(x) + g_0'(x)(x^3 - \lambda_0)^\omega], \end{aligned}$$

where $f_0'(x) = f_0(x)(x^3 - \lambda_0)^{j-\omega} \in \mathbb{F}_{p^m}[x]$, $g_0'(x) = f_u(x)(x^3 - \lambda_0)^{j-\omega} + g_0(x) \in \mathbb{F}_{p^m}[x]$.

Hence,

$$\begin{aligned} \mathrm{wt_{sp}}(c(x)) &\geq \max\Big\{\mathrm{wt_{sp}}(f_0'(x)(x^3 - \lambda_0)^\omega), \mathrm{wt_{sp}}(h'(x))\Big\} \\ &\geq \max\Big\{\mathrm{wt_{sp}}(f_0'(x)(x^3 - \lambda_0)^\omega), \mathrm{wt_{sp}}(f_0'(x)(x^3 - \lambda_0)^\omega)\Big\} \\ &\geq \mathrm{d_{sp}}(\langle (x^3 - \lambda_0)^\omega \rangle_F), \end{aligned}$$

where $h'(x) = f_0(x)(x^3 - \lambda_0)^r v(x) + g_0'(x)(x^3 - \lambda_0)^\omega$. Thus, $\mathrm{d_{sp}}(C_4) = 2(\delta + 1)p^\xi$, where $p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq \omega \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, $1 \leq \delta \leq p - 1$ and $0 \leq \xi \leq s - 1$. □

If $\lambda = \alpha + u\beta$ is not a cube in $\mathcal{R}$, then there is an $\alpha_1 \in \mathbb{F}_{p^m}$ satisfying $\alpha = \alpha_1^{p^s}$. As in part 3 of Theorem 1, $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$ are the ideals $\langle (x^3 - \alpha_1)^i \rangle \subseteq \mathcal{R}_{\alpha,\beta}$, where $0 \leq i \leq 2p^s$. When $(\alpha + u\beta)$ is not a cube in $\mathcal{R}$, we determine the symbol-pair distance of all $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$ in the following theorem.

Theorem 6. *Let C be a $(\alpha + u\beta)$-constacyclic code of length $3p^s$ over $\mathcal{R}$, where $(\alpha + u\beta)$ is not a cube in $\mathcal{R}$, i.e., $C = \langle (x^3 - \alpha_1)^j \rangle$ for $j \in \{0, 1, \ldots, 2p^s\}$, where $\alpha_1 \in \mathbb{F}_{p^m}$ such that $\alpha = \alpha_1^{p^s}$. Then*

$$\mathrm{d_{sp}}(C) = \begin{cases} 2, & \text{if } 0 \leq j \leq p^s \\ 2(\delta + 1)p^\xi, & \text{if } 2p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq j \leq 2p^s - p^{s-\xi} + \delta p^{s-\xi-1} \\ 0, & \text{if } j = 2p^s \end{cases}$$

where $1 \leq \delta \leq p - 1$, $0 \leq \xi \leq s - 1$.

Proof. We consider three cases.

Case 1: If $j = 0$ and $j = 2p^s$, then $C = \langle 1 \rangle$ and $C = \langle 0 \rangle$. It is simple to verify that $\mathrm{d_{sp}}(C) = 2$ and $\mathrm{d_{sp}}(C) = 0$, respectively.

Case 2: If $1 \le j \le p^s$. In $\mathcal{R}_{\alpha+u\beta}$, $\mathcal{R}_{\alpha+u\beta} = \langle 1 \rangle \supsetneq \langle (x^3-\alpha_1) \rangle \supsetneq \cdots \supsetneq \langle (x^3-\alpha_1)^{p^s} \rangle \supsetneq \cdots \supsetneq \langle (x^3-\alpha_1)^{2p^s} \rangle = \langle 0 \rangle$. Thus, we have $u \in \langle (x^3-\alpha_1)^j \rangle$. It implies that $\mathrm{d_{sp}}(C) = 2$.

Case 3: If $p^s + 1 \le j \le 2p^s - 1$, then we see that $\langle (x^3-\alpha_1)^j \rangle = \langle u(x^3-\alpha_1)^{j-p^s} \rangle$. Hence, $n_{\langle (x^3-\alpha_1)^j \rangle}$ in $\mathcal{R}_{\alpha+u\beta}$ is exactly the same as $n_{\langle (x^3-\alpha_1)^{j-p^s} \rangle}$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s}-\alpha \rangle}$ multiplied by u. Hence, $\mathrm{wt_{sp}}(\langle (x^3-\alpha_1)^j \rangle) = \mathrm{wt_{sp}}(\langle (x^3-\alpha_1)^{j-p^s} \rangle)$. By Theorem 1, we can determine the symbol-pair distance of $\langle (x^3-\alpha_1)^{j-p^s} \rangle$. Therefore, $\mathrm{d_{sp}}(C) = 2(\delta+1)p^{\xi}$ when $2p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le j \le 2p^s - p^{s-\xi} + \delta p^{s-\xi-1}$. □

Example 1. *We present some examples of symbol-pair λ-constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $\lambda \in \mathbb{F}_p^*$ and λ is not a cube. In Table 1, we compute the symbol-pair distances for $p = 7, m = 1, s = 1$ and in Table 2, symbol-pair distances have been computed by taking $p = 13, m = 1, s = 1$.*

Table 1. λ-constacyclic codes over $\mathbb{F}_7 + u\mathbb{F}_7$.

n	s	λ	$\langle g(x) \rangle$	$[n, M, \mathrm{d_{sp}}]$
21	1	3	$\langle u(x^3-3) \rangle$	$[21, 7^{18}, 4]$
21	1	3	$\langle (x^3-3)^2 \rangle$	$[21, 7^{30}, 6]$
21	1	3	$\langle (x^3-3)^2, u(x^3-3) \rangle$	$[21, 7^{33}, 4]$
147	2	2	$\langle u(x^3-2)^{44} \rangle$	$[147, 7^{15}, 42]$
147	2	2	$\langle (x^3-2)^{44} \rangle$	$[147, 7^{30}, 42]$

Table 2. λ-constacyclic codes over $\mathbb{F}_{13} + u\mathbb{F}_{13}$.

n	s	b	λ	$\langle g(x) \rangle$	$[n, M, \mathrm{d_{sp}}]$
39	1	3	2	$\langle u(x^3-2)^2 \rangle$	$[39, 13^{33}, 6]$
39	1	3	2	$\langle (x^3-2)^2 \rangle$	$[39, 13^{66}, 6]$
39	1	3	2	$\langle (x^3-2)^{15} + u(x^3-2), u(x^3-2) \rangle$	$[39, 13^{30}, 4]$

4. MDS Symbol-Pair Codes

In 2018, Ding et al. [34] discussed the Singleton bound with respect to the b-symbol distance $\mathrm{d_b}(C)$. Following them, the Singleton bound with respect to the b-symbol distance is given as $|C| \le q^{n-\mathrm{d_b}(C)+b}$. In order to determine MDS symbol-pair codes, we need to have Singleton Bound with respect to symbol-pair distance.

Theorem 7. *Let C be a linear symbol-pair code of length n over $\mathcal{R}$ with symbol-pair distance $\mathrm{d_{sp}}(C)$. Then, the Singleton bound with respect to the symbol-pair distance $\mathrm{d_{sp}}(C)$ is given by $|C| \le p^{2m(n-\mathrm{d_{sp}}(C)+2)}$.*

Proof. Let C be a $(n, M, \mathrm{d_{sp}}(C))$ symbol-pair code such that $2 \le \mathrm{d_{sp}}(C) \le n$. If we delete the last $\mathrm{d_{sp}}(C) - 2$ coordinates from all the codewords in C, then any $\mathrm{d_{sp}}(C) - 2$ consecutive coordinates contribute, at most, $\mathrm{d_{sp}}(C) - 1$ to the symbol-pair distance. Since C has symbol-pair distance $\mathrm{d_{sp}}(C)$, the resulting vectors of length $n - \mathrm{d_{sp}}(C) + 2$ are still distinct. The conclusion follows on from the fact that the maximum number of distinct vectors of length $n - \mathrm{d_{sp}}(C) + 2$ over $\mathcal{R}$ is $p^{2m(n-\mathrm{d_{sp}}(C)+2)}$. □

Definition 1. *Let C be a symbol-pair code of length n over $\mathcal{R}$. Then, C is said to be a MDS symbol-pair code with respect to the symbol-pair distance if $|C| = p^{2m(n-\mathrm{d_{sp}}(C)+2)}$.*

We will identify the MDS symbol-pair codes for each type of λ-constacyclic code, one by one.

Theorem 8. *Let C be a symbol-pair λ-constacyclic code of Type 1 of length $3p^s$ over $\mathcal{R}$. Then $C = \langle 1 \rangle$ is an MDS symbol-pair code.*

Proof. If $C = \langle 1 \rangle$, then $\mathrm{d_{sp}}(C) = 2$. Thus, C is MDS when $|C| = p^{2m(3p^s - \mathrm{d_{sp}}(C)+2)}$, i.e., $p^{6mp^s} = p^{2m(3p^s)}$, which is a contradiction. Therefore, the code $C = \langle 1 \rangle$ is an MDS code. □

Now we give the MDS condition for symbol-pair λ-constacyclic codes of Type 2 of length $3p^s$ over $\mathcal{R}$.

Theorem 9. *Let $C_2 = \langle u(x^3 - \lambda_0)^j \rangle$ be a symbol-pair λ-constacyclic code of Type 2 of length $3p^s$ over $\mathcal{R}$, where $0 \leq j \leq p^s - 1$. Then C_2 is not an MDS symbol-pair code λ-constacyclic code of Type 2 of length $3p^s$ over $\mathcal{R}$.*

Proof. Case 1: If $j = 0$, then $\mathrm{d_{sp}}(C_2) = 2$. If C_2 is an MDS symbol-pair code, then $|C_2| = p^{2m(3p^s - \mathrm{d_{sp}}(C_2)+2)}$, which is equivalent to $p^{3mp^s} = p^{2m(3p^s - \mathrm{d_{sp}}(C_2)+2)}$, i.e., $p^{3mp^s} = p^{6mp^s}$, which is a contradiction. Thus, C_2 is not MDS.

Case 2: If $p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \leq j \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then $\mathrm{d_{sp}}(C_2) = 2(\delta+1)p^{\xi}$. Thus, C_2 is an MDS symbol-pair code if $|C_2| = p^{2m(3p^s - \mathrm{d_{sp}}(C_2)+2)}$, i.e., $p^{3m(p^s - j)} = p^{2m(3p^s - \mathrm{d_{sp}}(C_2)+2)}$, i.e., $3j = 2\,\mathrm{d_{sp}}(C_2) - 3p^s - 4$.

Now,

$$
\begin{aligned}
3j &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1) \\
&\geq 3p^{\xi+1} - 3p + 3(\delta-1) + 3 \text{ (equality when } \xi = s-1) \\
&\geq 6(\delta+1)p^{\xi} - 3p^s - 3(\delta+1) + 3(\delta-1) + 3 \\
&\quad \text{(equality when } p - 1 = \delta) \\
&\geq 2\,\mathrm{d_{sp}}(C_2) - 3p^s - 4 + 1 + 4(\delta+1)p^{\xi} \\
&> 2\,\mathrm{d_{sp}}(C_2) - 3p^s - 4.
\end{aligned}
$$

Hence, C_2 is not MDS. □

Next, we consider the symbol-pair λ-constacyclic codes of Type 3 to verify the MDS condition for these codes.

Theorem 10. *Let $C_3 = \langle (x^3 - \lambda_0)^j + u(x^3 - \lambda_0)^r v(x) \rangle$ be a symbol-pair λ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$, where $1 \leq j \leq p^s - 1$, $0 \leq r < j$, and either $v(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \lambda \rangle}$ or 0. Then, C_3 is not an MDS symbol-pair λ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$.*

Proof. We consider two cases as follows:

Case 1: If $v(x) = 0$ and $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq T \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then we have $\mathrm{d_{sp}}(C_3) = 2(\delta+1)p^{\xi}$. Thus, C_3 is an MDS symbol-pair code if $|C_3| = p^{2m(3p^s - \mathrm{d_{sp}}(C_3)+2)}$, i.e., $p^{6m(p^s - j)} = p^{2m(3p^s - \mathrm{d_{sp}}(C_3)+2)}$, i.e., $3j = \mathrm{d_{sp}}(C_3) - 2$, i.e., $3T = \mathrm{d_{sp}}(C_3) - 2$.

We see that

$$\begin{aligned}3T &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\ &\geq 3(p^{\xi+1} - p + (\delta-1) + 1) \text{ (equality when } \xi = s-1)\\ &\geq 3(\delta+1)p^{\xi} - 3(\delta+1) + 3(\delta-1) + 3\\ &\quad \text{(equality when } p-1=\delta)\\ &\geq 2(\delta+1)p^{\xi} - 3 + (\delta+1)p^{\xi}\\ &> 2(\delta+1)p^{\xi} - 2.\end{aligned}$$

Thus, C_3 is not MDS.

Case 2: If $v(x) \neq 0$ and $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq T \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then we consider two subcases as follows:

Subcase 2.1: If $1 \leq j \leq \frac{p^s+r}{2}$, then $T = j$. Thus, C_3 is an MDS symbol-pair code if $|C_3| = p^{2m(3p^s - \mathrm{d}_{\mathrm{sp}}(C_3)+2)}$, i.e., $p^{6m(p^s-j)} = p^{2m(3p^s-\mathrm{d}_{\mathrm{sp}}(C_3)+2)}$, i.e., $3j = \mathrm{d}_{\mathrm{sp}}(C_3) - 2$, i.e., $3T = \mathrm{d}_{\mathrm{sp}}(C_3) - 2$. We see that

$$\begin{aligned}3T &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\ &\geq 3p^{\xi+1} - 3p + 3(\delta-1) + 3 \text{ (equality when } \xi = s-1)\\ &\geq 3(\delta+1)p^{\xi} - 3(\delta+1) + 3(\delta-1) + 3\\ &\quad \text{(equality when } p-1=\delta)\\ &> \mathrm{d}_{\mathrm{sp}}(C_3) - 2.\end{aligned}$$

Thus, C_3 is not MDS.

Subcase 2.2: If $\frac{p^s+r}{2} < j \leq p^s - 1$, then $T = p^s - j + r$. Hence, C_3 is MDS if $|C_3| = p^{2m(3p^s-\mathrm{d}_{\mathrm{sp}}(C_3)+2)}$, i.e., $p^{3m(p^s-r)} = p^{2m(3p^s-\mathrm{d}_{\mathrm{sp}}(C_3)+2)}$, i.e., $3r = 2\,\mathrm{d}_{\mathrm{sp}}(C_3) - 3p^s - 4$, i.e., $3p^s + 3r = 2\,\mathrm{d}_{\mathrm{sp}}(C_3) - 4$, i.e., $3p^s - 3j + 3r = 2\,\mathrm{d}_{\mathrm{sp}}(C_3) - 3j - 4$, i.e., $3T = 2\,\mathrm{d}_{\mathrm{sp}}(C_3) - 3j - 4$. We have

$$\begin{aligned}3T &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\ &\geq 3p^{\xi+1} - 3p^s + 3(\delta-1) + 3 \text{ (equality when } \xi = s-1)\\ &\geq 6(\delta+1)p^{\xi} - 3p^s - (\delta+1) + 3(\delta-1) + 3\\ &\geq 6(\delta+1)p^{\xi} - 3p^s + 2\delta - 1\\ &\geq 2\,\mathrm{d}_{\mathrm{sp}}(C_3) - 3(j+1) + 2\delta - 1 + 2(\delta+1)p^{\xi}\\ &\geq 2\,\mathrm{d}_{\mathrm{sp}}(C_3) - 3j - 4 + 2\delta\\ &> 2\,\mathrm{d}_{\mathrm{sp}}(C_3) - 3j - 4.\end{aligned}$$

Therefore, C_3 is not MDS. □

Next, we explore the MDS λ-constacyclic codes of Type 4.

Theorem 11. *Let $C_4 = \langle (x^3-\lambda_0)^j + u(x^3-\lambda_0)^r v(x), u(x^3-\lambda_0)^{\omega} \rangle$ be a symbol-pair λ-constacyclic code of Type 4 of length $3p^s$ over $\mathcal{R}$, where $1 \leq j \leq p^s - 1$, $0 \leq r < j$, either $v(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s}-\lambda \rangle}$ or 0, $\deg(v) \leq \omega - r - 1$, $\omega < T$, and T is the smallest integer satisfying $u(x^3-\lambda_0)^T \in \langle (x^3-\lambda_0)^j + u(x^3-\lambda_0)^r v(x) \rangle$, i.e., $T = j$, if $v(x) = 0$, otherwise $T = \min\{j, p^s - j + r\}$. Then, C_4 is not MDS.*

Proof. If $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq \omega \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then symbol-pair distance is $\mathrm{d}_{\mathrm{sp}}(C_4) = 2(\delta+1)p^{\xi}$. Thus, C_4 is an MDS symbol-pair λ-constacyclic code if

$|C_4| = p^{2m(3p^s - d_{sp}(C_4)+2)}$, i.e., $p^{3m(2p^s-j-\omega)} = p^{2m(3p^s-d_{sp}(C_4)+2)}$, i.e., $3\omega = 2\, d_{sp}(C) - 3j - 4$. Now,

$$\begin{aligned}
3\omega &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\
&\geq 3p^{\xi+1} - 3p + 3(\delta-1) + 3 \text{ (equality when } \xi = s-1)\\
&\geq 6(\delta+1)p^{\xi} - 3p^{\xi+1} - 3(\delta+1) + 3(\delta-1) + 3\\
&\text{(equality when } p-1=\delta)\\
&\geq 6(\delta+1)p^{\xi} - 3p^s - 3\\
&\geq 4(\delta+1)p^{\xi} - 3p^s - 3 + 2(\delta+1)p^{\xi}\\
&> 2\, d_{sp}(C) - 3j - 4.
\end{aligned}$$

Therefore, C_4 is not MDS, as required. □

In Theorem 6, we compute the symbol-pair distance of $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where $(\alpha + u\beta)$ is not a cube in $\mathcal{R}$. Using Theorem 6, as an application, we have the following theorem.

Theorem 12. *Let C be a $(\alpha + u\beta)$-constacyclic code of length $3p^s$ over $\mathcal{R}$, where $(\alpha+u\beta)$ is not a cube in $\mathcal{R}$, i.e., $C = \langle (x^3 - \alpha_1)^j \rangle$ for $j \in \{0, 1, \ldots, 2p^s\}$, where $\alpha_1 \in \mathbb{F}_{p^m}$ such that $\alpha = \alpha_1^{p^s}$. Then, C is not an MDS symbol-pair $(\alpha + u\beta)$-constacyclic code of length $3p^s$ over $\mathcal{R}$.*

Proof. By using the result in Theorem 1 (part 3), we have $|C| = p^{3m(2p^s-j)}$.

Case 1: When $0 \leq j \leq p^s$, by Theorem 6, $d_{sp}(C) = 3$. Hence, C is an MDS symbol-pair code if $|C| = p^{2m(3p^s-d_{sp}(C)+3)}$ i.e., $p^{3m(2p^s-j)} = p^{2m(3p^s-3+3)}$, i.e., $6p^s - 3j = 6p^s$, i.e., $j = 0$. Thus, $C = \langle 1 \rangle$ is MDS.

Case 2: When $2p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq j \leq 2p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then $d_{sp}(C) = 2(\delta+1)p^{\xi}$. Therefore, C is an MDS symbol-pair code if $|C| = p^{2m(3p^s-d_{sp}(C)+3)}$ i.e., $p^{3m(2p^s-j)} = p^{2m(3p^s-d_{sp}(C)+3)}$ i.e., $6p^s - 3j = 6p^s - d_{sp}(C) + 3)$ i.e., $3j = d_{sp}(C) - 3$. Now, we have

$$\begin{aligned}
3j &\geq 3(2p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\
&\geq 6p^{\xi+1} - 3p + 3(\delta-1) + 3 \text{ (equality when } \xi = s-1)\\
&\geq 6(\delta+1)p^{\xi} - 3(\delta+1) + 3(\delta-1) + 3\\
&\text{(equality when } p-1=\delta)\\
&\geq 2(\delta+1)p^{\xi} - 3 + 4(\delta+1)p^{\xi}\\
&> 2(\delta+1)p^{\xi} - 3.
\end{aligned}$$

Hence, $3j > d_{sp}(C) - 3$. Thus, C is not MDS. □

5. Conclusions

In this paper, the symbol-pair distance of all γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where γ is not a cube in $\mathbb{F}_{p^m}$ is given (Theorems 3–5). The symbol-pair distance of $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$ is obtained in Theorem 6, where $(\alpha + u\beta)$ is not a cube in $\mathcal{R}$. Example 1 gives us some examples of symbol-pair distance γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where γ is not a cube in $\mathbb{F}_{p^m}$. The necessary and sufficient conditions for MDS symbol-pair codes of length $3p^s$ over $\mathcal{R}$ are provided in Theorems 8–12.

For future work, it will be very interesting to study symbol-pair distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is a cube in $\mathcal{R}$. In the near future, we will discuss the b-symbol metrics for all constacyclic codes of length $3p^s$ over $\mathcal{R}$, and as an application, we will identify all MDS constacyclic codes of length $3p^s$, with respect to b-symbol distances.

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