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Article

A Note on Generalization of Combinatorial Identities Due to Gould and Touchard

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Abstract: Using a hypergeometric series approach, a general combinatorial identity is found in this note, and among its special cases are well-known and classical combinatorial identities due to Gould and Touchard.

Keywords: combinatorial identities; Gould; Touchard; hypergeometric series; Gauss summation theorem

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1. Introduction

Many collections of 500 combinatorial identities are provided in a well-known monogram published by Gould [1]; however, we are interested in the following identity in our current note [1] (Equation 3.99).

$$\sum_{k=0}^{\left[\frac{m}{2}\right]} \binom{m}{2k} 2^{m-2k} \binom{2k}{k} = \binom{2m}{m} = \frac{2^{2m}\left(\frac{1}{2}\right)_m}{(1)_m}, \quad m \geq 0, \tag{1}$$

where $\binom{2m}{m}$ is the central binomial coefficient defined by

$$\binom{2m}{m} = \frac{(2m)!}{(m!)^2}, \tag{2}$$

and $(a)_m$ is the well-known Pochammer symbol defined for any complex number $a(\neq 0)$ by

$$(a)_m = \begin{cases} a(a+1)\cdots(a+m-1), & m \in \mathbb{N} \\ 1, & m = 0. \end{cases}$$

A somewhat similar formula was given by Touchard [2]:

$$\sum_{k=0}^{\left[\frac{m}{2}\right]} \binom{m}{2k} 2^{m-2k} \frac{1}{k+1} \binom{2k}{k} = C_{m+1} = \frac{2^{2m}\left(\frac{3}{2}\right)_m}{(3)_m}, \quad m \geq 0, \tag{3}$$

where C_m is the m-th Catalan number defined by

$$C_m = \frac{1}{m+1} \binom{2m}{m}. \tag{4}$$

For several other proofs of the Touchard identity (3), we refer readers to the research papers by Izbicki [3], Riordan [4] and Shapiro [5]. These proofs are combinatorial in nature, and Shapiro [5,6] gives a combinatorial interpretation of the number $\binom{m}{2k}2^{m-2k}C_k$.

In 1977, Gould (p. 352, [7]) (Equation (8)) obtained a general combinatorial identity that includes (1) and (3) in the form

$$\sum_{k=0}^{[\frac{m}{2}]} \binom{m+x}{m-k} \binom{m-k}{k} 2^{m-2k} = \binom{2m+2x}{m}, \quad m \geq 0, \tag{5}$$

for arbitrary x.

Clearly, for $x = 0$ this yields (1), and for $x = 1$ it yields (3).

Motivated by this, the purpose of this note is to present another generic combinatorial identity using a hypergeometric series technique that encompasses both (1) and (3) via a hypergeometric series approach. For this, we recall the definition of hypergeometric series as follows (p. 45, [8]) (Equation (1)):

$${}_2F_1\left[\begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix} ; t\right] = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{t^m}{m!} \tag{6}$$

where α, β and γ are called the parameters of the series, which may be real or complex with the exception that γ is neither zero nor a negative integer, and t is called the variable of the series.

The series (6) is convergent for all values of t if $|t| < 1$ and divergent if $|t| > 1$. When $t = 1$, the series is convergent if $\Re(\gamma - \alpha - \beta) > 0$ and divergent if $\Re(\gamma - \alpha - \beta) \leq 0$. Additionally, when $t = -1$, the series is absolutely convergent if $\Re(\gamma - \alpha - \beta) > 0$ and is convergent but not absolutely if $-1 < \Re(\gamma - \alpha - \beta) \leq 0$ and divergent if $\Re(\gamma - \alpha - \beta) < 1$.

It should be remarked here that whenever a hypergeometric series reduces to the gamma function, the result is very important from the application point of view. Thus, the classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey play an important role. However, in our present investigation, we mention here the classical Gauss summation theorem [8]:

$${}_2F_1\left[\begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix} ; 1\right] = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma\{(\gamma - \alpha)\Gamma(\gamma - \beta)}, \tag{7}$$

provided $\Re(\gamma - \alpha - \beta) > 0$.

Hypergeometric series have numerous well-known applications in the fields of applied mathematics, number theory, probability, statistics, engineering mathematics and combinatorial analysis. In particular, the application of hypergeometric series to solve binomial sums was originally suggested by Andrews [9]. This approach starts by transforming the given binomial sum into a regular hypergeometric series. Any terms in the summation index that are polynomials are combined with the binomials to achieve this. The factorials are then converted into Pochhammer symbols after the binomials have been expanded into them. If this successfully converts, the resultant hypergeometric series is compared with well-known summation theorems found in the literature, and when a good match is found, a closed-form evaluation may be produced. The same method is also applicable in the case of generalized hypergeometric series ${}_pF_q$. The details about the generalized hypergeometric series ${}_pF_q$ can be seen in the standard text of Rainville [8].

It is interesting to mention here that the combinatorial identities due to Gould (1), Thouchard (3) and their generalization (5) can be easily obtained with the help of the Gauss summation theorem (7).

The natural generalization of the Gould identity (1) and Touchard identity (3) via a hypergeometric series approach—i.e., by making use of the Gauss summation theorem—is provided in the following section.

2. Generalization of Gould and Touchard Identities

The general combinatorial identity to be established is asserted in the following theorem.

Theorem 1. *For $\ell \in \mathbb{N}$, the following combinatorial identity holds true.*

$$\sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{\binom{m}{2k} 2^{m-2k} \binom{2k}{k}}{\binom{k+\ell-1}{k}} = 2^{2m} \frac{\left(\ell - \frac{1}{2}\right)_m}{(2\ell - 1)_m}. \tag{8}$$

Proof. The derivation of the combinatorial identity (8) asserted in the theorem is quite straightforward. For this, denoting the left-hand side of (8) by S and converting all binomial coefficients into Pochhammer symbols,

$$\begin{aligned} \binom{m}{2k} &= \frac{\left(-\frac{1}{2}m\right)_k \left(-\frac{1}{2}m + \frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_k (1)_k} \\ \binom{2k}{k} &= \frac{2^{2k}\left(\frac{1}{2}\right)_k}{(1)_k} \\ \text{and} \quad \binom{k+m-1}{k} &= \frac{(m)_k}{(1)_k}. \end{aligned}$$

We have, after some algebra,

$$S = 2^m \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{\left(-\frac{1}{2}m\right)_k \left(-\frac{1}{2}m + \frac{1}{2}\right)_k}{(\ell)_k \; k!};$$

summing up the series, we have

$$S = 2^m {}_2F_1\left[\begin{matrix} -\frac{1}{2}m, & -\frac{1}{2}m + \frac{1}{2} \\ & \ell \end{matrix}; 1\right].$$

We now observe that ${}_2F_1$ can be evaluated with the help of the Gauss summation theorem (3), and this yields

$$S = 2^m \frac{\Gamma(\ell)\Gamma\left(\ell + m - \frac{1}{2}\right)}{\Gamma\left(\ell + \frac{1}{2}m\right)\Gamma\left(\ell + \frac{1}{2}m - \frac{1}{2}\right)},$$

and upon using the duplication formula (pp. 23–24, [8])

$$\begin{aligned} \Gamma(2z) &= \frac{2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2})}{\sqrt{\pi}} \\ \text{and} \quad (\alpha)_m &= \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)}, \end{aligned}$$

we easily arrive at the right-hand side of (8). This completes the proof of our general combinatorial identity asserted in the theorem. □

We shall mention known as well as new results available in the literature from our main result (8).

Corollary 1. *For $\ell = 1$, we at once recover Gould's identity (1).*

Corollary 2. *For $\ell = 2$, we at once recover Touchard's identity (3).*

Corollary 3. *For $\ell = 3$ and 4, we get the following interesting identities:*

$$\sum_{k=0}^{[\frac{m}{2}]} \frac{\binom{m}{2k} 2^{m-2k} \binom{2k}{k}}{\binom{k+2}{k}} = 2^{2m} \frac{\left(\frac{5}{2}\right)_m}{(5)_m}$$

$$\text{and} \quad \sum_{k=0}^{[\frac{m}{2}]} \frac{\binom{m}{2k} 2^{m-2k} \binom{2k}{k}}{\binom{k+3}{k}} = 2^{2m} \frac{\left(\frac{7}{2}\right)_m}{(7)_m}.$$

3. Concluding Remark

In this note, a general combinatorial identity has been established via the Gauss summation formula for the hypergeometric series which includes among its special cases well-known identities due to Gould and Touchard. We hope that the result established in this note could be potentially useful in the area of combinatorial analysis and applied mathematics.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

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