axioms MDPI

Article

New Arithmetic Operations of Non-Normal Fuzzy Sets Using Compatibility

Hsien-Chung Wu

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan; hsien.chung.wu@gmail.com

Abstract: The new arithmetic operations of non-normal fuzzy sets are studied in this paper by using the extension principle and considering the general aggregation function. Usually, the aggregation functions are taken to be the minimum function or t-norms. In this paper, we considered a general aggregation function to set up the arithmetic operations of non-normal fuzzy sets. In applications, the arithmetic operations of fuzzy sets are always transferred to the arithmetic operations of their corresponding α-level sets. When the aggregation function is taken to be the minimum function, this transformation is clearly realized. Since the general aggregation function was adopted in this paper, the concept of compatibility with α-level sets is needed and is proposed, which can cover the conventional case using minimum functions as the special case.

Keywords: compatibility; extension principle; non-normal fuzzy sets

MSC: 03E72

check for updates

Citation: Wu, H.-C. New Arithmetic Operations of Non-Normal Fuzzy Sets Using Compatibility. *Axioms* **2023**, *12*, 277. https://doi.org/10.3390/axioms12030277

Academic Editor: Amit K. Shukla

Received: 24 December 2022
Revised: 17 February 2023
Accepted: 2 March 2023
Published: 7 March 2023

1. Introduction

In order to simplify the notations, the membership function $\xi_{\tilde{F}}$ of a fuzzy set $\tilde{F}$ is identified with $\tilde{F}$ by simply writing $\xi_{\tilde{F}}(x) = \tilde{F}(x)$. Let $\tilde{F}$ and $\tilde{G}$ be two fuzzy sets in $\mathbb{R}$, and let $\odot$ denote any one of the arithmetic operations $\oplus, \ominus, \otimes, \oslash$ between $\tilde{F}$ and $\tilde{G}$. According to the extension principle, the membership function of $\tilde{F} \odot \tilde{G}$ is defined by

$$\tilde{F} \odot \tilde{G}(u) = \sup_{\{(x,y):u=x\circ y\}} \min\{\tilde{F}(x), \tilde{G}(y)\} \tag{1}$$

for all $u \in \mathbb{R}$, where the arithmetic operations $\odot \in \{\oplus, \ominus, \otimes, \oslash\}$ correspond to the arithmetic operations $\circ \in \{+, -, *, \div\}$. The case of $\circ = \div$ should avoid the division of x/y for $y = 0$.

In general, we can consider the t-norm instead of the minimum function by referring to Dubois and Prade [1] and Weber [2]. For more detailed properties, we can refer to the monographs by Dubois and Prade [3] and Klir and Yuan [4]. In this paper, we used the general function to propose the arithmetic operations of fuzzy sets, and we present the compatibility with the conventional definition using the minimum functions. We can also refer to Gebhardt [5], Fullér and Keresztfalvi [6], Mesiar [7], Ralescu [8], and Yager [9] and Wu [10] for the arithmetic operations of fuzzy sets based on the extension principle.

The generalization of Zadeh's extension principle in (1) can also be used to set up the arithmetic operations without using the minimum function. Coroianua and Fuller [11,12] used the so-called joint probability distribution to generalize the extension principle (1), which is given by

$$\tilde{F} \odot_{\mathfrak{C}} \tilde{G}(u) = \sup_{\{(x,y):u=x\circ y\}} \mathfrak{C}(x,y) \tag{2}$$

for all $u \in \mathbb{R}$, where $\mathfrak{C} : \mathbb{R}^2 \to [0,1]$ is a joint probability distribution satisfying

$$\sup_{x\in\mathbb{R}} \mathfrak{C}(x,y) = \tilde{G}(y) \text{ and } \sup_{y\in\mathbb{R}} = \mathfrak{C}(x,y) = \tilde{F}(x). \tag{3}$$

Wu [10] considered a general function $\mathfrak{D} : [0,1] \times [0,1] \to [0,1]$ by defining the arithmetic as

$$\tilde{F} \odot_{\mathfrak{D}} \tilde{G}(u) = \sup_{\{(x,y):u=x\circ y\}} \mathfrak{D}\big(\tilde{F}(x), \tilde{G}(y)\big), \tag{4}$$

where $\mathfrak{D}$ does not need to satisfy some extra conditions. The main difference between (2) and (4) is that the domains of the joint probabilitydistribution $\mathfrak{C} : \mathbb{R}^2 \to [0,1]$ and function $\mathfrak{D} : [0,1]^2 \to [0,1]$ are different. We can also refer to Coroianua and Fuller [11] for the comparison between (2) and (4). Although $\mathfrak{D}$ in (4) is a general function, some sufficient conditions regarding $\mathfrak{D}$ are still needed to obtain some desired properties. Therefore, the second motivation of this paper was to propose the concept of compatibility. We shall say that the function $\mathfrak{D}$ is compatible with the arithmetic operations of α-level sets when the following equality:

$$\big(\tilde{F} \odot_{\mathfrak{D}} \tilde{G}\big)_\alpha = \tilde{F}_\alpha \circ \tilde{G}_\alpha$$

is satisfied for each $\alpha \in (0,1]$. The sufficient conditions imposed upon the function $\mathfrak{D}$ will be studied to guarantee the compatibility. Under the general function $\mathfrak{D}$, the associativity of the arithmetic operations is also an important issue. Therefore, many rules regarding the associativity were also studied.

There is some other interesting arithmetic of fuzzy numbers, which will be shown below. Holčapek, Škorupová, and Štěpnička [13,14] proposed the arithmetic of extensional fuzzy numbers based on a similarity relation $S : \mathbb{R}^2 \to [0,1]$ such that S satisfies some required conditions. On the other hand, based on the concept of the extensional hull, given a fixed real number $x \in \mathbb{R}$, the so-called extensional fuzzy number generated by x and a similarity relation S is a fuzzy set $\tilde{x}_S$ in $\mathbb{R}$ with membership degree

$$\tilde{x}_S(y) = S(x,y) \text{ for all } y \in \mathbb{R}.$$

Given any two extensional fuzzy numbers $\tilde{x}_S$ and $\tilde{y}_S$, the addition $\oplus_S$ and multiplication $\otimes_S$ are defined by

$$\tilde{x}_S \oplus_S \tilde{y}_S = (x+y)_S \text{ and } \tilde{x}_S \otimes_S \tilde{y}_S = (xy)_S,$$

where S is assumed to be the so-called separated similarity relation for the purpose of well-defined arithmetic. In general, based on a system $\mathcal{S}$ of so-called nested similarity relations, the addition $\oplus_{\mathcal{S}}$ and multiplication $\otimes_{\mathcal{S}}$ are defined by

$$\tilde{x}_S \oplus_{\mathcal{S}} \tilde{y}_T = (x+y)_{\max(S,T)} \text{ and } \tilde{x}_S \otimes_{\mathcal{S}} \tilde{y}_T = (xy)_{\max(S,T)} \text{ for } S,T \in \mathcal{S}.$$

Esmi et al. [15] and Pedro et al. [16] used the extension principle in (3) to study the fuzzy differential equations. They considered the interactivity between fuzzy numbers. Let $\tilde{P}$ be a fuzzy set in $\mathbb{R}$. Given any fuzzy numbers $\tilde{F}$ and $\tilde{G}$, we say that $\tilde{P}$ is a joint probability distribution of $\tilde{F}$ and $\tilde{G}$ when

$$\sup_{x\in\mathbb{R}} \tilde{P}(x,y) = \tilde{G}(y) \text{ and } \sup_{y\in\mathbb{R}} = \tilde{P}(x,y) = \tilde{F}(x).$$

We say that $\tilde{F}$ and $\tilde{G}$ are non-interactive when

$$\tilde{P}(x,y) = \min\{\tilde{F}(x), \tilde{G}(x)\}.$$

Otherwise, they are called interactive. The disadvantage is that the non-interactivity depends on their joint probability distributions. We cannot just say that $\tilde{F}$ and $\tilde{G}$ are non-interactive without considering the role of the joint probability distribution. Let $\odot$ denote any one of the arithmetic operations $\oplus_{\tilde{P}}, \ominus_{\tilde{P}}, \otimes_{\tilde{P}}, \oslash_{\tilde{P}}$ between fuzzy numbers $\tilde{F}$ and $\tilde{G}$ along with a joint probability distribution $\tilde{P}$. The membership function of $\tilde{F} \odot_{\tilde{P}} \tilde{G}$ is defined by

$$\tilde{F} \odot_{\tilde{P}} \tilde{G}(u) = \sup_{\{(x,y):u=x\circ y\}} \tilde{P}(x,y)$$

for all $u \in \mathbb{R}$, where the case of $\circ_{\tilde{P}} = \div$ should avoid the division of x/y for $y = 0$.

The arithmetic of fuzzy intervals is an important issue. Wu [17] considered the form of expression in the decomposition theorem to study the arithmetic of fuzzy intervals. Wu [18] also used the form of expression in the decomposition theorem to study the different types of binary operations of fuzzy sets, which were also applied to study the difference of fuzzy intervals and covered the so-called generalized differences proposed by Bede and Stefanini [19] and Gomes and Barros [20] as the special cases. The fuzzy axiom of choice, the fuzzy Zorn's lemma, and the fuzzy Hausdorff maximal principle studied by Zulqarnian et al. [21] were also based on normal fuzzy sets. It is also possible to extend those results based on the non-normal fuzzy sets.

The fuzzy sets considered in Wu [17,18] were implicitly assumed to be normal. Without using the form of expression in the decomposition theorem, in this paper, we shall use the extension principle based on a general function rather than the t-norm to study the arithmetic of non-normal fuzzy intervals. In this case, the concept of compatibility with α-level sets can be proposed and the equivalence with conventional arithmetic operations using the minimum function can also be established.

In Section 2, the concept and basic properties of non-normal fuzzy sets will be presented, and the arithmetic operations of non-normal fuzzy sets will be studied using the extension principle based on the general functions. In Section 3, we shall propose the concept of compatibility with the α-level sets, which can cover the conventional case using the minimum functions as the special case.

2. Arithmetic Operations of Fuzzy Sets

Let $\tilde{F}$ be a fuzzy set in $\mathbb{R}$. Recall that a fuzzy set $\tilde{F}$ in a universal set U is called normal when there exists $x \in U$ satisfying $\tilde{F}(x) = 1$. For $\alpha \in (0,1]$, the α-level set of $\tilde{F}$ is denoted and defined by

$$\tilde{F}_\alpha = \{x \in \mathbb{R} : \tilde{F}(x) \geq \alpha\}. \tag{5}$$

The support of a fuzzy set $\tilde{F}$ is the crisp set defined by

$$\tilde{F}_{0+} = \{x \in \mathbb{R} : \tilde{F}(x) > 0\}.$$

The 0-level set $\tilde{F}_0$ is defined to be the topological closure of the support of $\tilde{F}$, i.e., $\tilde{F}_0 = \text{cl}(\tilde{F}_{0+})$. We write $\mathcal{R}_{\tilde{F}}$ to denote the range of the membership function of $\tilde{F}$. In general, we have $\mathcal{R}_{\tilde{F}} \neq [0,1]$. The following result is very useful.

Proposition 1. *Let $\tilde{F}$ be a fuzzy set in $\mathbb{R}$ with membership function $\tilde{F}$. Define $\alpha^* = \sup \mathcal{R}_{\tilde{F}}$ and*

$$I_{\tilde{F}} = \begin{cases} [0,\alpha^*), & \text{if the supremum } \sup \mathcal{R}_{\tilde{F}} \text{ is not obtained} \\ [0,\alpha^*], & \text{if the supremum } \sup \mathcal{R}_{\tilde{F}} \text{ is obtained.} \end{cases} \tag{6}$$

Then, $\tilde{F}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{F}}$ and $\tilde{F}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{F}}$. Moreover, we have $\mathcal{R}_{\tilde{F}} \subseteq I_{\tilde{F}}$ and

$$\tilde{F}_{0+} = \bigcup_{\{\alpha \in I_{\tilde{F}}:\alpha>0\}} \tilde{F}_\alpha = \bigcup_{\{\alpha \in \mathcal{R}_{\tilde{F}}:\alpha>0\}} \tilde{F}_\alpha.$$

The interval $I_{\tilde{F}}$ is called an interval range of $\tilde{F}$.

We considered three arithmetic operations $\boxplus, \boxminus$ and $\boxtimes$ between any two fuzzy sets $\tilde{F}$ and $\tilde{G}$ in $\mathbb{R}$. The extension principle says that the membership functions are given by

$$\tilde{F} \boxplus \tilde{G}(u) = \sup_{\{(x,y):u=x+y\}} \min\{\tilde{F}(x), \tilde{G}(y)\} \tag{7}$$

$$\tilde{F} \boxminus \tilde{G}(u) = \sup_{\{(x,y):u=x-y\}} \min\{\tilde{F}(x), \tilde{G}(y)\} \tag{8}$$

$$\tilde{F} \boxtimes \tilde{G}(u) = \sup_{\{(x,y):u=xy\}} \min\{\tilde{F}(x), \tilde{G}(y)\} \tag{9}$$

for all $u \in \mathbb{R}$, where the arithmetic operations $\boxdot \in \{\boxplus, \boxminus, \boxtimes\}$ correspond to the arithmetic operations $\circ \in \{+, -, *\}$. The case of division was not considered in this paper, since it can be similarly obtained.

Instead of the minimum function, we can consider a general function $\mathfrak{D} : [0,1]^2 \to [0,1]$ defined on $[0,1]^2$. In this case, the membership functions are defined by

$$\tilde{F} \oplus_{EP} \tilde{G}(u) = \sup_{\{(x,y):u=x+y\}} \mathfrak{D}\big(\tilde{F}(x), \tilde{G}(y)\big);$$

$$\tilde{F} \ominus_{EP} \tilde{G}(u) = \sup_{\{(x,y):u=x-y\}} \mathfrak{D}\big(\tilde{F}(x), \tilde{G}(y)\big);$$

$$\tilde{F} \otimes_{EP} \tilde{G}(u) = \sup_{\{(x,y):u=x\cdot y\}} \mathfrak{D}\big(\tilde{F}(x), \tilde{G}(y)\big).$$

In general, the arithmetic operations are defined below.

Definition 1. *Given any fuzzy sets $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ in $\mathbb{R}$ and a function $\mathfrak{D}_n : [0,1]^n \to [0,1]$ defined on the product set $[0,1]^n$, regarding the operations $\odot_i \in \{\oplus, \ominus, \otimes\}$ for $i = 1, \cdots, n-1$, the membership function of $\tilde{F} = \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}$ is defined by*

$$\tilde{F}(u) = \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u) = \sup_{\{(a_1,\cdots,a_n):u=a_1\circ_1\cdots\circ_{n-1}a_n\}} \mathfrak{D}_n\Big(\tilde{F}^{(1)}(a_1), \cdots, \tilde{F}^{(n)}(a_n)\Big), \tag{10}$$

where the operations $\odot_i \in \{\oplus, \ominus, \otimes\}$ for $i = 1, \cdots, n-1$ correspond to the operations $\circ_i \in \{+, -, *\}$ for $i = 1, \cdots, n-1$.

When the function $\mathfrak{D}_n$ is taken to be the minimum function given by

$$\mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) = \min\{\alpha_1, \cdots, \alpha_n\},$$

the membership function of $\tilde{F}^{(1)} \boxdot_1 \cdots \boxdot_{n-1} \tilde{F}^{(n)}$ is given by

$$\tilde{F}^{(1)} \boxdot_1 \cdots \boxdot_{n-1} \tilde{F}^{(n)}(u) = \sup_{\{(x_1,\cdots,x_n):u=x_1\circ_1\cdots\circ_{n-1}x_n\}} \min\Big\{\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\Big\}, \tag{11}$$

where $\boxdot_i \in \{\boxplus, \boxminus, \boxtimes\}$ for $i = 1, \cdots, n-1$ can refer to (7), (8), and (9).

We can also insert the parentheses into the expression $\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}$. The following example shows the way of inserting parentheses.

Example 1. *Given fuzzy sets $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(7)}$ in $\mathbb{R}$, we can consider the membership functions of*

$$\tilde{G} \equiv \tilde{F}^{(1)} \otimes \Big(\tilde{F}^{(2)} \oplus \tilde{F}^{(3)}\Big) \ominus \Big(\tilde{F}^{(4)} \otimes \Big(\tilde{F}^{(5)} \oplus \tilde{F}^{(6)} \ominus \tilde{F}^{(7)}\Big)\Big)$$

and

$$\tilde{H} \equiv \tilde{F}^{(1)} \otimes \tilde{F}^{(2)} \oplus \tilde{F}^{(3)} \ominus \tilde{F}^{(4)} \otimes \tilde{F}^{(5)} \oplus \tilde{F}^{(6)} \ominus \tilde{F}^{(7)}$$

given by

$$\tilde{G}(u) = \sup_{\{(x_1,\cdots,x_7):u=x_1*(x_2+x_3)-(x_4*(x_5+x_6-x_7))\}} \mathfrak{D}_7\left(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(7)}(x_7)\right)$$

and

$$\tilde{H}(u) = \sup_{\{(x_1,\cdots,x_7):u=x_1*x_2+x_3-x_4*x_5+x_6-x_7\}} \mathfrak{D}_7\left(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(7)}(x_7)\right),$$

respectively. It is clear that $\tilde{G} \neq \tilde{H}$. *Since*

$$x_1 * x_2 + x_3 - x_4 * x_5 + x_6 - x_7 = (x_1 * x_2) + x_3 - (x_4 * x_5) + x_6 - x_7,$$

the fuzzy set $\tilde{H}$ *means the following form:*

$$\tilde{H} = \left(\tilde{F}^{(1)} \otimes \tilde{F}^{(2)}\right) \oplus \tilde{F}^{(3)} \ominus \left(\tilde{F}^{(4)} \otimes \tilde{F}^{(5)}\right) \oplus \tilde{F}^{(6)} \ominus \tilde{F}^{(7)}.$$

Example 2. *We present an example from mathematical finance. The well-known Black–Scholes formula (see Black and Scholes [22]) for the European call option on a stock is described as follows. Let the function* f *be given by the formula:*

$$f(s,t,K,r,\sigma) = s \cdot N(d_1) - K \cdot e^{-rt} \cdot N(d_2),$$

where s *denotes the stock price,* t *denotes the time,* K *denotes the strike price,* r *denotes the interest rate,* σ *denotes the volatility, and* N *stands for the cumulative distribution function of a standard normal random variable* $N(0,1)$. *The quantities* d_1 *and* d_2 *are given by*

$$d_1 = \frac{\ln(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma \cdot \sqrt{t}} \text{ and } d_2 = d_1 - \sigma \cdot \sqrt{t}.$$

Let T *be the expiry date, and let* C_t *denote the price of a European call option at time* $t \in [0,T]$. *Then, we have*

$$C_t = f(S_t, T-t, K, r, \sigma) \text{ for all } t \in [0,T], \tag{12}$$

where S_t *denotes the stock price at time* t. *On the other hand, the price* P_t *of a European put option at time* t *with the same expiry date* T *and strike price* K *can be obtained by the following put–call parity relationship (see Musiela and Rutkowski [23]):*

$$C_t - P_t = S_t - K \cdot e^{-r(T-t)} \text{ for all } t \in [0,T]. \tag{13}$$

Under the considerations of the fuzzy interest rate $\tilde{r}$, *fuzzy volatility* $\tilde{\sigma}$, *and fuzzy stock price* $\tilde{S}$, *we can obtain the fuzzy price* $\tilde{H}_t$ *of a European call option at time* t *according to (12) and the extension principle. Therefore, the membership function of* $\tilde{H}_t$ *is given by*

$$\tilde{H}_t(c) = \sup_{\{(s,r,\sigma):c=f(s,T-t,K,r,\sigma)\}} \mathfrak{D}_3\left(\tilde{S}_t(s), \tilde{r}(r), \tilde{\sigma}(\sigma)\right).$$

According to the put–call parity relationship in (13), we can also study the fuzzy price $\tilde{P}_t$ *of a European put option at time* t. *Let*

$$g(s,t,K,r,\sigma) = f(s,t,K,r,\sigma) - s + K \cdot e^{-rt}.$$

Then, we can obtain the fuzzy price $\tilde{P}_t$ *of a European put option at time* t *in which the membership function of* $\tilde{P}_t$ *is given by*

$$\tilde{P}_t(p) = \sup_{\{(s,r,\sigma):p=g(s,T-t,K,r,\sigma)\}} \mathfrak{D}_3\left(\tilde{S}_t(s), \tilde{r}(r), \tilde{\sigma}(\sigma)\right).$$

Let $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ be fuzzy sets in $\mathbb{R}$, and let $\alpha_i^* = \sup \mathcal{R}_{\tilde{F}^{(i)}}$. From Proposition 1, we see that $\tilde{F}_\alpha^{(i)} \neq \emptyset$ for all $\alpha \in I_{\tilde{F}^{(i)}}$ and $\tilde{F}_\alpha^{(i)} = \emptyset$ for all $\alpha \notin I_{\tilde{F}^{(i)}}$, where the interval range $I_{\tilde{F}^{(i)}}$ is given by

$$I_{\tilde{F}^{(i)}} = \begin{cases} [0, \alpha_i^*), & \text{if the supremum } \sup \mathcal{R}_{\tilde{F}^{(i)}} \text{ is not obtained} \\ [0, \alpha_i^*], & \text{if the supremum } \sup \mathcal{R}_{\tilde{F}^{(i)}} \text{ is obtained.} \end{cases} \tag{14}$$

Let $I^* = I_{\tilde{F}^{(1)}} \cap \cdots \cap I_{\tilde{F}^{(n)}}$. Then, I^* is also an interval of the form $[0, \alpha]$ or $[0, \alpha)$ for some $\alpha \in (0, 1]$. For $\alpha \in I^*$, we see that $\tilde{F}_\alpha^{(i)} \neq \emptyset$ for all $i = 1, \cdots, n$.

Let $\tilde{F} = \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}$, and let $I_{\tilde{F}}$ be the interval range of $\tilde{F}$. We also write $\mathcal{R}_i \equiv \mathcal{R}_{\tilde{F}^{(i)}}$ to denote the range of the membership function of $\tilde{F}^{(i)}$ for $i = 1, \cdots, n$. The supremum of the range $\mathcal{R}_{\tilde{F}}$ of the membership function of $\tilde{F}$ is given by

$$\begin{aligned} \sup \mathcal{R}_{\tilde{F}} &= \sup_{u \in \mathbb{R}} \tilde{F}(u) = \sup_{u \in \mathbb{R}} \sup_{\{(x_1, \cdots, x_n): u = x_1 \circ_1 \cdots \circ_{n-1} x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right) \\ &= \sup_{(\alpha_1, \cdots, \alpha_n) \in \mathcal{R}_1 \times \cdots \times \mathcal{R}_n} \mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) \equiv \alpha^*. \end{aligned} \tag{15}$$

Therefore, the definition of interval range says

$$I_{\tilde{F}} = \begin{cases} [0, \alpha^*] & \text{if the supremum } \mathcal{R}_{\tilde{F}} \text{ is obtained} \\ [0, \alpha^*) & \text{if the supremum } \mathcal{R}_{\tilde{F}} \text{ is not obtained} \end{cases} \tag{16}$$

Proposition 2. *Let $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ be fuzzy sets in $\mathbb{R}$, and let $\tilde{F} = \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}$ with interval range $I_{\tilde{F}}$. Suppose that the function $\mathfrak{D}_n : [0, 1]^n \to [0, 1]$ satisfies the following condition:*

$$\alpha_i \leq \beta_i \text{ for } i = 1, \cdots, n \text{ imply } \mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) \leq \mathfrak{D}_n(\beta_1, \cdots, \beta_n). \tag{17}$$

We also assumed that the supremum $\alpha_i^* = \sup \mathcal{R}_{\tilde{F}^{(i)}}$ is obtained for $i = 1, \cdots, n$. Then, the following supremum:

$$\sup \mathcal{R}_{\tilde{F}} = \alpha^* = \mathfrak{D}_n(\alpha_1^*, \cdots, \alpha_n^*)$$

is obtained. Moreover, we have

$$I_{\tilde{F}} = [0, \alpha^*] \text{ and } I^* = [0, \alpha^\bullet],$$

where

$$\alpha^\bullet = \min\{\alpha_1^*, \cdots, \alpha_n^*\}.$$

In particular, suppose that

$$\mathfrak{D}_n(\alpha_1^*, \cdots, \alpha_n^*) = \min\{\alpha_1^*, \cdots, \alpha_n^*\}. \tag{18}$$

Then, we have

$$I_{\tilde{F}} = I_{\tilde{F}^{(1)}} \cap \cdots \cap I_{\tilde{F}^{(n)}} = I^* = [0, \alpha^*].$$

Proof. Since the supremum $\sup \mathcal{R}_{\tilde{F}^{(i)}}$ is obtained for $i = 1, \cdots, n$, we have

$$I_{\tilde{F}^{(i)}} = [0, \alpha_i^*] \text{ and } \alpha_i^* = \tilde{F}^{(i)}(x_i^*) \in \mathcal{R}_{\tilde{F}^{(i)}} \equiv \mathcal{R}_i \tag{19}$$

for some $x_i^* \in \mathbb{R}$ and for all $i = 1, \cdots, n$. It is also clear that

$$I^* = I_{\tilde{F}^{(1)}} \cap \cdots \cap I_{\tilde{F}^{(n)}} = [0, \alpha^\bullet].$$

From (15), we have

$$\alpha^* = \sup_{(\alpha_1,\cdots,\alpha_n)\in\mathcal{R}_1\times\cdots\times\mathcal{R}_n} \mathfrak{D}_n(\alpha_1,\cdots,\alpha_n) \geq \mathfrak{D}_n(\alpha_1^*,\cdots,\alpha_n^*).$$

On the other hand, since $\tilde{F}^{(i)}(x_i) \leq \alpha_i^*$ for all $i = 1,\cdots,n$, from (15), again, we also have

$$\begin{aligned}
\alpha^* &= \sup_{u\in\mathbb{R}} \sup_{\{(x_1,\cdots,x_n):u=x_1\circ_1\cdots\circ_{n-1}x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(n)}(x_n)\right) \\
&\leq \sup_{u\in\mathbb{R}} \sup_{\{(x_1,\cdots,x_n):u=x_1\circ_1\cdots\circ_{n-1}x_n\}} \mathfrak{D}_n(\alpha_1^*,\cdots,\alpha_n^*) \text{ (using (17))} \\
&= \mathfrak{D}_n(\alpha_1^*,\cdots,\alpha_n^*),
\end{aligned}$$

which proves

$$\alpha^* = \mathfrak{D}_n(\alpha_1^*,\cdots,\alpha_n^*).$$

We take $u^* = x_1^* \circ_1 \cdots \circ_{n-1} x_n^*$. Then, we have

$$\begin{aligned}
\tilde{F}(u^*) &= \sup_{\{(x_1,\cdots,x_n):u^*=x_1\circ_1\cdots\circ_{n-1}x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(n)}(x_n)\right) \\
&\geq \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1^*),\cdots,\tilde{F}^{(n)}(x_n^*)\right) \text{ (since } u^* = x_1^*\circ_1\cdots\circ_{n-1}x_n^*) \\
&= \mathfrak{D}_n(\alpha_1^*,\cdots,\alpha_n^*) = \alpha^*
\end{aligned}$$

and

$$\begin{aligned}
\tilde{F}(u^*) &= \sup_{\{(x_1,\cdots,x_n):u^*=x_1\circ_1\cdots\circ_{n-1}x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(n)}(x_n)\right) \\
&\leq \sup_{\{(x_1,\cdots,x_n):u^*=x_1\circ_1\cdots\circ_{n-1}x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1^*),\cdots,\tilde{F}^{(n)}(x_n^*)\right) \text{ (using (17))} \\
&= \mathfrak{D}_n(\alpha_1^*,\cdots,\alpha_n^*) = \alpha^*
\end{aligned}$$

Therefore, we obtain $\tilde{F}(u^*) = \alpha^*$. From (15), we conclude that the supremum $\sup \mathcal{R}_{\tilde{F}}$ is obtained at u^*. From (16), it follows that $I_{\tilde{F}} = [0,\alpha^*]$. This completes the proof. □

3. Compatibility

Let $S_1,\cdots,S_n$ be subsets of $\mathbb{R}$. We write

$$S_1 \circ_1 \cdots \circ_{n-1} S_n = \{x_1 \circ_1 \cdots \circ_{n-1} x_n : x_i \in S_i \text{ for } i = 1,\cdots,n\},$$

where the arithmetic operations $\circ_i \in \{+,-,*\}$ for $i = 1,\cdots,n-1$.

Given any fuzzy sets $\tilde{F}^{(1)},\cdots,\tilde{F}^{(n)}$ in $\mathbb{R}$, let $\tilde{F} = \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}$ be defined in Definition 1. For any

$$\alpha \in I^* \cap I_{\tilde{F}} = I_{\tilde{F}^{(1)}} \cap \cdots \cap I_{\tilde{F}^{(n)}} \cap I_{\tilde{F}},$$

it is clear that the α-level sets $\tilde{F}_\alpha$ and $\tilde{F}_\alpha^{(i)}$ are nonempty for $i = 1,\cdots,n$. Therefore, we propose the following definition.

Definition 2. *Given any fuzzy sets $\tilde{F}^{(1)},\cdots,\tilde{F}^{(n)}$ in $\mathbb{R}$, we considered the arithmetic operations $\odot_i \in \{\oplus,\ominus,\otimes\}$, which correspond to the arithmetic operations $\circ_i \in \{+,-,*\}$ for $i = 1,\cdots,n-1$:*

- *The function $\mathfrak{D}_n : [0,1]^n \to [0,1]$ is said to be compatible with the arithmetic operations of α-level sets when the following equality is satisfied:*

$$\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha = \tilde{F}_\alpha^{(1)} \circ_1 \cdots \circ_{n-1} \tilde{F}_\alpha^{(n)} \text{ for all } \alpha \in I^* \cap I_{\tilde{F}} \text{ with } \alpha > 0.$$

- *The function $\mathfrak{D}_n : [0,1]^n \to [0,1]$ is said to be strongly compatible with the arithmetic operations of α-level sets when the following equality is satisfied:*

$$\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha = \tilde{F}^{(1)}_\alpha \circ_1 \cdots \circ_{n-1} \tilde{F}^{(n)}_\alpha \text{ for all } \alpha \in I^* \cap I_{\tilde{F}}.$$

The purpose of this paper was to present some sufficient conditions such that the compatibility with the arithmetic operations of α-level sets can be satisfied.

Recall that the real-valued function $f : \mathbb{R} \to \mathbb{R}$ is upper semi-continuous on $\mathbb{R}$ if and only if the set $\{x \in \mathbb{R} : f(x) \geq \alpha\}$ is a closed set in $\mathbb{R}$ for each $\alpha \in \mathbb{R}$. Especially, if $\tilde{F}$ is a fuzzy set in $\mathbb{R}$ such that its membership function $\tilde{F}$ is upper semi-continuous on $\mathbb{R}$, then each α-level set $\tilde{F}_\alpha$ is a closed subset of $\mathbb{R}$ for $\alpha \in I_{\tilde{F}}$.

Lemma 1 (Royden ([24] p. 161))**.** *Let K be a closed and bounded subset of $\mathbb{R}$, and let f be a real-valued function defined on $\mathbb{R}$. Suppose that f is upper semi-continuous on $\mathbb{R}$. Then, f assumes its maximum on K; that is, the supremum is obtained in the following sense:*

$$\sup_{x \in K} f(x) = \max_{x \in K} f(x).$$

Theorem 1. *Given any fuzzy sets $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ in $\mathbb{R}$, we considered the arithmetic operations $\odot_i \in \{\oplus, \ominus, \otimes\}$, which correspond to the arithmetic operations $\circ_i \in \{+, -, *\}$ for $i = 1, \cdots, n-1$. Then, we have the following properties:*

(i) *For any $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$, we assumed that the function $\mathfrak{D}_n$ satisfies the following condition:*

$$\alpha_i \geq \alpha \text{ for all } i = 1, \cdots, n \text{ imply } \mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) \geq \alpha. \tag{20}$$

Then, the following inclusion:

$$\tilde{F}^{(1)}_\alpha \circ_1 \cdots \circ_{n-1} \tilde{F}^{(n)}_\alpha \subseteq \left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha$$

holds true for all $\alpha \in I^* \cap I_{\tilde{F}}$.

(ii) *Suppose that the membership functions of $\tilde{F}^{(i)}$ are upper semi-continuous for all $i = 1, \cdots, n$. We also assumed that the function $\mathfrak{D}_n$ satisfies the following conditions:*

- *given any $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$,*

$$\mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) \geq \alpha \text{ if and only if } \alpha_i \geq \alpha \text{ for all } i = 1, \cdots, n. \tag{21}$$

- *Given any $\alpha \notin I^*$ with $\alpha \in (0,1]$,*

$$\alpha_i < \alpha \text{ for some } i \in \{1, \cdots, n\} \text{ imply } \mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) < \alpha \tag{22}$$

for any $\alpha_j \in [0,1]$ with $j \neq i$.

Then, the following equality:

$$\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha = \tilde{F}^{(1)}_\alpha \circ_1 \cdots \circ_{n-1} \tilde{F}^{(n)}_\alpha \tag{23}$$

holds true for all $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$. We further assumed that the supports $\tilde{F}^{(i)}_{0+}$ are bounded for all $i = 1, \cdots, n$. Then, the following equality:

$$\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_0 = \tilde{F}^{(1)}_0 \circ_1 \cdots \circ_{n-1} \tilde{F}^{(n)}_0. \tag{24}$$

regarding the 0-level sets holds true.

Proof. To prove Part (i), given any $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$, we have $\tilde{F}_\alpha \neq \emptyset$ and $\tilde{F}_\alpha^{(i)} \neq \emptyset$ for all $i = 1, \cdots, n$. Given any

$$u_\alpha \in \tilde{F}_\alpha^{(1)} \circ_1 \cdots \circ_{n-1} \tilde{F}_\alpha^{(n)}.$$

there exist $x_\alpha^{(i)} \in \tilde{F}_\alpha^{(i)}$ for all $i = 1, \cdots, n$ satisfying

$$u_\alpha = x_\alpha^{(1)} \circ_1 \cdots \circ_{n-1} x_\alpha^{(n)}.$$

We see that

$$\tilde{F}^{(i)}(x_\alpha^{(i)}) \geq \alpha \text{ for all } i = 1, \cdots, n.$$

Using the assumption (20) of $\mathfrak{D}_n$, we also have

$$\mathfrak{D}_n\left(\tilde{F}^{(1)}(x_\alpha^{(1)}), \cdots, \tilde{F}^{(n)}(x_\alpha^{(n)})\right) \geq \alpha \tag{25}$$

Therefore, we have

$$\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u_\alpha) = \sup_{\{(x_1,\cdots,x_n): u_\alpha = x_1 \circ_1 \cdots \circ_{n-1} x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right)$$
$$\geq \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_\alpha^{(1)}), \cdots, \tilde{F}^{(n)}(x_\alpha^{(n)})\right) \geq \alpha \text{ (using (25))}.$$

This shows

$$u_\alpha \in (\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)})_\alpha.$$

Therefore, we obtain the following inclusion:

$$\tilde{F}_\alpha^{(1)} \circ_1 \cdots \circ_{n-1} \tilde{F}_\alpha^{(n)} \subseteq \left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha$$

for all $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$.

Next, we considered the 0-level sets. For $\alpha = 0$, given any

$$u_0 \in \tilde{F}_0^{(1)} \circ_1 \cdots \circ_{n-1} \tilde{F}_0^{(n)},$$

there exist $x_0^{(i)} \in \tilde{F}_0^{(i)}$ for all $i = 1, \cdots, n$ satisfying

$$u_0 = x_0^{(1)} \circ_1 \cdots \circ_{n-1} x_0^{(n)}.$$

For each fixed i, since

$$x_0^{(i)} \in \tilde{F}_0^{(i)} = \text{cl}\left(\{x \in \mathbb{R} : \tilde{F}^{(i)}(x) > 0\}\right),$$

the concept of closure says that there exists a sequence

$$\{x_m^{(i)}\}_{m=1}^\infty \subseteq \{x \in \mathbb{R} : \tilde{F}^{(i)}(x) > 0\} \tag{26}$$

satisfying

$$\lim_{m\to\infty} x_m^{(i)} = x_0^{(i)}.$$

We considered a function $\eta : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\eta(x_1, \cdots, x_n) = x_1 \circ_1 \cdots \circ_{n-1} x_n,$$

where the binary operations $\circ_i \in \{+, -, *\}$ for $i = 1, \cdots, n$. Then, it is clear that η is continuous. We define

$$u_m = x_m^{(1)} \circ_1 \cdots \circ_{n-1} x_m^{(n)} = \eta\left(x_m^{(1)}, \cdots, x_m^{(n)}\right).$$

Using (26) and the continuity of η, we obtain

$$\lim_{m\to\infty} u_m = \lim_{m\to\infty} \eta\left(x_m^{(1)}, \cdots, x_m^{(n)}\right) = \eta\left(x_0^{(1)}, \cdots, x_0^{(n)}\right) = x_0^{(1)} \circ_1 \cdots \circ_{n-1} x_0^{(n)} = u_0. \quad (27)$$

Given any $\alpha_i \in I_{\tilde{F}^{(i)}}$ with $\alpha_i > 0$ for $i = 1, \cdots, n$ and any $\bar{\alpha} \in I_{\tilde{F}}$ with $\bar{\alpha} > 0$, let

$$\alpha = \min\{\bar{\alpha}, \alpha_1, \cdots, \alpha_n\}.$$

Then, we have $0 < \alpha \leq \bar{\alpha}$ and $0 < \alpha \leq \alpha_i$ for $i = 1, \cdots, n$. From (14), we also see $\alpha \in I_{\tilde{F}}$ and $\alpha \in I_{\tilde{F}^{(i)}}$ for all $i = 1, \cdots, n$, i.e., $\alpha \in I^* \cap I_{\tilde{F}}$. The assumption (20) of $\mathfrak{D}_n$ says

$$\mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) \geq \alpha > 0.$$

Therefore, the following statement holds true:

$$0 < \alpha_i \in I_{\tilde{F}^{(i)}} \text{ for all } i = 1, \cdots, n \text{ imply } \mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) > 0. \quad (28)$$

Now, we have

$$\begin{aligned} \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u_m) &= \sup_{\{(x_1,\cdots,x_n): u_m = x_1 \circ_1 \cdots \circ_{n-1} x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right) \\ &\geq \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_m^{(1)}), \cdots, \tilde{F}^{(n)}(x_m^{(n)})\right) > 0 \text{ (using (28))}, \end{aligned}$$

which also says

$$u_m \in \{u \in \mathbb{R} : \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u) > 0\} \text{ for all } m.$$

From (27), we obtain

$$u_0 \in \text{cl}\left(\{u \in \mathbb{R} : \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u) > 0\}\right) = \left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_0,$$

which shows the following inclusion:

$$\tilde{F}_0^{(1)} \circ_1 \cdots \circ_{n-1} \tilde{F}_0^{(n)} \subseteq \left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_0.$$

Therefore, we obtain the desired inclusion.

Proving Part (ii) means proving another direction of inclusion. Now, we further assumed that the membership functions of $\tilde{F}^{(i)}$ are upper semi-continuous for all $i = 1, \cdots, n$. In other words, the nonempty α-level sets $\tilde{F}_\alpha^{(i)}$ are closed sets in $\mathbb{R}$ for all $\alpha \in I^*$ and $i = 1, \cdots, n$. Given any $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$ and any

$$u_\alpha \in (\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)})_\alpha,$$

we have

$$\sup_{\{(x_1,\cdots,x_n): u_\alpha = x_1 \circ_1 \cdots \circ_{n-1} x_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right) = \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u_\alpha) \geq \alpha. \quad (29)$$

Since u_α is a finite number, we see that

$$F \equiv \{(x_1, \cdots, x_n) : u_\alpha = x_1 \circ_1 \cdots \circ_{n-1} x_n\}$$

is a bounded set in $\mathbb{R}^n$. We also see that the function

$$\eta(x_1, \cdots, x_n) = x_1 \circ_1 \cdots \circ_{n-1} x_n$$

is continuous on $\mathbb{R}^n$. Since the singleton set $\{u_\alpha\}$ is a closed set in $\mathbb{R}$, the continuity of η says that the inverse image $F = \eta^{-1}(\{u_\alpha\})$ of $\{u_\alpha\}$ is also a closed set in $\mathbb{R}^n$. This says that F is a bounded and closed set in $\mathbb{R}^n$. Next, we want to claim that the function

$$f(x_1, \cdots, x_n) = \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right)$$

is upper semi-continuous. In other words, we want to show that

$$\{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) \geq \alpha\}$$

is a closed set in $\mathbb{R}^n$ for any $\alpha \in \mathbb{R}$. We considered the different cases as follows:

- Suppose that $\alpha \leq 0$. Then, we have

  $$\{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) \geq \alpha\} = \mathbb{R}^n,$$

  which is a closed set in $\mathbb{R}^n$.
- Suppose that $\alpha > 1$. Then, we have

  $$\{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) \geq \alpha\} = \emptyset,$$

  which is also a closed set in $\mathbb{R}^n$.
- Suppose that $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$, i.e., $\tilde{F}_\alpha^{(i)} \neq \emptyset$ for all $i = 1, \cdots, n$. Then, we have

$$\begin{aligned}
\{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) \geq \alpha\} &= \left\{(x_1, \cdots, x_n) : \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right) \geq \alpha\right\} \\
&= \left\{(x_1, \cdots, x_n) : \tilde{F}^{(i)}(x_i) \geq \alpha \text{ for all } i = 1, \cdots, n\right\} \text{ (using (21))} \\
&= \left\{(x_1, \cdots, x_n) : x_i \in \tilde{F}_\alpha^{(i)} \text{ for all } i = 1, \cdots, n\right\} = \tilde{F}_\alpha^{(1)} \times \cdots \times \tilde{F}_\alpha^{(n)},
\end{aligned}$$

  which is a closed set in $\mathbb{R}^n$, since $\tilde{F}_\alpha^{(i)}$ are closed sets in $\mathbb{R}$ for all $i = 1, \cdots, n$.
- Suppose that $\alpha \notin I^*$ with $\alpha \in (0, 1]$. Then, we have $\tilde{F}_\alpha^{(i)} = \emptyset$ for some i, i.e., $\alpha \notin I_{\tilde{F}^{(i)}}$. By referring to (14), it follows that $\tilde{F}^{(i)}(x) < \alpha$ for all $x \in \mathbb{R}$. Therefore, using the assumption (22), we obtain

  $$f(x_1, \cdots, x_n) = \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right) < \alpha \text{ for all } (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

  This shows

  $$\{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) \geq \alpha\} = \emptyset,$$

  which is also a closed set in $\mathbb{R}^n$.
- Suppose that $\alpha \notin I_{\tilde{F}}$ with $\alpha \in (0, 1]$. Then, we have

  $$\begin{aligned}
  \emptyset &= \left\{(x_1, \cdots, x_n) : \mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right) \geq \alpha\right\} \\
  &= \{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) \geq \alpha\},
  \end{aligned}$$

  which is a closed set in $\mathbb{R}^n$.

The above cases conclude that the function $f(x_1, \cdots, x_n)$ is indeed upper semi-continuous. Lemma 1 says that the function f assumes the maximum on the set F. Therefore, using (29), we have

$$\begin{aligned}\max_{(x_1,\cdots,x_n)\in F} f(x_1,\cdots,x_n) &= \max_{\{(x_1,\cdots,x_n):u_\alpha = x_1\circ_1\cdots\circ_{n-1}x_n\}} f(x_1,\cdots,x_n)\\ &= \sup_{\{(x_1,\cdots,x_n):u_\alpha = x_1\circ_1\cdots\circ_{n-1}x_n\}} f(x_1,\cdots,x_n) \geq \alpha. \end{aligned} \tag{30}$$

Therefore, there exists $(x_1^*, \cdots, x_n^*) \in F$ satisfying

$$u_\alpha = x_1^* \circ_1 \cdots \circ_{n-1} x_n^*$$

and

$$\mathfrak{D}_n\left(\tilde{F}^{(1)}(x_1^*), \cdots, \tilde{F}^{(n)}(x_n^*)\right) = f(x_1^*, \cdots, x_n^*) = \max_{(x_1,\cdots,x_n)\in F} f(x_1,\cdots,x_n) \geq \alpha.$$

Using the assumption (21), we obtain $\tilde{F}^{(i)}(x_i^*) \geq \alpha$, which says $x_i^* \in \tilde{F}_\alpha^{(i)}$ for all $i = 1, \cdots, n$. Therefore, we obtain

$$u_\alpha \in \tilde{F}_\alpha^{(1)} \circ_1 \cdots \circ_{n-1} \tilde{F}_\alpha^{(n)},$$

which shows the following inclusion:

$$\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha \subseteq \tilde{F}_\alpha^{(1)} \circ_1 \cdots \circ_{n-1} \tilde{F}_\alpha^{(n)}$$

for all $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$. Using Part (i), we obtain the desired equality (23).

Considering the 0-level sets, for $\alpha = 0$, we further assumed that the supports $\tilde{F}_{0+}^{(i)}$ are bounded for all $i = 1, \cdots, n$. Suppose that $\mathfrak{D}_n(\alpha_1 \cdots, \alpha_n) > 0$ for $\alpha_i \in I_{\tilde{F}^{(i)}}$ and $i = 1, \cdots, n$. Since $I^* \cap I_{\tilde{F}}$ is an interval beginning from 0, using the denseness of $\mathbb{R}$, there exists $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$ satisfying

$$\mathfrak{D}_n(\alpha_1 \cdots, \alpha_n) \geq \alpha > 0.$$

Using the assumption (21), we have $\alpha_i \geq \alpha > 0$ for all $i = 1, \cdots, n$, which says that the following statement holds true:

$$\mathfrak{D}_n(\alpha_1 \cdots, \alpha_n) > 0 \text{ for } \alpha_i \in I_{\tilde{F}^{(i)}} \text{ and } i = 1, \cdots, n \text{ imply } \alpha_i > 0 \text{ for all } i = 1, \cdots, n. \tag{31}$$

Now, considering the 0-level set, we have

$$\begin{aligned} u_0 \in \left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_0 &= \text{cl}\left(\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_{0+}\right)\\ &= \text{cl}\left(\left\{u \in \mathbb{R} : \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u) > 0\right\}\right). \end{aligned}$$

Therefore, there exists a sequence $\{u_m\}_{m=1}^\infty$ in the following set:

$$\left\{u \in \mathbb{R} : \tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u) > 0\right\}$$

satisfying

$$\lim_{m\to\infty} u_m = u_0.$$

Using the above arguments by referring to (30), we can obtain

$$0<\tilde{F}^{(1)}\odot_1\cdots\odot_{n-1}\tilde{F}^{(n)}(u_m)=\sup_{\{(x_1,\cdots,x_n):u_m=x_1\circ_1\cdots\circ_{n-1}x_n\}}\mathfrak{D}_n\Big(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(n)}(x_n)\Big)$$
$$=\max_{\{(x_1,\cdots,x_n):u_m=x_1\circ_1\cdots\circ_{n-1}x_n\}}\mathfrak{D}_n\Big(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(n)}(x_n)\Big).$$

Therefore, there exist $x_{1m},\cdots,x_{nm}$ satisfying

$$u_m=x_{1m}\circ_1\cdots\circ_{n-1}x_{nm}$$

and

$$\mathfrak{D}_n\Big(\tilde{F}^{(1)}(x_{1m}),\cdots,\tilde{F}^{(n)}(x_{nm})\Big)=\max_{\{(x_1,\cdots,x_n):u_m=x_1\circ_1\cdots\circ_{n-1}x_n\}}\mathfrak{D}_n\Big(\tilde{F}^{(1)}(x_1),\cdots,\tilde{F}^{(n)}(x_n)\Big)>0,$$

Using (31), we have $\tilde{F}^{(i)}(x_{im})>0$ for all $i=1,\cdots,n$, which shows that the sequence $\{x_{im}\}_{m=1}^{\infty}$ is in the support $\tilde{F}^{(i)}_{0+}$ for all $i=1,\cdots,n$. Since each $\tilde{F}^{(i)}_{0+}$ is bounded for $i=1,\cdots,n$, it follows that $\{x_{im}\}_{m=1}^{\infty}$ is also a bounded sequence. Therefore, there exists a convergent subsequence $\{x_{im_k}\}_{k=1}^{\infty}$ of $\{x_{im}\}_{m=1}^{\infty}$. In other words, we have

$$\lim_{k\to\infty}x_{im_k}=x_{i0}\text{ for all }i=1,\cdots,n,$$

which also says $x_{i0}\in\text{cl}(\tilde{F}^{(i)}_{0+})=\tilde{F}^{(i)}_{0}$ for all $i=1,\cdots,n$. Let

$$u_{m_k}=x_{1m_k}\circ_1\cdots\circ_{n-1}x_{nm_k}.$$

Then, we see that $\{u_{m_k}\}_{k=1}^{\infty}$ is a subsequence of $\{u_m\}_{n=1}^{\infty}$, i.e.,

$$\lim_{k\to\infty}u_{m_k}=u_0.$$

Since

$$u_0=\lim_{k\to\infty}u_{m_k}=\lim_{k\to\infty}\left(x_{1m_k}\circ_1\cdots\circ_{n-1}x_{nm_k}\right)$$
$$=\left(\lim_{k\to\infty}x_{1m_k}\right)\circ_1\cdots\circ_{n-1}\left(\lim_{k\to\infty}x_{nm_k}\right)=x_{10}\circ_1\cdots\circ_{n-1}x_{n0},$$

which shows

$$u_0\in\tilde{F}^{(1)}_0\circ_1\cdots\circ_{n-1}\tilde{F}^{(n)}_0.$$

Therefore, we obtain the following inclusion:

$$\left(\tilde{F}^{(1)}\odot_1\cdots\odot_{n-1}\tilde{F}^{(n)}\right)_0\subseteq\tilde{F}^{(1)}_0\circ_1\cdots\circ_{n-1}\tilde{F}^{(n)}_0.$$

Using Part (i), we obtain the desired equality (24), and the proof is complete. □

Theorem 2. *Given any fuzzy sets $\tilde{F}^{(1)},\cdots,\tilde{F}^{(n)}$ in $\mathbb{R}$, we considered the arithmetic operations $\odot_i\in\{\oplus,\ominus,\otimes\}$, which correspond to the arithmetic operations $\circ_i\in\{+,-,*\}$ for $i=1,\cdots,n-1$. Suppose that the function $\mathfrak{D}_n$ satisfies the following conditions:*

- *Given any $\alpha\in I^*\cap I_{\tilde{F}}$ with $\alpha>0$,*

$$\mathfrak{D}_n(\alpha_1,\cdots,\alpha_n)\geq\alpha\text{ if and only if }\alpha_i\geq\alpha\text{ for all }i=1,\cdots,n.$$

- *Given any $\alpha\notin I^*$ with $\alpha\in(0,1]$,*

$$\alpha_i<\alpha\text{ for some }i\in\{1,\cdots,n\}\text{ imply }\mathfrak{D}_n(\alpha_1,\cdots,\alpha_n)<\alpha$$

for any $\alpha_j \in [0,1]$ with $j \neq i$.

Then, we have the following properties:

(i) *Suppose that the membership functions of $\tilde{F}^{(i)}$ are upper semi-continuous for all $i = 1, \cdots, n$. Then, the function $\mathfrak{D}_n$ is compatible with arithmetic operations of α-level sets. In other words, given any $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$, we have*

$$\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha = \tilde{F}^{(1)}_\alpha \circ_1 \cdots \circ_{n-1} \tilde{F}^{(n)}_\alpha. \tag{32}$$

In particular, if $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ are normal, the equality (32) holds true for all $\alpha \in (0,1]$.

(ii) *Suppose that the membership functions of $\tilde{F}^{(i)}$ are upper semi-continuous and that the supports $\tilde{F}^{(i)}_{0+}$ are bounded for all $i = 1, \cdots, n$. Then, the function $\mathfrak{D}_n$ is strongly compatible with the arithmetic operations of α-level sets. In other words, the equality (32) holds true for all $\alpha \in I^* \cap I_{\tilde{F}}$. In particular, if $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ are normal, the equality (32) holds true for all $\alpha \in [0,1]$.*

Proof. To prove Part (i), the equality (32) follows immediately from Part (ii) of Theorem 1. In particular, if each $\tilde{F}^{(i)}$ is assumed to be normal for $i = 1, \cdots, n$, then we have $I_{\tilde{F}^{(i)}} = [0,1]$ for all $i = 1, \cdots, n$, which also says $I^* = [0,1]$. Part (ii) can be easily realized from Part (ii) of Theorem 1 and Part (i) of this theorem. This completes the proof. □

Corollary 1. *Given any fuzzy sets $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ in $\mathbb{R}$, we considered the arithmetic operations $\boxdot_i \in \{\boxplus, \boxminus, \boxtimes\}$, which correspond to the arithmetic operations $\circ_i \in \{+, -, *\}$ for $i = 1, \cdots, n-1$. Then, we have the following properties:*

(i) *Suppose that the membership functions of $\tilde{F}^{(i)}$ are upper semi-continuous for all $i = 1, \cdots, n$. Then, given any $\alpha \in I^* \cap I_{\tilde{F}}$ with $\alpha > 0$, we have*

$$\left(\tilde{F}^{(1)} \boxdot_1 \cdots \boxdot_{n-1} \tilde{F}^{(n)}\right)_\alpha = \tilde{F}^{(1)}_\alpha \circ_1 \cdots \circ_{n-1} \tilde{F}^{(n)}_\alpha. \tag{33}$$

In particular, if $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ are normal, the equality (33) holds true for all $\alpha \in (0,1]$.

(ii) *Suppose that the membership functions of $\tilde{F}^{(i)}$ are upper semi-continuous and that the supports $\tilde{F}^{(i)}_{0+}$ are bounded for all $i = 1, \cdots, n$. Then, the equality (33) holds true for all $\alpha \in I^* \cap I_{\tilde{F}}$. In particular, if $\tilde{F}^{(1)}, \cdots, \tilde{F}^{(n)}$ are normal, the equality (33) holds true for all $\alpha \in [0,1]$.*

Proof. Since we considered the arithmetic operations $\boxdot_i \in \{\boxplus, \boxminus, \boxtimes\}$, this means that we take

$$\mathfrak{D}_n(\alpha_1, \cdots, \alpha_n) = \min\{\alpha_1, \cdots, \alpha_n\},$$

which clearly satisfies all the assumptions of Theorem 2. Therefore, the desired results follow immediately from Theorem 2. □

Definition 3. *We denote by $\mathcal{F}_{cc}(\mathbb{R})$ the family of all fuzzy sets in $\mathbb{R}$ such that each $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ satisfies the following conditions:*

- *The supremum $\sup \mathcal{R}_{\tilde{a}}$ is obtained, i.e., $\sup \mathcal{R}_{\tilde{a}} = \max \mathcal{R}_{\tilde{a}}$.*
- *The membership function of $\tilde{a}$ is upper semi-continuous and quasi-concave on $\mathbb{R}$.*
- *The 0-level set $\tilde{a}_0$ is a closed and bounded subset of $\mathbb{R}$.*

Each $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ is also called a fuzzy interval. *If the fuzzy interval $\tilde{a}$ is normal and the one-level set $\tilde{a}_1$ is a singleton set $\{a\}$, where $a \in \mathbb{R}$, then $\tilde{a}$ is also called a* fuzzy number *with* core value *a.*

If $\tilde{a}$ is a fuzzy interval, then its 0-level set $\tilde{a}_0$ is a closed and bounded subset of $\mathbb{R}$. The conditions in Definition 3 says that each α-level set $\tilde{a}_\alpha$ is a bounded closed interval for $\alpha \in [0,1]$. It is also clear that

$$\tilde{a}_\alpha = \begin{cases} \emptyset & \text{if } \alpha \notin I_{\tilde{a}} \\ [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha] & \text{if } \alpha \in I_{\tilde{a}}, \end{cases}$$

where $I_{\tilde{a}}$ denotes the interval range of $\tilde{a}$ and $[\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ is a bounded closed interval with endpoints $\tilde{a}^L_\alpha$ and $\tilde{a}^U_\alpha$. The α-level set $\tilde{a}_\alpha$ can be interpreted as a bounded closed interval $[\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ with degree α, which explains the terminology of the fuzzy interval.

Proposition 3. *Given any fuzzy intervals $\tilde{a}$ and $\tilde{b}$ with interval ranges $I_{\tilde{a}}$ and $I_{\tilde{b}}$, respectively, let $I_{\tilde{a}\boxdot\tilde{b}}$ denote the interval range of $\tilde{a} \boxdot \tilde{b}$ for $\boxdot \in \{\boxplus, \boxminus, \boxtimes\}$. Then, $\tilde{a} \boxdot \tilde{b}$ is also a fuzzy interval, and its α-level set is given by*

$$(\tilde{a} \boxdot \tilde{b})_\alpha = \tilde{a}_\alpha \circ \tilde{b}_\alpha \text{ for all } \alpha \in I_{\tilde{a}} \cap I_{\tilde{b}} \cap I_{\tilde{a}\boxdot\tilde{b}}.$$

More precisely, we have

$$\begin{aligned} (\tilde{a} \boxplus \tilde{b})_\alpha &= \left[\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha\right] \\ (\tilde{a} \boxminus \tilde{b})_\alpha &= \left[\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - \tilde{b}^L_\alpha\right], \\ (\tilde{a} \boxtimes \tilde{b})_\alpha &= \left[\min\left\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\right\}, \max\left\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\right\}\right], \end{aligned}$$

for any $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}} \cap I_{\tilde{a}\boxdot\tilde{b}}$. We further assumed that the suprema:

$$\alpha^* = \sup \mathcal{R}_{\tilde{a}} \text{ and } \beta^* = \sup \mathcal{R}(\tilde{b})$$

are obtained. Then,

$$I_{\tilde{a}} \cap I_{\tilde{b}} = I_{\tilde{a}\boxdot\tilde{b}} = [0, \min\{\alpha^*, \beta^*\}]$$

is a closed interval.

Proof. Given any $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}} \cap I_{\tilde{a}\boxdot\tilde{b}}$, it is clear that the α-level sets $(\tilde{a} \boxdot \tilde{b})_\alpha$, $\tilde{a}_\alpha$, and $\tilde{b}_\alpha$ are nonempty. Therefore, the desired results follow immediately from Corollary 1 and Proposition 2. This completes the proof. □

4. Conclusions

The arithmetic operations of non-normal fuzzy sets using the extension principle based on general functions were investigated in this paper. The membership function of arithmetic operation $\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}$ is defined by

$$\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}(u) = \sup_{\{(a_1,\cdots,a_n):u=a_1\circ_1\cdots\circ_{n-1}a_n\}} \mathfrak{D}_n\left(\tilde{F}^{(1)}(a_1), \cdots, \tilde{F}^{(n)}(a_n)\right),$$

where the way of calculation $\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}$ for $\odot_i \in \{\oplus, \ominus, \otimes\}$ and $i = 1, \cdots, n-1$ corresponds to the way of calculation for $a_1 \circ_1 \cdots \circ_{n-1} a_n$ for $\circ_i \in \{+, -, *\}$ and $i = 1, \cdots, n-1$. This kind of arithmetic operation generalizes the conventional one given by

$$\tilde{F}^{(1)} \boxdot_1 \cdots \boxdot_{n-1} \tilde{F}^{(n)}(u) = \sup_{\{(x_1,\cdots,x_n):u=x_1\circ_1\cdots\circ_{n-1}x_n\}} \min\left\{\tilde{F}^{(1)}(x_1), \cdots, \tilde{F}^{(n)}(x_n)\right\},$$

where $\boxdot_i \in \{\boxplus, \boxminus, \boxtimes\}$ for $i = 1, \cdots, n-1$.

The main issue of arithmetic operations is studying their α-level sets. Therefore, the concept of compatibility with α-level sets is proposed by saying that the function

$\mathfrak{D}_n : [0,1]^n \to [0,1]$ is (strongly) compatible with the arithmetic operations of α-level sets when

$$\left(\tilde{F}^{(1)} \odot_1 \cdots \odot_{n-1} \tilde{F}^{(n)}\right)_\alpha = \tilde{F}^{(1)}_\alpha \circ_1 \cdots \circ_{n-1} \tilde{F}^{(n)}_\alpha \text{ for all } \alpha \in I^* \cap I_{\tilde{F}} \text{ with } \alpha > 0.$$

It is clear that the minimum function:

$$\mathfrak{D}(\alpha_1, \cdots, \alpha_n) = \min\{\alpha_1, \cdots, \alpha_n\}$$

considered in the conventional case is compatible with arithmetic operations of α-level sets.

Theorems 1 and 2 present the sufficient conditions to guarantee the compatibility with the arithmetic operations of α-level sets. This means that Theorems 1 and 2 are the general situation. Therefore, Corollary 1 and Proposition 3, which are the conventional cases, are the special cases of Theorems 1 and 2. This was the main purpose of this paper: to generalize the conventional cases. In other words, from some other functions $\mathfrak{D}_n$ that can satisfy the sufficient conditions, the desired results can be obtained as the conventional cases. The main focus was on the functions $\mathfrak{D}_n$ and the non-normal fuzzy sets, rather than the t-norm and the normal fuzzy sets. As we can see in Part (i) of Theorem 2, the equality (32) holds true for non-normal fuzzy sets. The case of normal fuzzy sets is just the special case of (32). Therefore, Theorems 1 and 2 indeed generalize the conventional cases. The limitation of Theorems 1 and 2 is checking the assumptions of general function $\mathfrak{D}_n$. Since those assumptions are satisfied for the conventional cases, as shown in Corollary 1 and Proposition 3, this also means that those assumptions are not too strong to be used in real applications.

The interval ranges of non-normal fuzzy sets comprise an important tool to handle the arithmetic of non-normal fuzzy sets. The future research will focus on the applications by using non-normal fuzzy sets and will solve the difficulty caused by the different forms of the interval ranges of non-normal fuzzy sets.

Funding: The APC was funded by NSTC Taiwan.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Dubois, D.; Prade, H. A Review of Fuzzy Set Aggregation Connectives. *Inf. Sci.* **1985**, *36*, 85–121. [CrossRef]
2. Weber, S. A General Concept of Fuzzy Connectives, Negations and Implications Based on t-Norms and t-Conorms. *Fuzzy Sets Syst.* **1983**, *11*, 115–134. [CrossRef]
3. Dubois, D.; Prade, H. *Possibility Theory*; Springer: New Yourk, NY, USA, 1988.
4. Klir, G.J.; Yuan, B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*; Prentice-Hall: New York, NY, USA, 1995.
5. Gebhardt, A. On Types of Fuzzy Numbers and Extension Principles. *Fuzzy Sets Syst.* **1995**, *75*, 311–318. [CrossRef]
6. Fullér, R.; Keresztfalvi, T. On Generalization of Nguyen's Theorem. *Fuzzy Sets Syst.* **1990**, *41*, 371–374. [CrossRef]
7. Mesiar, R. Triangular-Norm-Based Addition of Fuzzy Intervals. *Fuzzy Sets Syst.* **1997**, *91*, 231–237. [CrossRef]
8. Ralescu, D.A. A generalization of the representation theorem. *Fuzzy Sets Syst.* **1992**, *51*, 309–311. [CrossRef]
9. Yager, R.R. A Characterization of the Extension Principle. *Fuzzy Sets Syst.* **1986**, *18*, 205–217. [CrossRef]
10. Wu, H.-C. Generalized Extension Principle for Non-Normal Fuzzy Sets. *Fuzzy Optim. Decis. Mak.* **2019**, *18*, 399–432. [CrossRef]
11. Coroianua, L.; Fuller, R. Nguyen Type Theorem For Extension Principle Based on a Joint Possibility Distribution. *Int. J. Approx. Reason.* **2018**, *95*, 22–35. [CrossRef]
12. Coroianua, L.; Fuller, R. Necessary and Sufficient Conditions for The Equality of Interactive and Non-Interactive Extensions of Continuous Functions. *Fuzzy Sets Syst.* **2018**, *331*, 116–130. [CrossRef]
13. Holčapek, M.; Štěpnixcxka, M. MI-Algebras: A New Framework for Arithmetics of (Extensional) Fuzzy Numbers. *Fuzzy Sets Syst.* **2014**, *257*, 102–131. [CrossRef]
14. Holčapek, M.; Škorupová, N.; xSxtěpnixcxka, M. Fuzzy Interpolation with Extensional Fuzzy Numbers. *Symmetry* **2021**, *13*, 170. [CrossRef]
15. Esmi, E.; Sánchez, D.E.; Wasques, V.F.; de Barros, L.C. Solutions of Higher Order Linear Fuzzy Differential Equations with Interactive Fuzzy Values. *Fuzzy Sets Syst.* **2021**, *419*, 122–140. [CrossRef]
16. Pedro, F.S.; de Barros, L.C.; Esmi, E. Population Growth Model via Interactive Fuzzy Differential Equation. *Inf. Sci.* **2019**, *481*, 160–173. [CrossRef]

17. Wu, H.-C. Decomposition and Construction of Fuzzy Sets and Their Applications to the Arithmetic Operations on Fuzzy Quantities. *Fuzzy Sets Syst.* **2013**, *233*, 1–25. [CrossRef]
18. Wu, H.-C. Compatibility between Fuzzy Set Operations and Level Set Operations: Applications to Fuzzy Difference. *Fuzzy Sets Syst.* **2018**, *353*, 1–43. [CrossRef]
19. Bede, B.; Stefanini, L. Generalized Differentiability of Fuzzy-Valued Functions. *Fuzzy Sets Syst.* **2013**, *230*, 119–141. [CrossRef]
20. Gomes, L.T.; Barros, L.C. A Note on the Generalized Difference and the Generalized Differentiability. *Fuzzy Sets Syst.* **2015**, *280*, 142–145. [CrossRef]
21. Zulqarnian, R.M.; Xin, X.L.; Jun, Y.B. Fuzzy axiom of choice, fuzzy Zorn's lemma and fuzzy Hausdorff maximal principle. *Soft Comput.* **2021**, *25*, 11421–11428. [CrossRef]
22. Black, F.; Scholes, M. The Pricing of Options and Corporate Liabilities. *J. Political Econ.* **1973**, *81*, 637–659. [CrossRef]
23. Musiela, M.; Rutkowski, M. *Martingale Methods in Financial Modelling*; Springer: New York, NY, USA, 1997.
24. Royden, H.L. *Real Analysis*, 2nd ed.; Macmillan: London, UK, 1968.