axioms

Article
Further Closed Formulae of Exotic ${}_3F_2$-Series

Wenchang Chu [1,2]

[1] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China; wenchang.chu@unisalento.it
[2] Department of Mathematics and Physics, University of Salento, 73100 Lecce, Italy

Abstract: By making use of the linearization method, we examine a class of nonterminating ${}_3F_2$-series with five free integer parameters that yields twenty summation formulae. Under the Kummer and Thomae transformations, six classes of exotic ${}_3F_2$-series are consequently evaluated in closed forms. There are overall 100 identities recorded in the present paper.

Keywords: hypergeometric series; nonterminating exotic ${}_3F_2$-series linearization method; Thomae transformation; Kummer transformation

MSC: Primary 33C20; Secondary 33F10

check for updates

Citation: Chu, W. Further Closed Formulae of Exotic ${}_3F_2$-Series. *Axioms* **2023**, *12*, 291. https://doi.org/10.3390/axioms12030291

Academic Editors: Gradimir V. Milovanović and Clemente Cesarano

Received: 20 December 2022
Revised: 24 February 2023
Accepted: 9 March 2023
Published: 10 March 2023

1. Introduction and Outline

Denote by $\mathbb{N}$ and $\mathbb{Z}$, respectively, the sets of natural numbers and integers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The shifted factorials are given by $(x)_0 = \langle x \rangle_0 =\equiv 1$ and

$$\left.\begin{aligned}(x)_n &= x(x+1)\cdots(x+n-1)\\ \langle x\rangle_n &= x(x-1)\cdots(x-n+1)\end{aligned}\right\}\quad \text{for}\quad n\in\mathbb{N}.$$

We can express them, even when $n \in \mathbb{Z}$, as the quotients

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{and} \quad \langle x\rangle_n = \frac{\Gamma(1+x)}{\Gamma(1+x-n)},$$

where the Γ-function is defined by the Euler integral

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\mathrm{d}u \quad \text{for} \quad \Re(x) > 0.$$

For brevity, their fractional forms are concisely shortened as

$$\begin{aligned}\left[\begin{matrix}\alpha,\beta,\cdots,\gamma\\ A,B,\cdots,C\end{matrix}\right]_n &= \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n},\\ \Gamma\left[\begin{matrix}\alpha,\beta,\cdots,\gamma\\ A,B,\cdots,C\end{matrix}\right] &= \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.\end{aligned}$$

According to Bailey [1], the generalized hypergeometric series is defined by

$${}_{1+p}F_p\left[\begin{matrix}a_0, & a_1, & \cdots, & a_p\\ & b_1, & \cdots, & b_p\end{matrix}\middle|\, z\right] = \sum_{n=0}^{\infty}\frac{(a_0)_n(a_1)_n\cdots(a_p)_n}{n!(b_1)_n\cdots(b_p)_n}z^n.$$

When $z = 1$, this series is convergent only if the "parameter excess" (i.e., the difference between the sum of the denominator parameters and that of the numerator ones) has a positive real part.

There exist many strange evaluations of hypergeometric series (cf. [2–8] for example). Recently, Campbell, D'Aurizio and Sondow [9,10] discovered two mysterious-looking formulae (see **D1** and **D12**)

$$
{}_3F_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & \frac{1}{2} \\ & 1, & \frac{3}{2}\end{matrix}\,\middle|\,1\right] = \frac{4\ln(1+\sqrt{2})}{\pi},
$$

$$
{}_3F_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & -\frac{1}{2} \\ & 1, & \frac{1}{2}\end{matrix}\,\middle|\,1\right] = \frac{\sqrt{2}+\ln(1+\sqrt{2})}{\pi}.
$$

Campbell and Abrarov [11] found, among the others, the following two further ones (see **F10** and **G8**)

$$
{}_3F_2\left[\begin{matrix}\frac{3}{2}, & \frac{3}{4}, & -\frac{1}{4} \\ & 1, & \frac{7}{4}\end{matrix}\,\middle|\,1\right] = \frac{3\sqrt{\pi}\{3\sqrt{2}-\log(1+\sqrt{2})\}}{2\Gamma(\frac{1}{4})^2},
$$

$$
{}_3F_2\left[\begin{matrix}\frac{3}{2}, & \frac{1}{4}, & \frac{5}{4} \\ & 1, & \frac{9}{4}\end{matrix}\,\middle|\,1\right] = \frac{5\sqrt{\pi}\{3\sqrt{2}-\log(1+\sqrt{2})\}}{8\Gamma(\frac{3}{4})^2}.
$$

These series are said "exotic" because one numerator parameter minus a denominator parameter results in a negative integer. By examining carefully these seemingly unrelated series, we find that they are connected, under the Thomae and Kummer transformation (cf. Bailey [1] §3.2 and Page 98), to the following ${}_3F_2$-series

$$
\mathcal{F}(a,c,e;b,d) := {}_3F_2\left[\begin{matrix}1+a, & c, & \frac{1}{2}+e \\ & \frac{3}{4}+b, & \frac{5}{4}+d\end{matrix}\,\middle|\,1\right], \quad \left\{\begin{matrix}\Delta:=\frac{1}{2}+b+d-a-c-e>0 \\ \sigma:=b+d-a-c-e\geq 0\end{matrix}\right\},
$$

where $a,b,c,d,e \in \mathbb{Z}$ satisfying the conditions $a \geq 0$ and $c > 0$ so that the both series involved are nonterminating. When $\sigma = b+d-a-c-e \geq 0$, the series is convergent, because in this case the parameter excess $\Delta = \sigma + \frac{1}{2} > 0$ (i.e., the sum of the denominator parameters minus that of the numerator ones).

Classically, there are three typical summation theorems (for the ${}_3F_2$-series) discovered by Dixon, Watson and Whipple (cf. Bailey [1] §3.1, §3.3 and §3.4). However, neither of them can evaluate the afore-displayed series in closed form. In particular, the formulae for the ${}_3F_2$-series presented in this paper are not present in the recent paper by the author [12], and two useful compendiums: ([13] §8.1.2 and [14] §7.4.4), where numerous closed formulae are collected for the ${}_3F_2(1)$ series with numerical parameters.

By applying the linearization method (cf. [15–18]), we shall transform, in the next section, the evaluation of $\mathcal{F}$-series into the $\Omega_{m,n}$-series treated recently by the author [19]. The main results are summarized in the conclusive theorem as well as twenty closed formulae for the $\mathcal{F}$-series. Finally in Section 3, analytic formulae for six further classes of exotic ${}_3F_2$-series will be provided by employing the Thomae and Kummer transformations (cf. Bailey [1] §3.2 and Page 98) to the $\mathcal{F}$-series.

In order to ensure the accuracy, all the formulae appearing in this paper have been checked numerically by appropriately devised Mathematica commands.

2. Linearization Procedure for the $\mathcal{F}$-Series

In this section, we shall reduce, by means of the linearization method (cf. [15–18]), the $\mathcal{F}$-series to specific instances of a known $\Omega_{m,n}(x,y)$ function, that has recently been examined by the author [19].

2.1. $a = 0$

According to the Chu–Vandermonde convolution identity on binomial coefficients, it is routine to establish the following lemma.

Lemma 1 (Linear relation: $m \in \mathbb{N}_0$)**.**

$$(A+n)_m = \sum_{k=0}^{m} (B+n)_k X_k \quad \textit{where} \quad X_k = \binom{m}{k}(A-B)_{m-k}.$$

Specifying the above relation to the equality

$$(1+n)_a = \sum_{k=0}^{a} (c+n)_k X_k(a) \quad \text{where} \quad X_k(a) = \binom{a}{k}(1-c)_{a-k}$$

and then substituting it into the $\mathcal{F}$-series, we have the double series

$$\begin{aligned}\mathcal{F}(a,c,e;b,d) &= \sum_{n=0}^{\infty} \begin{bmatrix} 1+a, & c, & \frac{1}{2}+e \\ 1, & \frac{3}{4}+b, & \frac{5}{4}+d \end{bmatrix}_n \sum_{k=0}^{a} \frac{(c+n)_k}{(1+n)_a} X_k(a) \\ &= \sum_{k=0}^{a} \frac{(c)_k}{(1)_a} X_k(a) \sum_{n=0}^{\infty} \begin{bmatrix} c+k, & \frac{1}{2}+e \\ \frac{3}{4}+b, & \frac{5}{4}+d \end{bmatrix}_n.\end{aligned}$$

This results in the reduction formula as below.

Proposition 1 (Reduction formula from $a>0$ to $a=0$)**.**

$$\mathcal{F}(a,c,e;b,d) = \sum_{k=0}^{a} (-1)^a \binom{-c}{k}\binom{c-1}{a-k} \mathcal{F}(0,c+k,e;b,d).$$

2.2. $b=d$

The $\mathcal{F}$-series can further be reduced to the case $b=d$.

When $b>d$, we can specify Lemma 1 to the equality

$$(\tfrac{5}{4}+d+n)_{b-d} = \sum_{k=0}^{b-d} (c+n)_k Y_k(b,d) \quad \text{where} \quad Y_k(b,d) = \binom{b-d}{k}(\tfrac{5}{4}-c+d)_{b-d-k}.$$

Putting this inside the $\mathcal{F}$-series, we have the double series

$$\begin{aligned}\mathcal{F}(0,c,e;b,d) &= \sum_{n=0}^{\infty} \begin{bmatrix} c, \frac{1}{2}+e \\ \frac{3}{4}+b, \frac{5}{4}+d \end{bmatrix}_n \sum_{k=0}^{b-d} \frac{(c+n)_k}{(\frac{5}{4}+d+n)_{b-d}} Y_k(b,d) \\ &= \sum_{k=0}^{b-d} \frac{(c)_k}{(\frac{5}{4}+d)_{b-d}} Y_k(b,d) \sum_{n=0}^{\infty} \begin{bmatrix} c+k, \frac{1}{2}+e \\ \frac{3}{4}+b, \frac{5}{4}+b \end{bmatrix}_n.\end{aligned}$$

This yields the following reduction formula.

Proposition 2 (Reduction formula from $b>d$ to $b=d$)**.**

$$\mathcal{F}(0,c,e;b,d) = \sum_{k=0}^{b-d} \binom{b-d}{k} \frac{(c)_k(\frac{5}{4}-c+d)_{b-d-k}}{(\frac{5}{4}+d)_{b-d}} \mathcal{F}(0,c+k,e;b,b).$$

Alternatively, for $b<d$, we can specify Lemma 1 to the equality

$$(\tfrac{3}{4}+b+n)_{d-b} = \sum_{k=0}^{d-b} (c+n)_k \mathcal{Y}_k(b,d) \quad \text{where} \quad \mathcal{Y}_k(b,d) = \binom{d-b}{k}(\tfrac{3}{4}+b-c)_{d-b-k}.$$

Substituting this into the $\mathcal{F}$-series, we have the double series

$$\begin{aligned}\mathcal{F}(0,c,e;b,d) &= \sum_{n=0}^{\infty}\left[\begin{matrix} c, \frac{1}{2}+e \\ \frac{3}{4}+b, \frac{5}{4}+d \end{matrix}\right]_n \sum_{k=0}^{d-b} \frac{(c+n)_k}{(\frac{3}{4}+b+n)_{d-b}} \mathcal{Y}_k(b,d) \\ &= \sum_{k=0}^{d-b} \frac{(c)_k}{(\frac{3}{4}+b)_{d-b}} \mathcal{Y}_k(b,d) \sum_{n=0}^{\infty}\left[\begin{matrix} c+k, \frac{1}{2}+e \\ \frac{3}{4}+d, \frac{5}{4}+d \end{matrix}\right]_n .\end{aligned}$$

This gives rise to another reduction formula.

Proposition 3 (Reduction formula from $b < d$ to $b = d$)**.**

$$\mathcal{F}(0,c,e;b,d) = \sum_{k=0}^{d-b} \binom{d-b}{k} \frac{(c)_k(\frac{3}{4}+b-c)_{d-b-k}}{(\frac{3}{4}+b)_{d-b}} \mathcal{F}(0,c+k,e;d,d).$$

2.3. $c = e$

The $\mathcal{F}$-series can further be reduced to the case $c = e$. For this purpose, we have to show the following linearization lemma.

Lemma 2 (Linear relation: $m \in \mathbb{N}_0$)**.**

$$(A+n)_m = \sum_{k=0}^{m} \langle B+2n \rangle_k Z_k \quad \textit{where} \quad Z_k = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{k!} \binom{k}{i} (A - \tfrac{B-i}{2})_m.$$

Proof. By substitution, it suffices to evaluate the double sum

$$\mathcal{S} := \sum_{k=0}^{m} \langle B+2n \rangle_k \sum_{i=0}^{k} \frac{(-1)^{k-i}}{k!} \binom{k}{i} (A - \tfrac{B-i}{2})_m = (A+n)_m.$$

By exchanging the order of summations, we can reformulate it as

$$\begin{aligned}\mathcal{S} &= \sum_{i=0}^{m} \frac{\langle B+2n \rangle_i}{i!} (A - \tfrac{B-i}{2})_m \sum_{k=i}^{m} (-1)^{k-i} \binom{B+2n-i}{k-i} \\ &= \sum_{i=0}^{m} (-1)^{m-i} \frac{\langle B+2n \rangle_i}{i!} (A - \tfrac{B-i}{2})_m \binom{B+2n-i-1}{m-i} \\ &= \frac{\langle B+2n \rangle_{m+1}}{m!} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \frac{(A - \frac{B-i}{2})_m}{B+2n-i} \\ &= \frac{(B+2n)_{m+1}}{m!} \times \frac{m!(A+n)_m}{\langle B+2n \rangle_{m+1}} = (A+n)_m,\end{aligned}$$

where the last line is justified by finite difference calculus (cf. [20,21]). □

First for $c < e$, we have from Lemma 2 the equality

$$(\tfrac{1}{2}+c+n)_{e-c} = \sum_{k=0}^{e-c} \left\langle 2b+2n+\tfrac{1}{2} \right\rangle_k \mathcal{Z}_k(b,c,e),$$

$$\text{where} \quad \mathcal{Z}_k(b,c,e) = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{k!} \binom{k}{i} (c-b+\tfrac{1+2i}{4})_{e-c}.$$

By inserting this into the $\mathcal{F}$-series, we obtain the double series below

$$\begin{aligned}\mathcal{F}(0,c,e;b,b) &= \sum_{n=0}^{\infty}\left[\begin{matrix} c, \frac{1}{2}+e \\ \frac{3}{4}+b, \frac{5}{4}+b \end{matrix}\right]_n \sum_{k=0}^{e-c} \frac{\left\langle 2b+2n+\frac{1}{2}\right\rangle_k}{(\frac{1}{2}+c+n)_{e-c}} \mathcal{Z}_k(b,c,e) \\ &= \sum_{k=0}^{e-c}(-1)^k \frac{(-\frac{1}{2}-2b)_k}{(\frac{1}{2}+c)_{e-c}} \mathcal{Z}_k(b,c,e) \sum_{n=0}^{\infty}\left[\begin{matrix} c, \frac{1}{2}+c \\ \frac{3-2k}{4}+b, \frac{5-2k}{4}+b \end{matrix}\right]_n.\end{aligned}$$

Writing the inner sum concerning n in terms of the $\mathcal{F}$-series, we immediately establish the reduction formula as in the following proposition.

Proposition 4 (Reduction formula from $c < e$ to $c = e$)**.**

$$\mathcal{F}(0,c,e;b,b) = \sum_{k=0}^{e-c}(-1)^k \frac{(-\frac{1}{2}-2b)_k}{(\frac{1}{2}+c)_{e-c}} \mathcal{Z}_k(b,c,e)\mathcal{F}(0,c,c;b-\tfrac{k}{2},b-\tfrac{k}{2}).$$

When $c > e$ and $e > 0$, we infer from Lemma 2 that

$$(e+n)_{c-e} = \sum_{k=0}^{c-e}\left\langle 2b+2n+\tfrac{1}{2}\right\rangle_k \mathcal{Z}_k(b,c,e),$$

where $$\mathcal{Z}_k(b,c,e) = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{k!}\binom{k}{i}(e-b+\tfrac{2i-1}{4})_{c-e}. \tag{1}$$

Putting this inside the $\mathcal{F}$-series, we can analogously treat the double series

$$\begin{aligned}\mathcal{F}(0,c,e;b,b) &= \sum_{n=0}^{\infty}\left[\begin{matrix} c, \frac{1}{2}+e \\ \frac{3}{4}+b, \frac{5}{4}+b \end{matrix}\right]_n \sum_{k=0}^{c-e} \frac{\left\langle 2b+2n+\frac{1}{2}\right\rangle_k}{(e+n)_{c-e}} \mathcal{Z}_k(b,c,e) \\ &= \sum_{k=0}^{c-e}(-1)^k \frac{(-\frac{1}{2}-2b)_k}{(e)_{c-e}} \mathcal{Z}_k(b,c,e) \sum_{n=0}^{\infty}\left[\begin{matrix} e, \frac{1}{2}+e \\ \frac{3-2k}{4}+b, \frac{5-2k}{4}+b \end{matrix}\right]_n.\end{aligned}$$

Instead, for $c > e$ and $e \leq 0$, reformulate first the $\mathcal{F}$-series by reindexing

$$\begin{aligned}\mathcal{F}(0,c,e;b,b) &= \mathcal{F}(0,1+c-e,1;1+b-e,1+b-e) \\ &\times \left[\begin{matrix} c, \frac{1}{2}+e \\ \frac{3}{4}+b, \frac{5}{4}+b \end{matrix}\right]_{1-e} + \sum_{n=0}^{-e}\left[\begin{matrix} c, \frac{1}{2}+e \\ \frac{3}{4}+b, \frac{5}{4}+b \end{matrix}\right]_n.\end{aligned}$$

Then according to Lemma 2, we have another equality

$$(1+n)_{c-e} = \sum_{k=0}^{c-e}\left\langle \tfrac{5}{2}+2b-2e+2n\right\rangle_k \mathcal{Z}_k(b,c,e),$$

where the connection coefficients $\mathcal{Z}_k(b,c,e)$ coincide with those given by (1). Now, by substitution, we have another double series

$$\begin{aligned}&\mathcal{F}(0,1+c-e,1;1+b-e,1+b-e) \\ &= \sum_{n=0}^{\infty}\left[\begin{matrix} 1+c-e, \frac{3}{2} \\ \frac{7}{4}+b-e, \frac{9}{4}+b-e \end{matrix}\right]_n \sum_{k=0}^{c-e} \frac{\left\langle \frac{5}{2}+2b-2e+2n\right\rangle_k}{(1+n)_{c-e}} \mathcal{Z}_k(b,c,e) \\ &= \sum_{k=0}^{c-e}(-1)^k \frac{(2e-2b-\frac{5}{2})_k}{(c-e)!} \mathcal{Z}_k(b,c,e) \sum_{n=0}^{\infty}\left[\begin{matrix} 1, \quad \frac{3}{2} \\ \frac{7-2k}{4}+b-e, \frac{9-2k}{4}+b-e \end{matrix}\right]_n.\end{aligned}$$

Summing up, we have established the reduction formula to the case $c = e$.

Proposition 5 (Reduction formula from $c > e$ to $c = e$)**.**

$$\begin{aligned}
e>0:\ \mathcal{F}(0,c,e;b,b) &= \sum_{k=0}^{c-e}(-1)^k\frac{(-\frac{1}{2}-2b)_k}{(e)_{c-e}}\mathcal{Z}_k(b,c,e)\mathcal{F}(0,e,e;b-\tfrac{k}{2},b-\tfrac{k}{2}),\\
e\le 0:\ \mathcal{F}(0,c,e;b,b) &= \sum_{n=0}^{-e}\left[\begin{matrix} c,\frac{1}{2}+e\\ \frac{3}{4}+b,\frac{5}{4}+b\end{matrix}\right]_n+\sum_{k=0}^{c-e}(-1)^k\frac{(2e-2b-\frac{5}{2})_k}{(c-e)!}\mathcal{Z}_k(b,c,e)\\
&\times\left[\begin{matrix} c,\frac{1}{2}+e\\ \frac{3}{4}+b,\frac{5}{4}+b\end{matrix}\right]_{1-e}\mathcal{F}(0,1,1;1+b-e-\tfrac{k}{2},1+b-e-\tfrac{k}{2}).
\end{aligned}$$

Observe that the parameter excess $\Delta \ge \frac{1}{2}$ for the $\mathcal{F}$-series is not diminished hitherto by the established reduction formulae. Consequently, all the $\mathcal{F}$-series displayed on the right hand sides of Propositions 4 and 5 have the parameter excess $\Delta \ge \frac{1}{2}$, and can be expressed as the following bisection series

$$\begin{aligned}
\mathcal{F}(0,c,c;b,b)=\sum_{n=0}^{\infty}\frac{(2c)_{2n}}{(2b+\frac{3}{2})_{2n}} &= \frac{1}{2}\times{}_2F_1\left[\begin{matrix}1,\ 2c\\ \frac{3}{2}+2b\end{matrix}\,\middle|\,1\right]\\
&+\frac{1}{2}\times{}_2F_1\left[\begin{matrix}1,\ 2c\\ \frac{3}{2}+2b\end{matrix}\,\middle|\,-1\right],
\end{aligned}$$

where $b,c\in\mathbb{N}$ subject to the condition $b\ge c$. Therefore, to evaluate the $\mathcal{F}$-series explicitly, it suffices to do that for the above bisection series.

2.4. $\Omega_{m,n}$-Series

In a recent paper [19], the author examined a more general series

$$\Omega_{m,n}(x,y) := {}_2F_1\left[\begin{matrix}x,\quad m-x\\ n+\frac{1}{2}\end{matrix}\,\middle|\,y^2\right]\quad\text{where}\quad m,n\in\mathbb{Z} \tag{2}$$

and proved the following evaluation formula.

Theorem 1 (Chu [19] Theorems 4 and 8: Recurrence formula)**.** *For the two natural numbers m and n satisfying $m<n$, there holds the following formula*

$$\begin{aligned}
\Omega_{m,n}(x,y) = \frac{(\frac{1}{2})_n}{y^{2n}}&\sum_{i=0}^{n-m}\binom{n-m}{i}\frac{(x)_i(m-x)_{n-m-i}}{(2x-n+i)_i(m-2x-i)_{n-m-i}}\\
&\times\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2x+2i-2k}{(2x+2i-n-k)_{n+1}}\Omega_{0,0}(x+i-k,y),
\end{aligned}$$

where the series $\Omega_{0,0}$ is evaluated by

$$\Omega_{0,0}(x,y)={}_2F_1\left[\begin{matrix}x,\quad -x\\ \frac{1}{2}\end{matrix}\,\middle|\,y^2\right]=\cos(2x\arcsin y).$$

Hence, the $\mathcal{F}$-series can be evaluated in terms of the Ω-series by the theorem below.

Theorem 2 ($b\ge c:\ b,c\in\mathbb{N}$)**.**

$$\mathcal{F}(0,c,c;b,b)=\frac{1}{2}\lim_{x\to 1}\Omega_{2c+1,2b+1}(x,1)+\frac{1}{2}\lim_{x\to 1}\Omega_{2c+1,2b+1}(x,\sqrt{-1})$$

with $\quad\Omega_{0,0}(x,1)=\cos(\pi x)\quad$ *and* $\quad\Omega_{0,0}(x,\sqrt{-1})=\cosh\left(2x\ln(1+\sqrt{2})\right).$

2.5. Conclusive Theorem and Examples (Class-A)

Based on the preceding reduction formulae, we may evaluate, for any quintuple integers $a, b, c, d, e \in \mathbb{Z}$ subject to $a \geq 0$, $c > 0$ and $\sigma = b + d - a - c - e > 0$, the $\mathcal{F}$-series by carrying out the following procedure:

- **Step-A**: If $a = 0$, go directly to **Step-B**. Otherwise for $a > 0$, according to Proposition 1, express $\mathcal{F}(a, c, e; b, d)$ in terms of $\mathcal{F}(0, c, e; b, d)$, and then go to **Step-B**.
- **Step-B**: By means of Propositions 2 and 3, express $\mathcal{F}(0, c, e; b, d)$ in terms of $\mathcal{F}(0, c, e; b, b)$, and then go to **Step-C**.
- **Step-C**: In virtu of Propositions 4 and 5, express $\mathcal{F}(0, c, e; b, b)$ in terms of $\mathcal{F}(0, c, c; b, b)$, and then go to **Step-D**.
- **Step-D**: Finally by applying Theorems 1 and 2, evaluate $\mathcal{F}(0, c, c; b, b)$ explicitly in terms of the Ω-series.

Therefore, we have validated the conclusive theorem as below.

Theorem 3 (Conclusion)**.** *For any quintuple integers*

$$a, b, c, d, e \in \mathbb{Z} \quad \textit{subject to} \quad a \geq 0,\ c > 0 \quad \textit{and} \quad \sigma = b + d - a - c - e > 0,$$

the nonterminating $\mathcal{F}(a, c, e; b, d)$ *series can always be evaluated by finitely linear sums of trigonometric function* $\cos(\pi x)$ *and hyperbolic function* $\cosh\left(2x \ln(1 + \sqrt{2})\right)$*, where* $x \in \mathbb{Z}$ *and the coefficients are rational numbers.*

According to the afore-described procedure, we have written appropriate *Mathematica* commands to determine explicitly closed form expressions for $\mathcal{F}(a, c, e; b, d)$ series. Twenty summation formulae are displayed below, where the argument "1" will be suppressed from the notation of ${}_3F_2$-series for the sake of brevity. We shall call these series "Class-A". Among them, an equivalent form of **A5** has been obtained by Campbell and Abrarov ([11] Equation (18)).

A1. $${}_3F_2\left[1,\ 1,\ \tfrac{1}{2};\ \tfrac{5}{4},\ \tfrac{7}{4}\right] = \tfrac{3}{\sqrt{2}} \log(1 + \sqrt{2}).$$

A2. $${}_3F_2\left[1,\ 1,\ \tfrac{1}{2};\ \tfrac{7}{4},\ \tfrac{9}{4}\right] = -5\left\{1 - \sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A3. $${}_3F_2\left[1,\ 1,\ \tfrac{1}{2};\ \tfrac{5}{4},\ \tfrac{11}{4}\right] = \tfrac{-7}{15}\left\{1 - 3\sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A4. $${}_3F_2\left[1,\ 1,\ \tfrac{3}{2};\ \tfrac{5}{4},\ \tfrac{11}{4}\right] = \tfrac{7}{12}\left\{2 + 3\sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A5. $${}_3F_2\left[1,\ 1,\ \tfrac{3}{2};\ \tfrac{7}{4},\ \tfrac{9}{4}\right] = \tfrac{15}{4}\left\{2 - \sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A6. $${}_3F_2\left[1,\ 1,\ \tfrac{3}{2};\ \tfrac{9}{4},\ \tfrac{11}{4}\right] = \tfrac{35}{6}\left\{4 - 3\sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A7. $${}_3F_2\left[1,\ 1,\ -\tfrac{1}{2};\ \tfrac{3}{4},\ \tfrac{5}{4}\right] = \tfrac{1}{3}\left\{1 - \sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A8. $${}_3F_2\left[1,\ 1,\ -\tfrac{1}{2};\ \tfrac{1}{4},\ \tfrac{7}{4}\right] = \tfrac{3}{5}\left\{1 - 3\sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A9. $${}_3F_2\left[1,\ 1,\ -\tfrac{1}{2};\ \tfrac{7}{4},\ \tfrac{9}{4}\right] = \tfrac{-3}{7}\left\{3 - 4\sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A10. $${}_3F_2\left[1,\ 1,\ -\tfrac{3}{2};\ \tfrac{3}{4},\ \tfrac{5}{4}\right] = \tfrac{1}{35}\left\{3 - 4\sqrt{2}\log(1 + \sqrt{2})\right\}.$$

A11. ${}_3F_2\left[1,\ 2,\ \tfrac{1}{2};\ \tfrac{7}{4},\ \tfrac{9}{4}\right] = \tfrac{5}{8}\left\{2+\sqrt{2}\log(1+\sqrt{2})\right\}.$
A12. ${}_3F_2\left[1,\ 2,\ -\tfrac{1}{2};\ \tfrac{5}{4},\ \tfrac{7}{4}\right] = \tfrac{3}{20}\left\{2-\sqrt{2}\log(1+\sqrt{2})\right\}.$
A13. ${}_3F_2\left[1,\ 2,\ -\tfrac{1}{2};\ \tfrac{3}{4},\ \tfrac{9}{4}\right] = \tfrac{5}{84}\left\{2-5\sqrt{2}\log(1+\sqrt{2})\right\}.$
A14. ${}_3F_2\left[1,\ 2,\ -\tfrac{1}{2};\ \tfrac{7}{4},\ \tfrac{9}{4}\right] = \tfrac{1}{14}\left\{8+\sqrt{2}\log(1+\sqrt{2})\right\}.$
A15. ${}_3F_2\left[1,\ 2,\ -\tfrac{3}{2};\ \tfrac{3}{4},\ \tfrac{9}{4}\right] = \tfrac{-5}{77}\left\{1+\sqrt{2}\log(1+\sqrt{2})\right\}.$
A16. ${}_3F_2\left[2,\ 2,\ \tfrac{1}{2};\ \tfrac{7}{4},\ \tfrac{13}{4}\right] = \tfrac{135}{224}\left\{6-\sqrt{2}\log(1+\sqrt{2})\right\}.$
A17. ${}_3F_2\left[2,\ 2,\ -\tfrac{1}{2};\ \tfrac{5}{4},\ \tfrac{11}{4}\right] = \tfrac{7}{48}\left\{2-3\sqrt{2}\log(1+\sqrt{2})\right\}.$
A18. ${}_3F_2\left[2,\ 2,\ -\tfrac{1}{2};\ \tfrac{9}{4},\ \tfrac{11}{4}\right] = \tfrac{1}{24}\left\{2+9\sqrt{2}\log(1+\sqrt{2})\right\}.$
A19. ${}_3F_2\left[2,\ 2,\ -\tfrac{3}{2};\ \tfrac{7}{4},\ \tfrac{13}{4}\right] = \tfrac{1}{22}\left\{8-3\sqrt{2}\log(1+\sqrt{2})\right\}.$
A20. ${}_3F_2\left[2,\ 2,\ -\tfrac{3}{2};\ \tfrac{11}{4},\ \tfrac{13}{4}\right] = \tfrac{-1}{13}\left\{13-15\sqrt{2}\log(1+\sqrt{2})\right\}.$

3. The Thomae and Kummer Transformations

In the classical theory of hypergeometric series, the Thomae and Kummer transformations are fundamental (cf. Bailey [1] §3.2 and Page 98 , where $\sigma = b+d-a-c-e$):

$${}_3F_2\left[\begin{matrix}a,c,e\\ b,d\end{matrix}\,\middle|\,1\right] = {}_3F_2\left[\begin{matrix}\sigma,b-a,d-a\\ c+\sigma,e+\sigma\end{matrix}\,\middle|\,1\right]\Gamma\left[\begin{matrix}\sigma,\quad b,\quad d\\ a,c+\sigma,e+\sigma\end{matrix}\right] \tag{3}$$

$${}_3F_2\left[\begin{matrix}a,c,e\\ b,d\end{matrix}\,\middle|\,1\right] = {}_3F_2\left[\begin{matrix}a,b-c,b-e\\ \sigma+a,\ b\end{matrix}\,\middle|\,1\right]\Gamma\left[\begin{matrix}\sigma,\quad d\\ \sigma+a,d-a\end{matrix}\right]. \tag{4}$$

They will be applied to the $\mathcal{F}$-series to evaluete six classes of exotic ${}_3F_2$-series.

3.1. Class B

Applying the Kummer transformation (4), we can express the following "Class-B" series in terms of the $\mathcal{F}$-series (where $\sigma = b+d-a-c-e$):

$${}_3F_2\left[\begin{matrix}1+a,\quad c+\frac{1}{4},\quad e+\frac{3}{4}\\ b+\frac{3}{2},\quad d+\frac{5}{4}\end{matrix}\,\middle|\,1\right] = \Gamma\left[\begin{matrix}b+\frac{3}{2},\sigma+\frac{3}{4}\\ b-a+\frac{1}{2},\sigma+a+\frac{7}{4}\end{matrix}\right] \times {}_3F_2\left[\begin{matrix}1+a,\quad d-c+1,\quad d-e+\frac{1}{2}\\ d+\frac{5}{4},\quad \sigma+a+\frac{7}{4}\end{matrix}\,\middle|\,1\right].$$

Then we can derive the following closed formulae for these series (except for divergent series) from those displayed in "Class A".

B1. ${}_3F_2\left[1,\ \tfrac{1}{4},\ \tfrac{3}{4};\ \tfrac{3}{2},\ \tfrac{5}{4}\right] = \sqrt{2}\log(1+\sqrt{2}).$
B2. ${}_3F_2\left[1,\ \tfrac{1}{4},\ \tfrac{7}{4};\ \tfrac{5}{2},\ \tfrac{5}{4}\right] = \tfrac{2}{5}\left\{1+2\sqrt{2}\log(1+\sqrt{2})\right\}.$
B3. ${}_3F_2\left[1,\ \tfrac{1}{4},\ \tfrac{7}{4};\ \tfrac{5}{2},\ \tfrac{9}{4}\right] = \tfrac{3}{2}\left\{2-\sqrt{2}\log(1+\sqrt{2})\right\}.$
B4. ${}_3F_2\left[1,\ \tfrac{3}{4},\ \tfrac{5}{4};\ \tfrac{3}{2},\ \tfrac{9}{4}\right] = \tfrac{5}{2}\left\{2-\sqrt{2}\log(1+\sqrt{2})\right\}.$
B5. ${}_3F_2\left[1,\ \tfrac{3}{4},\ \tfrac{5}{4};\ \tfrac{5}{2},\ \tfrac{9}{4}\right] = 5\left\{4-3\sqrt{2}\log(1+\sqrt{2})\right\}.$
B6. ${}_3F_2\left[1,\ \tfrac{7}{4},\ \tfrac{9}{4};\ \tfrac{5}{2},\ \tfrac{13}{4}\right] = \tfrac{9}{5}\left\{8-5\sqrt{2}\log(1+\sqrt{2})\right\}.$
B7. ${}_3F_2\left[2,\ \tfrac{1}{4},\ \tfrac{7}{4};\ \tfrac{7}{2},\ \tfrac{5}{4}\right] = \tfrac{2}{9}\left\{4+3\sqrt{2}\log(1+\sqrt{2})\right\}.$
B8. ${}_3F_2\left[2,\ \tfrac{3}{4},\ \tfrac{5}{4};\ \tfrac{5}{2},\ \tfrac{9}{4}\right] = \tfrac{5}{4}\left\{-2+3\sqrt{2}\log(1+\sqrt{2})\right\}.$
B9. ${}_3F_2\left[2,\ \tfrac{5}{4},\ \tfrac{7}{4};\ \tfrac{7}{2},\ \tfrac{9}{4}\right] = 5\left\{2-\sqrt{2}\log(1+\sqrt{2})\right\}.$
B10. ${}_3F_2\left[2,\ \tfrac{9}{4},\ \tfrac{11}{4};\ \tfrac{9}{2},\ \tfrac{13}{4}\right] = 30\left\{4-3\sqrt{2}\log(1+\sqrt{2})\right\}.$

3.2. Class C

By means of the Kummer transformation (4), we can express the "Class-C" series below in terms of the $\mathcal{F}$-series (where $\sigma = b + d - a - c - e$):

$$ {}_3F_2\left[\begin{matrix} 1+a, & c+\frac{1}{4}, & e+\frac{3}{4} \\ & b+\frac{3}{2}, & d+\frac{3}{4} \end{matrix}\middle| 1\right] = \Gamma\left[\begin{matrix} \sigma+\frac{1}{4}, b+\frac{3}{2} \\ \sigma+a+\frac{5}{4}, b-a+\frac{1}{2} \end{matrix}\right] \times {}_3F_2\left[\begin{matrix} 1+a, & d-e, & d-c+\frac{1}{2} \\ & d+\frac{3}{4}, & \sigma+a+\frac{5}{4} \end{matrix}\middle| 1\right]. $$

Then the closed formulae below for these series (except for divergent series) follow directly from those recorded in "Class A".

C1. ${}_3F_2\left[1, \frac{1}{4}, \frac{3}{4}; \frac{3}{2}, \frac{7}{4}\right] = \frac{3}{2}\left\{2 - \sqrt{2}\log(1+\sqrt{2})\right\}.$
C2. ${}_3F_2\left[1, \frac{1}{4}, \frac{3}{4}; \frac{5}{2}, \frac{7}{4}\right] = \frac{3}{5}\left\{8 - 5\sqrt{2}\log(1+\sqrt{2})\right\}.$
C3. ${}_3F_2\left[1, \frac{3}{4}, \frac{5}{4}; \frac{3}{2}, \frac{7}{4}\right] = 3\sqrt{2}\log(1+\sqrt{2}).$
C4. ${}_3F_2\left[1, \frac{3}{4}, \frac{9}{4}; \frac{5}{2}, \frac{7}{4}\right] = \frac{6}{5}\left\{1 + 2\sqrt{2}\log(1+\sqrt{2})\right\}.$
C5. ${}_3F_2\left[1, \frac{5}{4}, -\frac{1}{4}; \frac{3}{2}, \frac{3}{4}\right] = \frac{2}{3}\left\{1 - \sqrt{2}\log(1+\sqrt{2})\right\}.$
C6. ${}_3F_2\left[1, \frac{5}{4}, -\frac{1}{4}; \frac{3}{2}, \frac{7}{4}\right] = \frac{1}{4}\left\{2 + \sqrt{2}\log(1+\sqrt{2})\right\}.$
C7. ${}_3F_2\left[1, \frac{5}{4}, -\frac{1}{4}; \frac{5}{2}, \frac{3}{4}\right] = \frac{2}{7}\left\{5 - 2\sqrt{2}\log(1+\sqrt{2})\right\}.$
C8. ${}_3F_2\left[2, \frac{3}{4}, \frac{5}{4}; \frac{5}{2}, \frac{7}{4}\right] = \frac{3}{2}\left\{2 + \sqrt{2}\log(1+\sqrt{2})\right\}.$
C9. ${}_3F_2\left[2, \frac{3}{4}, \frac{9}{4}; \frac{7}{2}, \frac{7}{4}\right] = \frac{6}{7}\left\{8 + \sqrt{2}\log(1+\sqrt{2})\right\}.$
C10. ${}_3F_2\left[2, \frac{3}{4}, \frac{13}{4}; \frac{7}{2}, \frac{11}{4}\right] = \frac{1}{2}\left\{2 + 9\sqrt{2}\log(1+\sqrt{2})\right\}.$

3.3. Class D

By virtue of the Thomae transformation (3), we can express the following "Class-D" series in terms of the $\mathcal{F}$-series (where $\sigma = b + d - a - c - e$):

$$ {}_3F_2\left[\begin{matrix} a+\frac{1}{2}, & c+\frac{1}{4}, & e+\frac{3}{4} \\ & b+1, & d+\frac{1}{2} \end{matrix}\middle| 1\right] = \Gamma\left[\begin{matrix} \sigma, & b+1, & d+\frac{1}{2} \\ a+\frac{1}{2}, \sigma+c+\frac{1}{4}\sigma+e+\frac{3}{4} \end{matrix}\right] \times {}_3F_2\left[\begin{matrix} \sigma, & d-a, & b-a+\frac{1}{2} \\ & \sigma+c+\frac{1}{4}, & \sigma+e+\frac{3}{4} \end{matrix}\middle| 1\right]. \tag{5} $$

Then we find the closed formulae below for these series (except for divergent series) as consequences of those produced in "Class A".

D1. ${}_3F_2\left[\frac{1}{4}, \frac{3}{4}, \frac{1}{2}; 1, \frac{3}{2}\right] = \frac{4\log(1+\sqrt{2})}{\pi}.$
D2. ${}_3F_2\left[\frac{1}{4}, \frac{7}{4}, \frac{1}{2}; 2, \frac{3}{2}\right] = \frac{8\left\{\sqrt{2}+3\log(1+\sqrt{2})\right\}}{9\pi}.$
D3. ${}_3F_2\left[\frac{1}{4}, \frac{7}{4}, \frac{1}{2}; 1, \frac{5}{2}\right] = \frac{2\left\{\sqrt{2}+9\log(1+\sqrt{2})\right\}}{5\pi}.$
D4. ${}_3F_2\left[\frac{1}{4}, \frac{7}{4}, \frac{3}{2}; 1, \frac{7}{2}\right] = \frac{8\left\{2\sqrt{2}+3\log(1+\sqrt{2})\right\}}{9\pi}.$
D5. ${}_3F_2\left[\frac{3}{4}, \frac{5}{4}, \frac{1}{2}; 1, \frac{5}{2}\right] = \frac{2\left\{\sqrt{2}+\log(1+\sqrt{2})\right\}}{\pi}.$

D6. $${}_3F_2\left[\tfrac{3}{4},\quad \tfrac{5}{4},\quad \tfrac{1}{2};\quad 2,\quad \tfrac{3}{2}\right] = \frac{8\left\{\sqrt{2}-\log(1+\sqrt{2})\right\}}{\pi}.$$

D7. $${}_3F_2\left[\tfrac{3}{4},\quad \tfrac{5}{4},\quad \tfrac{3}{2};\quad 1,\quad \tfrac{7}{2}\right] = \frac{8\left\{4\sqrt{2}+\log(1+\sqrt{2})\right\}}{7\pi}.$$

D8. $${}_3F_2\left[\tfrac{5}{4},\quad -\tfrac{1}{4},\quad \tfrac{1}{2};\quad 1,\quad \tfrac{3}{2}\right] = \frac{4\left\{2\sqrt{2}-\log(1+\sqrt{2})\right\}}{3\pi}.$$

D9. $${}_3F_2\left[\tfrac{5}{4},\quad -\tfrac{1}{4},\quad \tfrac{3}{2};\quad 1,\quad \tfrac{5}{2}\right] = \frac{4\left\{5\sqrt{2}-4\log(1+\sqrt{2})\right\}}{7\pi}.$$

D10. $${}_3F_2\left[\tfrac{5}{4},\quad \tfrac{7}{4},\quad \tfrac{3}{2};\quad 2,\quad \tfrac{7}{2}\right] = \frac{16\left\{\sqrt{2}-\log(1+\sqrt{2})\right\}}{\pi}.$$

Observing that the parameter excess of the ${}_3F_2$-series displayed on the right hand side of (5) equals $\Delta = \frac{1}{2} + a$, the equality (5) valid only when $a \geq 0$ and $\sigma \geq 0$. It remains a problem to evaluate, for $a < 0$, the ${}_3F_2$-series on the left of (5). This can also be resolved by the linearization method.

According to the Pfaff–Saalschütz summation theorem (cf. Bailey [1] §2.2), it is not hard to confirm the linear relation in the following lemma.

Lemma 3 (Linear relation: $m \in \mathbb{N}_0$)**.**

$$(A+n)_m = \sum_{k=0}^{m} \langle n \rangle_k (B+n)_{m-k} X_k, \quad \textit{where} \quad X_k = (-1)^k \binom{m}{k} \frac{(A)_m (A-B)_k}{(B)_m (A)_k}.$$

By specializing this to the equality

$$(1+b+n)_{-a} = \sum_{k=0}^{-a} \langle n \rangle_k (\tfrac{1}{2}+a+n)_{-a-k} X_a^k,$$
$$\text{where} \quad X_k(a) = \frac{(-a)!}{\left\langle -\frac{1}{2} \right\rangle_{-a}} \binom{a-b-\frac{1}{2}}{k} \binom{b-a}{-a-k}$$

and then substituting it into the ${}_3F_2$-series, we may manipulate the double sum

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} \frac{1}{2}+a, & \frac{1}{4}+c, & \frac{3}{4}+e \\ & 1+b, & \frac{1}{2}+d \end{matrix} \,\middle|\, 1\right] \\
&= \sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{2}+a, \frac{1}{4}+c, \frac{3}{4}+e \\ 1,\ 1+b, \frac{1}{2}+d \end{matrix}\right]_n \sum_{k=0}^{-a} \frac{\langle n \rangle_k (\frac{1}{2}+a+n)_{-a-k}}{(1+b+n)_{-a}} X_k(a) \\
&= \sum_{k=0}^{-a} \frac{(\frac{1}{2}+a)_{-a-k}}{(1+b)_{-a}} X_k(a) \sum_{n=0}^{\infty} \frac{\langle n \rangle_k}{n!} \left[\begin{matrix} \frac{1}{2}-k, \frac{1}{4}+c, \frac{3}{4}+e \\ 1-a+b, \frac{1}{2}+d \end{matrix}\right]_n.
\end{aligned}$$

Performing the replacement $n \to n+k$, we can express the last sum with respect to n as

$$\left[\begin{matrix} \frac{1}{2}-k, \frac{1}{4}+c, \frac{3}{4}+e \\ 1-a+b, \frac{1}{2}+d \end{matrix}\right]_k {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{4}+c+k, \frac{3}{4}+e+k \\ 1-a+b+k, \frac{1}{2}+d+k \end{matrix} \,\middle|\, 1\right].$$

Therefore, we have established, after some simplifications, the following transformation formula.

Theorem 4 (Reduction formula from $a<0$ to $a=0$)**.**

$$
{}_3F_2\left[\begin{matrix}\frac{1}{2}+a, & \frac{1}{4}+c, & \frac{3}{4}+e\\ & 1+b, & \frac{1}{2}+d\end{matrix}\,\Big|\,1\right]=\sum_{k=0}^{-a}\left[\begin{matrix}a,\frac{1}{2}-a+b,\frac{1}{4}+c,\frac{3}{4}+e\\ 1,\,1-a+b,1+b,\frac{1}{2}+d\end{matrix}\right]_k
\times{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{4}+c+k,\frac{3}{4}+e+k\\ 1-a+b+k,\frac{1}{2}+d+k\end{matrix}\,\Big|\,1\right].
$$

It should be emphasized that under this transformation, the parameter excess $\Delta=\sigma=b+d-a-c-e$ remains invariant for all the ${}_3F_2$-series. However the ${}_3F_2$-series on the right belongs to **Class-D** and can therefore be evaluated by (5). Ten more formulae are recorded below.

D11. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{1}{4},\ -\frac{1}{4};\ 1,\ \frac{3}{2}\right]=\frac{13\sqrt{2}+\log(1+\sqrt{2})}{6\pi}.$

D12. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4};\ 1,\ \frac{1}{2}\right]=\frac{\sqrt{2}+\log(1+\sqrt{2})}{\pi}.$

D13. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4};\ 1,\ \frac{3}{2}\right]=\frac{\sqrt{2}+5\log(1+\sqrt{2})}{2\pi}.$

D14. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{1}{4},\ \frac{7}{4};\ 1,\ \frac{3}{2}\right]=\frac{5\sqrt{2}+9\log(1+\sqrt{2})}{6\pi}.$

D15. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{3}{4},\ \frac{5}{4};\ 1,\ \frac{3}{2}\right]=\frac{3\sqrt{2}-\log(1+\sqrt{2})}{2\pi}.$

D16. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{3}{4},\ -\frac{3}{4};\ 1,\ \frac{1}{2}\right]=\frac{5\sqrt{2}+9\log(1+\sqrt{2})}{3\pi}.$

D17. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{3}{4},\ -\frac{3}{4};\ 2,\ \frac{1}{2}\right]=\frac{34\sqrt{2}+42\log(1+\sqrt{2})}{21\pi}.$

D18. $\quad {}_3F_2\left[-\frac{1}{2},\ \frac{5}{4},\ \frac{7}{4};\ 2,\ \frac{3}{2}\right]=\frac{7\sqrt{2}+3\ln(1+\sqrt{2})}{9\pi}.$

D19. $\quad {}_3F_2\left[-\frac{1}{2},\ -\frac{3}{4},\ \frac{7}{4};\ 1\ \ \frac{3}{2}\right]=\frac{43\sqrt{2}+87\log(1+\sqrt{2})}{30\pi}.$

D20. $\quad {}_3F_2\left[-\frac{3}{2},\ -\frac{1}{4},\ -\frac{3}{4};\ 1,\ \frac{1}{2}\right]=\frac{31\sqrt{2}-37\log(1+\sqrt{2})}{8\pi}.$

Campbell, D'Aurizio and Sondow [9,10,22] discovered some formulae in **Class-D**.

- The formula **D1** has been found by them in ([9] Equation (10)), where they also conjectured **D12**. For this last evaluation, five different proofs have been provided by the same authors [10].
- By making use of beta integrals, Campbell recoded in ([22] Theorems 2,3,7 and Example 12) four formulae. The first one ([22] Theorem 2) is corrected by **D18**. The second one ([22] Theorem 3) is incorrect. The third one ([22] Theorem 7) is simplified by **D2**. The fourth one ([22] Example 12) is too complicated to reproduce here.

3.4. Class E

Again in view of the Thomae transformation (3), we can express the "Class-E" series below in terms of the $\mathcal{F}$-series (where $\sigma=b+d-a-c-e$):

$$
{}_3F_2\left[\begin{matrix}a+\frac{1}{2}, & c+\frac{1}{4}, & e+\frac{3}{4}\\ & b+\frac{1}{2}, & d+\frac{3}{2}\end{matrix}\,\Big|\,1\right]=\Gamma\left[\begin{matrix}\sigma+\frac{1}{2},b+\frac{1}{2},d+\frac{3}{2}\\ a+\frac{1}{2},\sigma+c+\frac{3}{4},\sigma+e+\frac{5}{4}\end{matrix}\right]
\times{}_3F_2\left[\begin{matrix}1+d-a, & b-a, & \sigma+\frac{1}{2}\\ & \sigma+c+\frac{3}{4}, & \sigma+e+\frac{5}{4}\end{matrix}\,\Big|\,1\right]. \tag{6}
$$

Consequently, the closed formulae below for these series (except for divergent series) can be deduced from those exhibited in "Class A". Among them, **E2** simplifies a formula of Campbell ([22] Example 5).

$$\begin{array}{lllllll}
\textbf{E1}. & {}_3F_2\big[\frac14, & \frac34, & \frac12; & \frac32, & \frac32\big] & = 2\sqrt{2} - 2\log(1+\sqrt{2}). \\
\textbf{E2}. & {}_3F_2\big[\frac14, & \frac74, & \frac12; & \frac32, & \frac32\big] & = \frac13\big\{4\sqrt{2} - 2\log(1+\sqrt{2})\big\}. \\
\textbf{E3}. & {}_3F_2\big[\frac14, & \frac74, & \frac32; & \frac52, & \frac52\big] & = \frac{12}{25}\big\{8\sqrt{2} - 10\log(1+\sqrt{2})\big\}. \\
\textbf{E4}. & {}_3F_2\big[\frac34, & \frac54, & \frac12; & \frac32, & \frac32\big] & = 2\log(1+\sqrt{2}). \\
\textbf{E5}. & {}_3F_2\big[\frac34, & \frac54, & \frac32; & \frac52, & \frac52\big] & = 4\big\{4\sqrt{2} - 6\log(1+\sqrt{2})\big\}. \\
\textbf{E6}. & {}_3F_2\big[\frac34, & \frac94, & \frac12; & \frac32, & \frac52\big] & = \frac15\big\{\sqrt{2} + 9\log(1+\sqrt{2})\big\}. \\
\textbf{E7}. & {}_3F_2\big[\frac54, & \frac74, & \frac12; & \frac32, & \frac52\big] & = \sqrt{2} + \log(1+\sqrt{2}). \\
\textbf{E8}. & {}_3F_2\big[\frac54, & \frac{11}{4}, & \frac12; & \frac52, & \frac52\big] & = \frac{9}{14}\big\{3\sqrt{2} - \log(1+\sqrt{2})\big\}. \\
\textbf{E9}. & {}_3F_2\big[\frac74, & \frac94, & \frac32; & \frac52, & \frac72\big] & = 12\big\{\sqrt{2} - \log(1+\sqrt{2})\big\}. \\
\textbf{E10}. & {}_3F_2\big[\frac94, & \frac{11}{4}, & \frac52; & \frac72, & \frac92\big] & = 40\big\{4\sqrt{2} - 6\log(1+\sqrt{2})\big\}.
\end{array}$$

Analogous to the series in **Class-D**, the parameter excess of the ${}_3F_2$-series displayed on the right hand side of (6) equals $\Delta = \frac12 + a$, which converges only when $a \geq 0$. We can also evaluate that ${}_3F_2$-series by reducing the case $a < 0$ to $a = 0$.

By means of Lemma 3, we have the equality

$$\left(\tfrac12 + b + n\right)_{-a} = \sum_{k=0}^{-a} \langle n\rangle_k \left(\tfrac12 + a + n\right)_{-a-k} X_a^k,$$
$$\text{where}\quad X_k(a) = \frac{(-a)!}{\left\langle -\frac12 \right\rangle_{-a}} \binom{a-b}{k}\binom{b-a-\frac12}{-a-k}$$

and then insert it in the ${}_3F_2$-series, we can handle the double sum

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}\frac12+a, & \frac14+c, & \frac34+e \\ & \frac12+b, & \frac32+d\end{matrix}\,\middle|\, 1\right] \\
&= \sum_{n=0}^{\infty}\left[\begin{matrix}\frac12+a, \frac14+c, \frac34+e \\ 1,\ \frac12+b, \frac32+d\end{matrix}\right]_n \sum_{k=0}^{-a} \frac{\langle n\rangle_k (\frac12+a+n)_{-a-k}}{(\frac12+b+n)_{-a}} X_k(a) \\
&= \sum_{k=0}^{-a} \frac{(\frac12+a)_{-a-k}}{(\frac12+b)_{-a}} X_k(a) \sum_{n=0}^{\infty} \frac{\langle n\rangle_k}{n!}\left[\begin{matrix}\frac12-k, \frac14+c, \frac34+e \\ \frac12-a+b, \frac32+d\end{matrix}\right]_n.
\end{aligned}$$

Making the replacement $n \to n+k$, we can express the last sum as

$$\left[\begin{matrix}\frac12-k, \frac14+c, \frac34+e \\ \frac12-a+b, \frac32+d\end{matrix}\right]_k {}_3F_2\left[\begin{matrix}\frac12, \frac14+c+k, \frac34+e+k \\ \frac12-a+b+k, \frac32+d+k\end{matrix}\,\middle|\, 1\right].$$

After some simplifications, we establish the transformation below.

Theorem 5 (Reduction formula from $a < 0$ to $a = 0$)**.**

$$\begin{aligned}
{}_3F_2\left[\begin{matrix}\frac12+a, & \frac14+c, & \frac34+e \\ & \frac12+b, & \frac32+d\end{matrix}\,\middle|\, 1\right] &= \sum_{k=0}^{-a}\left[\begin{matrix}a, b-a, \frac14+c, \frac34+e \\ 1,\ \frac12-a+b, \frac12+b, \frac32+d\end{matrix}\right]_k \\
&\times {}_3F_2\left[\begin{matrix}\frac12, \frac14+c+k, \frac34+e+k \\ \frac12-a+b+k, \frac32+d+k\end{matrix}\,\middle|\, 1\right].
\end{aligned}$$

Under this transformation, the parameter excess $\Delta = \sigma = b + d - a - c - e$ remains invariant for all the ${}_3F_2$-series involved. However the ${}_3F_2$-series on the right belongs to **Class-E** and can therefore be evaluated by (6). We record ten more examples.

E11. $_3F_2\left[-\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}; \tfrac{1}{2}, \tfrac{3}{2}\right] = \frac{6-\sqrt{2}\log(1+\sqrt{2})}{4\sqrt{2}}.$

E12. $_3F_2\left[-\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{7}{4}; \tfrac{1}{2}, \tfrac{3}{2}\right] = \frac{14-5\sqrt{2}\log(1+\sqrt{2})}{12\sqrt{2}}.$

E13. $_3F_2\left[-\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{4}; \tfrac{1}{2}, \tfrac{3}{2}\right] = \frac{2-3\sqrt{2}\log(1+\sqrt{2})}{4\sqrt{2}}.$

E14. $_3F_2\left[-\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{4}; \tfrac{3}{2}, \tfrac{3}{2}\right] = \frac{2+5\sqrt{2}\log(1+\sqrt{2})}{8\sqrt{2}}.$

E15. $_3F_2\left[-\tfrac{1}{2}, \tfrac{5}{4}, \tfrac{7}{4}; \tfrac{1}{2}, \tfrac{5}{2}\right] = \frac{3\left\{2+5\sqrt{2}\log(1+\sqrt{2})\right\}}{-16\sqrt{2}}.$

E16. $_3F_2\left[-\tfrac{1}{2}, \tfrac{5}{4}, \tfrac{7}{4}; \tfrac{3}{2}, \tfrac{3}{2}\right] = \frac{10-7\sqrt{2}\log(1+\sqrt{2})}{24\sqrt{2}}.$

E17. $_3F_2\left[-\tfrac{3}{2}, \tfrac{3}{4}, \tfrac{5}{4}; \tfrac{1}{2}, \tfrac{3}{2}\right] = \frac{10-39\sqrt{2}\log(1+\sqrt{2})}{128\sqrt{2}}.$

E18. $_3F_2\left[-\tfrac{3}{2}, \tfrac{3}{4}, \tfrac{5}{4}; \tfrac{1}{2}, \tfrac{5}{2}\right] = \frac{3\left\{62-37\sqrt{2}\log(1+\sqrt{2})\right\}}{256\sqrt{2}}.$

E19. $_3F_2\left[-\tfrac{3}{2}, \tfrac{5}{4}, \tfrac{7}{4}; \tfrac{3}{2}, \tfrac{3}{2}\right] = \frac{62-37\sqrt{2}\log(1+\sqrt{2})}{512\sqrt{2}}.$

E20. $_3F_2\left[-\tfrac{3}{2}, \tfrac{5}{4}, -\tfrac{1}{4}; \tfrac{1}{2}, \tfrac{3}{2}\right] = \frac{3\left\{42+41\sqrt{2}\log(1+\sqrt{2})\right\}}{128\sqrt{2}}.$

3.5. Class F

By invoking the Kummer transformation (4), we can express the "Class-F" series below in terms of the $\mathcal{F}$-series (where $\sigma = b + d - a - c - e$):

$$
{}_3F_2\left[\begin{matrix} a+\frac{1}{2}, & c+\frac{3}{4}, & e+\frac{3}{4} \\ & b+1, & d+\frac{7}{4} \end{matrix}\,\middle|\, 1\right] = \Gamma\left[\begin{matrix} b+1, \sigma+\frac{3}{4} \\ b-a+\frac{1}{2}, \sigma+a+\frac{5}{4} \end{matrix}\right] \times {}_3F_2\left[\begin{matrix} 1+d-c, & 1+d-e, & a+\frac{1}{2} \\ & \sigma+a+\frac{5}{4}, & d+\frac{7}{4} \end{matrix}\,\middle|\, 1\right].
$$

Then the closed formulae below for these series (except for divergent series) can be established from those shown in "Class A". Among them, the formula **F10** is due to Campbell and Abrarov ([11] Corollary 5).

F1. $_3F_2\left[-\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{4}; 1, \tfrac{3}{4}\right] = \frac{2\sqrt{\pi}\left\{5\sqrt{2}-4\log(1+\sqrt{2})\right\}}{\Gamma(\frac{1}{4})^2}.$

F2. $_3F_2\left[-\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{4}; 1, \tfrac{7}{4}\right] = \frac{3\sqrt{\pi}\left\{8\sqrt{2}+2\log(1+\sqrt{2})\right\}}{5\Gamma(\frac{1}{4})^2}.$

F3. $_3F_2\left[-\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{3}{4}; 1, \tfrac{7}{4}\right] = \frac{6\sqrt{\pi}\left\{\sqrt{2}+4\log(1+\sqrt{2})\right\}}{5\Gamma(\frac{1}{4})^2}.$

F4. $_3F_2\left[-\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{4}; 1, \tfrac{11}{4}\right] = \frac{7\sqrt{\pi}\left\{4\sqrt{2}+6\log(1+\sqrt{2})\right\}}{15\Gamma(\frac{1}{4})^2}.$

F5. $_3F_2\left[\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{4}; 1, \tfrac{3}{4}\right] = \frac{2\sqrt{\pi}\left\{4\sqrt{2}-2\log(1+\sqrt{2})\right\}}{\Gamma(\frac{1}{4})^2}.$

F6. $_3F_2\left[\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{4}; 1, \tfrac{7}{4}\right] = \frac{9\sqrt{\pi}\left\{3\sqrt{2}-\log(1+\sqrt{2})\right\}}{4\Gamma(\frac{1}{4})^2}.$

F7. $_3F_2\left[\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{4}; 1, \tfrac{7}{4}\right] = \frac{3\sqrt{\pi}\left\{\sqrt{2}+\log(1+\sqrt{2})\right\}}{\Gamma(\frac{1}{4})^2}.$

F8. $_3F_2\left[\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{3}{4}; 1, \tfrac{7}{4}\right] = \frac{12\sqrt{\pi}\log(1+\sqrt{2})}{\Gamma(\frac{1}{4})^2}.$

F9. $_3F_2\left[\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{4}; 1, \tfrac{11}{4}\right] = \frac{7\sqrt{\pi}\left\{\sqrt{2}+9\log(1+\sqrt{2})\right\}}{5\Gamma(\frac{1}{4})^2}.$

F10 $_3F_2\left[\tfrac{3}{2}, \tfrac{3}{4}, -\tfrac{1}{4}; 1, \tfrac{7}{4}\right] = \frac{3\sqrt{\pi}\left\{3\sqrt{2}-\log(1+\sqrt{2})\right\}}{2\Gamma(\frac{1}{4})^2}.$

3.6. Class G

Finally, by employing the Kummer transformation (4), we can express the "Class-G" series below in terms of the $\mathcal{F}$-series (where $\sigma = b + d - a - c - e$):

$$ {}_3F_2\left[\begin{matrix} a+\frac{1}{2}, & c+\frac{1}{4}, & e+\frac{1}{4} \\ & b+1, & d+\frac{1}{4} \end{matrix} \middle| 1\right] = \Gamma\left[\begin{matrix} b+1, \sigma+\frac{1}{4} \\ b-a+\frac{1}{2}, \sigma+a+\frac{3}{4} \end{matrix}\right] \times {}_3F_2\left[\begin{matrix} d-c, & d-e, & a+\frac{1}{2} \\ & d+\frac{1}{4}, & \sigma+a+\frac{3}{4} \end{matrix} \middle| 1\right]. $$

Then the closed formulae below for these series (except for divergent series) can be shown from those displayed in "Class A". Among them, the formula **G8** is due to Campbell and Abrarov ([11] Corollary 4), who evaluated also another similar series ([11] Corollary 6).

G1. $\;{}_3F_2\left[-\frac{3}{2}, \frac{5}{4}, \frac{5}{4}; 1, \frac{13}{4}\right] = \frac{5\sqrt{\pi}\left\{4\sqrt{2}-3\log(1+\sqrt{2})\right\}}{44\Gamma(\frac{3}{4})^2}.$

G2. $\;{}_3F_2\left[-\frac{1}{2}, \frac{1}{4}, -\frac{3}{4}; 1, \frac{5}{4}\right] = \frac{\sqrt{\pi}\left\{2\sqrt{2}+3\log(1+\sqrt{2})\right\}}{6\Gamma(\frac{3}{4})^2}.$

G3. $\;{}_3F_2\left[\frac{1}{2}, \frac{1}{4}, -\frac{3}{4}; 1, \frac{5}{4}\right] = \frac{\sqrt{\pi}\left\{\sqrt{2}+9\log(1+\sqrt{2})\right\}}{12\Gamma(\frac{3}{4})^2}.$

G4. $\;{}_3F_2\left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}; 1, \frac{5}{4}\right] = \frac{\sqrt{\pi}\log(1+\sqrt{2})}{\Gamma(\frac{3}{4})^2}.$

G5. $\;{}_3F_2\left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}; 2, \frac{5}{4}\right] = \frac{2\sqrt{\pi}\left\{-\sqrt{2}+6\log(1+\sqrt{2})\right\}}{9\Gamma(\frac{3}{4})^2}.$

G6. $\;{}_3F_2\left[\frac{1}{2}, \frac{1}{4}, \frac{5}{4}; 2, \frac{9}{4}\right] = \frac{5\sqrt{\pi}\left\{\sqrt{2}-\log(1+\sqrt{2})\right\}}{3\Gamma(\frac{3}{4})^2}.$

G7. $\;{}_3F_2\left[\frac{1}{2}, \frac{5}{4}, \frac{9}{4}; 2, \frac{13}{4}\right] = \frac{3\sqrt{\pi}\left\{-\sqrt{2}+5\log(1+\sqrt{2})\right\}}{7\Gamma(\frac{3}{4})^2}.$

G8. $\;{}_3F_2\left[\frac{3}{2}, \frac{1}{4}, \frac{5}{4}; 1, \frac{9}{4}\right] = \frac{5\sqrt{\pi}\left\{3\sqrt{2}-\log(1+\sqrt{2})\right\}}{8\Gamma(\frac{3}{4})^2}$

G9. $\;{}_3F_2\left[\frac{3}{2}, \frac{5}{4}, \frac{5}{4}; 2, \frac{9}{4}\right] = \frac{5\sqrt{\pi}\left\{2\sqrt{2}-2\log(1+\sqrt{2})\right\}}{\Gamma(\frac{3}{4})^2}.$

G10. $\;{}_3F_2\left[\frac{3}{2}, \frac{5}{4}, \frac{9}{4}; 3, \frac{13}{4}\right] = \frac{6\sqrt{\pi}\left\{-6\sqrt{2}+10\log(1+\sqrt{2})\right\}}{\Gamma(\frac{3}{4})^2}.$

Concluding Comments

By combining the linearization method with the Kummer and Thomae transformations, we present 100 explicit formulae for 7 classes of nonterminating ${}_3F_2(1)$-series. They may potentially find applications in mathematics and physics as other mathematical formulae. Further explorations are encouraged to enrich this bank database of hypergeometric series identities.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bailey, W.N. *Generalized Hypergeometric Series*; Cambridge University Press: Cambridge, UK, 1935
2. Asakura, M.; Yabu, T. Explicit logarithmic formulas of special values of hypergeometric functions ${}_3F_2$. *Commun. Contemp. Math.* **2020**, *22*, 1950040. [CrossRef]
3. Chen, K.-W. Explicit formulas for some infinite ${}_3F_2(1)$-Series. *Axioms* **2021**, *10*, 125. [CrossRef]
4. Chu, W. Inversion techniques and combinatorial identities: A unified treatment for the ${}_7F_6$-series identities. *Collect. Math.* **1994**, *45*, 13–43

5. Ekhad, S.B. Forty Strange Computer–Discovered and Computer–Proved (Of Course) Hypergeometric Series Evaluations. Available online: http://www.math.rutgers.edu/~zeilberg/ekhad/ekhad.html (accessed on 10 October 2021).
6. Gessel, I.M. Finding identities with the WZ method. *J. Symbolic Comput.* **1995**, *20*, 537–566 [CrossRef]
7. Gessel, I.M.; Stanton, D. Strange evaluations of hypergeometric series. *SIAM J. Math. Anal.* **1982**, *13*, 295–308 [CrossRef]
8. Zudilin, W. Arithmetic hypergeometric series. *Russ. Math. Surv.* **2011**, *66*, 1–51 [CrossRef]
9. Campbell, J.M.; D'Aurizio, J.; Sondow, J. On the interplay among hypergeometric functions, complete elliptic integrals, and Fourier–Legendre expansions. *J. Math. Anal. Appl.* **2019**, *479*, 90–121. [CrossRef]
10. Campbell, J.M.; D'Aurizio, J.; Sondow, J. Hypergeometry of the parbelos. *Am. Math. Mon.* **2020**, *127*, 23–32 [CrossRef]
11. Campbell, J.M.; Abrarov, S. An Analogue of Ramanujan's Master Theorem. Available online: https://hal.archives-ouvertes.fr/hal-01897255 (accessed on 9 September 2020).
12. Chu, W. Seven hundreds of exotic ${}_3F_2$-series evaluated in π, $\sqrt{2}$ and $\log(1+\sqrt{2})$. *Contemp. Math.* **2021**, *2*, 327–398. [CrossRef]
13. Brychkov, Y.A. *Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas*; CRC Press: Boca Raton, FL, USA, 2008.
14. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *More Special Functions*; Integrals and Series; Gordon & Breach: New York, NY, USA, 1990; Volume 3.
15. Chu, W. Analytical formulae for extended ${}_3F_2$-series of Watson–Whipple–Dixon with two extra integer parameters. *Math. Comp.* **2012**, *81*, 467–479 [CrossRef]
16. Chu, W. Terminating ${}_4F_3$-series extended with two integer parameters. *Integral Transform. Spec. Funct.* **2016**, *27*, 794–805 [CrossRef]
17. Fields, J.L.; Wimp, J. Expansions of hypergeometric functions in hypergeometric functions. *Math. Comp.* **1961**, *15*, 390–395 [CrossRef]
18. Lewanowicz, S. Generalized Watson's summation formula for ${}_3F_2(1)$. *J. Comput. Appl. Math.* **1997**, *86*, 375–386. [CrossRef]
19. Chu, W. Trigonometric expressions for Gaussian ${}_2F_1$-series. *Turk. J. Math.* **2019**, *43*, 1823–1836. [CrossRef]
20. Gessel, I.M. A simple proof of Andrews's ${}_5F_4$ evaluation. *Ramanujan J.* **2015**, *36*, 165–170. [CrossRef]
21. Chu, W. Finite differences and terminating hypergeometric series. *Bull. Irish Math. Soc.* **2016**, *78*, 31–45. [CrossRef]
22. Campbell, J.M. Ramanujan–type series related to Clausen's product. *arXiv* **2011**, arXiv:1010.1941v3.