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New Contributions to Fixed Point Theory for Multi-Valued Feng–Liu Contractions

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Abstract: In this paper, we will prove several new results related to the concept of the multi-valued Feng–Liu contraction. An existence, approximation and localization fixed point theorem for a generalized multi-valued nonself Feng–Liu contraction and a new fixed point theorem for multi-valued Feng–Liu contractions in vector-valued metric spaces are proved. Stability results and an application to a system of operatorial inclusions are also given.

Keywords: multi-valued operator; fixed point; strict fixed point; vector-valued metric; local fixed point theorem; stability properties; system of operatorial inclusions; altering points

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1. Introduction and Preliminary Notions and Results

Let (X,d) be a metric space and $P(X)$ be the set of all nonempty subsets of X. We denote

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\}, P_{b,cl}(X) := \{Y \in P(X) |\ Y \text{ is bounded and closed}\}.$$

We recall first the following notions:
(1) The distance from a point $x \in X$ to a set $Y \in P(X)$:

$$D(x,Y) := \inf\{d(x,y) \mid y \in Y\};$$

(2) The excess of Y over Z (where $Y, Z \in P(X)$):

$$e(Y,Z) := \sup\{D(y,Z), y \in Y\};$$

(3) The Hausdorff–Pompeiu distance between two sets $Y, Z \in P(X)$:

$$H(Y,Z) = \max\{e(Y,Z), e(Z,Y)\}.$$

Notice that H is a generalized metric (in the sense that $H(Y,Z) \in \mathbb{R}_+ \cup \{\infty\}$) on $P_{cl}(X)$, and it is a classical metric on $P_{b,cl}(X)$.
For $x \in X$ and $r > 0$, we denote by

$$B(x,r) := \{z \in X : d(z,x) < r\} \text{ respectively by } \tilde{B}(x;r) := \{z \in X : d(z,x) \leq r\}$$

the open ball (respectively, the closed ball) centered in x with radius r.
If X is a nonempty set and $T : X \to P(X)$ is a multi-valued operator, then $x \in X$ is called a fixed point for T if $x \in T(x)$. The set

$$Fix(T) := \{x \in X |\ x \in T(x)\}$$

is the fixed point set of T, while

$$SFix(T) := \{x \in X |\ T(x) = \{x\}\}$$

is the strict fixed point set of T. We also denote by

$$Graph(T) := \{(u, v) \in X \times X |\ v \in T(u)\}$$

the graph of the multi-valued operator T.

Remark 1. *If X is a nonempty set and $T : X \to P(X)$, then the sequence $(x_n)_{n\in\mathbb{N}}$ satisfying*

$$x_0 \in X,\ x_{n+1} \in T(x_n), \text{ for each } n \in \mathbb{N}$$

is called an iterative sequence of the Picard type for T starting from x_0.

The multi-valued contraction principle was proved in 1969 by Nadler (see [1]), while a slight extension of it was presented by Covitz and Nadler in 1970 (see [2]).

There are several generalizations of the above multi-valued contraction principle of Nadler/Covitz–Nadler. A consistent extension of it appeared in a paper of Feng and Liu (see [3]), as follows.

Definition 1. *Let (X, d) be a metric space, $T : X \to P(X)$ be a multi-valued operator, $\beta \in]0, 1[$ and $x \in X$. Consider the set*

$$I^x_\beta := \{y \in F(x) : \beta d(x, y) \le D(x, T(x))\}.$$

Then, T is called a multi-valued α-contraction of the Feng–Liu type if there exists $\alpha \in]0, \beta[$, such that for each $x \in X$ there is $y \in I^x_\beta$, for which the following assumption holds

$$D(y, T(y)) \le \alpha d(x, y).$$

Theorem 1. ([3]) *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued α-contraction of the Feng–Liu type. Suppose that the mapping $h : X \to \mathbb{R}_+$ defined by $h(x) = D(x, T(x))$ is lower and semi-continuous. Then, $Fix(T) \neq \emptyset$.*

Another generalization of the multi-valued contraction principle involves the notion of the multi-valued graph contraction.

Definition 2. *Ref. [4] Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. Then, T is called a multi-valued graph α-contraction if $\alpha \in]0, 1[$ and*

$$H(T(x), T(y)) \le \alpha d(x, y), \text{ for all } (x, y) \in Graph(T). \tag{1}$$

The following theorem, proved in [4], is the main result for multi-valued graph contractions.

Theorem 2. ([4]) *Let (X, d) be a complete metric space and $T : X \to P(X)$ be a multi-valued graph α-contraction with a closed graph. Then, T is a $\frac{1}{1-\alpha}$-multi-valued weak Picard operator; i.e., for each $(x, y) \in Graph(T)$, there exists in X a sequence $(x_n)_{n\in\mathbb{N}}$, such that:*

(i) $x_0 = x, x_1 = y$;
(ii) $x_{n+1} \in F(x_n)$, *for each* $n \in \mathbb{N}$;
(iii) $(x_n)_{n\in\mathbb{N}}$ *is convergent to a fixed point* $x^*(x, y)$ *of* T;
(iv) $d(x, x^*(x, y)) \le \frac{1}{1-\alpha} d(x, y)$, *for all* $(x, y) \in Graph(T)$.

Notice that any multi-valued α-contraction is a multi-valued graph α-contraction and any multi-valued graph α-contraction is a multi-valued α-contraction of the Feng–Liu type. For examples and applications of the fixed point theory for the multi-valued graph contraction, see [4].

In this paper, we will prove several new results related to the concept of the multi-valued Feng–Liu contraction. An existence, approximation and localization fixed point theorem for a generalized multi-valued nonself Feng–Liu contraction and a new fixed point theorem for multi-valued Feng–Liu contractions in vector-valued metric spaces are proved. Stability results and an application to a system of operatorial inclusions are also given. For the role and the importance of vector-valued metrics in a nonlinear analysis, see [5–8].

2. A Local Fixed Point Theorem for a Generalized Multi-Valued Feng–Liu Operator

We start this section by proving a local fixed point theorem for a generalized multi-valued α-contraction of the Feng–Liu type. For this purpose, we adapt the notion of the multi-valued α-contraction of the Feng–Liu type to a nonself setting.

Definition 3. *Let (X, d) be a metric space, and $Y \in P(X)$ and $T : Y \to P(X)$ be a multi-valued operator. Consider $\beta \in]0, 1[$ and $x \in Y$. Define*

$$I_\beta^x := \{y \in T(x) : \beta d(x, y) \leq D(x, T(x))\}.$$

Then, T is called a generalized multi-valued nonself (α, γ)-contraction of the Feng–Liu type if there exist $\gamma \in]0, 1[$ and $\alpha \in]0, \beta - \gamma(1 + \beta)[$, such that, for each $x \in Y$ there is $y \in I_\beta^x$, for which the following implication holds

$$y \in Y \;\Rightarrow\; D(y, T(y)) \leq \alpha d(x, y) + \gamma D(x, T(y)).$$

Notice that for $Y = X$ and $\gamma = 0$, we obtain the classical concept of the multi-valued α-contraction of the Feng–Liu type.

We present now an existence, approximation and localization fixed point theorem for a generalized multi-valued nonself contraction of the Feng–Liu type on a closed ball.

Theorem 3. *Let (X, d) be a complete metric space, let $x_0 \in X$ and $r > 0$. We consider $T : \tilde{B}(x_0; r) \to P_{cl}(X)$ a multi-valued nonself (α, γ)-contraction of the Feng–Liu type, such that $D(x_0, T(x_0)) \leq \left(\beta - \frac{\alpha+\gamma}{1-\gamma}\right) r$. Suppose that the mapping of $h : \tilde{B}(x_0; r) \to \mathbb{R}_+$ defined by $h(x) = D(x, T(x))$ is lower and semi-continuous on $\tilde{B}(x_0; r)$, or that T has a closed graph. Then, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard iterates for T, starting from x_0, which converges to a fixed point $x^*(x_0) \in \tilde{B}(x_0; r)$ of T. Moreover, if $k := \frac{\alpha+\gamma}{\beta(1-\gamma)}$, then the following relations hold:*

(a) $d(x_n, x^*(x_0)) \leq \frac{k^n}{1-k} d(x_0, x_1)$, *for each $n \in \mathbb{N}$;*
(b) $d(x_0, x^*(x_0)) \leq \frac{1}{1-k} d(x_0, x_1)$.

Proof. We will show that there exists in $\tilde{B}(x_0; r)$ a sequence $\{x_n\}_{\mathbb{N}}$ of Picard iterates for T, starting from x_0, which converges to a fixed point of T. For x_0 there exists $x_1 \in I_\beta^{x_0}$, having the properties that $x_1 \in T(x_0)$ and $\beta d(x_0, x_1) \leq D(x_0, T(x_0))$. Then, we observe that

$$\beta d(x_0, x_1) \leq D(x_0, T(x_0)) \leq \beta \left(1 - \frac{\alpha + \gamma}{\beta(1 - \gamma)}\right) r,$$

which shows that

$$d(x_0, x_1) \leq \left(1 - \frac{\alpha + \gamma}{\beta(1 - \gamma)}\right) r.$$

Hence, since $x_1 \in \tilde{B}(x_0; r)$, by the definition of the multi-valued nonself (α, γ)-contraction of the Feng–Liu type, we obtain that

$$D(x_1, T(x_1)) \leq \alpha d(x_0, x_1) + \gamma D(x_0, T(x_1)) \leq \alpha d(x_0, x_1) + \gamma(d(x_0, x_1) + D(x_1, T(x_1))).$$

Thus, we have

$$D(x_1, T(x_1)) \leq \frac{\alpha + \gamma}{1 - \gamma} d(x_0, x_1).$$

For $x_1 \in \tilde{B}(x_0; r)$, there exists $x_2 \in I_\beta^{x_1}$, such that $x_2 \in T(x_1)$ and $\beta d(x_1, x_2) \leq D(x_1, T(x_1))$. Additionally, $D(x_2, F(x_2)) \leq \alpha d(x_1, x_2) + \gamma D(x_1, T(x_2))$. As a consequence, we obtain

$$d(x_1, x_2) \leq \frac{1}{\beta} D(x_1, T(x_1)) \leq \frac{\alpha + \gamma}{\beta(1 - \gamma)} d(x_0, x_1).$$

By our assumptions, we have that $k := \frac{\alpha+\gamma}{\beta(1-\gamma)} < 1$.
Now, we observe that

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq d(x_0, x_1) + \frac{\alpha + \gamma}{\beta(1 - \gamma)} d(x_0, x_1) \leq$$

$$\left(1 + \frac{\alpha + \gamma}{\beta(1 - \gamma)}\right) \frac{1}{\beta} D(x_0, T(x_0)) \leq \left[1 - \left(\frac{\alpha + \gamma}{\beta(1 - \gamma)}\right)^2\right] r.$$

As a consequence, $x_2 \in \tilde{B}(x_0; r)$.
By this procedure, we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ of Picard iterates for T, starting from x_0 with the following properties:

(1) $x_{n+1} \in T(x_n) \cap \tilde{B}(x_0; r), n \in \mathbb{N}$;
(2) $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), n \in \mathbb{N}$;
(3) $D(x_n, T(x_n)) \leq k^n D(x_0, T(x_0)), n \in \mathbb{N}$.

By (2), we find that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy. Thus, the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to an element $x^*(x_0) \in \tilde{B}(x_0; r)$. We only need to demonstrate that $x^*(x_0)$ is a fixed point of T. If T has a closed graph, the conclusion follows immediately by the fact that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is of Picard iterates for T. If we suppose that the mapping $h(x) = D(x, T(x))$ is a lower semi-continuous on $\tilde{B}(x_0; r)$, then the conclusion follows by (3), observing that the sequence $\{D(x_n, T(x_n))\}_{n \in \mathbb{N}}$ is convergent to 0.
The conclusions (a) and (b) follow (2), taking into account that, for $n, p \in \mathbb{N}$ with $p \geq 1$, we have

$$d(x_n, x_{n+p}) \leq \sum_{i=0}^{n+p-1} k^i d(x_0, x_1) - \sum_{i=0}^{n-1} k^i d(x_0, x_1).$$

Letting $p \to \infty$, we obtain (a). Then, taking $n = 1$, we obtain

$$d(x_1, x^*(x_0)) \leq \sum_{i=0}^{\infty} k^i d(x_0, x_1) - d(x_0, x_1)$$

which immediately gives the conclusion (b). □

Remark 2. *For related results, see [9–13]. For complementary results, see also [14–18].*

3. A Fixed Point Theorem for Multi-Valued Feng–Liu Contractions in Vector-Valued Metric Spaces

In this section, we will prove a fixed point result for multi-valued Feng–Liu contractions in complete vector-valued metric spaces. For this purpose, we recall some notions and results.

If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition

$$x \preceq y \text{ if and only if } x_i \leq y_i, \text{ for each } i \in \{1, 2, \cdots, m\}.$$

We will make an identification between row and column vectors in $\mathbb{R}^m$.

We now recall the concept of vector-valued metric space in the sense of Perov, see, e.g., [19]. If X is a nonempty set, then a functional $d : X \times X \to \mathbb{R}^m_+$ satisfying the usual axioms of a metric with respect to the above mentioned relation $\preceq$ is called a vector-valued metric in the sense of Perov. In this case, the pair (X, d) is a vector-valued metric space.

We may suppose that

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_m(x,y) \end{pmatrix}, \text{ for } x,y \in X.$$

We denote by

$$D(x,Y) := \begin{pmatrix} D_{d_1}(x,Y) \\ ... \\ D_{d_m}(x,Y) \end{pmatrix}, \text{ the vectorial distance from a point } x \in X \text{ to a set } Y \in P(X)$$

and by

$$H(A,B) := \begin{pmatrix} H_{d_1}(A,B) \\ ... \\ H_{d_m}(A,B) \end{pmatrix}, \text{ the vectorial Pompeiu–Hausdorff distance on } P(X).$$

In this section, $M_{m,m}(\mathbb{R}_+)$ denotes the set of all $m \times m$ matrices with positive elements, I_m is the identity $m \times m$ matrix and O_m denotes the null $m \times m$ matrix.

By definition, a matrix $K \in M_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if $K^n \to O_m$ as $n \to \infty$. The following characterization theorem is useful for the proof of our main results, see, e.g., [20].

Theorem 4. *Let $K \in M_{m,m}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) $K^n \to O_m$ *as* $n \to \infty$;

(ii) *The spectral radius $\rho(K)$ of K is strictly less than 1, i.e., the eigenvalues of K are in the open unit disc;*

(iii) *The matrix $(I_m - K)$ is nonsingular and*

$$(I_m - K)^{-1} = I_m + K + ... + K^n + ...; \tag{2}$$

(iv) *The matrix $(I_m - K)$ is nonsingular and $(I_m - K)^{-1}$ has nonnegative elements.*

The following theorem is the main fixed point result for K-contractions in complete vector-valued metric spaces, see [19].

Theorem 5 (Perov)**.** *Let (X, d) be a complete vector-valued metric space and let $f : X \to X$ be an K-contraction; i.e., $K \in M_{m,m}(\mathbb{R}_+)$ converges to zero and*

$$d(f(x), f(y)) \preceq K d(x,y), \text{ for all } x,y \in X.$$

Then:

(1) *$Fix(f) = \{x^*\}$, i.e., there exists a unique solution $x^* \in X$ of the fixed point equation $x = f(x)$;*

(2) *the sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := f^n(x_0)$ of successive approximations for f starting from any $x_0 \in X$ is convergent to x^*;*

(3) *the following estimation holds*

$$d(x_n, x^*) \preceq K^n (I_m - K)^{-1} d(x_0, x_1), \text{ for every } n \in \mathbb{N}; \tag{3}$$

A multi-valued variant of Perov's theorem was given in [21].

Theorem 6. *Let (X, d) be a complete vector-valued metric space and let $F : X \to P_{cl}(X)$ be a multi-valued K-contraction, i.e., $K \in M_{m,m}(\mathbb{R}_+)$ is convergent to zero and*

$$H(F(x), F(y)) \preceq Kd(x, y), \text{ for all } x, y \in X. \tag{4}$$

Then:

(i) $Fix(F) \neq \emptyset$;
(ii) *For each $(x, y) \in Graph(F)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ (with $x_0 = x$, $x_1 = y$ and $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}^*$), such that $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point $x^* := x^*(x, y)$ of F, and the following relations hold:*

$$d(x_n, x^*) \preceq K^n(I - K)^{-1}d(x_0, x_1), \text{ for each } n \in \mathbb{N}^*$$

and

$$d(x, x^*) \preceq (I - K)^{-1}d(x, y).$$

Our next result is a generalization of the previous theorem in terms of a multi-valued Feng–Liu contraction.

We introduce first the following notion.

Definition 4. *Let (X, d) be a vector-valued metric space, $F : X \to P(X)$ be a multi-valued operator, $B \in M_{m,m}(\mathbb{R}_+)$ be a diagonal matrix with elements $b_1, \cdots, b_m \in]0, 1[$ and $x \in X$. Consider the set*

$$I_B^x := \{y \in F(x) : Bd(x, y) \preceq D(x, F(x))\}.$$

Then, F is called a multi-valued vectorial contraction of the Feng–Liu type if there exists a matrix $A \in M_{m,m}(\mathbb{R}_+)$, such that the matrix $B^{-1}A$ is convergent to zero, and for each $x \in X$, there is $y \in I_B^x$, for which the following relation holds

$$D(y, F(y)) \preceq Ad(x, y).$$

Theorem 7. *Let (X, d) be a complete vector-valued metric space and $F : X \to P(X)$ be a multi-valued vectorial contraction of the Feng–Liu type. Suppose that F has closed graph. Then, for each $x_0 \in X$, there exists an iterative sequence $(x_n)_{n\in\mathbb{N}}$ of the Picard type for F starting from x_0 with the following properties:*

(1) $(x_n)_{n\in\mathbb{N}}$ *converges to* $x^*(x_0) \in Fix(F)$;
(2) $d(x_n, x^*(x_0)) \preceq (B^{-1}A)^n \left(I_m - B^{-1}A\right)^{-1} d(x_0, x_1), n \in \mathbb{N}$;
(3) $d(x_0, x^*(x_0)) \preceq \left(I_m - B^{-1}A\right)^{-1} d(x_0, x_1) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1}D(x_0, F(x_0))$.

Proof. Let $x_0 \in X$ be arbitrarily chosen. Then, there exists $x_1 \in I_B^{x_0}$ such that

$$D(x_1, F(x_1)) \preceq Ad(x_0, x_1).$$

For $x_1 \in X$, by the multi-valued vectorial contraction condition of the Feng–Liu type, there exists $x_2 \in I_B^{x_1}$, such that

$$D(x_2, F(x_2)) \preceq Ad(x_1, x_2).$$

Since $Bd(x_1, x_2) \preceq D(x_1, F(x_1))$, we obtain that

$$d(x_1, x_2) \preceq B^{-1}D(x_1, F(x_1)) \preceq B^{-1}Ad(x_0, x_1).$$

Let us denote $K := B^{-1}A$. Notice that $B^{-1}A = AB^{-1}$. By the above procedure, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X with the following properties:

(a) $x_{n+1} \in I_B^{x_n}$, for each $n \in \mathbb{N}$;
(b) $d(x_n, x_{n+1}) \preceq Kd(x_{n-1}, x_n) \preceq \cdots \preceq K^n d(x_0, x_1)$, for each $n \in \mathbb{N}^*$;

(c) $D(x_{n+1}, F(x_{n+1})) \preceq KD(x_n, F(x_n)) \preceq \cdots \preceq K^{n+1}D(x_0, F(x_0))$, for each $n \in \mathbb{N}$.

Then, by (b), the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Hence, it is convergent to an element $x^* := x^*(x_0) \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is an iterative sequence of the Picard type for F starting from x_0, and F has a closed graph, we obtain that $x^* \in Fix(F)$. Moreover, by the relation

$$d(x_n, x_{n+p}) \preceq K^n\left(I_m + K + \cdots K^{p-1}\right)d(x_0, x_1) \preceq K^n(I_m - K)^{-1}d(x_0, x_1),$$

letting $p \to \infty$, we obtain the following a priori approximation for the fixed point:

$$d(x_n, x^*(x_0)) \preceq K^n(I_m - K)^{-1}d(x_0, x_1), n \in \mathbb{N}.$$

Taking $n = 0$ in the above relation, we obtain the following retraction–displacement condition:

$$d(x_0, x^*(x_0)) \preceq (I_m - K)^{-1}d(x_0, x_1) \preceq (I_m - K)^{-1}B^{-1}D(x_0, F(x_0)).$$

The proof is complete. □

We now present some stability concepts for the fixed point inclusion $x \in F(x)$ in the setting of a vector-valued metric space.

The concept of the Ulam–Hyers stability is now introduced; see also [22].

Definition 5. *Let (X, d) be a vector-valued metric space and $F : X \to P(X)$ be a multi-valued operator. The fixed point inclusion $x \in F(x)$ is called Ulam–Hyers stable if there exists a matrix $C \in M_{m,m}(\mathbb{R}^*_+)$, such that for every $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m)$ (with $\varepsilon_i > 0$ for each $i \in \{1, 2, \cdots, m\}$) and for each ε-fixed point $\tilde{x} \in X$ of F (i.e., $D(\tilde{x}, F(\tilde{x})) \preceq \varepsilon$), there exists $x^* \in Fix(F)$, such that*

$$d(\tilde{x}, x^*) \preceq C\varepsilon.$$

The well-posedness of the fixed point inclusion $x \in F(x)$ in a vector-valued metric space is defined, as follows. The concept is inspired by the single-valued case; see the papers of Reich and Zaslavski [23,24].

Definition 6. *Let (X, d) be a vector-valued metric space. Let $F : X \to P(X)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$ and let $r : X \to Fix(F)$ be a set retraction. Then, the fixed point inclusion $x \in F(x)$ is called well-posed in the sense of Reich and Zaslavski if for each $x^* \in Fix(F)$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that $\{D(y_n, F(y_n))\}_{n\in\mathbb{N}}$ converges to zero as $n \to \infty$; we have that*

$$y_n \to x^* \text{ as } n \to \infty.$$

The data dependence property is given in our next definition.

Definition 7. *Let (X, d) be a vector-valued metric space and $F : X \to P(X)$ be a multi-valued operator. Let $G : X \to P(X)$ be a multi-valued operator satisfying the following conditions:*

(i) $Fix(G) \neq \emptyset$;

(ii) There exists $\eta := (\eta_1, \cdots, \eta_m)$ (with $\eta_i > 0$ for each $i \in \{1, 2, \cdots, m\}$), such that $H(F(x), G(x)) \preceq \eta$, for all $x \in X$.

Then, the fixed point inclusion $x \in F(x), x \in X$ has the data dependence property if for each $g^* \in Fix(G)$ there exists $x^* \in Fix(F)$, such that

$$d(g^*, x^*) \preceq S\eta, \text{ for some matrix } S \in M_{m,m}(\mathbb{R}^*_+).$$

The notion of the Ostrowski stability property for a fixed point inclusion in the vector-valued metric space is now presented; see also [12].

Definition 8. *Let (X, d) be a vector-valued metric space. Let $F : X \to P(X)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$ and let $r : X \to Fix(F)$ be a set retraction. Then, the fixed point inclusion $x \in F(x)$ is said to have the Ostrowski stability property if for each $x^* \in Fix(F)$ and for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that $\{D(z_{n+1}, F(z_n))\}_{n\in\mathbb{N}}$ converges to zero as $n \to \infty$; we have that*

$$z_n \to x^* \text{ as } n \to \infty.$$

The following retraction–displacement condition will be important for our main results.

Definition 9. *Let (X, d) be a vector-valued metric space and let $F : X \to P(X)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$. Then, we say that F satisfies the strong vectorial retraction–displacement condition if there exist a matrix $Q \in M_{m,m}(\mathbb{R}^*_+)$ and a set retraction $r : X \to Fix(F)$, such that*

$$d(x, r(x)) \preceq QD(x, F(x)), \text{ for all } x \in X. \tag{5}$$

An abstract result concerning some stability properties of a multi-valued operator is given in our next result.

Theorem 8. *Let (X, d) be a vector-valued metric space and let $F : X \to P(X)$ be a multi-valued operator satisfying the strong vectorial retraction–displacement condition, such that $Fix(F) \neq \emptyset$. Then, the fixed point inclusion $x \in F(x)$ has the Ulam–Hyers stability property; it is well-posed and satisfies the data dependence property.*

Proof. Suppose that there exists a matrix $Q \in M_{m,m}(\mathbb{R}^*_+)$ and a set retraction $r : X \to Fix(F)$, such that

$$d(x, r(x)) \preceq QD(x, F(x)), \text{ for all } x \in X.$$

In order to prove the Ulam–Hyers stability property, let us consider $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m)$ (with $\varepsilon_i > 0$ for each $i \in \{1, 2, \cdots, m\}$) and $\tilde{x} \in X$ such that $D(\tilde{x}, F(\tilde{x})) \preceq \varepsilon$. Then, by the strong vectorial retraction–displacement condition, we have

$$d(\tilde{x}, r(\tilde{x})) \preceq QD(\tilde{x}, F(\tilde{x})) \preceq Q\varepsilon.$$

Thus, the fixed point inclusion $x \in F(x)$ is Ulam–Hyers stable.

For the well-posedness property of the fixed point inclusion, let us consider the sequence $\{y_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that the sequence $D(y_n, F(y_n))$ converges to zero as $n \to \infty$. Then, for each $n \in \mathbb{N}$, we have $r(y_n) = x^*$ and, again by the strong vectorial retraction–displacement condition, we conclude that

$$d(y_n, x^*) = d(y_n, r(y_n)) \preceq QD(y_n, F(y_n)) \to 0 \text{ as } n \to \infty.$$

Let us now prove the data dependence of the fixed point set. Let us consider a multi-valued operator $G : X \to P(X)$ to have the properties:

(i) $Fix(G) \neq \emptyset$;

(ii) There exists $\eta := (\eta_1, \cdots, \eta_m)$ (with $\eta_i > 0$ for each $i \in \{1, 2, \cdots, m\}$), such that

$$H(F(x), G(x)) \preceq \eta, \text{ for all } x \in X.$$

Take any $g^* \in Fix(G)$ and denote $x^* := r(g^*)$. Then, by the strong vectorial retraction–displacement condition, we have that

$$d(g^*, x^*) = d(g^*, r(g^*)) \preceq QD(g^*, F(g^*)) \preceq QH(G(g^*), F(g^*)) \leq Q\eta.$$

The proof is now complete.

□

The following result shows that any multi-valued vectorial contraction of the Feng–Liu type has a strong vectorial retraction–displacement condition.

Theorem 9. *Let (X, d) be a complete vector-valued metric space and $F : X \to P(X)$ be a multi-valued vectorial contraction of the Feng–Liu type. Suppose that F has a closed graph. Then, F satisfies the strong vectorial retraction–displacement condition.*

Proof. By Theorem 7, we know that $Fix(F) \neq \emptyset$ and, for every $x \in X$, there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of the Picard type for F starting from the arbitrary $x_0 \in X$, which converges to a fixed point $x^*(x_0)$ of F. Moreover, the following relation holds

$$d(x_0, x^*(x_0)) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1} D(x_0, F(x_0)).$$

Thus, we can define the set-retraction $r : X \to Fix(F)$, $x \longmapsto r(x) := x^*(x)$ with the property

$$d(x, r(x)) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1} D(x, F(x)), x \in X.$$

Hence, the strong vectorial retraction–displacement condition from Definition 9 is satisfied. □

By combining the above two theorems, we obtain the following stability properties for the multi-valued vectorial contraction of the Feng–Liu type.

Theorem 10. *Let (X, d) be a complete vector-valued metric space and $F : X \to P(X)$ be a multi-valued vectorial contraction of the Feng–Liu type. Suppose that F has a closed graph. Then, the fixed point inclusion $x \in F(x)$ is well-posed in the sense of Reich and Zaslavsi, has the Ulam–Hyers stability property and satisfies the data dependence property.*

Proof. By Theorem 7, we have that $Fix(F) \neq \emptyset$, while Theorem 9 implies that F has the strong vectorial retraction–displacement property. The conclusions follow by Theorem 8. □

Remark 3. *It is an open question to prove the Ostrowski stability property for a multi-valued vectorial contraction of the Feng–Liu type.*

4. An Application to a System of Operatorial Inclusions

In this section, we will present an existence result for a system of operatorial inclusion in complete metric spaces. The approach is based on the vectorial technique for multi-valued Feng–Liu operators.

Let (X, d_1) and (Y, d_2) be two complete metric spaces and let $G_1 : X \times Y \to P(X)$ and $G_2 : X \times Y \to P(Y)$ be two multi-valued operators with a closed graph. We consider the following system of operatorial inclusions

$$\begin{cases} x \in G_1(x, y) \\ y \in G_2(x, y). \end{cases} \tag{6}$$

Denote by $Z := X \times Y$ and define on Z the vectorial metric $\tilde{d} : Z \times Z \to \mathbb{R}^2_+$ given by

$$\tilde{d}((x, y), (u, v)) := \begin{pmatrix} d_1(x, u) \\ d_2(y, v) \end{pmatrix}, \text{ for each } (x, y), (u, v) \in Z.$$

Let $b_1, b_2 \in]0, 1[$ and define the following nonempty sets:

$$I_{b_1}^{(x,y)} := \{u \in G_1(x, y) : b_1 d_1(x, u) \leq D_1(x, G_1(x, y))\} \subset X$$

and
$$I_{b_2}^{(x,y)} := \{v \in G_2(x,y) : b_2 d_2(y,v) \le D_2(y, G_2(x,y))\} \subset Y,$$
where D_1 and D_2 are the distances from a point to a set with respect to d_1 and d_2, respectively.

Denote also
$$B := \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \tag{7}$$
We suppose that for every $(x,y) \in X \times Y$ there exist $u \in I_{b_1}^{(x,y)}$ and $v \in I_{b_2}^{(x,y)}$, such that
$$D_1(u, G_1(u,v)) \le a_1 d_1(x,u) + a_2 d_2(y,v) \tag{8}$$
and
$$D_2(v, G_2(u,v)) \le a_3 d_1(x,u) + a_4 d_2(y,v), \tag{9}$$
where
$$A := \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}. \tag{10}$$
is a matrix with nonnegative elements. We also suppose that the matrix
$$B^{-1}A = \begin{pmatrix} \frac{a_1}{b_1} & \frac{a_2}{b_1} \\ \frac{a_3}{b_2} & \frac{a_4}{b_2} \end{pmatrix} \tag{11}$$
is convergent to zero.

Under the above assumptions, we have the following existence and approximation result.

Theorem 11. *Let us consider the system of operatorial inclusions (6). Under the above assumptions, the system (6) has at least one solution $(x^*, y^*) \in X \times Y$. Moreover, for each $(x_0, y_0) \in X \times Y$, there exist two sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$ with the following properties:*

(A) $x_{n+1} \in G_1(x_n, y_n)$ and $y_{n+1} \in G_2(x_n, y_n)$, for each $n \in \mathbb{N}$;

(B) $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* and $\{y_n\}_{n\in\mathbb{N}}$ converges to y^* as $n \to \infty$;

(C) $\begin{pmatrix} d_1(x_n, x^*) \\ d_2(y_n, y^*) \end{pmatrix} \preceq (B^{-1}A)^n (I_m - B^{-1}A)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix}, n \in \mathbb{N}$;

(D) $$\begin{pmatrix} d_1(x_0, x^*) \\ d_2(y_0, y^*) \end{pmatrix} \preceq (I_m - B^{-1}A)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix} \preceq (I_m - B^{-1}A)^{-1} B^{-1} \begin{pmatrix} D_1(x_0, G_1(x_0, y_0)) \\ D_2(y_0, G_2(x_0, y_0)) \end{pmatrix}.$$

Proof. We denote $Z := X \times Y$ and, for $z := (x,y) \in Z$, consider the multi-valued operator $G : Z \to P(Z)$ given by $G(z) := G_1(z) \times G_2(z)$. Notice that the fixed points $z^* = (x^*, y^*)$ of G are solutions for the operatorial inclusion (6).

Let $(x,y) \in X \times Y$ and $(u,v) \in I_{b_1}^{(x,y)} \times I_{b_2}^{(x,y)}$, such that
$$D_1(u, G_1(u,v)) \le a_1 d_1(x,u) + a_2 d_2(y,v)$$
and
$$D_2(v, G_2(u,v)) \le a_3 d_1(x,u) + a_4 d_2(y,v).$$
We consider the vectorial gap function
$$\tilde{D}(z, Z) := \begin{pmatrix} D_1(x, X) \\ D_2(y, Y) \end{pmatrix}.$$

We denote $I_B^{(x,y)} := \{(u,v) \in G(x,y) : B\tilde{d}((x,y),(u,v)) \preceq \tilde{D}((x,y),G(x,y))\}$. Then, by our assumptions, the set $I_B^{(x,y)}$ is nonempty for each $(x,y) \in Z$. Moreover, by (8) and (9), we obtain that for each $z := (x,y) \in Z$, there exists $w := (u,v) \in I_B^{(x,y)}$, such that

$$\tilde{D}(w, G(w))) \preceq A\tilde{d}(z,w),$$

where

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

Thus, G satisfies all the assumptions of the Theorem 7. As a consequence, for each $z_0 := (x_0, y_0) \in Z$ there exists an iterative sequence of the Picard type $\{z_n := (x_n, y_n)\}_{n\in\mathbb{N}}$, which converges to a fixed point $z^*(z_0) \in Z$ of G, and the following relations hold:

(I) $\tilde{d}(z_n, z^*(z_0)) \preceq (B^{-1}A)^n \left(I_m - B^{-1}A\right)^{-1} \tilde{d}(z_0, z_1), n \in \mathbb{N}$;

(II) $\tilde{d}(z_0, z^*(z_0)) \preceq \left(I_m - B^{-1}A\right)^{-1} \tilde{d}(z_0, z_1) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1}\tilde{D}(z_0, G(z_0))$.

Thus, the proof is complete. □

Remark 4. *In the above mentioned conditions, some stability results (well-posedness, Ulam–Hyers stability and data dependence property) for the system of operatorial inclusions (6) can be established by applying the abstract results proved in Section 3.*

In particular, an existence and approximation result for the multi-valued altering points problem can be obtained. We notice that, if (X, d_1) and (Y, d_2) are two metric spaces and $G_1 : Y \to P(X)$ and $G_2 : X \to P(Y)$ are two multi-valued operators, then the following system of operatorial inclusions is called an altering points problem for multi-valued operators:

$$\begin{cases} x \in G_1(y) \\ y \in G_2(x). \end{cases} \tag{12}$$

The above problem has important applications in the theory of generalized/multivalued variational inequalities, see, e.g., [25].

Theorem 12. *Let (X, d_1) and (Y, d_2) be two complete metric spaces and let $G_1 : Y \to P(X)$ and $G_2 : X \to P(Y)$ be two multi-valued operators with a closed graph. Let $b_1, b_2 \in]0, 1[$ and the sets*

$$J_{b_1}^{(x,y)} := \{u \in G_1(y) : b_1 d_1(x,u) \le D_1(x, G_1(y))\}$$

and

$$J_{b_2}^{(x,y)} := \{v \in G_2(x) : b_2 d_2(y,v) \le D_2(y, G_2(x))\}.$$

We suppose that for every $(x,y) \in X \times Y$, there exists $(u,v) \in J_{b_1}^{(x,y)} \times J_{b_2}^{(x,y)}$, such that

$$D_1(u, G_1(v)) \le a_1 d_1(x,u) + a_2 d_2(y,v)$$

and

$$D_2(v, G_2(u)) \le a_3 d_1(x,u) + a_4 d_2(y,v),$$

where

$$A := \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

is a matrix with nonnegative elements. We also suppose that the matrix

$$B^{-1}A = \begin{pmatrix} \frac{a_1}{b_1} & \frac{a_2}{b_1} \\ \frac{a_3}{b_2} & \frac{a_4}{b_2} \end{pmatrix}$$

is convergent to zero, where

$$B := \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Then, the altering points problem (12) has at least one solution in $X \times Y$*. Moreover, for each* $(x_0, y_0) \in X \times Y$*, there exist two sequences* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *and* $\{y_n\}_{n\in\mathbb{N}} \subset Y$ *with the following properties:*

(A) $x_{n+1} \in G_1(y_n)$ *and* $y_{n+1} \in G_2(x_n)$*, for each* $n \in \mathbb{N}$*;*
(B) $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x^* *and* $\{y_n\}_{n\in\mathbb{N}}$ *converges to* y^* *as* $n \to \infty$*;*
(C) $\begin{pmatrix} d_1(x_n, x^*) \\ d_2(y_n, y^*) \end{pmatrix} \preceq (B^{-1}A)^n (I_m - B^{-1}A)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix}, n \in \mathbb{N};$
(D) $\begin{pmatrix} d_1(x_0, x^*) \\ d_2(y_0, y^*) \end{pmatrix} \preceq \left(I_m - B^{-1}A\right)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix}$

$$\preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1} \begin{pmatrix} D_1(x_0, G_1(y_0)) \\ D_2(y_0, G_2(x_0)) \end{pmatrix}.$$

Example. Let $X := \{0, \frac{1}{2}, \cdots, \frac{1}{2^n}, \cdots\}$ and $Y := \{0, 1, \frac{1}{2}, \cdots, \frac{1}{2^n}, \cdots\}$, both endowed with the absolute value metric. Let $G_1 : Y \to P(X)$ and $G_2 : X \to P(Y)$ be given by

$$G_1(y) := \begin{cases} \{\frac{1}{2^{n+1}}\}, & y \in \{\frac{1}{2^n} : n \in \mathbb{N}\} \\ \{0, \frac{1}{2}\}, & y = 0 \end{cases}$$

and

$$G_2(x) := \begin{cases} \{\frac{1}{2^{n+1}}\}, & x \in \{\frac{1}{2^n} : n \in \mathbb{N}, n \geq 2\} \\ \{0, \frac{1}{2}, 1\}, & x \in \{0, \frac{1}{2}\} \end{cases}$$

The above operators satisfy the multi-valued vectorial contraction condition of the Feng–Liu type, having as solutions of the altering point problem the pairs $(0, 0)$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, 1)$.

The above results generalize some altering points theorems, as given for the single-valued case in [26].

5. Conclusions

In this paper, some new contributions to the study of the fixed point inclusion are given. The two main results of this work are:

1. An existence, approximation and localization result for the fixed points of a multi-valued Feng–Liu contraction;
2. A study of the fixed point inclusion $x \in T(x)$ with a multi-valued Feng–Liu contraction $T : X \to P(X)$ in the context of a vector-valued metric space; the study includes existence, approximation and stability results for the fixed point inclusion $x \in T(x)$; the importance of the fixed point theory in vector-valued metric spaces is illustrated by an application to a system of operatorial inclusions. The particular case of altering points for multi-valued operators is also considered.

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References

1. Nadler, S.B., Jr. Multi-valued contraction mappings. *Pacific J. Math.* **1969**, *30*, 475–488. [CrossRef]
2. Covitz, H.; Nadler, S.B., Jr. Multi-valued contraction mappings in generalized metric spaces. *Israel J. Math.* **1970**, *8*, 5–11. [CrossRef]
3. Feng, Y.; Liu, S. Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings. *J. Math. Anal. Appl.* **2006**, *317*, 103–112. [CrossRef]
4. Petruşel, A.; Petruşel, G.; Yao, J.-C. Multi-valued graph contraction principle with applications. *Optimization* **2020**, *69*, 1541–1556. [CrossRef]
5. Precup, R. The role of matrices that are convergent to zero in the study of semilinear operator systems. *Math. Comput. Model.* **2009**, *49*, 703–708. [CrossRef]
6. Agratini, O.; Precup, R. Iterates of multidimensional approximation operators via Perov theorem. *Carpathian J. Math.* **2022**, *38*, 539–546. [CrossRef]
7. Cvetković, M.; Karapinar, E.; Rakococević, V.; Yeşilkaya, S.S. Perov-type contractions. In *Approximation and Computation in Science and Engineering*; Springer Optim. Appl., 180; Springer: Cham, Switzerland, 2022; pp. 167–215.
8. Zaslavski, A.J. Existence and convergence results for contractive mappings of Perov type. *J. Nonlinear Convex Anal.* **2022**, *23*, 2677–2685.
9. Agarwal, R.P.; Meehan, M.; Donal O'Regan, D. *Fixed Point Theory and Applications*; Cambridge University Press: Cambridge, UK, 2001.
10. Chifu, C.; Petruşel, G. Existence and data dependence of fixed points and strict fixed points for contractive-type multivalued operators. *Fixed Point Theory Appl.* **2007**, *2007*, 034248. [CrossRef]
11. Chifu, C.; Petruşel, A.; Petruşel, G. Fixed point results for non-self nonlinear graphic contractions in complete metric spaces with applications. *J. Fixed Point Theory Appl.* **2020**, *22*, 16. [CrossRef]
12. Petruşel, A.; Petruşel, G. Some variants of the contraction principle for multi-valued operators, generalizations and applications. *J. Nonlinear Convex Anal.* **2019**, *20*, 2187–2203.
13. Xu, S.; Han, Y.; Aleksić, S.; Radenović, S. Fixed point results for nonlinear contractions of Perov type in abstract metric spaces with applications, *AIMS Math.* **2022**, *7*, 14895–14921. [CrossRef]
14. Ćirić, L.j.B. Fixed points for generalized multivalued contractions. *Mat. Vesn.* **1972**, *9*, 265–272.
15. Minak, G.; Acar, O.; Altun, I. Multivalued pseudo-Picard operators and fixed point results. *J. Funct. Spaces Appl.* **2013**, *2013*, 827458. [CrossRef]
16. Debnath, P.; Konwar, N.; Radenović, S. (Eds.) *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences*; Springer: Singapore, 2021.
17. Reich, S. Fixed points of contractive functions. *Boll. Unione Mat. Ital.* **1972**, *5*, 26–42.
18. Wang, T.X. Fixed point theorems and fixed point stability for multi-valued mappings on metric spaces. *Nanjing Daxue Xuebao Shuxue Bannian Kan* **1989**, *6*, 16–23.
19. Perov, A.I.; Kibenko, A.V. On a certain general method for investigation of boundary value problems. *Izv. Akad. Nauk SSSR* **1966**, *30*, 249–264. (In Russian)
20. Varga, R.S. *Matrix Iterative Analysis*; Springer Series in Computational Mathematics; Springer: Berlin, Germany, 2000; Volume 27.
21. Petruşel, A.; Petruşel, G.; Urs, C. Vector-valued metrics, fixed points and coupled fixed points for nonlinear operators. *Fixed Point Theory Appl.* **2013**, *2013*, 218. [CrossRef]
22. Petru, T.P.; Petruşel, A.; Yao, J.-C. Ulam-Hyers stability for operatorial equations and inclusions via nonself operators. *Taiwan. J. Math.* **2011**, *15*, 2195–2212. [CrossRef]
23. Reich, S.; Zaslavski, A.J. Well-posedness of fixed point problems. *Far East J. Math. Sci.* **2011**, *46* 393–401. [CrossRef]
24. Reich, S.; Zaslavski, A.J. *Genericity in Nonlinear Analysis*; Springer: New York, NY, USA, 2014.
25. Sahu, D.R.; Kang, S.M.; Kumar, A. Convergence analysis of parallel S-iteration process for system of generalized variational inequalities. *J. Funct. Spaces* **2017,** *2017*, 5847096. [CrossRef]
26. Sahu, D.R. Altering points and applications. *Nonlinear Stud.* **2014**, *21*, 349–365.