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Solving Feasibility Problems with Infinitely Many Sets

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Abstract: In this paper, we study a feasibility problem with infinitely many sets in a metric space. We present a novel algorithm and analyze its convergence. The algorithms used for the feasibility problem in the literature work for finite collections of sets and cannot be applied if the collection of sets is infinite. The main feature of these algorithms is that, for iterative steps, we need to calculate the values of all the operators belonging to our family of maps and even their sums with weighted coefficients. This is impossible if the family of maps is not finite. In the present paper, we introduce a new algorithm for solving feasibility problems with infinite families of sets and study its convergence. It turns out that our results hold for feasibility problems in a general metric space.

Keywords: complete metric space; convergence analysis; iteration; non-expansive mapping

MSC: 47H04; 47H09; 47H10

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1. Introduction

The convex feasibility problem is used to obtain a common element of a finite family of convex and closed sets or at least its approximation. This problem, investigated in [1–23], is very important in the optimization with constraints. It is also used in engineering, medical, and natural sciences.

Assume that C_i, $i = 1, \dots, m$, where $m \geq 2$ is a natural number, are closed and convex sets in a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a complete norm $\| \cdot \|$, which is induced by the inner product. We consider the problem

$$\text{Find } z \in \cap_{i=1}^m C_i$$

under the assumption that $\cap_{i=1}^m C_i$ is nonempty. It is well-known [3] that, for each $i \in \{1, \dots, m\}$ and each $x \in X$, there exists a unique element $P_{C_i}(x) \in C_i$ such that

$$\|x - P_{C_i}(x)\| = \inf\{\|x - y\| : \ y \in C_i\},$$

$$\|P_{C_i}(x) - P_{C_i}(y)\| \leq \|x - y\|, \ x, y \in X$$

and

$$\|z - x\|^2 \geq \|z - P_i(x)\|^2 + \|x - P_i(x)\|^2$$

for each $x \in X$ and each $z \in C_i$. For each $i \in \{1, \dots, m\}$ and each $x \in X$ set,

$$d(x, C_i) = \|x - P_{C_i}(x)\|.$$

This convex feasibility problem can be written as the optimization problem

$$\sum_{i=1}^{m} d(x, C_i) \to \min, \ x \in C.$$

This is a convex minimization problem and one can try to solve it using some optimization methods. However, in the practice for solving convex feasibility problems, the following iterative method is used.

Fix an integer $\bar{N} \geq 1$ and denote by $\mathcal{R}$ the collection of all maps $r : \{1, 2, \ldots, \} \to \{1, \ldots, m\}$ such that for every positive integer s,

$$\{1, \ldots, m\} \subset \{r(s), \ldots, r(s + \bar{N} - 1)\}.$$

We associate with any map $r \in \mathcal{R}$ the following iterative algorithm:
Initialization: choose any starting point x_0 of the space X.
Iterative step: given a current iterate $x_k \in X$ calculate

$$x_{k+1} = P_{r(k+1)}(x_k).$$

It is known that iterates obtained using this method weakly converge to a solution of our feasibility problem. The same result is also guaranteed by the well-known Cimmino algorithm described below:
Initialization: choose any starting point x_0 of the space X.
Iterative step: given a current iterate $x_k \in X$ calculate

$$x_{k+1} = \sum_{i=1}^{m} m^{-1} P_i(x_k).$$

Recently, Y. Censor, T. Elfving, and G. T. Herman in [24] introduced dynamic string-averaging methods, which are, in some sense, a combination of the iterative algorithm and the Cimmino algorithm. In these dynamic string-averaging methods, which became very popular in the literature, a family of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel.

In the present paper, we study a feasibility problem with a collection of sets that is not necessarily finite. Clearly, the algorithms described above cannot be applied if the collection of sets is infinite. The main feature of these algorithms is that, for iterative steps, we need to calculate the values of all the operators belonging to our family of maps and even their sums with weighted coefficients. Of course, this is impossible if the family of maps is not finite. In the present paper, we introduce a new algorithm for solving feasibility problems with infinite families of sets and study its convergence. It turns out that our results hold for feasibility problems in a general metric space.

2. Preliminaries and the First Main Result

Let (X, ρ) be a metric space endowed with a metric ρ. For every element $x \in X$ and every positive number r, put

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\}.$$

For every element $x \in X$ and every nonempty set $D \subset X$, define

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

Fix $\theta \in X$. Denote by $\mathrm{Card}(E)$ the cardinality of a set E. We assume that the sum over an empty set is zero.

Assume that $\mathcal{A}$ is a nonempty set, for each $\alpha \in \mathcal{A}$, $C_\alpha \subset X$ is a nonempty, closed set and that there exists $P_\alpha : X \to C_\alpha$ such that

$$P_\alpha(x) = x, \ x \in C_\alpha. \tag{1}$$

In the sequel, we use the following assumption.
(A1) There exists $\bar{c} \in (0, 1)$ such that, for each $\alpha \in \mathcal{A}$, each $z \in C_\alpha$ and each $x \in X$,

$$\rho(z, x)^2 \geq \rho(z, P_\alpha(x))^2 + \bar{c}\rho(x, P_\alpha(x))^2. \tag{2}$$

Assume that there exists

$$\widehat{z} \in \cap_{\alpha \in \mathcal{A}} C_\alpha. \tag{3}$$

We consider the problem

$$\text{Find } z \in \cap_{\alpha \in \mathcal{A}} C_\alpha$$

and use the following algorithm.

Let a sequence $\{\Delta_i\}_{i=1}^{+\infty} \subset (0, +\infty)$ satisfy

$$\lim_{i \to +\infty} \Delta_i = 0.$$

Initialization: choose any element $x_0 \in X$.

Iterative step: given a current iterate x_k calculate $\alpha(k) \in \mathcal{A}$ such that

$$\rho(x_k, P_{\alpha(k)}(x_k)) \geq \sup\{\rho(x_k, P_\alpha(x_k)) : \alpha \in \mathcal{A}\} - \Delta_{k+1}$$

and calculate

$$x_{k+1} = P_{\alpha(k)}(x_k).$$

The following theorem is our first main result.

Theorem 1. *Let (A1) hold,*

$$M > \max\{1, \rho(\theta, \widehat{z})\}, \tag{4}$$

$\epsilon \in (0,1)$*, a natural number* Q *satisfy*

$$Q \geq 16M^2 \bar{c}^{-1} \epsilon^{-2}, \tag{5}$$

a sequence $\{\Delta_i\}_{i=1}^{+\infty} \subset (0, \infty)$ *satisfy*

$$\lim_{i \to +\infty} \Delta_i = 0 \tag{6}$$

and let an integer $n_0 \geq 1$ *satisfy*

$$\Delta_i \leq \epsilon/2 \textit{ for each integer } i \geq n_0. \tag{7}$$

Assume that a sequence $\{x_t\}_{t=0}^{+\infty} \subset X$ *satisfies*

$$\rho(x_0, \theta) \leq M \tag{8}$$

and that, for each integer $t \geq 0$*, there exists* $\alpha(t) \in \mathcal{A}$ *such that*

$$x_{t+1} = P_{\alpha(t)}(x_t) \tag{9}$$

and

$$\rho(x_t, x_{t+1}) \geq \sup\{\rho(x_t, P_\alpha(x_t)) : \alpha \in \mathcal{A}\} - \Delta_{i+1}. \tag{10}$$

Then,

$$\rho(x_t, \theta) \leq 3M, \ t = 0, 1, \dots,$$

$$Card(\{k \in \{0, 1, \dots,\} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) \leq Q,$$

if an integer $t \geq n_0$ *satisfies* $\rho(x_t, x_{t+1}) < \epsilon/2$*, then*

$$\rho(x_t, P_\alpha(x_t)) \leq \epsilon, \ \alpha \in \mathcal{A}$$

and

$$Card(\{k \in \{0, 1, \dots,\} : \sup\{\rho(x_k, P_\alpha(x_k)) : \alpha \in \mathcal{A}\} > \epsilon\}) \leq Q + n_0.$$

Proof. Assumption (A1) and Equations (2) and (3) imply that, for each integer $t \geq 0$,

$$\rho(\widehat{z}, x_{t+1}) \leq \rho(\widehat{z}, x_t). \tag{11}$$

It follows from (4), (8), and (11) that, for each integer $t \geq 0$,

$$\rho(\widehat{z}, x_t) \leq \rho(\widehat{z}, x_0) \leq 2M, \ \rho(\theta, x_t) \leq 3M. \tag{12}$$

Assumption (A1) and Equations (3), (9) and (12) imply that for each integer $t \geq 0$ satisfying

$$\rho(x_t, x_{t+1}), \geq \epsilon/2$$

we have

$$\rho(x_t, \widehat{z})^2 - \rho(x_{t+1}, \widehat{z})^2 \geq \bar{c}\rho(x_t, x_{t+1})^2 \geq \bar{c}\epsilon^2/4. \tag{13}$$

Let n be a natural number. By (4), (5), (8), and (13),

$$\begin{aligned}
4M^2 &\geq (\rho(\widehat{z}, \theta) + M)^2 \geq (\rho(\widehat{z}, \theta) + \rho(\theta, x_0))^2 \\
&\geq \rho(\widehat{z}, x_0)^2 \geq \rho(\widehat{z}, x_0)^2 - \rho(\widehat{z}, x_n)^2 \\
&= \sum_{k=0}^{n-1} (\rho(\widehat{z}, x_k)^2 - \rho(\widehat{z}, x_{k+1})^2) \\
&\sum \{ (\rho(\widehat{z}, x_k)^2 - \rho(\widehat{z}, x_{k+1})^2) : \ k \in \{0, \dots, n-1\}, \\
&\quad \rho(x_k, x_{k+1}) \geq 2^{-1}\epsilon \} \\
&\geq 4^{-1}\bar{c}\epsilon^2 Card(\{k \in \{0, \dots, n-1\} : \rho(x_k, x_{k+1}) \geq \epsilon/2\})
\end{aligned}$$

and

$$Card(\{k \in \{0, \dots, n-1\} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) \leq 16M^2\bar{c}^{-1}\epsilon^{-2} \leq Q.$$

Since n is any natural number, we conclude that

$$Card(\{k \in \{0, 1, \dots, \} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) \leq Q. \tag{14}$$

Since ϵ is any element of $(0, 1)$, we have

$$\lim_{t \to +\infty} \rho(x_t, x_{t+1}) = 0.$$

Set

$$E = \{k \in \{0, 1, \dots, \} : \rho(x_k, x_{k+1}) < \epsilon/2 \ \text{and} \ k \geq n_0\}. \tag{15}$$

In view of (14) and (15),

$$\text{Card}(\{k \in \{0, 1, \dots, \} \setminus E\}) \leq Q + n_0.$$

Assume that

$$t \in E. \tag{16}$$

By (15) and (16),

$$\rho(x_t, x_{t+1}) < \epsilon/2. \tag{17}$$

It follows from (7) and (15)–(17) that, for each $\alpha \in \mathcal{A}$,

$$\rho(x_t, P_\alpha(x_t)) \leq \Delta_{t+1} + \rho(x_t, x_{t+1}) \leq \epsilon/2 + \epsilon/2$$

and

$$B(x_t, \epsilon) \cap C_\alpha \neq \emptyset.$$

Theorem 1 is proved. □

We say that the family $C_\alpha,\ \alpha \in \mathcal{A}$ has a bounded regularity property (or (BRP) for short) [3] if, for each $M, \epsilon > 0$, there exists $\delta > 0$ such that, for each $x \in B(\theta, M)$ satisfying $\rho(x, C_\alpha) \le \delta$, $\alpha \in \mathcal{A}$, the inequality $\rho(\cap_{\alpha \in \mathcal{A}} C_\alpha) \le \epsilon$ holds.

Clearly, (BRP) holds if the space X is finite dimensional or if there is a set in the collection such that all its bounded, closed subsets are compact.

Theorem 1 implies the following result.

Proposition 1. *Let (BRP) and (A1) hold,*

$$M > \max\{1, \rho(\theta, \widehat{z})\};$$

$\epsilon \in (0,1)$ *and a let sequence* $\{\Delta_i\}_{i=1}^{+\infty} \subset (0, +\infty)$ *satisfy*

$$\lim_{i \to +\infty} \Delta_i = 0.$$

Then, there exists a natural number Q *such that for each sequence* $\{x_t\}_{t=0}^{+\infty} \subset X$*, which satisfies*

$$\rho(x_0, \theta), \le M$$

and such that for each integer $t \ge 0$ *there exists* $\alpha(t) \in \mathcal{A}$ *satisfying*

$$x_{t+1} = P_{\alpha(t)}(x_t)$$

and

$$\rho(x_t, x_{t+1}) \ge \sup\{\rho(x_t, P_\alpha(x_t)) :\ \alpha \in \mathcal{A}\} - \Delta_{i+1}$$

the equations

$$Card(\{k \in \{0, 1, \dots, \} : \rho(x_k, \cap_{\alpha \in \mathcal{A}} C_\alpha) > \epsilon\}) \le Q$$

and

$$\lim_{t \to +\infty} \rho(x_t, \cap_{\alpha \in \mathcal{A}} C_\alpha) = 0$$

hold.

Example 1. *The results of this section can be applied for the feasibility problem, where* X *is the Hilbert space* l^2 *of square-summable sequences of the real numbers* $x = (x_1, x_2, \dots, x_n, x_{n+1}, \dots)$*, for each integer* $i \ge 1$, $C_i = \{x \in l^2 :\ x_{2i} = 0\}$*, and for each* $x \in l^2$, $P_i(x) = y \in C_i$ *such that* $y_j = x_j$ *for each natural number* $j \ne 2i$*. It is easy to see that the assumptions posed in this section as well as its results hold for this family of sets.*

3. The Second Main Result

We use the notation and definitions introduced in Section 2.

We continue to assume that $\mathcal{A}$ is a nonempty set, for each $\alpha \in \mathcal{A}$, $C_\alpha \subset X$ is a nonempty, closed set and that there exists $P_\alpha : X \to C_\alpha$ such that

$$P_\alpha(x) = x,\ x \in C_\alpha. \tag{18}$$

In the sequel, we use the following assumption.

(A2) For each $M, \gamma > 0$, there exists $\delta > 0$ such that for each $\alpha \in \mathcal{A}$, each $z \in C_\alpha \cap B(\theta, M)$ and each $x \in B(\theta, M)$ satisfying

$$\rho(x, P_\alpha(x)) \ge \epsilon,$$

the inequality

$$\rho(z, x) - \delta \ge \rho(z, P_\alpha(x))$$

holds.

Assume that there exists
$$\widehat{z} \in \cap_{\alpha \mathcal{A}} C_\alpha. \tag{19}$$

The following theorem is our second main result.

Theorem 2. *Let (A2) hold,*
$$M > \max\{1, \rho(\theta, \widehat{z})\}, \tag{20}$$
a sequence $\{\Delta_i\}_{i=1}^{+\infty} \subset (0, +\infty)$ *satisfy*
$$\lim_{i \to +\infty} \Delta_i = 0,$$
$\epsilon \in (0,1)$, *and let an integer* $n_0 \geq 1$ *satisfy*
$$\Delta_i \leq \epsilon/2 \text{ for each integer } i \geq n_0. \tag{21}$$

Then, there exists a natural number Q *depending on* M, ϵ *such that, for each sequence,* $\{x_t\}_{t=0}^{\infty} \subset X$, *which satisfies*
$$\rho(x_0, \theta), \leq M \tag{22}$$
and such that, for each integer $t \geq 0$, *there exists* $\alpha(t) \in \mathcal{A}$ *satisfying*
$$x_{t+1} = P_{\alpha(t)}(x_t) \tag{23}$$
and
$$\rho(x_t, x_{t+1}) \geq \sup\{\rho(x_t, P_\alpha(x_t)) : \ \alpha \in \mathcal{A}\} - \Delta_{i+1}. \tag{24}$$

The inequalities
$$\rho(x_t, \theta) \leq 3M, \ t = 0, 1, \ldots$$
and
$$Card(\{k \in \{0, 1, \ldots, \} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) \leq Q$$
hold, if an integer $t \geq n_0$ *satisfies* $\rho(x_t, x_{t+1}) \leq \epsilon/2$; *then,*
$$\rho(x_t, P_\alpha(x_t)) \leq \epsilon, \ \alpha \in \mathcal{A}$$
and
$$Card(\{k \in \{0, 1, \ldots, \} : \sup\{\rho(x_k, P_\alpha(x_k)) : \ \alpha \in \mathcal{A}\} > \epsilon\}) \leq Q + n_0.$$

Proof. Assumption (A2) implies that there exists $\delta \in (0, \epsilon/2)$ such that the following property holds:

(i) For each $\alpha \in \mathcal{A}$, with each $z \in C_\alpha \cap B(\theta, 3M)$ and each $x \in B(\theta, 3M)$ satisfying
$$\rho(x, P_\alpha(x)) \geq \epsilon/2,$$
we have
$$\rho(z, x) - \delta \geq \rho(z, P_\alpha(x)).$$

Fix an integer
$$Q > 2M\delta^{-1}. \tag{25}$$

Assume that $\{x_t\}_{t=0}^{+\infty} \subset X$ and $\{\alpha(t)\}_{t=0}^{+\infty} \subset \mathcal{A}$ satisfy (22)–(24) for each integer $t \geq 0$. By (A2) and Equations (19), (20) and (22), for each integer $t \geq 0$,
$$\rho(\widehat{z}, x_{t+1}) \leq \rho(\widehat{z}, x_t), \tag{26}$$
$$\rho(\widehat{z}, x_t) \leq \rho(\widehat{z}, x_0) \leq 2M \tag{27}$$
and
$$\rho(\theta, x_t) \leq 3M. \tag{28}$$

Property (i) and Equations (19), (20), (23) and (28) imply that, for each integer $t \geq 0$ satisfying

$$\rho(x_t, x_{t+1}), \geq \epsilon/2 \tag{29}$$

we have

$$\rho(x_{t+1}, \widehat{z}) = \rho(\widehat{z}, P_{\alpha(t)}(x_t)) \leq \rho(x_t, \widehat{z}) - \delta. \tag{30}$$

Thus, the following property holds:

(ii) If $t \geq 0$ is an integer and (29) holds, then (30) is true.

Let n be a natural number. Property (ii) and Equations (20), (22), (26), (29) and (30) imply that

$$\begin{aligned} 2M &\geq \rho(\widehat{z}, \theta) + \rho(\theta, x_0) \geq \rho(\widehat{z}, x_0) \\ &\geq \rho(\widehat{z}, x_0) - \rho(\widehat{z}, x_n) \\ &= \sum_{k=0}^{n-1} (\rho(\widehat{z}, x_k) - \rho(\widehat{z}, x_{k+1})) \\ &\sum \{ (\rho(\widehat{z}, x_k) - \rho(\widehat{z}, x_{k+1})) : \ k \in \{0, \dots, n-1\}, \\ &\rho(x_k, x_{k+1}) \geq 2^{-1}\epsilon \} \\ &\geq \delta Card(\{k \in \{0, \dots, n-1\} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) \end{aligned}$$

and

$$Card(\{k \in \{0, \dots, n-1\} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) \leq 2M\delta^{-1}.$$

Since n is any natural number, we conclude using (25) that

$$Card(\{k \in \{0, 1, \dots, \} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) \leq 2M\delta^{-1} < Q. \tag{31}$$

Since ϵ is any element of $(0, 1)$, we have

$$\lim_{t \to +\infty} \rho(x_t, x_{t+1}) = 0.$$

Assume that $t \geq n_0$ is an integer and that

$$\rho(x_t, x_{t+1}) \leq \epsilon/2.$$

It follows from (21) and (24) that, for each $\alpha \in \mathcal{A}$,

$$\rho(x_t, P_\alpha(x_t)) \leq \Delta_{t+1} + \rho(x_t, x_{t+1}) \leq \epsilon/2 + \epsilon/2.$$

Together with (31), this implies that

$$\begin{aligned} &\text{Card}(\{k \in \{0, 1, \dots, \} : \sup\{\rho(x_k, P_\alpha(x_k)) : \ \alpha \in \mathcal{A}\} > \epsilon\}) \\ &\leq \text{Card}(\{k \in \{n_0, n_0 + 1, \dots, \} : \rho(x_k, x_{k+1}) \geq \epsilon/2\}) + n_0 < Q + n_0. \end{aligned}$$

Theorem 3 is proved. □

Theorem 3 implies the following result.

Proposition 2. *Let (BRP) and (A2) hold,*

$$M > \max\{1, \rho(\theta, \widehat{z})\};$$

$\epsilon \in (0, 1)$ *and a sequence* $\{\Delta_i\}_{i=1}^{+\infty} \subset (0, +\infty)$ *satisfy*

$$\lim_{i \to +\infty} \Delta_i = 0.$$

Then, there exists a natural number Q such that, for each integer sequence $\{x_t\}_{t=0}^{+\infty} \subset X$, *which satisfies*

$$\rho(x_0, \theta), \le M$$

and such that, for each integer $t \ge 0$, *there exists* $\alpha(t) \in \mathcal{A}$ *satisfying*

$$x_{t+1} = P_{\alpha(t)}(x_t)$$

and

$$\rho(x_t, x_{t+1}) \ge \sup\{\rho(x_t, P_\alpha(x_t)) : \alpha \in \mathcal{A}, \} - \Delta_{i+1}$$

the equations

$$Card(\{k \in \{0, 1, \dots, \} : \rho(x_k, \cap_{\alpha \in \mathcal{A}} C_\alpha) > \epsilon\}) \le Q$$

and

$$\lim_{t \to +\infty} \rho(x_t, \cap_{\alpha \in \mathcal{A}} C_\alpha) = 0$$

hold.

4. The Third Main Result

We use the notation and definitions introduced in Section 2.

We continue to assume that $\mathcal{A}$ is a nonempty set, for each $\alpha \in \mathcal{A}$, $C_\alpha \subset X$ is a nonempty, closed set, that there exists $P_\alpha : X \to C_\alpha$, and that (18) and (19) hold.

In the sequel, we use the following assumption.

(A3) For each $M, \gamma > 0$, there exists $\delta > 0$ such that, for each $\alpha \in \mathcal{A}$, each $z \in C_\alpha \cap B(\theta, M)$ and each $x \in B(\theta, M)$ satisfying

$$\rho(x, C_\alpha) \ge \gamma$$

the inequality

$$\rho(z, x) - \delta \ge \rho(z, P_\alpha(x))$$

holds.

The following theorem is our third main result.

Theorem 3. *Let (A3) hold,*

$$M > \max\{1, \rho(\theta, \hat{z})\}, \tag{32}$$

$\epsilon \in (0, 1)$, *a sequence* $\{\Delta_i\}_{i=1}^{+\infty} \subset (0, +\infty)$ *satisfy*

$$\lim_{i \to +\infty} \Delta_i = 0$$

and an integer $n_0 \ge 1$ *satisfy*

$$\Delta_i < \epsilon/2 \text{ for each integer } i \ge n_0. \tag{33}$$

Then, there exists a natural number Q depending on M, ϵ *such that, for each sequence* $\{x_t\}_{t=0}^{+\infty} \subset X$, *which satisfies*

$$\rho(x_0, \theta) \le M \tag{34}$$

and such that, for each integer $t \ge 0$, *there exists* $\alpha(t) \in \mathcal{A}$ *satisfying*

$$x_{t+1} = P_{\alpha(t)}(x_t) \tag{35}$$

and

$$\rho(x_t, x_{t+1}) \ge \sup\{\rho(x_t, P_\alpha(x_t)) : \alpha \in \mathcal{A}\} - \Delta_{i+1}, \tag{36}$$

the inequalities

$$\rho(x_t, \theta) \le 3M, \ t = 0, 1, \dots$$

and

$$Card(\{k \in \{0,1,\dots,\} : \rho(x_k, C_{\alpha(k)}) \geq \epsilon/4\}) < Q$$

hold; if an integer $t \geq n_0$ *satisfies* $\rho(x_t, C_{\alpha(t)}) \leq \epsilon/4$, *then*

$$\rho(x_t, P_\alpha(x_t)) \leq \epsilon,\ \alpha \in \mathcal{A}$$

and

$$Card(\{k \in \{0,1,\dots,\} : \sup\{\rho(x_k, P_\alpha(x_k)) :\ \alpha \in \mathcal{A}\} > \epsilon\}) \leq Q + n_0.$$

Proof. Assumption (A3) implies that there exists $\delta \in (0, \epsilon/4)$ such that the following property holds:

(i) For each $\alpha \in \mathcal{A}$, with each $z \in C_\alpha \cap B(\theta, 3M)$ and each $x \in B(\theta, 3M)$ satisfying

$$\rho(x, C_\alpha) \geq, \epsilon/4$$

we have

$$\rho(z, x) - \delta \geq \rho(z, P_\alpha(x)).$$

Fix an integer

$$Q > 2M\delta^{-1}. \tag{37}$$

Assume that $\{x_t\}_{t=0}^{+\infty} \subset X$ and $\{\alpha(t)\}_{t=0}^{+\infty} \subset \mathcal{A}$ satisfy (34)–(36) for each integer $t \geq 0$. By (A3) and Equations (18), (19), (32), and (34), for each integer $t \geq 0$,

$$\rho(\widehat{z}, x_{t+1}) \leq \rho(\widehat{z}, x_t), \tag{38}$$

$$\rho(\widehat{z}, x_t) \leq \rho(\widehat{z}, x_0) \leq 2M, \tag{39}$$

and

$$\rho(\theta, x_t) \leq 3M. \tag{40}$$

Property (i) and Equations (19), (32) and (40) imply that, for each integer $t \geq 0$ satisfying

$$\rho(x_t, C_{\alpha(t)}) \geq \epsilon/4, \tag{41}$$

we have

$$\rho(\widehat{z}, P_{\alpha(t)}(x_t)) \leq \rho(x_t, \widehat{z}) - \delta. \tag{42}$$

Thus, the following property holds:

(ii) If $t \geq 0$ is an integer and (41) holds, then (42) is true.

Assume that $t \geq n_0$ is an integer and that

$$\rho(x_t, C_{\alpha(t)}) < \epsilon/4.$$

Then, there exists $z \in X$ such that

$$z \in C_\alpha(t),\ \rho(x_t, z) < \epsilon/4. \tag{43}$$

By (A3), (35) and (43),

$$\rho(x_{t+1}, z) = \rho(P_{\alpha(t)}(x_t), z) \leq \rho(x_t, z) < \epsilon/4. \tag{44}$$

In view of (43) and (44),

$$\rho(x_t, x_{t+1}) \leq \epsilon/2. \tag{45}$$

By (33), (36), (45), and the inequality $t \geq n_0$, for each $\alpha \in \mathcal{A}$,

$$\rho(x_t, P_\alpha(x_t)) \leq \epsilon.$$

Therefore, the following property holds:

(iii) If $t \geq n_0$ is an integer and

$$\rho(x_t, C_{\alpha(t)}) < \epsilon/4,$$

then

$$\rho(x_t, P_\alpha(x_t)) \leq \epsilon, \ \alpha \in \mathcal{A}.$$

Let n be a natural number. Property (ii) and Equations (38), (39), (41) and (42) imply that

$$\begin{aligned} 2M &\geq \rho(\widehat{z}, x_0) \\ &\geq \rho(\widehat{z}, x_0) - \rho(\widehat{z}, x_n) \\ &= \sum_{k=0}^{n-1} (\rho(\widehat{z}, x_k) - \rho(\widehat{z}, x_{k+1})) \\ &\sum\{\rho(\widehat{z}, x_k) - \rho(\widehat{z}, x_{k+1}) : \ k \in \{0, \dots, n-1\}, \\ &\rho(x_k, C_{\alpha(k)}) \geq 4^{-1}\epsilon\} \\ &\geq \delta Card(\{k \in \{0, \dots, n-1\} : \rho(x_k, C_{\alpha(k)}) \geq 4^{-1}\epsilon\}) \end{aligned}$$

and

$$Card(\{k \in \{0, \dots, n-1\} : \rho(x_k, C_{\alpha(k)}) \geq 4^{-1}\epsilon\}) \leq 2M\delta^{-1}.$$

Since n is any natural number, we conclude using (37) that

$$Card(\{k \in \{0, 1, \dots, \} : \rho(x_k, C_{\alpha(k)}) \geq 4^{-1}\epsilon\}) \leq 2M\delta^{-1} < Q. \tag{46}$$

Property (iii) and (46) imply that

$$\begin{aligned} &Card(\{k \in \{0, 1, \dots, \} : \sup\{\rho(x_k, P_\alpha(x_k)) : \ \alpha \in \mathcal{A}\} > \epsilon\}) \\ &\leq Card(\{k \in \{0, 1, \dots, \} : \rho(x_k, C_{\alpha(k)}) \geq 4^{-1}\epsilon\} + n_0 \leq n_0 + Q. \end{aligned}$$

Theorem 5 is proved. □

Theorem 5 implies the following result.

Proposition 3. *Let (BRP) and (A3) hold,*

$$M > \max\{1, \rho(\theta, \widehat{z})\};$$

$\epsilon \in (0, 1)$ *and a sequence* $\{\Delta_i\}_{i=1}^{+\infty} \subset (0, +\infty)$ *satisfy*

$$\lim_{i \to +\infty} \Delta_i = 0.$$

Then, there exists a natural number Q *such that for each sequence* $\{x_t\}_{t=0}^{+\infty} \subset X$ *satisfying*

$$\rho(x_0, \theta) \leq M$$

and such that, for each integer $t \geq 0$*, there exists* $\alpha(t) \in \mathcal{A}$ *satisfying (35) and (36), the inequality*

$$Card(\{k \in \{0, 1, \dots, \} : \rho(x_k, \cap_{\alpha \in \mathcal{A}} C_\alpha) > \epsilon\}) \leq Q$$

holds.

5. Conclusions

In this paper, we study a feasibility problem with infinitely many sets in a metric space. Usually, in the literature, the feasibility problem is studied with a finite family of sets using the iterative method, the Cimmino algorithm, and the dynamic string-averaging methods, which are, in some sense, a combination of the iterative algorithm and the Cimmino algorithm. These algorithms work well for problems with finite families of sets but cannot be applied when a family of sets is infinite. The main feature of these algorithms is that, for iterative steps, we need to calculate the values of all the operators belonging to our family of maps and even their sums with weighted coefficients. Of course, this is impossible if the family of maps is not finite. In our paper, we introduce a new algorithm that can be applied for feasibility problems with infinite families of sets and analyze its convergence.

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