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A Space-Time Legendre-Petrov-Galerkin Method for Third-Order Differential Equations

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Abstract: In this article, a space-time spectral method is considered to approximate third-order differential equations with non-periodic boundary conditions. The Legendre-Petrov-Galerkin discretization is employed in both space and time. In the theoretical analysis, rigorous proof of error estimates in the weighted space-time norms is obtained for the fully discrete scheme. We also formulate the matrix form of the fully discrete scheme by taking appropriate test and trial functions in both space and time. Finally, extensive numerical experiments are conducted for linear and nonlinear problems, and spectral accuracy is derived for both space and time. Moreover, the numerical results are compared with those computed by other numerical methods to confirm the efficiency of the proposed method.

Keywords: third-order differential equations; Legendre-Petrov-Galerkin methods; space-time spectral methods; exponential convergence

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1. Introduction

Spectral methods [1–5] play an increasingly important role in numerical methods for solving partial differential equations (PDEs) of classical and fractional orders. In most applications of time-dependent problems [6,7], spectral methods are applied in space combined with time discretization using finite difference methods. However, such a combination leads to a mismatch in the accuracy in the space and time of the fully discrete scheme. In recent years, some researchers proposed space-time spectral methods [8–11] for time-dependent problems to attain high-order accuracy in space and time simultaneously. In [12], Shen and Wang presented a new space-time spectral method based on a Legendre-Galerkin method in space and a dual Petrov-Galerkin formulation in time. In [13], new space-time spectral and structured spectral element methods were described for approximating solutions of fourth order problems with homogeneous boundary conditions, and the spectral accuracy was proved in both space and time. In [14], a novel multi-implicit space-time spectral element method was proposed for advection-diffusion-reaction problems characterized by multiple time scales. Authors in [15] investigated a space-time spectral approximation to handle multi-dimensional space-time variable-order fractional Schrödinger equations. In [16], Legendre spectral methods were employed in both space and time discretization for solving multi-term time-fractional diffusion equations, and Fourier-like basis functions were constructed in space.

In this paper, a space-time Legendre-Petrov-Galerkin method is considered for the linear third-order differential equations, see [17]:

$$\begin{cases} \partial_t u + \partial_x^3 u = f, \quad (x,t) \in I_x \times I_t, \\ u(\pm 1, t) = \partial_x u(1,t) = 0, \quad t \in \bar{I}_t, \\ u(x,-1) = u_0(x), \quad x \in I_x, \end{cases} \tag{1}$$

where the spatial domain is $I_x = (-1,1)$ and time interval is $I_t = (-1,1]$. Without loss of generality, we assume that $u_0(x) = 0$.

The study of the above model is of great significance for general third-order problems. The Korteweg-de Vries (KdV) equation [18–23] is a typical third-order nonlinear differential equation with an important physics application background. There are some efficient numerical methods to solve the problem (1). Considering the nonsymmetric property of the third-order differential operator, Ma and Sun proposed a Legendre-Petrov-Galerkin method in space and the Crank-Nicolson method in time for solving third-order differential equations in [17]. They also analyzed in detail the stability and L^2-norm convergence of the fully discrete schemes and extended the application of the proposed method to the KdV equation by performing Chebyshev collocation treatment on the nonlinear terms. In [24], Shen developed a new dual Petrov-Galerkin method to solve third-order differential equations, where the main feature of the proposed method is to choose trail functions and test functions satisfying underlying and "dual" boundary conditions, respectively. The author not only presented the error estimates of the fully discrete numerical schemes in the theoretical analysis, but also verified the theoretical results and demonstrated the efficiency of the proposed method through extensive numerical experiments. In [25], a new pseudo-spectral method was investigated for third-order differential equations in which zeros of $P^{(2,1)}_{N-2}(x)$ were used as collocation points. In [26], the author proposed spectral Chebyshev collocation algorithms for the approximation of the KdV equation with non-periodic boundary conditions, and discussed single- and multi-domain approaches combined with the backward Euler/Crank-Nicolson schemes in time. In [27], the error estimates of semi-discrete and fully discrete schemes were given for the KdV equations by applying the Legendre pseudo-spectral methods in space and finite difference methods in time. Qin and Ma developed a Legendre-tau-Galerkin method in time for solving nonlinear evolution equations in [28] and also considered the multi-interval forms. In numerical examples, the generalized KdV equations was considered, and its fully discrete scheme was given by combining the Legendre-Petrov-Galerkin methods in space with multi-interval forms of the Legendre-tau-Galerkin schemes in time. Furthermore, numerical results were compared with those computed by other methods in [17] under the same conditions. In this paper, in order to solve the linear third-order differential Equation (1) with high order accuracy in both space and time, we investigate a space-time spectral method. It is worth noting that considering the nonsymmetric property of third-order and first-order differential operators in space and time, respectively, we apply Petrov-Galerkin methods in both space and time.

The organization of this paper is as follows. In Section 2, on the basis of introducing some notations and definitions, we give the weak form and space-time Legendre-Petrov-Galerkin scheme for the problem (1). In Section 3, some corresponding lemmas are provided, and error estimates in the weighted space-time norms are obtained for the fully discrete scheme. In Section 4, a detailed implementation of the proposed method is presented by selecting the appropriate test and trial functions in both space and time. In Section 5, extensive numerical tests including nonlinear problems are conducted, and the numerical results are compared with those obtained by other methods to assess the efficiency and accuracy of our method. Finally, some conclusions are drawn in Section 6.

2. Space-Time Legendre-Petrov-Galerkin Scheme

We now introduce some notations. Let ω be a positive weight function in a bounded domain Ω. The norm is denoted by $\|\cdot\|_{\Omega,\omega}$ in $L^2_\omega(\Omega)$ whose inner product is given by $(u,v)_{\Omega,\omega} = \int_\Omega uv\omega d\Omega$. Additionally, ω is dropped if $\omega \equiv 1$. Let X be a Banach space with norm $\|\cdot\|_X$ and define $L^2((a,b);X) = \{v : \int_a^b \|v\|_X dt < +\infty\}$. Throughout this paper, C denotes a generic positive constant. In the following, denote $\tilde{\omega}_{\alpha,\beta} = (1-t)^\alpha(1+t)^\beta$ and $\omega_{\alpha,\beta} = (1-x)^\alpha(1+x)^\beta$ to distinguish the weight functions on I_t from I_x, where α, β are two parameters. Specific values are used in the article.

Definition 1 ([29])**.** *Let $A = (a_{ij})_{m\times n}$ and $B_{p\times q}$, then the tensor product of A and B is defined as the matrix*

$$A\otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp\times nq}.$$

Moreover, for arbitrary matrices A, B and C, the following properties hold:

$$\mathrm{vec}(ABC) = (C^T\otimes A)\mathrm{vec}(B), \quad (A\otimes B)^T = A^T\otimes B^T,$$

where $\mathrm{vec}(A) = [a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}]^T$.

Denote $\Omega := I_x \times I_t$. Let

$$\begin{cases} H_0^{1,2}(I_x) = \{v\in H^2(I_x) : v(\pm 1) = \partial_x v(1) = 0\}, \\ H_0^1(I_x) = \{v\in H^1(I_x) : v(\pm 1) = 0\}, \end{cases} \tag{2}$$

and

$$\begin{cases} H^1_{-0}(I_t) = \{v\in H^1(I_t) : v(-1) = 0\}, \\ H^1_{+0}(I_t) = \{v\in H^1(I_t) : v(1) = 0\}. \end{cases} \tag{3}$$

Then, applying the integration by parts, the weak form of Problem (1) is to find $u(x,t) \in H_0^{1,2}(I_x)\otimes H^1_{-0}(I_t)$ such that

$$(\partial_t u, v)_\Omega - (\partial_x^2 u, \partial_x v)_\Omega = (f, v)_\Omega, \quad \forall v\in H_0^1(I_x)\otimes H^1_{+0}(I_t). \tag{4}$$

Let P_κ denote the set of polynomials of degree $\le \kappa$ on $[-1,1]$ and denote $L = (M, N)$, where M and N are a pair of given positive integers. In order to present the fully discrete space-time spectral scheme, we introduce the following finite-dimensional spaces:

$$\begin{cases} V_N = P_N(I_x)\cap H_0^{1,2}(I_x), \\ W_{N-1} = P_{N-1}(I_x)\cap H_0^1(I_x), \end{cases} \tag{5}$$

and

$$\begin{cases} S_M = P_M(I_t)\cap H^1_{-0}(I_t), \\ S_M^* = P_M(I_t)\cap H^1_{+0}(I_t). \end{cases} \tag{6}$$

Then we obtain the following space-time Legendre-Petrov-Galerkin scheme of (1): Find $u_L \in V_N\otimes S_M$ satisfying

$$(\partial_t u_L, v)_\Omega - (\partial_x^2 u_L, \partial_x v)_\Omega = (f, v)_\Omega, \quad \forall v\in W_{N-1}\otimes S_M^*. \tag{7}$$

Remark 1. *Since Equation (1) is first-order in time, it is natural to use a "dual Petrov-Galerkin method". The key idea of the method is to use trial functions satisfying the underlying boundary conditions of the differential equations and test functions satisfying the "dual" boundary conditions, namely, S_M and S_M^* are a pair of "dual" approximation spaces.*

3. Error Estimate

Now, in order to analyze the error estimates for the scheme (7), we give a suitable comparison function introduced in [17,30]:

$$\boldsymbol{P}_N u(x) = \bar{\partial}_x^{-2}\mathcal{P}_{N-2}\partial_x^2 u(x), \qquad \forall v\in H^2(I_x), \tag{8}$$

which satisfies $\boldsymbol{P}_N u(\pm 1) = \partial_x \boldsymbol{P}_N u(1) = 0$, $\partial_x \boldsymbol{P}_N u(-1) = \partial_x u(-1)$ and $\boldsymbol{P}_N : H_0^{1,2}(I_x) \to V_N$ such that

$$(\partial_x^2(\boldsymbol{P}_N u - u), \partial_x v) = 0, \quad \forall v \in W_{N-1}, \tag{9}$$

where

$$\bar{\partial}_x^{-1} v(x) = -\int_x^1 v(y)dy, \quad \bar{\partial}_x^{-m} v(x) = (\bar{\partial}_x^{-1})^m v(x), \tag{10}$$

and $\mathcal{P}_{N-2}$ is the Legendre-Galerkin projection operator.

Lemma 1 ([17,30])**.** *If $u \in H_0^{1,2}(I_x) \cap H^r(I_x)$ and $r \geq 2$,*

$$\|\partial_x^l(\boldsymbol{P}_N u - u)\|_{I_x,\omega_{l-2,l-2}} \leq CN^{l-r}\|\partial_x^r u\|_{I_x,\omega_{r-2,r-2}}, \quad 0 \leq l \leq 2. \tag{11}$$

Orthogonal projection operator in time introduced in [12] is given by $\boldsymbol{\Pi}_M : L^2_{\tilde{\omega}_{0,-1}}(I_t) \to S_M$ such that

$$(\boldsymbol{\Pi}_M u - u, v)_{I_t,\tilde{\omega}_{0,-1}} = 0, \qquad \forall v \in S_M. \tag{12}$$

Define $\hat{H}^1(I_t) := \{v : v \in H^1(I_t) \cap L^2_{\tilde{\omega}_{0,-2}}(I_t)\}$, then considering the fact that $\tilde{\omega}_{0,1}\partial_t v \in S_M$ ($\forall v \in S_M^*$), one can observe $\forall u \in \hat{H}^1(I_t)$,

$$\big(\partial_t(\boldsymbol{\Pi}_M u - u), v\big)_{I_t} = -(\boldsymbol{\Pi}_M u - u, \tilde{\omega}_{0,1}\partial_t v)_{I_t,\tilde{\omega}_{0,-1}} = 0, \quad \forall v \in S_M^*. \tag{13}$$

Lemma 2 ([12])**.** *If $u \in L^2_{\tilde{\omega}_{0,-1}}(I_t)$ and $\partial_t^k u \in L^2_{\tilde{\omega}_{k,k-1}}(I_t)$ with $1 \leq k \leq s$, we have*

$$\|\partial_t^l(\boldsymbol{\Pi}_M u - u)\|_{I_t,\tilde{\omega}_{l,l-1}} \leq CM^{l-s}\|\partial_t^s u\|_{I_t,\tilde{\omega}_{s,s-1}}, \quad l \leq s, l = 0, 1. \tag{14}$$

Let $A^r(\Omega)$ and $B^s(\Omega)$ denote the sets of measurable functions satisfying $\|u\|_{A^r(\Omega)} < +\infty$ and $\|u\|_{B^s(\Omega)} < +\infty$, respectively, where for integers $r \geq 2$, $s \geq 0$,

$$\begin{aligned} \|u\|_{A^r(\Omega)} &= (\|\partial_x^r \partial_t u\|^2_{L^2_{\tilde{\omega}_{2,0}}(I_t;L^2_{\omega_{r-2,r-2}}(I_x))} + \|\partial_x^r u\|^2_{L^2_{\tilde{\omega}_{0,-1}}(I_t;L^2_{\omega_{r-2,r-2}}(I_x))})^{\frac{1}{2}}, \\ \|u\|_{B^s(\Omega)} &= (\|\partial_x^2 \partial_t^s u\|^2_{L^2_{\tilde{\omega}_{s,s-1}}(I_t;L^2(I_x))} + \|\partial_t^s u\|^2_{L^2_{\tilde{\omega}_{s,s-1}}(I_t;L^2_{\omega_{-2,-2}}(I_x))})^{\frac{1}{2}}. \end{aligned} \tag{15}$$

Theorem 1. *Suppose u_L and u are the solutions of (1) and (7), respectively. If $u \in A^r(\Omega) \cap B^s(\Omega) \cap L^2_{\tilde{\omega}_{1,-1}}(I_t; H_0^{1,2}(I_x)) \cap \hat{H}^1(I_t; L^2(I_x))$ for integers $r \geq 2$, $s \geq 0$, then*

$$\|u - u_L\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-1}} \leq C(N^{-r}\|u\|_{A^r(\Omega)} + M^{-s}\|u\|_{B^s(\Omega)}). \tag{16}$$

Proof. Denote $U = \boldsymbol{P}_N \boldsymbol{\Pi}_M u$. In order to derive the error estimates, we decompose the error into two parts: $u_L - u = (u_L - U) + (U - u)$ and denote $\tilde{u} = u_L - U$. Then according to (4) and (7), we have $\forall v \in W_{N-1} \otimes S_M^*$,

$$(\partial_t \tilde{u}, v)_\Omega - (\partial_x^2 \tilde{u}, \partial_x v)_\Omega = \big(\partial_t(u - U), v\big)_\Omega + \big(\partial_x^2(U - u), \partial_x v\big)_\Omega. \tag{17}$$

According to the definition of $\boldsymbol{P}_N$ and $\boldsymbol{\Pi}_M$, we can see for the right-hand terms of (17)

$$\begin{aligned} \big(\partial_t(u - U), v\big)_\Omega &= \big(\partial_t(\boldsymbol{P}_N u - \boldsymbol{P}_N \boldsymbol{\Pi}_M u), v\big)_\Omega + \big(\partial_t(u - \boldsymbol{P}_N u), v\big)_\Omega \\ &= \big(\partial_t(u - \boldsymbol{P}_N u), v\big)_\Omega, \end{aligned} \tag{18}$$

$$\begin{aligned}\left(\partial_x^2(U-u),\partial_x v\right)_\Omega &= \left(\partial_x^2(\boldsymbol{P}_N\boldsymbol{\Pi}_M u-\boldsymbol{\Pi}_M u),\partial_x v\right)_\Omega+\left(\partial_x^2(\boldsymbol{\Pi}_M u-u),\partial_x v\right)_\Omega \\ &= \left(\partial_x^2(\boldsymbol{\Pi}_M u-u),\partial_x v\right)_\Omega.\end{aligned} \tag{19}$$

Then (17) can be simplified as follows:

$$(\partial_t\tilde{u},v)_\Omega-(\partial_x^2\tilde{u},\partial_x v)_\Omega=\left(\partial_t(u-\boldsymbol{P}_N u),v\right)_\Omega+\left(\partial_x^2(\boldsymbol{\Pi}_M u-u),\partial_x v\right)_\Omega. \tag{20}$$

Furthermore, taking $v=\omega_{-1,0}\tilde{\omega}_{1,-1}\tilde{u}$ ($\in W_{N-1}\otimes S_M^*$) in (20), we obtain for the first term

$$\begin{aligned}(\partial_t\tilde{u},v)_\Omega=(\partial_t\tilde{u},\omega_{-1,0}\tilde{\omega}_{1,-1}\tilde{u})_\Omega &= \int_{I_x}\left((\omega_{-1,0}\tilde{\omega}_{1,-1}\tilde{u}^2)|_{-1}^{1}-(\tilde{u},\omega_{-1,0}\partial_t[\tilde{\omega}_{1,-1}\tilde{u}])_{I_t}\right)dx \\ &= -(\tilde{u},\omega_{-1,0}\tilde{\omega}_{1,-1}\partial_t\tilde{u})_\Omega-(\tilde{u},\tilde{u}\omega_{-1,0}\partial_t\tilde{\omega}_{1,-1})_\Omega,\end{aligned} \tag{21}$$

namely,

$$(\partial_t\tilde{u},v)_\Omega=(\tilde{u},\omega_{-1,0}\tilde{\omega}_{0,-2}\tilde{u})_\Omega=\|\tilde{u}\|^2_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}}. \tag{22}$$

For the second term of (20), we denote $\eta(x,t)=\omega_{-1,0}\tilde{u}$, then

$$\begin{aligned}-(\partial_x^2\tilde{u},\partial_x v)_\Omega &= -(\partial_x^2\tilde{u},\tilde{\omega}_{1,-1}\partial_x[\omega_{-1,0}\tilde{u}])_\Omega=-(\partial_x^2[\eta\omega_{1,0}],\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega \\ &= -(\partial_x[\omega_{1,0}\partial_x\eta+\eta\partial_x\omega_{1,0}],\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega \\ &= -(\partial_x[\omega_{1,0}\partial_x\eta],\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega+(\partial_x\eta,\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega \\ &= -(\partial_x\omega_{1,0}\partial_x\eta+\omega_{1,0}\partial_x^2\eta,\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega+(\partial_x\eta,\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega \\ &= (\partial_x\eta,\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega-(\omega_{1,0}\partial_x^2\eta,\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega+(\partial_x\eta,\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega \\ &= -(\partial_x^2\eta,\omega_{1,0}\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega+2(\partial_x\eta,\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega,\end{aligned} \tag{23}$$

where

$$\begin{aligned}-(\partial_x^2\eta,\omega_{1,0}\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega &= -\int_{I_t}\left((\omega_{1,0}\tilde{\omega}_{1,-1}(\partial_x\eta)^2)|_{-1}^{1}-\left(\partial_x\eta,\tilde{\omega}_{-1,1}(\partial_x\omega_{1,0}\partial_x\eta+\omega_{1,0}\partial_x^2\eta)\right)_{I_x}\right)dt \\ &= -\int_{I_t}\left(0-2\tilde{\omega}_{1,-1}(\partial_x\eta)^2(-1)\right)dt-(\partial_x\eta,\tilde{\omega}_{1,-1}\partial_x\eta)_\Omega+(\partial_x\eta,\tilde{\omega}_{-1,1}\omega_{1,0}\partial_x^2\eta)_\Omega,\end{aligned} \tag{24}$$

namely

$$-(\partial_x^2\eta,\omega_{1,0}\tilde{\omega}_{-1,1}\partial_x\eta)_\Omega=\|\partial_x\eta(-1)\|^2_{I_t,\tilde{\omega}_{-1,1}}-\frac{1}{2}(\partial_x\eta,\tilde{\omega}_{1,-1}\partial_x\eta)_\Omega, \tag{25}$$

then we can obtain for (23)

$$\begin{aligned}-(\partial_x^2\tilde{u},\partial_x v)_\Omega &= \|\partial_x\eta(-1)\|^2_{I_t,\tilde{\omega}_{-1,1}}+\frac{3}{2}(\partial_x\eta,\tilde{\omega}_{1,-1}\partial_x\eta)_\Omega \\ &= \|\partial_x\eta(-1)\|^2_{I_t,\tilde{\omega}_{-1,1}}+\frac{3}{2}\|\partial_x\eta\|^2_{\Omega,\tilde{\omega}_{-1,1}} \\ &= \|\partial_x[\omega_{-1,0}\tilde{u}](-1)\|^2_{I_t,\tilde{\omega}_{-1,1}}+\frac{3}{2}\|\partial_x[\omega_{-1,0}\tilde{u}]\|^2_{\Omega,\tilde{\omega}_{-1,1}}.\end{aligned} \tag{26}$$

By the Cauchy-Schwarz inequality and Young's inequality, other terms of (20) can be estimated as follows:

$$\begin{aligned}\big(\partial_t(u-\boldsymbol{P}_N u),v\big)_\Omega &= \big(\partial_t(u-\boldsymbol{P}_N u),\omega_{-1,0}\tilde{\omega}_{1,-1}\tilde{u}\big)_\Omega \\ &\le \|\partial_t(u-\boldsymbol{P}_N u)\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{2,0}}\|\tilde{u}\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}} \\ &\le \frac{1}{2}\|\partial_t(u-\boldsymbol{P}_N u)\|^2_{\Omega,\omega_{-1,0}\tilde{\omega}_{2,0}}+\frac{1}{2}\|\tilde{u}\|^2_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}},\end{aligned} \tag{27}$$

$$\begin{aligned}\big(\partial_x^2(\boldsymbol{\Pi}_M u-u),\partial_x v\big)_\Omega &= \big(\partial_x^2(\boldsymbol{\Pi}_M u-u),\tilde{\omega}_{1,-1}\partial_x[\omega_{-1,0}\tilde{u}]\big)_\Omega \\ &\le \|\partial_x^2(\boldsymbol{\Pi}_M u-u)\|_{\Omega,\tilde{\omega}_{1,-1}}\|\partial_x[\omega_{-1,0}\tilde{u}]\|_{\Omega,\tilde{\omega}_{1,-1}} \\ &\le \frac{1}{2}\|\partial_x^2(\boldsymbol{\Pi}_M u-u)\|^2_{\Omega,\tilde{\omega}_{1,-1}}+\frac{1}{2}\|\partial_x[\omega_{-1,0}\tilde{u}]\|^2_{\Omega,\tilde{\omega}_{1,-1}}.\end{aligned} \tag{28}$$

Collecting (21) to (28) leads to

$$\begin{aligned}&\|\tilde{u}\|^2_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}}+\|\partial_x[\omega_{-1,0}\tilde{u}](-1)\|^2_{I_t,\tilde{\omega}_{-1,1}}+\frac{3}{2}\|\partial_x[\omega_{-1,0}\tilde{u}]\|^2_{\Omega,\tilde{\omega}_{-1,1}} \\ &\le \frac{1}{2}\|\partial_t(u-\boldsymbol{P}_N u)\|^2_{\Omega,\omega_{-1,0}\tilde{\omega}_{2,0}}+\frac{1}{2}\|\tilde{u}\|^2_{\Omega,\omega_{-1,1}\tilde{\omega}_{0,-2}}+\frac{1}{2}\|\partial_x^2(\boldsymbol{\Pi}_M u-u)\|^2_{\Omega,\tilde{\omega}_{1,-1}}+\frac{1}{2}\|\partial_x[\omega_{-1,0}\tilde{u}]\|^2_{\Omega,\tilde{\omega}_{1,-1}},\end{aligned} \tag{29}$$

namely,

$$\|\tilde{u}\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}} \le C\big(\|\partial_t(u-\boldsymbol{P}_N u)\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{2,0}}+\|\partial_x^2(\boldsymbol{\Pi}_M u-u)\|_{\Omega,\tilde{\omega}_{1,-1}}\big). \tag{30}$$

Furthermore, according to Lemmas 1 and 2, for the right-hand terms of (30), we can obtain estimates

$$\begin{aligned}\|\partial_t(u-\boldsymbol{P}_N u)\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{2,0}} &\le C\|\partial_t(u-\boldsymbol{P}_N u)\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{2,0}} \\ &\le CN^{-r}\|\partial_x^r\partial_t u\|^2_{\Omega,\omega_{r-2,r-2}\tilde{\omega}_{2,0}},\end{aligned} \tag{31}$$

$$\|\partial_x^2(\boldsymbol{\Pi}_M u-u)\|_{\Omega,\tilde{\omega}_{1,-1}} \le C\|\partial_x^2(\boldsymbol{\Pi}_M u-u)\|_{\Omega,\tilde{\omega}_{0,-1}} \le CM^{-s}\|\partial_t^s\partial_x^2 u\|_{\Omega,\tilde{\omega}_{s,s-1}}, \tag{32}$$

then

$$\|\tilde{u}\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}} \le C\big(N^{-r}\|\partial_x^r\partial_t u\|^2_{\Omega,\omega_{r-2,r-2}\tilde{\omega}_{2,0}}+M^{-s}\|\partial_t^s\partial_x^2 u\|_{\Omega,\tilde{\omega}_{s,s-1}}\big). \tag{33}$$

On the other hand, we obtain

$$\begin{aligned}\|U-u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{0,-1}} &\le \|\boldsymbol{P}_N\boldsymbol{\Pi}_M u-\boldsymbol{P}_N u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{0,-1}}+\|\boldsymbol{P}_N u-u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{0,-1}} \\ &\le C\|\boldsymbol{\Pi}_M u-u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{0,-1}}+\|\boldsymbol{P}_N u-u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{0,-1}} \\ &\le C\big(M^{-s}\|\partial_t^s u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{s,s-1}}+N^{-r}\|\partial_x^r u\|_{\Omega,\omega_{r-2,r-2}\tilde{\omega}_{0,-1}}\big).\end{aligned} \tag{34}$$

According to the above estimates (33) and (34) and the triangle inequality, we derive the final error estimate

$$\begin{aligned}\|u-u_L\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-1}} &\le \|\tilde{u}\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-1}}+\|U-u\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-1}} \\ &\le C(\|\tilde{u}\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}}+\|U-u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{0,-1}}) \\ &\le C\big(N^{-r}(\|\partial_x^r\partial_t u\|_{\Omega,\omega_{r-2,r-2}\tilde{\omega}_{2,0}}+\|\partial_x^r u\|_{\Omega,\omega_{r-2,r-2}\tilde{\omega}_{0,-1}}) \\ &\quad + M^{-s}(\|\partial_x^2\partial_t^s u\|_{\Omega,\tilde{\omega}_{s,s-1}}+\|\partial_t^s u\|_{\Omega,\omega_{-2,-2}\tilde{\omega}_{s,s-1}})\big).\end{aligned} \tag{35}$$

□

Remark 2. *Similar to the proof of error estimates, by taking* $v = \omega_{-1,0}\tilde{\omega}_{1,-1}u_L$ $(\in W_{N-1} \otimes S_M^*)$ *in scheme (7), we can easily obtain the stability results. Assume* $f \in L^2_{\tilde{\omega}_{2,0}}(I_t; L^2_{\omega_{-1,0}}(I_x))$*, then* u_L *is the solution of scheme (7) satisfying* $\|u_L\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{0,-2}} \le C\|f\|_{\Omega,\omega_{-1,0}\tilde{\omega}_{2,0}}$*. Furthermore, there exists a zero solution if* $f = 0$*, namely, the existence and uniqueness of* u_L *can be proved easily.*

4. Implementation

In order to present the detailed implementation of the Equation (1) with initial condition $u(x,-1) = u_0(x) \neq 0$, we take $w = u - u_0$ such that $w_0 := w(x,-1) = 0$ and reformulate the equation of unknown solution $w(x,t)$, then we obtain the discrete scheme: Find $w_L \in V_N \otimes S_M$ satisfying

$$(\partial_t w_L, v)_\Omega - (\partial_x^2 w_L, \partial_x v)_\Omega = (f, v)_\Omega + (\partial_x^2 u_0, \partial_x v)_\Omega, \quad \forall v \in W_{N-1} \otimes S_M^*. \tag{36}$$

We define the following basis functions in space and time:

$$\begin{aligned} &V_N = \text{span}\{(1-x)\phi_i(x)\}, \;\; W_{N-1} = \text{span}\{\phi_i(x)\}, \quad 0 \le i \le N-3, \\ &S_M = \text{span}\{\psi_j\}, \;\; S_M^* = \text{span}\{\psi_j^*\}, \quad 0 \le j \le M-1, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \phi_i(x) = c_{i+1}(L_i - L_{i+2}), \;\; c_i = \frac{1}{2i+1}, \\ \psi_j = L_j + L_{j+1}, \;\; \psi_j^* = L_j - L_{j+1}. \end{aligned} \tag{38}$$

Taking w_L and v in scheme (36) as

$$w_L = (1-x)\sum_{i=0}^{N-3}\sum_{j=0}^{M-1} w_{ji}\phi_i(x)\psi_j(t), \;\; v = \phi_n(x)\psi_s^*(t), \tag{39}$$

then we can obtain the matrix form of the scheme (36)

$$CWA + DWB = F, \tag{40}$$

where $A = (a_{in})_{0 \le i,n \le N-3}$ is a symmetric matrix with elements

$$a_{in} = ((1-x)\phi_i, \phi_n)_{I_x} = \begin{cases} 2c_{i+1}^2(c_i + c_{i+2}), & i = n, \\ -2c_i c_{i+1}(ic_{i-1}c_i - (i+1)c_i c_{i+1} + (i+2)c_{i+1}c_{i+2}), & i = n+1, \\ -2c_i c_{i-1} c_{i+1}, & i = n+2, \\ 2ic_{i-2}c_{i-1}c_i c_{i+1}, & i = n+3, \end{cases} \tag{41}$$

and matrix $B = (b_{in})_{0 \le i,n \le N-3}$ with elements

$$b_{in} = (2L_{i+1} - (1-x)L'_{i+1}, L_{n+1})_{I_x} = \begin{cases} 0, & i \le n-1, \\ 3c_{i+1} + 1, & i = n, \\ (-1)^{i+n}2, & i \ge n+1. \end{cases} \tag{42}$$

For details about matrices C and D, see [12], and $F = (f_{sn})_{0 \le s \le M-1, 0 \le n \le N-3}$ with $f_{sn} = (f, \phi_n\psi_s^*)_\Omega + (\partial_x^2 u_0, \phi'_n \psi_s^*)_\Omega$, which can be computed by the Legendre-Gauss-type quadrature formulas.

Then according to the properties of matrix multiplication introduced in Section 2, Equation (36) can be formulated as

$$(A' \otimes C + B' \otimes D)\text{vec}(W) = \text{vec}(F). \tag{43}$$

5. Numerical Experiments

In this section, we present some numerical examples, including nonlinear problems, to demonstrate the accuracy and efficiency of the proposed method for third-order partial differential equations. Some numerical results are compared with those computed in [17,27].

5.1. Example 1

Consider the Equation (1) in time interval $(0, T]$ with exact solution [17] :

$$u(x,t) = \sin^2(\pi x)\sin(12x + 12t). \tag{44}$$

In Figure 1, we plot the time evolution of exact solutions and numerical solutions obtained by the proposed method and one can observe that numerical solutions well simulate the image of exact solutions. In [17], third-order partial differential Equations (1) were considered by combining the Legendre-Petrov-Galerkin methods in space with the Crank-Nicolson scheme for time advancing (see scheme (2.2) in [17]) and L^2-errors in both space and time were given for the Equation (1) with exact solution (44) in numerical examples. So in order to show the efficiency of the proposed method in this paper, we present the spatial and temporal errors respectively in Tables 1 and 2, where we compare the numerical results by our methods with those in [17] under the same conditions. By comparison, we can find that the numerical solutions obtained in this paper attain higher accuracy. Moreover, we report the L^2-error in time and space by semi-log coordinates at $t = 2$ for the equation in Figure 2, and we can see that the straight lines indicate that the errors decay like $\exp(-cM)$ and $\exp(-cN)$.

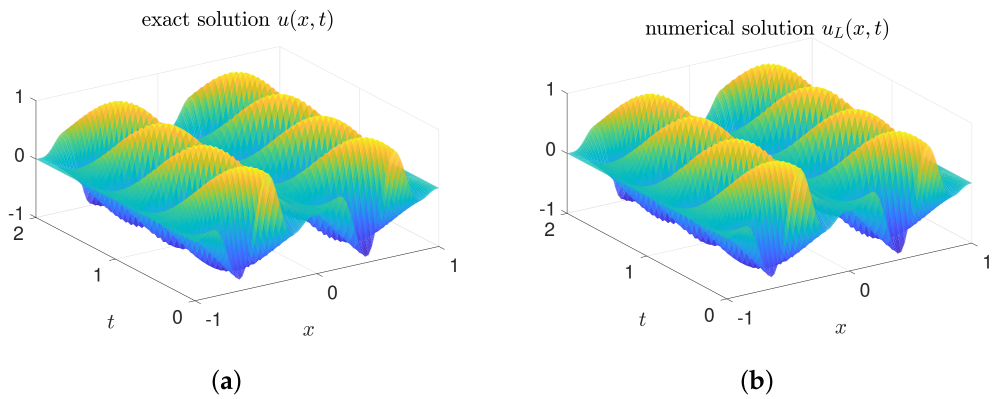

Figure 1. Time evolution of exact solutions and numerical solutions for Example 1. (**a**) Exact solutions $u(x,t) = \sin^2(\pi x)\sin(12x + 12t)$; (**b**) numerical solutions $u_L(x,t)$ by scheme (7).

Table 1. Temporal errors at $t = 1$ for Example 1.

N	Scheme (7)		Scheme (2.2) in [17]	
	M	L^2-Error	τ	L^2-Error
64	16	1.4063×10^{-6}	10^{-1}	3.9301×10^{-3}
64	18	4.0923×10^{-8}	10^{-2}	3.1014×10^{-5}
64	21	1.3429×10^{-10}	10^{-3}	3.0959×10^{-7}
64	23	2.3504×10^{-12}	10^{-4}	3.0959×10^{-9}
64	25	5.0067×10^{-14}	10^{-5}	3.0819×10^{-11}

Table 2. Spatial errors at $t = 1$ for Example 1.

Scheme (7) with $M = 30$		Scheme (2.2) with $\tau = 10^{-5}$ in [17]	
N	L^2-Error	N	L^2-Error
32	8.9457×10^{-7}	16	3.0866×10^{-1}
39	5.0037×10^{-11}	32	1.5288×10^{-7}
44	6.1458×10^{-14}	64	3.0819×10^{-11}

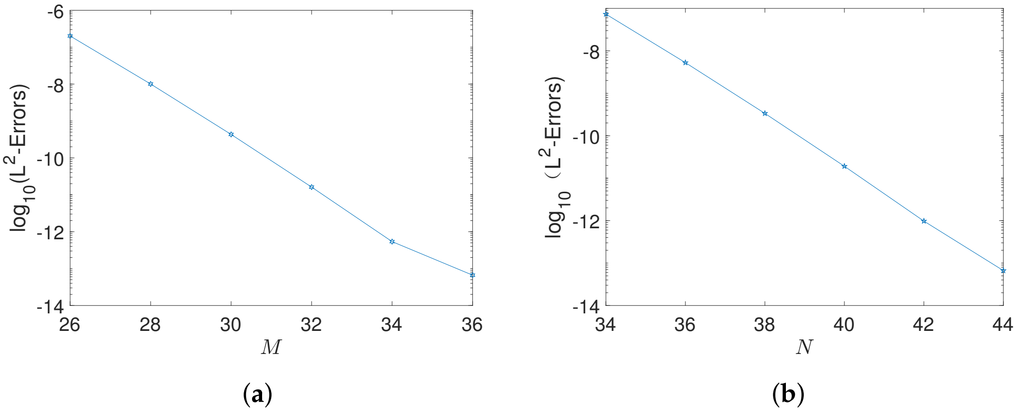

Figure 2. Spectral errors in time and space, respectively, at $t = 2$ for Example 1. (**a**) Temporal errors versus M with $N = 50$; (**b**) spatial errors versus N with $M = 40$.

5.2. Example 2

Consider the following KdV equation [24,27]:

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, & x \in (a,b),\ t > 0, \\ u(x,0) = u_0(x), & x \in (a,b), \end{cases} \tag{45}$$

with exact solution

$$u(x,t) = 12\kappa^2 \mathrm{sech}^2(\kappa(x - 4\kappa^2 t - x_0)), \tag{46}$$

where κ and x_0 are given parameters.

In Figure 3, we show temporal and spatial L^2-error using semi-log coordinates at t = 6 for the KdV Equation (45) with $a = -50, b = 50, \kappa = 0.3, x_0 = -20$. We can obviously observe that the errors are of the form e^{-cK}, where $K = N$ for spatial errors, and $K = M$ for temporal errors, which indicate the exponential convergence in both time and space.

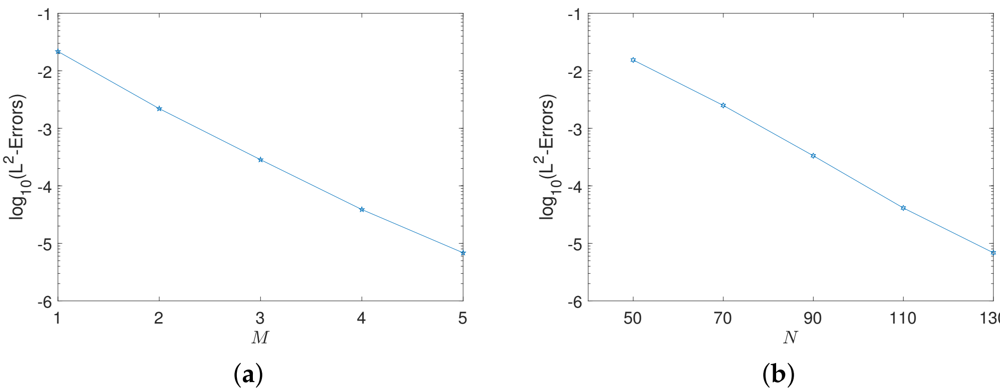

Figure 3. Spectral errors in time and space, respectively, at $t = 6$ for Example 2. Take $a = -50$, $b = 50, \kappa = 0.3, x_0 = -20$. (**a**) Temporal errors versus M with $N = 5$; (**b**) spatial errors versus N with $M = 130$.

In [27], a Legendre pseudo-spectral method was proposed for the KdV Equation (45) combined with finite difference methods in time (see scheme (2.3) in [27]), and temporal and spatial errors at $t = 1$ were given for the equation with exact solution (46) by taking $a = -40, b = 40, \kappa = 0.3, x_0 = 0$ in the numerical examples. In Tables 3 and 4, we compare the temporal and spatial errors, respectively, obtained by our method with those in [27] under the same conditions. We can observe from the results of comparison that the proposed method in this paper attains higher accuracy with smaller N.

Moreover, in Figure 4, we plot the time evolution of numerical solutions for the KdV Equation (45) with exact solution (46) by taking $a = -30, b = 30, \kappa = 0.3, x_0 = -20$ and $a = -30, b = 30, \kappa = 0.3, x_0 = 0$ respectively.

Table 3. Spatial errors at $t = 1$ for Example 2. Take $a = -40, b = 40, \kappa = 0.3, x_0 = 0$.

Scheme (7) with M = 3		**Scheme (2.3) with τ = 1e-06 in [27]**	
N	L^2-Error	N	L^2-Error
40	4.0000×10^{-3}	10	1.4367×10^{0}
60	6.7240×10^{-4}	20	6.9376×10^{-1}
80	8.3138×10^{-5}	40	9.2131×10^{-2}
100	9.1547×10^{-6}	80	1.3874×10^{-3}
120	9.6709×10^{-7}	160	2.0651×10^{-7}

Table 4. Temporal errors at $t = 1$ for Example 2. Take $a = -40, b = 40, \kappa = 0.3, x_0 = 0$.

Scheme (7) with N = 140		**Scheme (2.3) with N = 160 in [27]**	
M	L^2-Error	τ	L^2-Error
2	1.1230×10^{-5}	10^{-3}	8.3542×10^{-5}
3	2.4398×10^{-7}	10^{-4}	8.3626×10^{-6}
4	9.1633×10^{-8}	10^{-5}	8.6122×10^{-7}

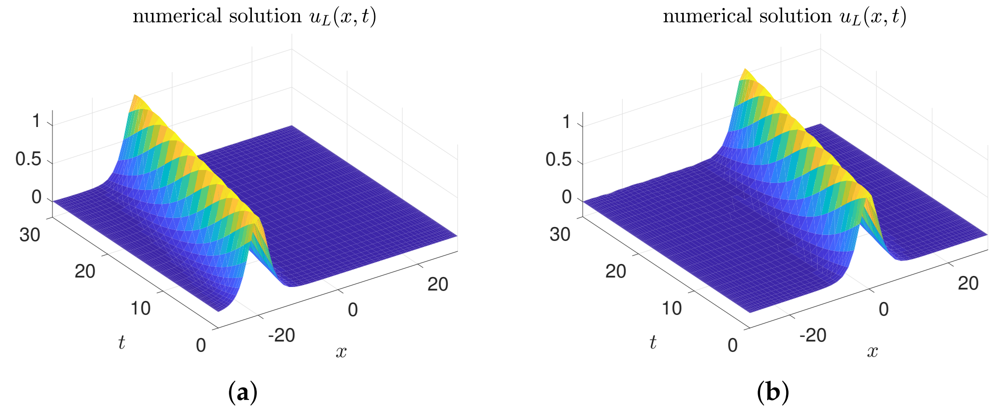

Figure 4. Time evolution of the numerical solutions for Example 2 with different constant x_0. Take $a = -30, b = 30, \kappa = 0.3$. (**a**) $x_0 = -20$; (**b**) $x_0 = 0$.

5.3. Example 3

Consider the following KdV equation [17,27]:

$$\begin{cases} \partial_t u + (1 + 12u)\partial_x u + \partial_x^3 u = 0, & x \in (a,b),\ t > 0, \\ u(x,0) = u_0(x), & x \in (a,b), \end{cases} \tag{47}$$

with exact solution

$$u(x,t) = \frac{1}{4}\kappa^2 \mathrm{sech}^2 \frac{1}{2}(\kappa x - (\kappa + \kappa^3)t + x_0), \tag{48}$$

where κ and x_0 are given parameters.

In [27], the KdV Equation (47) was considered with exact solution (48) by taking $a = -12, b = 12, \kappa = 1, x_0 = 0$ in numerical examples, where relative L^∞-errors were presented in numerical results compared with those computed by backward Euler/forward Euler methods in [26]. However, no specific error estimates were reported neither in space nor time. In this example, we present the spatial L^2-error estimates by taking $M = 6$ and temporal L^2-error estimates by taking $N = 54$ conducted by our method in Table 5. One can easily observe that exponential convergence is obtained both in space and time.

Table 5. Spectral errors at $t = 1$ for Example 3. Take $a = -12, b = 12, \kappa = 1, x_0 = 0$.

Temporal Errors with $N = 54$		**Spatial Errors with $M = 6$**	
M	L^2-Error	N	L^2-Error
1	1.8300×10^{-2}	14	3.1500×10^{-2}
2	2.7000×10^{-3}	24	4.5000×10^{-3}
3	4.9201×10^{-4}	34	3.8479×10^{-4}
4	9.4463×10^{-5}	44	3.2627×10^{-5}
5	9.4819×10^{-6}	54	9.4819×10^{-6}

6. Conclusions

In this article, we investigate a space-time Legendre-Petrov-Galerkin method for solving the linear third-order differential equations with non-periodic boundary conditions. In the theoretical analysis, rigorous proof of error estimates in the weighted space-time norms is presented. In the numerical experiments, L^2-error estimates are obtained by the proposed method and numerical results indicate the exponential convergence both in space and time. In addition, some numerical results are compared with those given by other numerical methods. By noticing the results, we can see that the proposed space-time spectral method in this paper can attain higher accuracy.

It is pointed out in some papers that multi-domain spectral methods in space and multi-interval spectral methods in time not only reduce the scale of the problems but also reach better accuracy. So, we try to investigate multi-domain forms in space or multi-interval forms in time of the space-time Legendre-Petrov-Galerkin methods in the forthcoming study.

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