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Entire Gaussian Functions: Probability of Zeros Absence

Andriy Kuryliak *,† and Oleh Skaskiv †

Faculty of Mechanics and Mathematics, Ivan Franko National University of Lviv, 79000 Lviv, Ukraine
* Correspondence: andriykuryliak@gmail.com
† These authors contributed equally to this work.

Abstract: In this paper, we consider a random entire function of the form $f(z,\omega)=\sum_{n=0}^{+\infty}\varepsilon_n(\omega_1)\times\xi_n(\omega_2)f_nz^n$, where (ε_n) is a sequence of independent Steinhaus random variables, (ξ_n) is the a sequence of independent standard complex Gaussian random variables, and a sequence of numbers $f_n\in\mathbb{C}$ is such that $\overline{\lim\limits_{n\to+\infty}}\sqrt[n]{|f_n|}=0$ and $\#\{n\colon f_n\neq 0\}=+\infty$. We investigate asymptotic estimates of the probability $P_0(r)=P\{\omega\colon f(z,\omega)$ has no zeros inside $r\mathbb{D}\}$ as $r\to+\infty$ outside of some set E of finite logarithmic measure, i.e., $\int_{E\cap[1,+\infty)}d\ln r<+\infty$. The obtained asymptotic estimates for the probability of the absence of zeros for entire Gaussian functions are in a certain sense the best possible result. Furthermore, we give an answer to an open question of A. Nishry for such random functions.

Keywords: Gaussian entire functions; Steinhaus entire functions; zeros distribution of random entire functions

MSC: 30B20; 30D35; 30E15

1. Introduction: Notations and Preliminaries

One of the problems of random functions is investigation of value distribution of such functions and also the asymptotic properties of the probability of the absence of zeros in a disc ("hole probability"). These problems were considered in the papers of J. E. Littlewood and A. C. Offord [1–6]; M. Sodin and B. Tsirelson [7–9]; Yu. Peres and B. Virag [10]; P. V. Filevych and M. P. Mahola [11–13]; M. Sodin [14,15]; F. Nazarov, M. Sodin, and A. Volberg [16,17]; M. Krishnapur [18]; A. Nishry [19–25]; and many others [26].

So, in [9] they considered a random entire function of the form

$$\psi(z,\omega)=\sum_{k=0}^{+\infty}\xi_k(\omega)\frac{z^k}{\sqrt{k!}},\tag{1}$$

where $\{\xi_k(\omega)\}$ are independent complex valued random variables defined on the Steinhaus probability space $(\Omega,\mathcal{A},P)$, that is $\Omega=[0,1]$, P is the Lebesgue measure on $\mathbb{R}$ and $\mathcal{A}$ is the σ-algebra of Lebesgue measurable subsets of Ω.

We denote by $\mathcal{N}_{\mathbb{C}}(0,1)$ the class of sequences of independent random complex-valued variables (ξ_k) with standard Gaussian distribution in the complex plane, i.e., this is the distribution with the density function of the form

$$p_{\xi_k}(z)=\frac{1}{\pi}e^{-|z|^2},\ z\in\mathbb{C},\ k\in\mathbb{Z}_+.$$

Let (c_n), $c_n=c_n(\omega)$, be the zeros of of the function $\psi(z,\omega)$ of form (1). For $r>0$ let us denote $n_\psi(r,\omega)=\sum\limits_{|c_n|\le r}1$ as the counting function of zeros of the function $\psi(z,\omega)$ in the disk $r\mathbb{D}:=\{z\colon |z|<r\}$. Then [9] for any $\delta\in(0,1/4]$ and all $r\ge 1$ the following inequality holds

$$P\left\{\omega\colon\left|\frac{n(r,\omega)}{r^2}-1\right|\ge\delta\right\}\le\exp(-c(\delta)r^4),$$

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where the constant $c(\delta)$ depends only on δ. Furthermore, in [9] it was investigated the probability of absence of zeros of the function $\psi(z,\omega)$,

$$P_0(r) = P\{\omega\colon n_\psi(r,\omega) = 0\}, \quad p_0(r) = \ln^- P_0(r),$$

where $\ln^- x := -\min\{\ln x; 0\}$. In particular, it was proved in [9] that there exist constants $c_1, c_2 > 0$ such that

$$\exp(-c_1 r^4) \le P_0(r) \le \exp(-c_2 r^4) \quad (r \ge 1).$$

Furthermore, in [9] the authors put the following question: Does the limit exist?

$$\lim_{r\to+\infty} \frac{\ln^- P_0(r)}{r^4} \ ?$$

We find the answer to this question in [20]. For the function $\psi(z,\omega)$ it was proved that

$$\lim_{r\to+\infty} \frac{\ln^- P_0(r)}{r^4} = \frac{e^2}{4}.$$

Let $K \subset \mathbb{C}$ be some compact such that $0 \notin K$. In [19], it was proved that if all of $\xi_n(\omega)\colon \xi_n(\omega) \subset K$, there exists $r_0(K) < +\infty$ such that $\psi(z,\omega)$ must vanish somewhere in the disc $r_0\mathbb{D}$.

For the function of the form (1) one can fix the disc of radius r and ask for the asymptotic behaviour of $P\{\omega\colon n_\psi(r,\omega) \ge m\}$ as $m \to +\infty$. So in [18] it was proved~~,~~ that for any $r > 0$, we obtain

$$\ln P\{\omega\colon n_\psi(r,\omega) \ge m\} = -\frac{1}{2} m^2 \ln m (1 + o(1)) \quad (m \to +\infty).$$

Very large deviations of zeros of function (1) were also considered in [17]. There we find such a relation

$$\lim_{r\to+\infty} \frac{\ln\Big(-\ln\big(P\{\omega\colon |n_\psi(r,\omega) - r^2| > r^\alpha\}\big)\Big)}{\ln r} = \begin{cases} 2\alpha - 1, & \frac{1}{2} \le \alpha \le 1; \\ 3\alpha - 2, & 1 \le \alpha \le 2; \\ 2\alpha, & \alpha \ge 2. \end{cases}$$

In the papers [21,23] an Gaussian entire functions of the following general form

$$f(z,\omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) f_n z^n,$$

were considered, where $f_0 \ne 0$, $\overline{\lim\limits_{n\to+\infty}} \sqrt[n]{|f_n|} = 0$, $(\xi_n(\omega_2)) \in \mathcal{N}_{\mathbb{C}}(0,1)$ is a sequence of the independent standard Gaussian random variables. For $\varepsilon > 0$ there exists [21,23] a set of finite logarithmic measure $E \subset (1,+\infty)$ ($\int_E \frac{dr}{r} < +\infty$) such that

$$q(r) - q^{1/2+\varepsilon}(r) \le p_0(r) \le q(r) + q^{1/2+\varepsilon}(r) \tag{2}$$

for all $r \in (1,+\infty) \setminus E$, where $q(r) = 2\sum\limits_{n=0}^{+\infty} \ln^+(|f_n| r^n)$. Remark [22], that there is a Gaussian entire function $f(z,\omega)$ and a set E of infinite Lebesgue's measure such that

$$p_0(r) \ge 2q(r) - c\sqrt{q(r)},\ r \in E,\ C > 0,$$

that is, the finiteness of the Lebesgue measure of the exceptional set in the above statement is a necessary condition.

Similar results for Gaussian analytic functions in the unit disc can be found in [10,15,18,23,27].

Furthermore, in [23] (p. 119) they formulated the following question: *Is the error term in inequality (2) optimal for a regular sequence of coefficients* $\{f_n\}$? In this paper, we obtain instead of inequalities (2) the following asymptotic estimates

$$0 \le \varliminf_{\substack{r\to+\infty\\ r\notin E}} \frac{\ln(p_0(r)-q(r))}{\ln q(r)}, \quad \varlimsup_{\substack{r\to+\infty\\ r\notin E}} \frac{\ln(p_0(r)-q(r))}{\ln q(r)} \le \frac{1}{2}, \tag{3}$$

$$\lim_{\substack{r\to+\infty\\ r\notin E}} \frac{\ln(p_0(r)-q(r))}{\ln N(r)} = 1 \tag{4}$$

in the case of general coefficients $f_n \in \mathbb{C}$ $(n \in \mathbb{Z}_+)$, $f_0 \ne 0$, such that $\lim\limits_{n\to+\infty} \sqrt[n]{|f_n|} = 0$, $\#\{n\colon f_n \ne 0\} = +\infty$. However, this inequality is proved for the functions of the form

$$f(z,\omega) = \sum_{n=0}^{+\infty} \varepsilon_n(\omega_1)\xi_n(\omega_2) f_n z^n. \tag{5}$$

Here, $\varepsilon_n(\omega_1) = e^{i\theta_n(\omega_1)}$, (θ_n) is a sequence of the independent random variables uniformly distributed on $[-\pi,\pi)$, $(\xi_n(\omega_2)) \in \mathcal{N}_{\mathbb{C}}(0,1)$. We prove that there exists a set E of finite logarithmic measure such that inequalities (3) hold.

An earlier version of the main statement of this paper (Theorem 5) is available in our preprint [28] and was obtained for random entire functions of the form

$$f(z,\omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) f_n z^n. \tag{6}$$

However, the proof in the preprint [28] contains gaps in reasoning.

2. Notations

For $r>0$, $\delta \in \mathbb{R}$ denote

$$\begin{gathered}
\mathcal{N}' = \{n\colon f_n = 0\}, \quad \mathcal{N}_\delta(r) = \{n\colon \ln(|f_n|r^n) > -\delta n\},\\
N_\delta(r) = \#\mathcal{N}_\delta(r), \quad \mathcal{N}(r) = \mathcal{N}_0(r), \quad N(r) = N_0(r),\\
m_\delta(r) = \sum_{n\in\mathcal{N}_\delta(r)} n, \quad m(r) = m_0(r) = \sum_{n\in\mathcal{N}(r)} n,\\
\mu_f(r) = \max\{|f_n|r^n\colon n \in \mathbb{Z}_+\}, \quad \nu_f(r) = \max\{n\colon \mu_f(r) = |f_n|r^n\},\\
M_f(r) = \max\{|f(z)|\colon |z| \le r\}, \quad \mathfrak{M}_f^2(r) = \sum_{n=0}^{+\infty} |f_n|^2 r^{2n}.
\end{gathered}$$

Remark, $q(r) = 2\sum_{n=0}^{+\infty} \ln^+(|f_n|r^n) = 2\sum_{n\in\mathcal{N}(r)} \ln(|f_n|r^n)$.

3. Auxiliary Statements

Lemma 1 (Borel–Nevanlinna, [29] (p. 90))**.** *Let* $u(r)$ *be a nondecreasing continuous function on* $[r_0;+\infty)$ *and* $\lim\limits_{r\to+\infty} u(r) = +\infty$, *and* $\varphi(u)$ *be a continuous nonincreasing positive function defined on* $[u_0;+\infty)$ *and (1)* $u_0 = u(r_0)$; *(2)* $\lim\limits_{u\to+\infty} \varphi(u) = 0$; *(3)* $\int_{u_0}^{+\infty} \varphi(u)du < +\infty$.

Then, the set

$$E = \{r \ge r_0\colon u(r+\varphi(u(r))) < u(r)+1\}.$$

has a finite measure.

We need the following elementary corollary of this lemma.

Lemma 2. *There exists a set $E \subset (1;+\infty)$ of finite logarithmic measure such that*

$$m(re^{\delta})\exp\{-2\sqrt{\ln m(r)}\} < em(r) < m(re^{-\delta})\exp\{2\sqrt{\ln m(r)}\}$$

for all $r \in (1;+\infty) \setminus E$, where $\delta = \frac{1}{2\ln m(r)}$.

Lemma 3. *Let $\varepsilon > 0$. There is a set $E \subset (1;+\infty)$ of finite logarithmic measure such that*

$$N(r) < q^{1/2}(r)\exp\{(1+\varepsilon)\sqrt{\ln q(r)}\} \tag{7}$$

for all $r \in (1;+\infty) \setminus E$.

Proof. Remark that (see also [20])

$$N_{-\delta}(r) = \#\{n\colon |f_n|r^n \geq e^{\delta n}\} = \#\{n\colon |f_n|(re^{-\delta})^n \geq 1\} = N(re^{-\delta}).$$

If $\mathcal{N}(r) = \{n_k\colon 1 \leq k \leq N(r)\}$, where $n_k < n_{k+1}$ $(1 \leq k \leq N(r)-1)$, then $n_k \geq k-1$ $(1 \leq k \leq N(r))$ and

$$m(r) \geq \sum_{k=0}^{N(r)-1} k = \frac{(N(r)-1)N(r)}{2} > \frac{N^2(r)}{e}$$

for all $r > r_0$, where r_0 such that $N(r_0) > 4$. So, by Lemma 2 we obtain

$$\frac{q(r)}{2} = \sum_{n\in\mathcal{N}(r)} \ln(|f_n|r^n) \geq \sum_{n\in\mathcal{N}_{-\delta}(r)} \ln(|f_n|r^n) \geq \sum_{n\in\mathcal{N}_{-\delta}(r)} n\delta =$$
$$= \delta m(re^{-\delta}) > \frac{e}{2\ln m(r)} m(r)\exp\{-2\sqrt{\ln m(r)}\}.$$

for $r \in (r_0;+\infty) \setminus E$. Then,

$$\ln q(r) > 1 + \ln m(r) - 2\sqrt{\ln m(r)} - \ln\ln m(r)$$

and for $r \in (r_2;+\infty) \setminus E$, where r_2 is large enough, we obtain $\ln m(r) < 2\ln q(r)$. Therefore, for any $\varepsilon > 0$

$$q(r) > em(r)\exp\{-2\sqrt{\ln m(r)} - \ln\ln m(r)\} >$$
$$> e\frac{N^2(r)}{e}\exp\{-2\sqrt{(1+\varepsilon)\ln q(r) - \ln((1+\varepsilon)\ln q(r))}\} >$$
$$> N^2(r)\exp\{-(2+2\varepsilon)\sqrt{\ln q(r)}\}$$

as $r \to +\infty$ outside some set of finite logarithmic measure. □

The exponent $1/2$ in the inequality (7) can not be replaced by a smaller number.

Lemma 4. *There exist a random entire function of form (5) and a set $E \subset (1;+\infty)$ of finite logarithmic measure such that*

$$N(r) > \frac{q^{1/2}(r)}{\ln^{5/2} q(r)}$$

for all $r \in (1;+\infty) \setminus E$.

Proof. We will consider the following entire function

$$f(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{(\frac{n}{2})^{\frac{n}{2}}}.$$

The function $y(n) = \ln f_n = -\frac{n}{2}\ln(\frac{n}{2})$ is concave function and the sequence (f_n) is log-concave ([21,27]). Since $m!e^m > m^m$ $(m \geq 1)$, one has

$$M_f(r) > 1 + \sum_{m=1}^{+\infty} \frac{r^{2m}}{m^m} > 1 + \sum_{m=1}^{+\infty} \frac{r^{2m}}{m!e^m} = \exp\Big\{\frac{r^2}{e}\Big\}, \ \ln M_f(r) > \frac{r^2}{e}.$$

By Wiman–Valiron's theorem there exists a set E_1 of finite logarithmic measure such that $M_f(r) \leq \mu_f(r)\ln^{1/2+\varepsilon}\mu_f(r)$ for all $r \in (1;+\infty)\setminus E_1$. Thus, for all $r \in (1;+\infty)\setminus E_1$ we obtain $\ln\mu_f(r) + \ln\ln\mu_f(r) > \ln M_f(r) > r^2/e$, $\ln\mu_f(r) > r^2/2e$ and finally

$$\frac{r^2}{2e} < \ln\mu_f(r) = \ln f_\nu + \nu_f(r)\ln r, \ \nu_f(r) > \frac{1}{\ln r}\Big(\frac{r^2}{2e} - \ln f_\nu\Big) > r, \ r \to +\infty.$$

Therefore, outside some set E of finite logarithmic measure we obtain ([21])

$$q(r) < 2(N(r)+1)\ln\mu_f(r) < \ln^2\mu_f(r)(\ln\ln\mu_f(r))^2 = \ln^3 r \frac{\ln^2\mu_f(r)}{\ln^2 r}\frac{(\ln\ln\mu_f(r))^2}{\ln r} <$$
$$< \ln^3 r\nu_f^2(r)\ln^2\nu_f(r) < \nu_f^2(r)\ln^5\nu_f(r) < N^2(r)\ln^5 N(r) < N^2(r)\ln^5 q(r).$$

Hence,

$$N(r) > \sqrt{\frac{q(r)}{\ln^5 q(r)}}.$$

□

By $\mathbf{E}\xi$ we denote the mathematical expectation of a random variable ξ. Furthermore, we will use the following lemma.

Lemma 5. *Let $\big(\eta_n(\omega)\big)$ be a sequence of independent non-negative identically distributed random variables, such that $\mathbf{E}\eta_n < +\infty$ and $\mathbf{E}(\frac{1}{\eta_n}) < +\infty$, $n \in \mathbb{Z}_+$. Then*

$$P\Big\{\omega\colon (\exists N^*(\omega))(\forall n > N^*(\omega))\ \Big[\frac{1}{n} \leq \eta_n(\omega) \leq n\Big]\Big\} = 1.$$

Proof. Let $F_\eta(t) = F_{\eta_n}(t)$ be the distribution function of the random variable η_n, $n \in \mathbb{Z}_+$. Denote $B_m = \{\omega\colon |\eta_m(w)| \geq m\}$, $m \in \mathbb{Z}_+$. Then

$$\sum_{m=1}^{+\infty} P\{\omega\colon |\eta_m(w)| \geq m\} = \sum_{m=1}^{+\infty}\int_{|t|\geq m} dF_{|\eta|}(t) = \sum_{m=1}^{+\infty}\sum_{s=m}^{+\infty}\int_{|t|\in[s,s+1)} dF_{|\eta|}(t) =$$
$$= \sum_{s=1}^{+\infty}\sum_{m=1}^{s}\int_{|t|\in[s,s+1)} dF_{|\eta|}(t) = \sum_{s=1}^{+\infty} s\int_{|t|\in[s,s+1)} dF_{|\eta|}(t) \leq$$
$$\leq \sum_{s=1}^{+\infty}\int_{|t|\in[s,s+1)} |t|dF_{|\eta|}(t) \leq \mathbf{E}|\eta| < +\infty.$$

Therefore, $\sum_{m=1}^{+\infty} P(B_m) < +\infty$. So, by the Borel–Cantelli lemma with probability that is equal to 1 only finite quantity of the events B_n can occur. That A_1 exists such that

$$P(A_1) = P\Big\{\omega\colon (\exists N_1^*(\omega))(\forall n > N_1^*(\omega))\ \Big[|\eta_n(\omega)| \leq n\Big]\Big\} = 1.$$

Since $\mathbf{E}(\frac{1}{|\eta|}) < +\infty$, we similarly obtain for the random variable $\frac{1}{|\eta(\omega)|}$

$$P(A_2) = P\Big\{\omega\colon (\exists N_2^*(\omega))(\forall n > N_2^*(\omega))\ \Big[\frac{1}{|\eta_n(\omega)|} \le n\Big]\Big\} =$$
$$= P\Big\{\omega\colon (\exists N_2^*(\omega))(\forall n > N_2^*(\omega))\ \Big[|\eta_n(\omega)| \ge \frac{1}{n}\Big]\Big\} = 1.$$

Finally,

$$P(A_1 \cap A_2) = P\Big\{\omega\colon (\exists N^*(\omega))(\forall n > N^*(\omega))\ \Big[\frac{1}{n} \le |\eta_n(\omega)| \le n\Big]\Big\} = 1.$$

□

4. Upper and Lower Bounds for $p_0(r)$

Theorem 1. *Let $\varepsilon > 0$ and $f(z,\omega)$ be random entire function of the form (5) with $f_0 \ne 0$. There exists a set $E \subset (1;+\infty)$ of finite logarithmic measure such that*

$$p_0(r) \le q(r) + N(r)\exp\{(2+\varepsilon)\sqrt{\ln N(r)}\} \tag{8}$$

for all $r \in (1;+\infty) \setminus E$.

Proof. Similarly as in [20], for fixed r we consider the event $A = \cap_{i=1}^4 A_i$, where

$$A_1 = \Big\{\omega\colon |\xi_0(\omega_2)| \ge \frac{2eN^{1/3}(r)\exp\{2\sqrt{\ln N(r)}\}}{|f_0|}\Big\},$$
$$A_2 = \Big\{\omega\colon (\forall n \in \mathcal{N}(r) \setminus \{0\})\Big[|\xi_n(\omega_2)| \le \frac{1}{|f_n| r^n N^{2/3}(r)}\Big]\Big\},$$
$$A_3 = \Big\{\omega\colon (\forall n \in \mathcal{N}_\delta(r) \setminus (\mathcal{N}(r) \cup \{0\}))\Big[|\xi_n(\omega_2)| \le \frac{1}{N^{2/3}(r)}\Big]\Big\},$$
$$A_4 = \Big\{\omega\colon (\forall n \notin \mathcal{N}_\delta(r) \cup \mathcal{N}' \cup \{0\})\Big[|\xi_n(\omega_2)| \le n\Big]\Big\},\ \delta = \frac{1}{2\ln N(r)}.$$

If A occurs, then for $r \notin E$ we obtain

$$|\varepsilon_0(\omega_1)\xi_0(\omega_2)f_0| - \Big|\sum_{n=1}^{+\infty} \varepsilon_n(\omega_1)\xi_n(\omega_2)f_n r^n\Big| \ge 2eN^{1/3}(r)\exp\{2\sqrt{\ln N(r)}\} -$$
$$- \sum_{n\in\mathcal{N}(r)} \frac{|f_n|r^n}{|f_n|r^n N^{2/3}(r)} - \sum_{n\in\mathcal{N}_\delta(r)\setminus\mathcal{N}(r)} \frac{|f_n|r^n}{N^{2/3}(r)} - \sum_{n\notin\mathcal{N}_\delta(r)\cup\mathcal{N}'} ne^{-n\delta} >$$
$$> 2eN^{1/3}(r)\exp\{2\sqrt{\ln N(r)}\} - \sum_{n\in\mathcal{N}_\delta(r)} \frac{1}{N^{2/3}(r)} - \int_1^{+\infty} xe^{-\delta x}dx >$$
$$> 2eN^{1/3}(r)\exp\{2\sqrt{\ln N(r)}\} - N^{1/3}(r) - eN^{1/3}(r)\exp\{2\sqrt{\ln N(r)}\} - 8\ln^2 N(r) > 0$$

as $r \to +\infty$, because

$$\int_1^{+\infty} xe^{-\delta x}dx = \frac{e^{-\delta}}{\delta^2}(\delta+1) < \frac{2}{\delta^2} = 8\ln^2 N(r).$$

So, we proved that first term dominants the sum of all the other terms inside $r\mathbb{D}$, i.e.,

$$|\varepsilon_0(\omega_1)\xi_0(\omega_2)f_0| > \Big|\sum_{n=1}^{+\infty} \varepsilon_n(\omega_1)\xi_n(\omega_2)f_n r^n\Big|. \tag{9}$$

If A occurs then the function $f(z,\omega)$ has no zeros inside $r\mathbb{D}$. Now we find a lower bound for the probability of the event A.

$$P(A_1) = \exp\Big\{-\frac{4e^2N^{2/3}(r)\exp\{4\sqrt{\ln N(r)}\}}{|f_0|^2}\Big\},$$
$$P(A_2) \geq \prod_{n\in\mathcal{N}(r)} \frac{1}{2|f_n|^2r^{2n}N^{4/3}(r)} = \prod_{n\in\mathcal{N}(r)} \frac{1}{2|f_n|^2r^{2n}}\times$$
$$\times\exp\{-N(r)\ln(N^{4/3}(r))\} = \exp\Big\{-q(r) - \frac{4}{3}N(r)\ln N(r) - N(r)\ln 2\Big\},$$
$$P(A_3) \geq \prod_{n\in\mathcal{N}(re^\delta)} \frac{1}{2N^{4/3}(r)} \geq \exp\Big\{-N(re^\delta)\ln(2N^{4/3}(r))\Big\}\geq$$
$$\geq \exp\Big\{-eN(r)\exp\{2\sqrt{N(r)}\}\ln(2N^{4/3}(r))\Big\},$$
$$P(A_4) = P\{\omega\colon (\forall n \notin \mathcal{N}_\delta(r)\cup\mathcal{N}'\cup\{0\})[|\xi_n(\omega_2)| < n]\} \geq$$
$$\geq 1 - \sum_{n\notin\mathcal{N}_\delta(r)\cup\mathcal{N}'\cup\{0\}} e^{-n^2} > \frac{1}{2},\ r\to+\infty\ (r\notin E).$$

From the definition of $\ln^- x$ and independence of events A_j, $j\in\{1,2,3,4\}$ we deduce
$$\ln^- P(A) = \sum_{n=1}^{4} \ln^- P(A_n).$$

Therefore, it follows from $A \subset \{\omega\colon n(r,\omega) = 0\}$ that for any $\varepsilon > 0$ and for every $r\in[r_0,+\infty)\setminus E$ we obtain

$$p_0(r) \leq \ln^- P(A) \leq$$
$$\leq \ln 2 + \frac{4e^2N^{2/3}(r)\exp\{4\sqrt{\ln N(r)}\}}{|f_0|^2} + q(r) + 2N(r)\ln N(r) + N(r)\ln 2+$$
$$+eN(r)\exp\{2\sqrt{N(r)}\}\ln(2N^{4/3}(r)) \leq q(r) + N(r)\exp\{(2+\varepsilon)\sqrt{N(r)}\}.$$

□

A random entire function of the form

$$g(z,\omega_1) = \sum_{n=0}^{+\infty} e^{i\theta_n(\omega_1)} f_n z^n, \tag{10}$$

where $f_0 \neq 0$ and independent random variables $\theta_n(\omega_1)$ are uniformly distributed on $[-\pi,\pi)$, was considered in [13]. For such functions there were proved the following statements.

Theorem 2 ([11])**.** *Let $g(z,\omega_1)$ be a random entire function of the form (10). Then, for $r > r_0$ and all ω_1 we obtain*

$$N_g(r,\omega_1) \leq \frac{1}{2e} + \ln\mathfrak{M}_g(r),$$

where

$$N_g(r,\omega_1) = \frac{1}{2\pi}\int_0^{2\pi} \ln|g(re^{i\alpha},\omega_1)|d\alpha - \ln|f_0|.$$

Theorem 3 ([13])**.** *There is an absolute constant $C > 0$ such that for a function $g(z,\omega_1)$ of the form (10) P_1-almost surely we have*

$$\ln\mathfrak{M}_g(r) \leq N_g(r,\omega_1) + C\ln N_g(r,\omega_1),\ r_0(\omega_1) \leq r < +\infty. \tag{11}$$

Let $P = P_1 \times P_2$ be a direct product of the probability measures P_1 and P_2 defined on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$. Here, $\mathcal{A}_1 \times \mathcal{A}_2$ is the minimal σ-algebra, which contains all $A_1 \times A_2$ such that $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Let $\varepsilon_n(\omega_1) = e^{i\theta_n(\omega_1)}$, (θ_n) is a sequence of the independent random variables uniformly distributed on $[-\pi, \pi)$ on $(\Omega_1, \mathcal{A}_1, P_1)$, $\xi_n(\omega_2) \in \mathcal{N}_{\mathbb{C}}(0, 1)$ on $(\Omega_2, \mathcal{A}_2, P_2)$, where $(\Omega_1, \mathcal{A}_1, P_1)$, $(\Omega_2, \mathcal{A}_2, P_2)$ are two probability spaces.

Corollary 1. *Let $(\zeta_n(\omega_2))$ be a sequence of independent identically distributed random variables such that for any $n \in \mathbb{Z}_+$ the density function of the distribution of the random variable $\eta = \zeta_n$ has the form $p_\eta(z) = q(|z|)$ and $\mathbf{E}|\eta| < +\infty$, $\mathbf{E}(\frac{1}{|\eta|}) < +\infty$. There exist an absolute constant $C > 0$ and a set $B \in \mathcal{A}$: $P(B) = 1$ such that for the functions $f(z, \omega) = \sum_{n=0}^{+\infty} \varepsilon_n(\omega_1)\zeta_n(\omega_2) f_n z^n$, $f_0 \neq 0$ and for all $\omega \in B$ and all $r \in [r_0(\omega); +\infty)$ we obtain*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\alpha}, \omega)|d\alpha - \ln|f_0\varepsilon_0(\omega_1)\zeta_0(\omega_2)| \geq$$
$$\geq \ln \mathfrak{M}_f(r, \omega_2) - (C+1)\ln\ln \mathfrak{M}_f(r, \omega_2).$$

Remark that, if density function of $\zeta_n(\omega_1)$ has the following form $p_{\zeta_n}(z) = q(|z|), n \in \mathbb{N}$, then $\arg \zeta_n(\omega_1)$ are uniformly distributed on $[-\pi, \pi)$. Really, for any $\alpha, \beta \in [-\pi, \pi)$: $\alpha < \beta$ we obtain

$$P_1(\omega_1\colon \zeta_n(\omega_1) \in \mathbb{C}) = \int\limits_{-\pi}^{\pi} d\varphi \int\limits_0^{+\infty} rq(r)dr = 2\pi \int\limits_0^{+\infty} rq(r)dr = 1,$$
$$P_1(\omega_1\colon \arg \zeta_n(\omega_1) \in (\alpha, \beta)) = \int\limits_{\alpha}^{\beta} d\varphi \int\limits_0^{+\infty} rq(r)dr = \frac{\beta - \alpha}{2\pi}.$$

Note that random variables $\xi_n(\omega_1)$ satisfies this condition (here $p_{\xi_k}(z) = q(|z|) = \frac{1}{\pi}e^{-|z|^2}$, $z \in \mathbb{C}$, $k \in \mathbb{Z}_+$ we have the following statement for the functions of the form (5).

Corollary 2. *There exist an absolute constant $C > 0$ and a set $B \in \mathcal{A}$: $P(B) = 1$ such that for the functions of the form (5) and for all $\omega \in B$ and all $r \in [r_0(\omega); +\infty)$ we obtain*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta}, \omega)|d\theta - \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \geq \ln \mathfrak{M}_f(r, \omega_2) - (C+1)\ln\ln \mathfrak{M}_f(r, \omega_2).$$

Proof of Corollary 1. It follows from Theorem 2 that $\ln N_g(r, \omega_1) \leq 1 + \ln\ln \mathfrak{M}_g(r)$ and by Theorem 3 we have ω_1

$$N_g(r, \omega_1) \geq \ln \mathfrak{M}_g(r) - C \ln N_g(r, \omega_1) \geq \ln \mathfrak{M}_g(r) - (C+1)\ln\ln \mathfrak{M}_g(r),$$

for $r_0(\omega_1) \leq r < +\infty$. Therefore,

$$P_1\{\omega\colon (\exists r_0(\omega_1))(\forall r > r_0(\omega_1))\ [N_g(r, \omega_1) \geq \ln \mathfrak{M}_g(r) - (C+1)\ln\ln \mathfrak{M}_g(r)]\} = 1.$$

Consider a random function $f(z, \omega_1, \omega_2)$ of the form (5). Define

$$A_f = \{(\omega_1, \omega_2)\colon (\exists r_0(\omega_1, \omega_2))(\forall r > r_0(\omega_1, \omega_2))$$
$$[N_f(r, \omega_1, \omega_2) \geq \ln \mathfrak{M}_f(r, \omega_2) - (C+1)\ln\ln \mathfrak{M}_f(r, \omega_2)]\},$$

where

$$\mathfrak{M}_f^2(r, \omega_2) = \sum_{n=0}^{+\infty} |\varepsilon_n(\omega_1)|^2 |\zeta_n(\omega_2)|^2 |a_n|^2 r^{2n} = \sum_{n=0}^{+\infty} |\zeta_n(\omega_2)|^2 |a_n|^2 r^{2n}.$$

Consider the events

$$F = \{\omega_2\colon (\forall n \in \mathbb{N})\ [\zeta_n(\omega_2) \neq 0]\},\ H = \Big\{\omega_2\colon \varlimsup_{n\to+\infty} \sqrt[n]{|f_n||\zeta_n(\omega_2)|} = 0\Big\}.$$

Then by Lemma 5 for $\eta_n = |\zeta_n|$, one has $P_2(H) = 1$. Since $\mathbf{E}(\frac{1}{\zeta_n}) < +\infty$, the probability of the event F

$$1 \geq P_2(F) \geq 1 - \sum_{n=0}^{+\infty} P_2\{\omega_2\colon \zeta_n(\omega_2) = 0\} = 1.$$

Denote $G = F \cap H$. So, $P_2(G) = 1$. Then, for fixed $\omega_2^0 \in G$

$$P_1(A_f(\omega_2^0)) := P_1\{\omega_1\colon (\exists r_0(\omega_1,\omega_2^0))(\forall r > r_0(\omega_1,\omega_2^0))$$
$$[N_f(r,\omega_1,\omega_2^0) \geq \ln \mathfrak{M}_f(r,\omega_2^0) - (C+1)\ln\ln \mathfrak{M}_f(r,\omega_2^0)]\} = 1.$$

It remains to use Fubini's theorem

$$P(A_f) = \int\limits_{\Omega_2} \Bigg(\int\limits_{A_f(\omega_2)} dP_1(\omega_1) \Bigg) dP_2(\omega_2) \geq \int\limits_{G} \Bigg(\int\limits_{A_f(\omega_2)} dP_1(\omega_1) \Bigg) dP_2(\omega_2) =$$
$$= \int\limits_{G} dP_2(\omega_2) = P_2(G) = 1.$$

□

Theorem 4. *Let f be a random entire function of the form (5) such that $f_0 \neq 0$. Then P_1-almost surely there is $r_0(\omega) > 0$ such that for all $r \in (r_0(\omega); +\infty)$ we obtain*

$$p_0(r) \geq q(r) + N(r)\ln N(r) - 4N(r).$$

Proof of Theorem 4. By Jensen's formula we reliably obtain

$$0 = \int_0^r \frac{n(t,\omega)}{t} dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta - \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|,$$
$$\ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta.$$

Therefore,

$$P\{\omega\colon n(r,\omega) = 0\} \leq P\Big\{\omega\colon \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta\Big\}.$$

We fix $r > r_0(\omega)$ and define

$$A = \Big\{\omega\colon \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta \geq$$
$$\geq \ln \mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln \mathfrak{M}_f(r,\omega_2) + \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\},$$
$$G_1(r) = \{\omega\colon \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \geq \ln\gamma(\omega_2)\},$$
$$G_2(r) = \Big\{\omega\colon \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta \leq \ln\gamma(\omega_2)\Big\},$$

where $r_0(\omega)$ is from Corollary 2 and $\gamma(\omega_2) > 1$. By this corollary we obtain that $P(A) = 1$.

Then, for $r > r_0(\omega)$

$$\overline{G_1}(r)\bigcap\overline{G_2}(r) = \Big\{\omega\colon \frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta},\omega)|d\theta > \ln\gamma(\omega_2) > \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\},$$
$$\overline{G_1}(r)\bigcap\overline{G_2}(r) \subset \Big\{\omega\colon \frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta},\omega)|d\theta \neq \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\},$$
$$G_1(r)\bigcup G_2(r) = \overline{\overline{G_1}\bigcap\overline{G_2}} \supset \Big\{\omega\colon \frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta},\omega)|d\theta = \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\}.$$

So, for $r > r_0(\omega)$

$$P\{\omega\colon n(r,\omega) = 0\} \le P(G_1\cup G_2) \le P(G_1) + P(G_2),\ r\to+\infty. \tag{12}$$

Put $\gamma(\omega_2) = C_1\cdot|f_0|\cdot|\xi_0(\omega_2)|,\ C_1 > 1$. Then we may calculate the probability of the event G_1

$$P(G_1) = P\Big\{\omega\colon\ \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \ge \ln C_1 + \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\} =$$
$$= P\Big\{\omega\colon\ \ln C_1 \le 0\Big\} = 0$$

and estimate the probability of the event G_2 as $r > r_0(\omega)$

$$P(G_2) = P(G_2\cap A) + P(G_2\cap\overline{A}) \le P(G_2\cap A) + P(\overline{A}) = P(G_2\cap A) =$$
$$= P\Big\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln\mathfrak{M}_f(r,\omega_2) +$$
$$+\ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \le \frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta},\omega)|d\theta \le \ln\gamma(r,\omega)\Big\} =$$
$$= P\Big\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln\mathfrak{M}_f(r,\omega_2) + \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \le$$
$$\le \ln C_1 + \ln|f_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\} =$$
$$= P\Big\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln\mathfrak{M}_f(r,\omega_2) \le \ln C_1\Big\} \le$$
$$\le P\Big\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) \le 2\ln C_1\Big\} = P\Big\{\omega\colon \mathfrak{M}_f(r,\omega_2) \le C_1^2\Big\} =$$
$$\le P\Big\{\omega\colon \sum_{n\in\mathcal{N}(r)}|\xi_n(\omega_2)|^2|f_n|^2r^{2n} \le C_1^4\Big\},\ r\to+\infty. \tag{13}$$

The distribution function of the random variable $|\xi_n(\omega_2)|$

$$F_{|\xi_n|}(x) = 1-\exp\{-x^2\},\ F_{|\xi_n|^2}(x) = F_{|\xi_n|}(\sqrt{x}) = 1-\exp\{-x\},$$
$$F_{|\xi_n|^2|f_n|^2r^{2n}}(x) = F_{|\xi_n|^2}\Big(\frac{x}{|f_n|^2r^{2n}}\Big) = 1-\exp\Big\{-\frac{x}{|f_n|^2r^{2n}}\Big\}$$

for $n\notin\mathcal{N}'$ and $x\in\mathbb{R}_+$. Then for the random vector $\eta(\omega_2) = (|\xi_1(\omega_2)|a_1r^{j_1},\ldots,|\xi_{j_k}(\omega_2)|a_{j_k}r^{j_k})$, $j_k\in\mathcal{N}(r)$, the density function

$$p_\eta(x) = \begin{cases}\prod\limits_{n\in\mathcal{N}(r)}\frac{1}{|f_n|^2r^{2n}}\exp\Big\{-\frac{x_n}{|f_n|^2r^{2n}}\Big\}, & x\in\mathbb{R}_+^{\mathcal{N}(r)},\\ 0, & x\notin\mathbb{R}_+^{\mathcal{N}(r)}.\end{cases}$$

So, for $r > r_0(\omega)$ we obtain

$$P\Big\{\omega\colon \sum_{n\in\mathcal{N}(r)} |\xi_n(\omega_2)|^2|f_n|^2 r^{2n} \le C_1^4\Big\} = P\{\omega\colon \eta(\omega_2)\in W(r)\} =$$
$$= \prod_{n\in\mathcal{N}(r)} \frac{1}{|f_n|^2 r^{2n}} \cdot \int\limits_{W(r)}\!\!\cdots\!\int \prod_{n\in\mathcal{N}(r)} \exp\Big\{-\frac{x_n}{|f_n|^2 r^{2n}}\Big\} dx_1\ldots dx_{N(r)} \le$$
$$\le \exp(-q(r))\cdot \mathrm{meas}_{N(r)} W(r), \tag{14}$$

where

$$W(r) = \Big\{x\in\mathbb{R}_+^{N(r)}\colon \sum_{n\in\mathcal{N}(r)} x_n \le C_1^4\Big\}.$$

For $C > 0$ by elementary calculation we obtain

$$\mathrm{meas}_n\Big\{x\in\mathbb{R}_+^n\colon \sum_{i=1}^n x_i \le C\Big\} = \frac{C^n}{n!}.$$

From this equality and Stirling's formula

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n \cdot \exp\Big\{-\frac{\theta_n}{12n}\Big\},\ \theta_n\in[0,1],\ n\in\mathbb{N},$$

it follows that the volume of the set $B(r)$

$$\ln\Big(\mathrm{meas}_{N(r)} W(r)\Big) \le -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln N(r) - N(r)\ln N(r) + \frac{1}{12N(r)} +$$
$$+N(r) + 4N(r)\ln C_1 \le -N(r)(\ln N(r) - 1 - 4\ln C_1).$$

Let us choose $C_1 = 2$. From (14) it follows $p_0(r) \ge q(r) + N(r)\ln N(r) - 4N(r)$, for $r > r_0(\omega)$. □

Using Lemma 3 from Theorems 1 and 4 we deduce such a statement.

Theorem 5. *Let $\varepsilon > 0$, and f be a random entire function of the form (5) such that $f_0 \ne 0$. Then P-almost surely there exist a nonrandom set E of finite logarithmic measure and $r_0(\omega) > 0$ such that for all $r\in(r_0(\omega),+\infty)\setminus E$ we obtain*

$$(1-\varepsilon)N(r)\ln N(r) \le p_0(r) - q(r) \le N(r)\exp\{(2+\varepsilon)\sqrt{\ln N(r)}\}, \tag{15}$$

in particular,

$$0 \le \varliminf_{\substack{r\to+\infty\\ r\notin E}} \frac{\ln(p_0(r)-q(r))}{\ln q(r)},\ \varlimsup_{\substack{r\to+\infty\\ r\notin E}} \frac{\ln(p_0(r)-q(r))}{\ln q(r)} \le \frac{1}{2} \tag{16}$$

and

$$\lim_{\substack{r\to+\infty\\ r\notin E}} \frac{\ln(p_0(r)-q(r))}{\ln N(r)} = 1.$$

Proof. It follows from Theorems 1 and 4 inequality (15). Furthermore, from (15) we deduce for $r\in(r_0(\omega);+\infty)\setminus E$

$$\frac{-\ln 2 + \ln N(r) + \ln\ln N(r)}{\ln N(r)} \le \frac{\ln(p_0(r)-q(r))}{\ln N(r)} \le \frac{\ln N(r) + 3\sqrt{\ln N(r)}}{\ln N(r)},$$
$$\lim_{\substack{r\to+\infty\\ r\notin E}} \frac{\ln(p_0(r)-q(r))}{\ln N(r)} = 1.$$

By Lemma 3 we obtain

$$\varlimsup_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-q(r))}{\ln q(r)}=\varlimsup_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-q(r))}{\ln N(r)}\cdot\frac{N(r)}{q(r)}=\varlimsup_{\substack{r\to+\infty\\ r\notin E}}\frac{N(r)}{q(r)}\le\frac{1}{2}.$$

Since $N(r)$ and $q(r)$ are non-negative functions

$$\varliminf_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-q(r))}{\ln q(r)}=\varliminf_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-q(r))}{\ln N(r)}\cdot\frac{N(r)}{q(r)}=\varliminf_{\substack{r\to+\infty\\ r\notin E}}\frac{N(r)}{q(r)}\ge 0.$$

□

5. Examples on Sharpness of Inequalities (16)

Theorem 6. *There is a random entire function of form (5) for which $f_0\neq 0$, a nonrandom set E of finite logarithmic measure and P–almost surely $r_0(\omega)>0$— such that for all $r\ge r_0(\omega)$ we obtain*

$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-q(r))}{\ln q(r)}=\frac{1}{2}.$$

Proof. Consider the entire function

$$f(z)=1+\sum_{n=1}^{+\infty}\frac{z^n}{(\frac{n}{2})^{\frac{n}{2}}}.$$

For this function and $r\in(r_0(\omega);+\infty)\setminus E$ we have

$$\frac{\sqrt{q(r)}}{\ln^3 q(r)}<N(r)<\sqrt{q(r)}\exp\{(1+\varepsilon)\sqrt{\ln q(r)}\},\ \lim_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln N(r)}{\ln q(r)}=\frac{1}{2}.$$

By Theorem 5 we have for $r\in(r_0(\omega);+\infty)\setminus E$

$$\frac{-\ln 2+\ln N(r)+\ln\ln N(r)}{\ln q(r)}\le\frac{\ln(p_0(r)-q(r))}{\ln q(r)}\le\frac{\ln N(r)+3\sqrt{\ln N(r)}}{\ln q(r)},$$
$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-q(r))}{\ln q(r)}=\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln N(r)}{\ln q(r)}=\frac{1}{2}.$$

□

Theorem 7. *There is a random entire function of form (5) for which $f_0\neq 0$, a nonrandom set E of finite logarithmic measure and P–almost surely $r_0(\omega)>0$— such that for all $r\ge r_0(\omega)$*

$$\varliminf_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-q(r))}{\ln q(r)}=0.$$

Proof. Consider the entire functions

$$f(z)=1+\sum_{n=1}^{+\infty}\frac{z^n}{(\frac{n}{2})^{\frac{n}{2}}},\quad h(z)=1+\sum_{n\in\mathcal{N}^*}\frac{z^n}{(\frac{n}{2})^{\frac{n}{2}}},$$

where $\mathcal{N}^*=\{n\colon n=[e^k]+1 \text{ for some } k\in\mathbb{Z}_+\}$. Here $[e^k]$ means the integral part of the real number e^k. We denote

$$\mathcal{N}_f(r) = \{n \in \mathbb{Z}_+ \colon \ln(|f_n|r^n) > 0\} \setminus \{0\},\ \mathcal{N}_h(r) = \{n \in \mathcal{N}^* \colon \ln(|f_n|r^n) > 0\},$$
$$q_f(r) = 2 \sum_{n \in \mathcal{N}_f(r)} \ln(|f_n|r^n),\ q_h(r) = 2 \sum_{n \in \mathcal{N}_h(r)} \ln(|f_n|r^n),\ f_n = \left(\frac{n}{2}\right)^{-\frac{n}{2}},\ n \in \mathbb{N}.$$

Remark that the sequence $\{(n/2)^{-n/2}\}$ is log-concave and

$$\mathcal{N}_f(r) = \{1, \ldots, N_f(r)\}.$$

Then by the definition of $N_h(r)$ we obtain $N_h(r) \le 2 \ln N_f(r),\ r \to +\infty$. For $r \in (r_0; +\infty) \setminus E$ we obtain

$$N_h(r) \le 2 \ln N_f(r) \le 2 \ln(\ln \mu_f(r) \ln^2(\ln \mu_f(r))) < 4 \ln \ln \mu_f(r).$$

Remark that $\min\{n \in \mathcal{N}' \colon n > \nu_h(r)\} \le [e\nu_h(r)] + 1 < (e+1) \ln \nu_h(r)$. Let us fix $r > 0$. Consider the function $y(t) = \ln(a(t)r^t) = -\frac{t}{2} \ln(\frac{t}{2}) + t \ln r$, for which $a(n) = f_n$. The graph of the function $y(t)$ passes through the points $(0;0)$ and $(\nu_h(r), \ln \mu_h(r))$. It follows from log-concavity of the function $y(t)$ that the point $(\nu_f(r), \ln \mu_f(r))$ belongs to the triangle with the vertices $(\nu_h(r), \ln \mu_h(r))$, $((e+1)\nu_h(r), \ln \mu_h(r))$ and $((e+1)\nu_h(r), (e+1) \ln \mu_h(r))$. Then,

$$\ln \mu_f(r) \le (e+1) \ln \mu_h(r),\ q_h(r) \ge 2 \ln \mu_h(r) \ge \frac{2}{e+1} \ln \mu_f(r).$$

For the function $h(z)$ and $r \in (r_0; +\infty) \setminus E$ we obtain

$$0 \le \varlimsup_{\substack{r \to +\infty \\ r \notin E}} \frac{\ln(p_0(r) - q_h(r))}{\ln q_h(r)} = \varlimsup_{\substack{r \to +\infty \\ r \notin E}} \frac{\ln N_h(r)}{\ln q_h(r)} \le \varlimsup_{\substack{r \to +\infty \\ r \notin E}} \frac{\ln(4 \ln \ln \mu_f(r))}{\ln(\frac{2}{e+1} \ln \mu_f(r))} = 0.$$

□

6. Discussion

Open Problem. Let $\varepsilon \in (0, 1/2)$. Note, that for random entire function of the form (6) $P_0(r) = P\{\omega \colon n_\psi(r, \omega) = 0\},\ p_0(r) = \ln^- P_0(r)$, we have ([23])

$$p_0(r) = q(r) + O((q(r))^{1/2+\varepsilon}),\ r \to +\infty,\ r \notin E.$$

Here, E is a non-random exceptional set of finite logarithmic measure. Is the error term in the previous inequality optimal?

Conjecture. Let $\varepsilon > 0$, and f be a random entire function of the form (6) such that $f_0 \ne 0$. Then, P–almost surely there is a nonrandom set E of finite logarithmic measure and $r_0(\omega) > 0$—such that for all $r \in (r_0(\omega), +\infty) \setminus E$ we obtain

$$(1 - \varepsilon) N(r) \ln N(r) \le p_0(r) - q(r) \le N(r) \exp\{(2 + \varepsilon)\sqrt{\ln N(r)}\},$$

in particular,

$$0 \le \varliminf_{\substack{r \to +\infty \\ r \notin E}} \frac{\ln(p_0(r) - q(r))}{\ln q(r)},\ \varlimsup_{\substack{r \to +\infty \\ r \notin E}} \frac{\ln(p_0(r) - q(r))}{\ln q(r)} \le \frac{1}{2}$$

and

$$\lim_{\substack{r \to +\infty \\ r \notin E}} \frac{\ln(p_0(r) - q(r))}{\ln N(r)} = 1.$$

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