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Between Soft θ-Openness and Soft ω^0-Openness

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Abstract: In this paper, we define and investigate soft ω_θ-open sets as a novel type of soft set. We characterize them and demonstrate that they form a soft topology that lies strictly between the soft topologies of soft θ-open sets and soft ω^0-open sets. Moreover, we show that soft ω_θ-open sets and soft ω^0-open sets are equivalent for soft regular spaces. Furthermore, we investigate the connections between particular types of soft sets in a given soft anti-locally countable space and the soft topological space of soft ω_θ-open sets generated by it. In addition to these, we define soft (ω_θ, ω)-sets and soft (ω_θ, θ)-sets as two classes of sets, and via these sets, we introduce two decompositions of soft θ-open sets and soft ω_θ-open sets, respectively. Finally, the relationships between these three new classes of soft sets and their analogs in general topology are examined.

Keywords: θ-open sets; ω_θ-open sets; soft θ-open sets; soft θ-interior; soft ω^0-sets; soft anti-locally countable; soft generated soft topological spaces

MSC: 54A10; 54A40; 54D1

1. Introduction

Mathematical models have been widely used in real-world data-based concerns in fields such as economics, engineering, computer science, medicine, and social sciences, among others. It is common to use mathematical tools to analyze a system's behavior and various properties, which leads to coping with uncertainties and incomplete data in various settings. Although some well-known mathematical methods, such as probability theory, fuzzy set theory, and rough set theory, are beneficial for understanding ambiguity, each has its inherent issues, as demonstrated in [1]. Soft sets were introduced in 1999 [1] as a new mathematical tool for dealing with uncertainties that are free of difficulties faced with pre-existing techniques. The authors of [2,3] then used soft sets in a decision-making problem and defined numerous soft set operators, including a soft subset, a soft equality relation, a soft intersection, and a union. The concept of a bijective soft set was presented and discussed in the context of a decision-making problem [4]. After comparing rough and fuzzy sets, the authors of [5] concluded that every rough and fuzzy set is a soft set. The authors in [6] improved on the results obtained in [3] by changing the necessary operators. It should be highlighted that the high potential for soft set theory applications in a variety of areas encourages rapid research progress (see, for example, [7–9]).

The concept of soft sets was used to define soft topological spaces in [10]. One established and explored fundamental concepts in soft topological spaces such as soft open sets, soft subspaces, and soft separation axioms. In [11], the author identified and corrected certain gaps in [10]. Many traditional topological concepts have been explored and expanded in soft set situations (see, [12–26]), but substantial additions remain possible. Thus, among topological scholars, the study of soft topology is a contemporary topic.

By defining a new class of soft sets in soft topological spaces, we hope to pave the way for multiple forthcoming research articles on the subject of soft topological spaces. In this paper, we define and investigate soft ω_θ-open sets as a novel type of soft set. We characterize them and demonstrate that they form a soft topology that lies strictly between

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the soft topologies of soft θ-open sets and soft ω^0-open sets. Moreover, we show that soft ω_θ-open sets and soft ω^0-open sets are equivalent to soft regular spaces. Furthermore, we investigate the connections between particular types of soft sets in a given soft anti-locally countable space and the soft topological space of soft ω_θ-open sets generated by it. In addition to these, we define soft (ω_θ, ω)-sets and soft (ω_θ, θ)-sets as two classes of sets, and via these sets, we introduce two decompositions of soft θ-open sets and soft ω_θ-open sets, respectively. Finally, the relationships between these three new classes of soft sets and their analogs in general topology are examined.

The arrangement of this article is as follows:

In Section 2, we recall several notions that will be employed in this paper.

In Section 3, we display the concept of "soft ω_θ-open sets", which is the main idea of this paper. We show that the family of soft ω_θ-open sets form a soft topology that lies between the soft topologies of soft θ-open sets and soft ω^0-open sets. We provide some interesting results regarding soft ω_θ-open sets in soft regular spaces, soft locally countable spaces, and soft anti-locally countable spaces. In addition to these, we examine the relationships between soft ω_θ-open sets and their analogs in general topology.

In Section 4, we display the concepts of "soft (ω_θ, ω)-sets" and "soft (ω_θ, θ)-sets" as two new classes of soft sets, and via them, we introduce decompositions of soft θ-open sets and soft ω_θ-open sets. Moreover, we examine the relationships between these classes and their analogs in general topology.

In Section 5, we summarize the main contributions and suggest some future work.

2. Preliminaries

In this section, we recall several notions that will be employed in the sequel to this paper.

In this paper, TS will be used to signify topological space.

Let (K, μ) be a TS and $T \subseteq K$. Throughout this paper, the collection of all closed sets of (K, μ) will be denoted by μ^c; the closure of T in (K, μ) and the interior of T in (K, μ) will be denoted by $Cl_\mu(T)$ and $Int_\mu(T)$, respectively.

Definition 1. *[27] Let (K, μ) be a TS and let $U \subseteq K$.*

(a) A point $k \in K$ is in the θ-closure of U ($k \in Cl^\theta_\mu(U)$) if for every $V \in \mu$ with $k \in V$, we have $Cl_\mu(V) \cap U \neq \emptyset$.
(b) U is θ-closed in (K, μ) if $Cl^\theta_\mu(U) = U$.
(c) U is θ-open in (K, μ) if $K - U$ is θ-closed in (K, μ).
(d) The family of all θ-open sets in (K, μ) is denoted by μ_θ.

Definition 2. *[28] Let (K, μ) be a TS and $T \subseteq K$. A point $k \in K$ is called a θ-interior point of T in (K, μ) if there exists $V \in \mu$ such that $k \in V \subseteq Cl_\mu(V) \subseteq T$. The set of all θ-interior points of T in (K, μ) is called the θ-interior of T in (K, μ) and is denoted by $Int^\theta_\mu(T)$.*

Definition 3. *[29] Let (K, μ) be an STS and let $U \subseteq K$. Then*

(a) U is called a ω_θ-open set in (K, μ) if for any $k \in U$, there is $V \in \mu$ such that $k \in V$ and $V - Int^\theta_\mu(U)$ is countable. The collection of all ω_θ-open set in (K, μ) will be denoted by μ_{ω_θ}.
(b) U is called a ω_θ-closed set in (K, μ) if $K - U \in \mu_{\omega_\theta}$.

Definition 4. *[1] Let K be an initial universe and S be a set of parameters. A soft set over K relative to S is a function $G : S \longrightarrow \mathcal{P}(K)$, where $\mathcal{P}(K)$ is the power set of K. The family of all soft sets over K relative to S will be denoted by $SS(K, S)$.*

Definition 5. *[3] Let $G \in SS(K, S)$.*

(a) G is called a null soft set over K relative to S, denoted by 0_S, if $G(s) = \emptyset$ for each $s \in S$.
(b) G is called an absolute soft set over K relative to S, denoted by 1_S, if $H(s) = K$ for each $s \in S$.

Definition 6. *[4] Let $H, G \in SS(K,S)$.*

(1) *H is a soft subset of G, denoted by $H\widetilde{\subseteq}G$, if $H(s) \subseteq G(s)$ for each $s \in S$.*
(2) *The soft union of H and G is denoted by $H\widetilde{\cup}G$ and defined to be the soft set $H\widetilde{\cup}G \in SS(K,S)$ where $\left(H\widetilde{\cup}G\right)(s) = H(s) \cup G(s)$ for each $s \in S$.*
(3) *The soft intersection of H and G is denoted by $H\widetilde{\cap}G$ and defined to be the soft set $H\widetilde{\cap}G \in SS(K,S)$ where $\left(H\widetilde{\cap}G\right)(s) = H(s) \cap G(s)$ for each $s \in S$.*
(4) *The soft difference of H and G is denoted by $H - G$ and defined to be the soft set $H - G \in SS(K,S)$ where $(H - G)(s) = H(s) - G(s)$ for each $s \in S$.*

Definition 7. *[30] Let Γ be an arbitrary index set and $\{H_r : r \in \Gamma\} \subseteq SS(K,S)$.*

(a) *The soft union of these soft sets is the soft set denoted by $\widetilde{\cup}_{r\in\Gamma}H_r$ and defined by $\left(\widetilde{\cup}_{r\in\Gamma}H_r\right)(s) = \cup_{r\in\Gamma}H_r(s)$ for each $s \in S$.*
(b) *The soft intersection of these soft sets is the soft set denoted by $\widetilde{\cap}_{r\in\Gamma}H_r$ and defined by $\left(\widetilde{\cap}_{r\in\Gamma}H_r\right)(s) = \cap_{r\in\Gamma}H_r(s)$ for each $s \in S$.*

Definition 8. *[6] Let $\Psi \subseteq SS(K,S)$. Then Ψ is called a soft topology on K relative to S if*

(1) *$0_S, 1_S \in \Psi$,*
(2) *the soft union of any number of soft sets in Ψ belongs to Ψ,*
(3) *the soft intersection of any two soft sets in Ψ belongs to Ψ.*

The triplet (K,Ψ,S) is called a soft topological space (STS) over K relative to S. The members of Ψ are called soft open sets in (K,Ψ,S) and their soft complements are called soft-closed sets in (K,Ψ,S). The family of all soft-closed sets in (K,Ψ,S) will be denoted by Ψ^c.

Definition 9. *[6] Let (K,Ψ,S) be a STS and let $N \in SS(K,S)$. Then*

(1) *the soft closure of N in (K,Ψ,S) is denoted by $Cl_\Psi(N)$ and defined by $Cl_\Psi(H) = \widetilde{\cap}\{M : M$ is soft closed in (K,Ψ,S) and $N\widetilde{\subseteq}M\}$.*
(2) *the soft interior of N in (K,Ψ,S) is denoted by $Int_\Psi(N)$ and defined by $Int_\Psi(N) = \widetilde{\cup}\{K : K \in \Psi$ and $K\widetilde{\subseteq}N\}$.*

Definition 10. *A soft set $H \in SS(K,S)$ defined by*

(1) *[31] $H(s) = \begin{cases} U & \text{if } s = a \\ \emptyset & \text{if } s \neq a \end{cases}$ is denoted by a_U.*
(2) *[32] $H(s) = U$ for all $s \in S$ is denoted by C_U.*
(3) *[33] $H(s) = \begin{cases} \{y\} & \text{if } s = a \\ \emptyset & \text{if } s \neq a \end{cases}$ is denoted by a_y and is called a soft point. The set of all soft points in $SS(K,S)$ is denoted by $SP(K,S)$.*

Definition 11. *[33] Let $H \in SS(K,S)$ and $a_y \in SP(K,S)$. Then a_y is said to belong to H (notation: $a_y\widetilde{\in}H$) if $y \in H(a)$.*

Definition 12. *[34] Let (K,Ψ,S) be a STS and let $H \in SS(K,S)$. Then H is called a soft ω-open set in (K,Ψ,S) if for each $s_k\widetilde{\in}H$, there exist $G \in \Psi$ and $N \in CSS(K,S)$ such that $s_k\widetilde{\in}G$ and $G - N\widetilde{\subseteq}H$. The family of all soft ω-open set in (K,Ψ,S) is denoted by Ψ_ω.*

Theorem 1. *[35] For any TS (K,μ), the family*
$\{H \in SS(K,S) : H(s) \in \mu$ for all $s \in S\}$
is a soft topology on K relative to S. This soft topology will be denoted by $\tau(\mu)$.

Theorem 2. *[31] For any collection of TSs $\{(K,\mu_s) : s \in S\}$, the family*

$$\{H \in SS(K,S) : H(s) \in \mu_s \text{ for all } s \in S\}$$

forms a soft topology on K relative to S. This soft topology is denoted by $\oplus_{s\in S}\mu_s$.

Definition 13. *[36] Let (K, Ψ, S) be a TS and let $H \in SS(K, S)$.*

(a) A soft point $s_k \in SP(K, S)$ is in the θ-closure of H ($s_k \widetilde{\in} Cl^{\theta}_{\Psi}(H)$) if for every $G \in \Psi$ with $s_k \widetilde{\in} G$, we have $Cl_{\Psi}(G) \cap H \neq 0_S$.

(b) H is soft θ-closed in (K, Ψ, S) if $Cl^{\theta}_{\Psi}(H) = H$.

(c) H is soft θ-open in (K, Ψ, S) if $1_S - H$ is soft θ-closed in (K, Ψ, S).

(d) The family of all soft θ-open sets in (K, Ψ, S) is denoted by Ψ_{θ}.

Definition 14. *[36] Let (K, Ψ, S) be a TS and let $H \in SS(K, S)$. A soft point $s_k \in SP(K, S)$ is called a soft θ-interior point of H in (K, Ψ, S) if there exists $G \in \Psi$ such that $s_k \widetilde{\in} G \widetilde{\subseteq} Cl_{\Psi}(G) \widetilde{\subseteq} H$. The soft set of all soft θ-interior points of H in (K, Ψ, S) is called the soft θ-interior of H in (K, Ψ, S) and is denoted by $Int^{\theta}_{\Psi}(H)$.*

Definition 15. *[37] Let (K, Ψ, S) be a STS and let $H \in SS(K, S)$. Then*

(a) H is called a soft ω^0-open set in (K, Ψ, S) if for any $s_k \widetilde{\in} H$, there is $G \in \Psi$ such that $s_k \widetilde{\in} G$ and $G - Int_{\Psi}(H) \in CSS(K, S)$. The collection of all soft ω^0-open set in (K, Ψ, S) will be denoted by Ψ_{ω^0}.

(b) H is called a soft ω^0-closed set in (K, Ψ, S) if $1_S - H \in \Psi_{\omega^0}$.

Theorem 3. *[37] For any STS (K, Ψ, S), $\Psi \subseteq \Psi_{\omega^0} \subseteq \Psi_{\omega}$.*

Definition 16. *A STS (K, Ψ, S) is called*

(1) [34] soft locally countable if for each $s_k \in SP(K, S)$, there exists $H \in \Psi \cap CSS(K, S)$ such that $s_k \widetilde{\in} H$.

(2) [34] soft anti-locally countable if for every $H \in \Psi - \{0_S\}$, $H \notin CSS(K, S)$.

(3) [38] soft regular if for each $s_k \in SP(K, S)$ and each $H \in \Psi$ such that $s_k \widetilde{\in} H$, there exists $G \in \Psi$ such that $s_k \widetilde{\in} G \widetilde{\subseteq} Cl_{\Psi}(G) \widetilde{\subseteq} H$.

(4) [39] soft Urysohn space if for any two soft points $s_k, t_n \in SP(K, S)$, there exist $H, G \in \Psi$ such that $s_k \widetilde{\in} H$, $t_n \widetilde{\in} G$ and $H \widetilde{\cap} G = 0_S$.

Definition 17. *Let (K, Ψ, S) be a STS and let $H \in SS(K, S)$. Then H is called soft α-open [40] (resp. soft β-open [41], soft regular open [42]) in (K, Ψ, S) if $H \widetilde{\subseteq} Int_{\Psi}(Cl_{\Psi}(Int_{\Psi}(H)))$ (resp. $H \widetilde{\subseteq} Cl_{\Psi}(Int_{\Psi}(Cl_{\Psi}(H)))$, $H = Int_{\Psi}(Cl_{\Psi}(H))$). The families of soft α-open sets, soft β-open sets, and soft regular open sets are denoted by $\alpha(K, \Psi, S)$, $\beta(K, \Psi, S)$, and $RO(K, \Psi, S)$, respectively.*

For the concepts and terminologies that have not appeared in this section, we shall follow [31,34].

3. Soft ω_θ-Open Sets

Herein, we display the concept of "soft ω_θ-open sets", which is the main idea of this paper. We show that the family of soft ω_θ-open sets form a soft topology that lies between the soft topologies of soft θ-open sets and soft ω^0-open sets. We provide some interesting results regarding soft ω_θ-open sets in soft regular spaces, soft locally countable spaces, and soft anti-locally countable spaces. In addition to these, we examine the relationships between soft ω_θ-open sets and their analogs in general topology.

Definition 18. *Let (K, Ψ, S) be a STS and let $G \in SS(K, S)$. Then*

(a) G is called a soft ω_θ-open set in (K, Ψ, S) if for any $s_k \widetilde{\in} G$, there is $H \in \Psi$ such that $s_k \widetilde{\in} H$ and $H - Int^{\theta}_{\Psi}(G) \in CSS(K, S)$. The collection of all soft ω_θ-open set in (K, Ψ, S) will be denoted by Ψ_{ω_θ}.

(b) G is called a soft ω_θ-closed set in (K, Ψ, S) if $1_S - G \in \Psi_{\omega_\theta}$.

Theorem 4. *Let (K, Ψ, S) be a STS and let $G \in SS(K, S)$. Then $G \in \Psi_{\omega_\theta}$ if and only if for each $s_k \widetilde{\in} G$, there are $H \in \Psi$ and $R \in CSS(K, S)$ such that $s_k \widetilde{\in} H$ and $H - R \widetilde{\subseteq} Int^{\theta}_{\Psi}(G)$.*

Proof. *Necessity*. Suppose that $G \in \Psi_{\omega_\theta}$. Let $s_k \widetilde{\in} G$. Then there is $H \in \Psi$ such that $s_k \widetilde{\in} H$ and $H - Int^\theta_\Psi(G) \in CSS(K,S)$. Let $R = H - Int^\theta_\Psi(G)$. Then $R \in CSS(K,S)$ and $H - R = Int^\theta_\Psi(G) \widetilde{\subseteq} Int^\theta_\Psi(G)$.

Sufficiency. Suppose that for each $s_k \widetilde{\in} G$, there is $H \in \Psi$ and $R \in CSS(K,S)$ such that $s_k \widetilde{\in} H$ and $H - R \widetilde{\subseteq} Int^\theta_\Psi(G)$. Let $s_k \widetilde{\in} G$. Then by assumption, there are $H \in \Psi$ and $R \in CSS(K,S)$ such that $s_k \widetilde{\in} H$ and $H - R \widetilde{\subseteq} Int^\theta_\Psi(G)$. Since $H - R \widetilde{\subseteq} Int^\theta_\Psi(G)$, then $H - Int^\theta_\Psi(G) = R \in CSS(K,S)$ and thus, $H - Int^\theta_\Psi(G) \in CSS(K,S)$. Therefore, $G \in \Psi_{\omega_\theta}$. □

Theorem 5. *For any STS (K,Ψ,S), $\Psi_\theta \subseteq \Psi_{\omega_\theta} \subseteq \Psi_{\omega^0}$.*

Proof. To see that $\Psi_\theta \subseteq \Psi_{\omega_\theta}$, let $G \in \Psi_\theta$ and let $s_k \widetilde{\in} G$. Since $G \in \Psi_\theta$, then $Int^\theta_\Psi(G) = G$. Thus, we have $s_k \widetilde{\in} G \in \Psi$ such that $G - Int^\theta_\Psi(G) = 0_S \in CSS(K,S)$, and hence $G \in \Psi_{\omega_\theta}$.

To see that $\Psi_{\omega_\theta} \subseteq \Psi_{\omega^0}$, let $G \in \Psi_{\omega_\theta}$ and let $s_k \widetilde{\in} G$. Then there is $H \in \Psi$ such that $s_k \widetilde{\in} H$ and $H - Int^\theta_\Psi(G) \in CSS(K,S)$. Since $Int^\theta_\Psi(G) \widetilde{\subseteq} Int_\Psi(G)$, then $H - Int_\Psi(G) \widetilde{\subseteq} H - Int^\theta_\Psi(G)$ and so $H - Int_\Psi(G) \in CSS(K,S)$. Hence, $G \in \Psi_{\omega^0}$. □

Theorem 6. *For any STS (K,Ψ,S), $(K,\Psi_{\omega_\theta},S)$ is a STS.*

Proof. Since by Proposition 5.7 of [36], (K,Ψ_θ,S) is a STS, then $0_S, 1_S \in \Psi_\theta$. Thus, by Theorem 5, $0_S, 1_S \in \Psi_{\omega_\theta}$.

Let $M, N \in \Psi_{\omega_\theta}$ and let $s_k \widetilde{\in} M \widetilde{\cap} N$. Then $s_k \widetilde{\in} M \in \Psi_{\omega_\theta}$ and $s_k \widetilde{\in} M \in \Psi_{\omega_\theta}$. So, there are $H, L \in \Psi$ such that $s_k \widetilde{\in} H \widetilde{\cap} L \in \Psi$ and $H - Int^\theta_\Psi(M), L - Int^\theta_\Psi(N) \in CSS(K,S)$. Since byProposition 5.4 of [36], $Int^\theta_\Psi(M \widetilde{\cap} N) = Int^\theta_\Psi(M) \widetilde{\cap} Int^\theta_\Psi(N)$, then $\left(H \widetilde{\cap} L\right) - \left(Int^\theta_\Psi(M \widetilde{\cap} N)\right)$

$$\begin{aligned}\left(H \widetilde{\cap} L\right) - \left(Int^\theta_\Psi(M \widetilde{\cap} N)\right) &= \left(H \widetilde{\cap} L\right) - \left(Int^\theta_\Psi(M) \widetilde{\cap} Int^\theta_\Psi(N)\right) \\ &= \left(\left(H \widetilde{\cap} L\right) - Int^\theta_\Psi(M)\right) \widetilde{\cup} \left(\left(H \widetilde{\cap} L\right) - Int^\theta_\Psi(N)\right) \in CSS(K,S).\end{aligned}$$

Hence, $M \widetilde{\cap} N \in \Psi_{\omega_\theta}$.

Let $\{G_\alpha : \alpha \in \Delta\} \subseteq \Psi_{\omega_\theta}$ and let $s_k \widetilde{\in} \cup_{\alpha \in \Delta} G_\alpha$. Then there exists $\alpha_\circ \in \Delta$ such that $s_k \widetilde{\in} G_{\alpha_\circ}$. Then by Theorem 4, there are $H \in \Psi$ and $R \in CSS(K,S)$ such that $s_k \widetilde{\in} H$ and $H - R \widetilde{\subseteq} Int^\theta_\Psi(G_{\alpha_\circ}) \widetilde{\subseteq} Int^\theta_\Psi(\widetilde{\cup}_{\alpha \in \Delta} G_{\alpha_\circ})$. Hence, $\widetilde{\cup}_{\alpha \in \Delta} G_{\alpha_\circ} \in \Psi_{\omega_\theta}$. □

Theorem 7. *If (K,Ψ,S) is a soft locally countable STS, then $\Psi_{\omega_\theta} = SS(K,S)$.*

Proof. Suppose that (K,Ψ,S) is soft locally countable. Let $G \in SS(K,S)$ and let $k_s \widetilde{\in} G$. By soft local countability of (K,Ψ,S), there is $H \in CSS(K,S) \cap \Psi$ such that $s_k \widetilde{\in} H \widetilde{\subseteq} G$. Thus, we have $s_k \widetilde{\in} H \in \Psi$, $H \in CSS(K,S)$ and $H - H = 0_S \widetilde{\subseteq} Int^\theta_\Psi(G)$. Hence, $G \in \Psi_{\omega_\theta}$. □

Lemma 1. *Let (K,Ψ,S) be a STS and let $N \in SS(K,S)$. Then for every $s \in S$, $\left(Int^\theta_\Psi(N)\right)(s) \subseteq Int^\theta_{\Psi_s}(N(s))$.*

Proof. Let $k \in \left(Int^\theta_\Psi(N)\right)(s)$. Then $s_k \widetilde{\in} Int^\theta_\Psi(N)$ and so, there is $H \in \Psi$ such that $s_k \widetilde{\in} H \widetilde{\subseteq} Cl_\Psi(H) \widetilde{\subseteq} N$. Since by Proposition 7 of [10], $Cl_{\Psi_s}(H(s)) \subseteq (Cl_\Psi(H))(s)$. Therefore, we have $H(s) \in \Psi_s$ and $k \in H(s) \subseteq Cl_{\Psi_s}(H(s)) \subseteq (Cl_\Psi(H))(s) \subseteq N(s)$. Hence, $k \in Int^\theta_{\Psi_s}(N(s))$. □

Theorem 8. *Let (K,Ψ,S) be a STS. Then for each $s \in S$, $(\Psi_{\omega_\theta})_s \subseteq (\Psi_s)_{\omega_\theta}$.*

Proof. Let $s \in S$. Let $U \in (\Psi_{\omega_\theta})_s$ and let $k \in U$. Choose $H \in \Psi_{\omega_\theta}$ such that $U = H(s)$. Since $s_k \widetilde{\in} H \in \Psi_{\omega_\theta}$, then by Theorem 4, there is $G \in \Psi$ and $R \in CSS(K,S)$ such that $s_k \widetilde{\in} G$ and $G - R \widetilde{\subseteq} Int^\theta_\Psi(H)$. Thus, we have $k \in G(s) \in \Psi_s$, $R(s)$ is a countable set, and $G(s) - R(s) = (G - R)(s) \subseteq \left(Int^\theta_\Psi(H)\right)(s)$. But by Lemma 1, $\left(Int^\theta_\Psi(H)\right)(s) \subseteq Int^\theta_{\Psi_s}(H(s)) = Int^\theta_{\Psi_s}(U)$. It follows that $U \in (\Psi_s)_{\omega_\theta}$. □

Corollary 1. *Let (K,Ψ,S) be a STS and let $G \in \Psi_{\omega_\theta}$. Then for every $s \in S$, $G(s) \in (\Psi_s)_{\omega_\theta}$.*

Proof. Let $s \in S$. Since $G \in \Psi_{\omega_\theta}$, then $G(s) \in (\Psi_{\omega_\theta})_s$. Thus, by Theorem 8, $G(s) \in (\Psi_s)_{\omega_\theta}$. $\square$

Lemma 2. *Let $\{(K, \lambda_s) : s \in S\}$. Then for every $t \in S$ and $U \subseteq K$, $t_{Int_{(\lambda_t)_\theta}(U)} \widetilde{\subseteq} Int_{(\oplus_{s\in S}\lambda_s)_\theta}(t_U)$.*

Proof. Let $t \in S$ and $U \subseteq K$. Let $t_k \widetilde{\in} t_{Int_{(\lambda_t)_\theta}(U)}$ where $k \in Int_{(\lambda_t)_\theta}(U)$. Since $k \in Int_{(\lambda_t)_\theta}(U)$, then there exists $V \in \lambda_t$ such that $k \in V \subseteq Cl_{\lambda_t}(V) \subseteq U$. So, we have $t_k \widetilde{\in} t_V \in \oplus_{s\in S}\lambda_s$, and $Cl_{\oplus_{s\in S}\lambda_s}(t_V) = t_{Cl_{\lambda_t}(V)} \widetilde{\subseteq} t_U$. Hence, $t_k \widetilde{\in} Int_{(\oplus_{s\in S}\lambda_s)_\theta}(t_U)$. $\square$

Theorem 9. *For any collection of TSs $\{(K, \lambda_s) : s \in S\}$, we have $(\oplus_{s\in S}\lambda_s)_{\omega_\theta} = \oplus_{s\in S}(\lambda_s)_{\omega_\theta}$.*

Proof. By Theorem 3.7 and Theorem 3.8 of [31], $\left((\oplus_{s\in S}\lambda_s)_{\omega_\theta}\right)_s \subseteq ((\oplus_{s\in S}\lambda_s)_s)_{\omega_\theta} = (\lambda_s)_{\omega_\theta}$ for all $s \in S$. Thus, $(\oplus_{s\in S}\lambda_s)_{\omega_\theta} \subseteq \oplus_{s\in S}(\lambda_s)_{\omega_\theta}$. To show that $\oplus_{s\in S}(\lambda_s)_{\omega_\theta} \subseteq (\oplus_{s\in S}\lambda_s)_{\omega_\theta}$, by Theorem 3.6 of [31], it is sufficient to show that $\left\{s_U : s \in S \text{ and } U \in (\lambda_s)_{\omega_\theta}\right\} \subseteq (\oplus_{s\in S}\lambda_s)_{\omega_\theta}$. Let $s \in S$ and $U \in (\lambda_s)_{\omega_\theta}$. Let $s_k \widetilde{\in} s_U$. Then $k \in U \in (\lambda_s)_{\omega_\theta}$. So, there are $V \in \lambda_s$ and a countable subset $D \subseteq K$ such that $k \in V$ and $V - D \subseteq Int_{(\lambda_s)_\theta}(U)$. Thus, we have $s_k \widetilde{\in} s_V \in \oplus_{s\in S}\lambda_s$, $s_D \in CSS(Y, B)$ and $s_V - s_D \widetilde{\subseteq} s_{Int_{(\lambda_s)_\theta}(U)}$. But by Lemma 2, $s_{Int_{(\lambda_s)_\theta}(U)} \widetilde{\subseteq} Int_{(\oplus_{s\in S}\lambda_s)_\theta}(s_U)$. Therefore, $s_V - s_D \widetilde{\subseteq} Int_{(\oplus_{s\in S}\lambda_s)_\theta}(s_U)$. Hence, $s_U \in (\oplus_{s\in S}\lambda_s)_{\omega_\theta}$. $\square$

Corollary 2. *For any TS (K, μ) and any set of parameters S, $(\tau(\mu))_{\omega_\theta} = \tau(\mu_{\omega_\theta})$.*

Proof. Let $\mu_s = \mu$ for every $s \in S$. Then $\tau(\mu) = \oplus_{s\in S}\lambda_s$. Thus, by Theorem 9,

$$\begin{aligned}(\tau(\mu))_{\omega_\theta} &= (\oplus_{s\in S}\lambda_s)_{\omega_\theta} \\ &= \oplus_{s\in S}(\lambda_s)_{\omega_\theta} \\ &= \tau(\mu_{\omega_\theta}).\end{aligned}$$

The examples below show that none of the two soft inclusions in Theorem 5 can be substituted by equality: $\square$

Example 1. *Let $K = \mathbb{Z}$, $S = \mathbb{R}$, $\Psi = \{0_S\} \cup \{F \in SS(K, S) : K - F(s) \text{ is finite for all } s \in S\}$. Since (K, Ψ, S) is soft locally countable, then $\Psi_{\omega_\theta} = SS(K, S)$. Therefore, $C_{\mathbb{N}} \in \Psi_{\omega_\theta} - \Psi_\theta$.*

Example 2. *Let $K = \mathbb{R}$, $S = \{a, b\}$, $\Psi = \left\{0_S, 1_S, C_{\mathbb{R}-(1,3)}\right\}$. Suppose that $Int^\theta_\Psi\left(C_{\mathbb{R}-(1,3)}\right) \neq 0_S$. Then there exists $s_k \widetilde{\in} Int^\theta_\Psi\left(C_{\mathbb{R}-(1,3)}\right)$ and so there is $G \in \Psi$ such that $s_k \widetilde{\in} G \widetilde{\subseteq} Cl_\Psi(G) \widetilde{\subseteq} C_{\mathbb{R}-(1,3)}$. Since $s_k \widetilde{\in} G \widetilde{\subseteq} C_{\mathbb{R}-(1,3)}$, then $G = C_{\mathbb{R}-(1,3)}$ and $Cl_\Psi(G) = 1_S \widetilde{\subseteq} C_{\mathbb{R}-(1,3)}$ which is impossible. Therefore, $Int^\theta_\Psi\left(C_{\mathbb{R}-(1,3)}\right) = 0_S$. If $C_{\mathbb{R}-(1,3)} \in \Psi_{\omega_\theta}$, then there are $N \in \Psi$ and $H \in CSS(K, S)$ such that $a_4 \widetilde{\in} N$ and $N - H \widetilde{\subseteq} Int^\theta_\Psi\left(C_{\mathbb{R}-(1,3)}\right) = 0_S$. Thus, $N \widetilde{\subseteq} H$ and hence $N \in CSS(K, S)$. On the other hand, since $a_4 \widetilde{\in} N \in \Psi$, then either $N = C_{\mathbb{R}-(1,3)}$ or $N = 1_S$ and in both cases $N \notin CSS(K, S)$. It follows that $C_{\mathbb{R}-(1,3)} \notin \Psi_{\omega_\theta}$. On the other hand, since $C_{\mathbb{R}-(1,3)} \in \Psi$, then by Theorem 3, $C_{\mathbb{R}-(1,3)} \in \Psi_{\omega^0}$.*

Example 2, shows also that Ψ need not be a subset of Ψ_{ω_θ} in general.

Theorem 10. *For any soft regular STS (K, Ψ, S), $\Psi_{\omega_\theta} = \Psi_{\omega^0}$.*

Proof. By Theorem 5, $\Psi_{\omega_\theta} \subseteq \Psi_{\omega^0}$. To see that $\Psi_{\omega^0} \subseteq \Psi_{\omega_\theta}$, let $G \in \Psi_{\omega_\theta}$ and let $s_k \widetilde{\in} G$. Then there are $L \in \Psi$ and $N \in CSS(K,S)$ such that $s_k \widetilde{\in} L$ and $L - N \widetilde{\subseteq} Int_\Psi(G)$. We are going to show that $Int_\Psi(G) \widetilde{\subseteq} Int^\theta_\Psi(G)$. Let $a_x \widetilde{\in} Int_\Psi(G)$. Since (K,Ψ,S) is soft regular and $a_x \widetilde{\in} Int_\Psi(G) \in \Psi$, then there exists $H \in \Psi$ such that $a_x \widetilde{\in} H \widetilde{\subseteq} Cl_\Psi(H) \widetilde{\subseteq} Int_\Psi(G) \widetilde{\subseteq} G$. Thus, $a_x \widetilde{\in} Int^\theta_\Psi(G)$. This ends the proof. □

Corollary 3. *For any soft regular STS (K,Ψ,S), $\Psi \subseteq \Psi_{\omega_\theta}$.*

Proof. Follows from Theorem 3 and Theorem 10. □

Theorem 11. *Let (K,Ψ,S) be a STS. If $C_U \in (\Psi \cap \Psi_{\omega_\theta}) - \{0_S\}$, then $(\Psi_{\omega_\theta})_U \subseteq (\Psi_U)_{\omega_\theta}$.*

Proof. Let $G \in (\Psi_{\omega_\theta})_U$ and let $s_u \widetilde{\in} G$. There exists $H \in \Psi_{\omega_\theta}$ such that $G = H \widetilde{\cap} C_U$. As $C_U \in \Psi_{\omega_\theta}$, then $G \in \Psi_{\omega_\theta}$. So, there are $L \in \Psi$ and $N \in CSS(K,S)$ such that $s_u \widetilde{\in} L$ and $L - N \widetilde{\subseteq} Int^\theta_\Psi(G)$. Thus, we have $s_u \widetilde{\in} L \widetilde{\cap} C_U \in \Psi_U$, $N \widetilde{\cap} C_U \in CSS(U,S)$, and $\left(L \widetilde{\cap} C_U\right) - \left(N \widetilde{\cap} C_U\right) \widetilde{\subseteq} (L-N) \widetilde{\cap} C_U \widetilde{\subseteq} Int^\theta_\Psi(G) \widetilde{\cap} C_U \widetilde{\subseteq} Int^\theta_{\Psi_U}(G)$. □

Corollary 4. *Let (K,Ψ,S) be a STS. If $C_U \in \Psi_\theta - \{0_S\}$, then $(\Psi_{\omega_\theta})_U \subseteq (\Psi_U)_{\omega_\theta}$.*

As can be shown by the following example, the condition '$C_U \in \Psi \cap \Psi_{\omega_\theta}$' is essential in Theorem 11.

Example 3. *Let $K = \mathbb{R}$, $U = \mathbb{Q}^c$, $S = \{a,b\}$, μ be the usual topology on K, and $\Psi = \{C_V : V \in \mu\}$. Since (K,Ψ,S) is soft regular and $C_{(2,\infty)} \in \Psi$, then by Corollary 3.15, $C_{(2,\infty)} \in \Psi_{\omega_\theta}$. So, $C_{(2,\infty)} \widetilde{\cap} C_U = C_{(2,\infty)\cap\mathbb{Q}^c} \in (\Psi_{\omega_\theta})_U$. If $C_{(2,\infty)\cap\mathbb{Q}^c} \in (\Psi_U)_{\omega_\theta}$, then there are $V \in \mu$ and $H \in CSS(U,S)$ such that $a_3 \in C_V$ and $C_V - H \widetilde{\subseteq} Int^\theta_{\Psi_U}\left(C_{(2,\infty)\cap\mathbb{Q}^c}\right) = 0_S$. Therefore, $C_V \widetilde{\subseteq} H$, and so $C_V \in CSS(U,S)$. Hence, V is a countable set. This is impossible.*

Theorem 12. *If (K,Ψ,S) is soft Lindelof, then for each $G \in \Psi_{\omega_\theta} \cap \Psi^c$, $G - Int^\theta_\Psi(G) \in CSS(K,S)$.*

Proof. Let (K,Ψ,S) be soft Lindelof and let $G \in \Psi_{\omega_\theta} \cap \Psi^c$. Since $G \in \Psi_{\omega_\theta}$, then for every $s_k \widetilde{\in} G$, there exists $H_{s_k} \in \Psi$ such that $s_k \widetilde{\in} H_{s_k}$ and $H_{s_k} - Int^\theta_\Psi(G) \in CSS(K,S)$. Since $G \in \Psi^c$, then G is a soft Lindelof subset of (K,Ψ,S). Put $\mathcal{R} = \left\{H_{s_k} : s_k \widetilde{\in} G\right\}$. Since $G \widetilde{\subseteq} \widetilde{\cup}_{R\in\mathcal{R}} R$, then there is a countable subcollection $\mathcal{R}_1 \subseteq \mathcal{R}$ such that $G \widetilde{\subseteq} \widetilde{\cup}_{R\in\mathcal{R}_1} R$. Since $\mathcal{R}_1$ is countable, then $\widetilde{\cup}\left\{R - Int^\theta_\Psi(G) : R \in \mathcal{R}_1\right\} \in CSS(K,S)$. Since $G - Int^\theta_\Psi(G) \widetilde{\subseteq} \widetilde{\cup}\left\{R - Int^\theta_\Psi(G) : R \in \mathcal{R}_1\right\}$, then $G - Int^\theta_\Psi(G) \in CSS(K,S)$. □

Theorem 13. *Let (K,Ψ,S) be a STS and let $H \in (\Psi_{\omega_\theta})^c$. Then there are $M \in \Psi^c$ and $N \in CSS(K,S)$ such that $Cl^\theta_\Psi(H) \widetilde{\subseteq} M \widetilde{\cup} N$.*

Proof. If $H = 1_S$, then $H \widetilde{\subseteq} 1_S \widetilde{\cup} 0_S$ with $1_S \in \Psi^c$ and $0_S \in CSS(K,S)$. If $H \neq 1_S$, then there exists $s_k \widetilde{\in} 1_S - H \in \Psi_{\omega_\theta}$. So, there are $G \in \Psi$ and $N \in CSS(K,S)$ such that $s_k \widetilde{\in} G$ and $G - N \widetilde{\subseteq} Int^\theta_\Psi(1_S - H) = 1_S - Cl^\theta_\Psi(H)$ and hence $Cl^\theta_\Psi(H) \widetilde{\subseteq} 1_S - (G - N) = (1_S - G) \widetilde{\cup} N$. Let $M = 1_S - G$. Then $M \in \Psi^c$ and $Cl^\theta_\Psi(H) \widetilde{\subseteq} M \widetilde{\cup} N$. □

Theorem 14. *A STS (K,Ψ,S) is soft anti-locally countable if and only if $(K,\Psi_{\omega_\theta},S)$ is soft anti-locally countable.*

Proof. *Necessity*. Suppose that (K,Ψ,S) is soft anti-locally countable and suppose to the contrary that there exists $G \in (\Psi_{\omega_\theta} \cap CSS(K,S)) - \{0_S\}$. Choose $s_k \widetilde{\in} G$. Since $G \in \Psi_{\omega_\theta}$, then there are $H \in \Psi$ and $R \in CSS(K,S)$ such that $s_k \widetilde{\in} H$ and $H - R \widetilde{\subseteq} Int^\theta_\Psi(G) \widetilde{\subseteq} G$. Thus, $H \widetilde{\subseteq} G \widetilde{\cup} R$ and hence $H \in CSS(K,S)$. Since $s_k \widetilde{\in} H$, then $H \in \Psi - \{0_S\}$. Since (K,Ψ,S) is soft anti-locally countable, then $H \notin CSS(K,S)$, a contradiction. *Sufficiency.* Obvious. □

Theorem 15. *Let* (K, Ψ, S) *be soft anti-locally countable and let* $H \in \Psi_{\omega_\theta}$*, then* $Cl_\Psi(H) = Cl_{\Psi_{\omega_\theta}}(H)$.

Proof. By Theorem 5, we have $\Psi_{\omega_\theta} \subseteq \Psi_{\omega^0}$ and so $Cl_{\Psi_{\omega^0}}(H) \widetilde{\subseteq} Cl_{\Psi_{\omega_\theta}}(H)$. Since (K, Ψ, S) is soft anti-locally countable and $H \in \Psi_{\omega_\theta} \subseteq \Psi_{\omega^0}$, then by Theorem 21 of [37], $Cl_{\Psi_{\omega^0}}(H) = Cl_\Psi(H)$. Therefore, $Cl_\Psi(H) \widetilde{\subseteq} Cl_{\Psi_{\omega_\theta}}(H)$. We will show that $1_S - Cl_{\Psi_{\omega_\theta}}(H) \widetilde{\subseteq} 1_S - Cl_\Psi(H)$. Let $s_k \widetilde{\in} 1_S - Cl_{\Psi_{\omega_\theta}}(H) \in \Psi_{\omega_\theta}$. Then there are $G \in \Psi$ and $L \in CSS(K, S)$ such that $s_k \widetilde{\in} G$ and $G - L \widetilde{\subseteq} Int^\theta_\Psi\left(1_S - Cl_{\Psi_{\omega_\theta}}(H)\right) \widetilde{\subseteq} 1_S - Cl_{\Psi_{\omega_\theta}}(H) \widetilde{\subseteq} 1_S - H$. Thus, $G \widetilde{\cap} H \widetilde{\subseteq} L$ and hence $G \widetilde{\cap} H \in CSS(K, S)$. Since $G \widetilde{\cap} H \in \Psi_{\omega^0}$ and by Theorem 18 of [37], (K, Ψ_{ω^0}, S) is soft anti-locally countable, then $G \widetilde{\cap} H = 0_S$. Therefore, we have $s_k \widetilde{\in} G \in \Psi$ such that $G \widetilde{\cap} H = 0_S$, and hence $s_k \widetilde{\in} 1_S - Cl_\Psi(H)$. □

Corollary 5. *Let* (K, Ψ, S) *be soft anti-locally countable and let* $H \in (\Psi_{\omega_\theta})^c$*, then* $Int_\Psi(H) = Int_{\Psi_{\omega_\theta}}(H)$.

In Theorem 15, the condition "soft anti-locally countable" is necessary, as the following example shows:

Example 4. *Let* $K = \mathbb{Z}$, $S = \{s, r\}$ *and* $\Psi = \{0_S, 1_S, r_1\}$*. Since* (K, Ψ, S) *is soft locally countable, then by Theorem 7,* $\Psi_{\omega_\theta} = SS(K, S)$*. Thus,* $r_1 \in \Psi_{\omega_\theta}$ *and* $Cl_{\Psi_{\omega_\theta}}(r_1) = r_1$ *while* $Cl_\Psi(r_1) = 1_S$.

Theorem 16. *If* (K, Ψ, S) *is a soft anti-locally countable STS such that* $\Psi \subseteq \Psi_{\omega_\theta}$*, then* $\alpha(K, \Psi, S) \subseteq \alpha(K, \Psi_{\omega_\theta}, S)$.

Proof. Let $H \in \alpha(K, \Psi, S)$. Then $H \widetilde{\subseteq} Int_\Psi(Cl_\Psi(Int_\Psi(H)))$. Since by assumption $\Psi \subseteq \Psi_{\omega_\theta}$, then $H \widetilde{\subseteq} Int_{\Psi_{\omega_\theta}}(Cl_\Psi(Int_{\Psi_{\omega_\theta}}(H)))$. On the other hand, since $Int_{\Psi_{\omega_\theta}}(H) \in \Psi_{\omega_\theta}$, then by Theorem 14, $Cl_\Psi(Int_{\Psi_{\omega_\theta}}(H)) = Cl_{\Psi_{\omega_\theta}}(Int_{\Psi_{\omega_\theta}}(H))$. Therefore, $H \widetilde{\subseteq} Int_{\Psi_{\omega_\theta}}(Cl_{\Psi_{\omega_\theta}}(Int_{\Psi_{\omega_\theta}}(H)))$ and hence $H \in \alpha(K, \Psi_{\omega_\theta}, S)$. □

Corollary 6. *If* (K, Ψ, S) *is soft regular and soft anti-locally countable, then* $\alpha(K, \Psi, S) \subseteq \alpha(K, \Psi_{\omega_\theta}, S)$.

Proof. Follows from Corollary 3 and Theorem 16. □

The inclusion in Theorem 16 cannot be replaced by equality in general, as will be shown in the following example:

Example 5. *Let* μ *be the usual topology on* $\mathbb{R}$*. Consider the STS* $(\mathbb{R}, \tau(\mu), [0,1])$*. Then* $(\mathbb{R}, \tau(\mu), [0,1])$ *is soft anti-locally countable. On the other hand,* $C_{\mathbb{Q}^c} \in \alpha(\mathbb{R}, (\tau(\mu))_{\omega_\theta}, [0,1]) - \alpha(\mathbb{R}, \tau(\mu), [0,1])$.

Theorem 17. *If* (K, Ψ, S) *is a soft anti-locally countable STS such that* $\Psi \subseteq \Psi_{\omega_\theta}$*, then* $RO(K, \Psi, S) = RO(K, \Psi_{\omega_\theta}, S)$.

Proof. To see that $RO(K, \Psi, S) \subseteq RO(K, \Psi_{\omega_\theta}, S)$, let $H \in RO(K, \Psi, S)$. Then $H = Int_\Psi(Cl_\Psi(H))$. Since $H \in \Psi \subseteq \Psi_{\omega_\theta}$, then by Theorem 15, $Cl_{\Psi_{\omega_\theta}}(H) = Cl_\Psi(H)$, and thus $H = Int_\Psi(Cl_{\Psi_{\omega_\theta}}(H))$. Also, since $Cl_{\Psi_{\omega_\theta}}(H) \in (\Psi_{\omega_\theta})^c$, then by Corollary 5, $H = Int_\Psi(Cl_{\Psi_{\omega_\theta}}(H)) = Int_{\Psi_{\omega_\theta}}(Cl_{\Psi_{\omega_\theta}}(H))$. Hence, $H \in RO(K, \Psi_{\omega_\theta}, S)$.

To see that $RO(K, \Psi_{\omega_\theta}, S) \subseteq RO(K, \Psi, S)$, let $H \in RO(K, \Psi_{\omega_\theta}, S)$. Then $H = Int_{\Psi_{\omega_\theta}}(Cl_{\Psi_{\omega_\theta}}(H))$. Since $Cl_{\Psi_{\omega_\theta}}(H) \in (\Psi_{\omega_\theta})^c$, then by Corollary 5, $Int_\Psi(Cl_{\Psi_{\omega_\theta}}(H)) = Int_{\Psi_{\omega_\theta}}(Cl_{\Psi_{\omega_\theta}}(H))$. Also, since $H \in \Psi \subseteq \Psi_{\omega_\theta}$, then by Theorem 15, $Cl_{\Psi_{\omega_\theta}}(H) = Cl_\Psi(H)$. Thus, $Int_\Psi(Cl_\Psi(H))$. Hence, $H \in RO(K, \Psi, S)$. □

Corollary 7. *If (K, Ψ, S) is soft regular and soft anti-locally countable, then $RO(K, \Psi, S) = RO(K, \Psi_{\omega_\theta}, S)$.*

Proof. Follows from Corollary 3 and Theorem 17. □

In Theorem 17, the condition in 'soft anti-locally countable' cannot be dropped:

Example 6. *Let $K = \{a, b, c, d, e\}$, $\mu = \{\emptyset, K, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$. Consider the STS $(K, \tau(\mu), [0, 1])$. Then $RO(K, \tau(\mu), [0, 1]) = \{0_S, 1_S\}$ but $RO(K, (\tau(\mu))_{\omega_\theta}, [0, 1]) = \tau(\mu)$.*

Theorem 18. *If (K, Ψ, S) is a soft anti-locally countable STS such that $\Psi \subseteq \Psi_{\omega_\theta}$, then $\beta(K, \Psi_{\omega_\theta}, S) \subseteq \beta(K, \Psi, S)$.*

Proof. Let $H \in \beta(K, \Psi_{\omega_\theta}, S)$. Then $H \widetilde{\subseteq} Cl_{\Psi_{\omega_\theta}}(Int_{\Psi_{\omega_\theta}}(Cl_{\Psi_{\omega_\theta}}(H)))$. Since $Cl_{\Psi_{\omega_\theta}}(H) \in (\Psi_{\omega_\theta})^c$, then by Corollary 5, $Int_\Psi(Cl_{\Psi_{\omega_\theta}}(H)) = Int_{\Psi_{\omega_\theta}}(Cl_{\Psi_{\omega_\theta}}(H))$ and thus $H \widetilde{\subseteq} Cl_{\Psi_{\omega_\theta}}(Int_\Psi(Cl_{\Psi_{\omega_\theta}}(H)))$. Also, since $\Psi \subseteq \Psi_{\omega_\theta}$, then $Cl_{\Psi_{\omega_\theta}}(H) \widetilde{\subseteq} Cl_\Psi(H)$, $Int_\Psi(Cl_{\Psi_{\omega_\theta}}(H)) \widetilde{\subseteq} Int_\Psi(Cl_\Psi(H))$, and $Cl_{\Psi_{\omega_\theta}}(Int_\Psi(Cl_\Psi(H))) \widetilde{\subseteq} Cl_\Psi(Int_\Psi(Cl_\Psi(H)))$. Therefore, $H \widetilde{\subseteq} Cl_\Psi(Int_\Psi(Cl_\Psi(H)))$. Hence, $H \in \beta(K, \Psi, S)$. □

Theorem 19. *If (K, Ψ, S) is soft anti-locally countable and soft Urysohn such that $\Psi \subseteq \Psi_{\omega_\theta}$, then $(K, \Psi_{\omega_\theta}, S)$ is soft Urysohn.*

Proof. Let $s_k, t_m \in SP(K, S)$ such that $s_k \neq t_m$. Since (K, Ψ, S) is soft Urysohn, then there are $L, M \in \Psi$ such that $s_k \widetilde{\in} L$, $t_m \widetilde{\in} M$, and $Cl_\Psi(L) \widetilde{\cap} Cl_\Psi(M) = 0_S$. Since $\Psi \subseteq \Psi_{\omega_\theta}$, then $L, M \in \Psi_{\omega_\theta}$ and by Theorem 15, $Cl_\Psi(L) = Cl_{\Psi_{\omega_\theta}}(L)$ and $Cl_\Psi(M) = Cl_{\Psi_{\omega_\theta}}(M)$. Thus, $Cl_{\Psi_{\omega_\theta}}(L) \widetilde{\cap} Cl_{\Psi_{\omega_\theta}}(M) = Cl_\Psi(L) \widetilde{\cap} Cl_\Psi(M) = 0_S$. Hence, $(K, \Psi_{\omega_\theta}, S)$ is soft Urysohn. □

Theorem 20. *If (K, Ψ, S) is soft anti-locally countable and soft regular, then $(K, \Psi_{\omega_\theta}, S)$ is soft Urysohn.*

Proof. Follows from Corollary 3 and Theorem 19. □

4. Decompositions of θ-Openness ω_θ-Openness

Herein, we display the concepts of "soft (ω_θ, ω)-sets" and "soft (ω_θ, θ)-sets" as two new classes of soft sets, and via them, we introduce decompositions of soft θ-open sets and soft ω_θ-open sets. Moreover, we examine the relationships between these classes and their analogs in general topology.

Definition 19. *Let (K, Ψ, S) be a STS and let $G \in SS(K, S)$. Then G is called*

(a) a soft (ω_θ, ω)-set in (K, Ψ, S) if $Int_{\Psi_{\omega_\theta}}(G) = Int_{\Psi_\omega}(G)$. The collection of all soft (ω_θ, ω)-sets in (K, Ψ, S) will be denoted by $\Psi_{(\omega_\theta, \omega)}$.

(b) a soft (ω_θ, θ)-set in (K, Ψ, S) if $Int_{\Psi_{\omega_\theta}}(G) = Int_{\Psi_\theta}(G)$. The collection of all soft (ω_θ, θ)-sets in (K, Ψ, S) will be denoted by $\Psi_{(\omega_\theta, \theta)}$.

Theorem 21. *Let $\{(K, \lambda_s) : s \in S\}$ be a collection of TSs and let $G \in SS(K, S)$. Then $G \in (\oplus_{s \in S} \lambda_s)_{(\omega_\theta, \omega)}$ if and only if $G(s) \in (\lambda_s)_{(\omega_\theta, \omega)}$ for all $s \in S$.*

Proof. *Necessity*. Suppose that $G \in (\oplus_{s \in S} \lambda_s)_{(\omega_\theta, \omega)}$ and let $t \in S$. Then $Int_{(\oplus_{s \in S} \lambda_s)_{\omega_\theta}}(G) = Int_{(\oplus_{s \in S} \lambda_s)_\omega}(G)$ and $\left(Int_{(\oplus_{s \in S} \lambda_s)_{\omega_\theta}}(G)\right)(t) = \left(Int_{(\oplus_{s \in S} \lambda_s)_\omega}(G)\right)(t)$. Since by Theorem 9, $(\oplus_{s \in S} \lambda_s)_{\omega_\theta} = \oplus_{s \in S} (\lambda_s)_{\omega_\theta}$ and by Theorem 8 of [34], $(\oplus_{s \in S} \lambda_s)_\omega = \oplus_{s \in S} (\lambda_s)_\omega$, then we have $\left(Int_{\oplus_{s \in S} (\lambda_s)_{\omega_\theta}}(G)\right)(t) = \left(Int_{\oplus_{s \in S} (\lambda_s)_\omega}(G)\right)(t)$. But by Lemma 4.9

of [26], $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G)\right)(t) = Int_{(\lambda_t)_{\omega_\theta}}(G(t))$ and $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega}}(G)\right)(t) = Int_{(\lambda_t)_{\omega}}(G(t))$. Therefore, $Int_{(\lambda_t)_{\omega_\theta}}(G(t)) = Int_{(\lambda_t)_{\omega}}(G(t))$ and hence $G(t) \in (\lambda_t)_{(\omega_\theta,\omega)}$.

Sufficiency. Suppose that $G(s) \in (\lambda_s)_{(\omega_\theta,\omega)}$ for all $s \in S$. Then for each $s \in S$ we have $Int_{(\lambda_s)_{\omega_\theta}}(G(s)) = Int_{(\lambda_s)_{\omega}}(G(s))$. But, by Lemma 4.9 of [26], $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G)\right)(t) = Int_{(\lambda_t)_{\omega_\theta}}(G(t))$ and $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega}}(G)\right)(t) = Int_{(\lambda_t)_{\omega}}(G(t))$ for all $t \in S$. Thus, $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G)\right)(t) = \left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega}}(G)\right)(t)$ for all $t \in S$ and hence $Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G) = Int_{\oplus_{s\in S}(\lambda_s)_{\omega}}(G)$. Since by Theorem 9, $(\oplus_{s\in S}\lambda_s)_{\omega_\theta} = \oplus_{s\in S}(\lambda_s)_{\omega_\theta}$ and by Theorem 8 of [34], $(\oplus_{s\in S}\lambda_s)_{\omega} = \oplus_{s\in S}(\lambda_s)_{\omega}$. Then we have $Int_{(\oplus_{s\in S}\lambda_s)_{\omega_\theta}}(G) = Int_{(\oplus_{s\in S}\lambda_s)_{\omega}}(G)$. Hence, $G \in (\oplus_{s\in S}\lambda_s)_{(\omega_\theta,\omega)}$. □

Corollary 8. *Let (K,μ) be a TS and let S be a set of parameters. Then $G \in (\tau(\mu))_{(\omega_\theta,\omega)}$ if and only if $G(s) \in (\tau(\mu))_{(\omega_\theta,\omega)}$ for all $s \in S$.*

Proof. Let $\mu_s = \mu$ for every $s \in S$. Then $\tau(\mu) = \oplus_{s\in S}\lambda_s$. Thus, by Theorem 21 we get the result. □

Theorem 22. *Let $\{(K,\lambda_s) : s \in S\}$ be a collection of TSs and let $G \in SS(K,S)$. Then $G \in (\oplus_{s\in S}\lambda_s)_{(\omega_\theta,\theta)}$ if and only if $G(s) \in (\lambda_s)_{(\omega_\theta,\theta)}$ for all $s \in S$.*

Proof. *Necessity*. Suppose that $G \in (\oplus_{s\in S}\lambda_s)_{(\omega_\theta,\theta)}$ and let $t \in S$. Then $Int_{(\oplus_{s\in S}\lambda_s)_{\omega_\theta}}(G) = Int_{(\oplus_{s\in S}\lambda_s)_{\theta}}(G)$ and $\left(Int_{(\oplus_{s\in S}\lambda_s)_{\omega_\theta}}(G)\right)(t) = \left(Int_{(\oplus_{s\in S}\lambda_s)_{\theta}}(G)\right)(t)$. Since by Theorem 9, $(\oplus_{s\in S}\lambda_s)_{\omega_\theta} = \oplus_{s\in S}(\lambda_s)_{\omega_\theta}$ and by Theorem 2.21 of [13], $(\oplus_{s\in S}\lambda_s)_{\theta} = \oplus_{s\in S}(\lambda_s)_{\theta}$, then we have $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G)\right)(t) = \left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega}}(G)\right)(t)$. But by Lemma 4.9 of [26], $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G)\right)(t) = Int_{(\lambda_t)_{\omega_\theta}}(G(t))$ and $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\theta}}(G)\right)(t) = Int_{(\lambda_t)_{\theta}}(G(t))$. Therefore, $Int_{(\lambda_t)_{\omega_\theta}}(G(t)) = Int_{(\lambda_t)_{\theta}}(G(t))$ and hence $G(t) \in (\lambda_t)_{(\omega_\theta,\theta)}$.

Sufficiency. Suppose that $G(s) \in (\lambda_s)_{(\omega_\theta,\theta)}$ for all $s \in S$. Then for each $s \in S$ we have $Int_{(\lambda_s)_{\omega_\theta}}(G(s)) = Int_{(\lambda_s)_{\theta}}(G(s))$. But, by Lemma 4.9 of [26], $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G)\right)(t) = Int_{(\lambda_t)_{\omega_\theta}}(G(t))$ and $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\theta}}(G)\right)(t) = Int_{(\lambda_t)_{\theta}}(G(t))$ for all $t \in S$. Thus, $\left(Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G)\right)(t) = \left(Int_{\oplus_{s\in S}(\lambda_s)_{\theta}}(G)\right)(t)$ for all $t \in S$ and hence $Int_{\oplus_{s\in S}(\lambda_s)_{\omega_\theta}}(G) = Int_{\oplus_{s\in S}(\lambda_s)_{\theta}}(G)$. Since by Theorem 9, $(\oplus_{s\in S}\lambda_s)_{\omega_\theta} = \oplus_{s\in S}(\lambda_s)_{\omega_\theta}$ and by Theorem 2.21 of [13], $(\oplus_{s\in S}\lambda_s)_{\theta} = \oplus_{s\in S}(\lambda_s)_{\theta}$. Thus, we have $Int_{(\oplus_{s\in S}\lambda_s)_{\omega_\theta}}(G) = Int_{(\oplus_{s\in S}\lambda_s)_{\theta}}(G)$. Hence, $G \in (\oplus_{s\in S}\lambda_s)_{(\omega_\theta,\theta)}$. □

Corollary 9. *Let (K,μ) be a TS and let S be a set of parameters. Then $G \in (\tau(\mu))_{(\omega_\theta,\theta)}$ if and only if $G(s) \in (\tau(\mu))_{(\omega_\theta,\theta)}$ for all $s \in S$.*

Proof. Let $\mu_s = \mu$ for every $s \in S$. Then $\tau(\mu) = \oplus_{s\in S}\lambda_s$. Thus, by Theorem 21 we get the result. □

Theorem 23. *For any STS (K,Ψ,S), $\Psi_{\omega_\theta} \subseteq \Psi_{(\omega_\theta,\omega)}$.*

Proof. Let $G \in \Psi_{\omega_\theta}$. Then $Int_{\Psi_{\omega_\theta}}(G) = G$. On the other hand, since by Theorem 5 and Theorem 3, $\Psi_{\omega_\theta} \subseteq \Psi_{\omega^0} \subseteq \Psi_{\omega}$, then $G = Int_{\Psi_{\omega_\theta}}(G)\widetilde{\subseteq}Int_{\Psi_{\omega}}(G)$. Therefore, $G = Int_{\Psi_{\omega_\theta}}(G) = Int_{\Psi_{\omega}}(G)$ and hence $G \in \Psi_{(\omega_\theta,\omega)}$. □

Theorem 24. *For any STS (K,Ψ,S), $\Psi_{\theta} \subseteq \Psi_{(\omega_\theta,\theta)}$.*

Proof. Let $G \in \Psi_\theta$. Then $Int_{\Psi_\theta}(G) = G$. On the other hand, since by Theorem 5, $\Psi_\theta \subseteq \Psi_{\omega_\theta}$, then $G = Int_{\Psi_\theta}(G) \widetilde{\subseteq} Int_{\Psi_{\omega_\theta}}(G)$. Therefore, $G = Int_{\Psi_\theta}(G) = Int_{\Psi_{\omega_\theta}}(G)$ and hence $G \in \Psi_{(\omega_\theta,\theta)}$. □

The following two examples show that Theorems 23 and 24 are not reversible in general:

Example 7. *Let $K = \mathbb{R}$, $S = \{s, r\}$ and $\Psi = \{0_S, 1_S, C_{\mathbb{R}-\mathbb{Q}}\}$. Let $G = C_{\mathbb{N}}$. Since (K, Ψ, S) is soft anti-locally countable and $G \in \Psi^c$, then by Theorem 14 of [34], $Int_{\Psi_\omega}(G) = Int_\Psi(G) = 0_S$. Also, since by Theorem 5 and Theorem 3, $\Psi_{\omega_\theta} \subseteq \Psi_{\omega^0} \subseteq \Psi_\omega$, then $Int_{\Psi_{\omega_\theta}}(G) \widetilde{\subseteq} Int_{\Psi_\omega}(G) = 0_S$. Thus, $Int_{\Psi_\omega}(G) = Int_{\Psi_{\omega_\theta}}(G) = 0_S$ and hence $G \in \Psi_{(\omega_\theta,\theta)}$. Suppose that $Int^\theta_\Psi(G) \neq 0_S$. Then there exists $s_k \widetilde{\in} Int^\theta_\Psi(G)$ and so there is $H \in \Psi$ such that $s_k \widetilde{\in} H \widetilde{\subseteq} Cl_\Psi(H) \widetilde{\subseteq} G$. Since $s_k \widetilde{\in} H \widetilde{\subseteq} G$, then $H = C_{\mathbb{R}-\mathbb{Q}}$ and $C_{\mathbb{R}-\mathbb{Q}} \widetilde{\subseteq} C_{\mathbb{N}}$ which is impossible. Therefore, $Int^\theta_\Psi(G) = 0_S$. If $G \in \Psi_{\omega_\theta}$, then there are $N \in \Psi$ and $M \in CSS(K, S)$ such that $s_1 \widetilde{\in} N$ and $N - M \widetilde{\subseteq} Int^\theta_\Psi(G) = 0_S$. Thus, $N \widetilde{\subseteq} M$ and hence $N \in CSS(K, S)$. On the other hand, since $s_1 \widetilde{\in} N \in \Psi$, then either $N = C_{\mathbb{R}-\mathbb{Q}}$ or $N = 1_S$ and in both cases $N \notin CSS(K, S)$. It follows that $G \notin \Psi_{\omega_\theta}$.*

Example 8. *Let $K = \mathbb{R}$, $S = \{s, r\}$ and $\Psi = \{0_S, 1_S, C_{(-\infty,1)}\}$. Let $G = C_{(2,\infty)}$. Since $\Psi_\theta \subseteq \Psi$ and $G \notin \Psi$, then $G \notin \Psi_\theta$. Again since $\Psi_\theta \subseteq \Psi$, then $Int_{\Psi_\theta}(G) \widetilde{\subseteq} Int_\Psi(G) = 0_S$. Suppose that $Int_{\Psi_{\omega_\theta}}(G) \neq 0_S$. Then there exists $s_k \widetilde{\in} Int_{\Psi_{\omega_\theta}}(G) \in \Psi_{\omega_\theta}$ and so there are $N \in \Psi$ and $M \in CSS(K, S)$ such that $s_3 \widetilde{\in} N$ and $N - M \widetilde{\subseteq} Int^\theta_\Psi(G) \widetilde{\subseteq} Int_\Psi(G) = 0_S$. Thus, $N \widetilde{\subseteq} M$ and hence $N \in CSS(K, S)$. On the other hand, since $s_3 \widetilde{\in} N \in \Psi$, then either $N = C_{(-\infty,1)}$ or $N = 1_S$ and in both cases $N \notin CSS(K, S)$. Therefore, $Int_{\Psi_\theta}(G) = Int_{\Psi_{\omega_\theta}}(G) = 0_S$ and hence, $G \in \Psi_{(\omega_\theta,\theta)}$.*

Theorem 25. *For any STS (K, Ψ, S), $\Psi_{\omega_\theta} = \Psi_\omega \cap \Psi_{(\omega_\theta,\omega)}$.*

Proof. By Theorem 5 and Theorem 3, we have $\Psi_{\omega_\theta} \subseteq \Psi_{\omega^0} \subseteq \Psi_\omega$. Also, by Theorem 23, we have $\Psi_{\omega_\theta} \subseteq \Psi_{(\omega_\theta,\omega)}$. Hence, $\Psi_{\omega_\theta} \subseteq \Psi_\omega \cap \Psi_{(\omega_\theta,\omega)}$. To see that $\Psi_\omega \cap \Psi_{(\omega_\theta,\omega)} \subseteq \Psi_{\omega_\theta}$, let $G \in \Psi_\omega \cap \Psi_{(\omega_\theta,\omega)}$. Then $G \in \Psi_\omega$ and $G \in \Psi_{(\omega_\theta,\omega)}$. Since $G \in \Psi_\omega$, then $Int_{\Psi_\omega}(G) = G$. Since $G \in \Psi_{(\omega_\theta,\omega)}$, then $Int_{\Psi_{\omega_\theta}}(G) = Int_{\Psi_\omega}(G)$. Thus, we have $Int_{\Psi_{\omega_\theta}}(G) = G$, and hence $G \in \Psi_{\omega_\theta}$. □

Theorem 26. *For any STS (K, Ψ, S), $\Psi_\theta = \Psi_{\omega_\theta} \cap \Psi_{(\omega_\theta,\theta)}$.*

Proof. By Theorem 5, we have $\Psi_\theta \subseteq \Psi_{\omega_\theta}$. Also, by Theorem 24, we have $\Psi_\theta \subseteq \Psi_{(\omega_\theta,\theta)}$. Hence, $\Psi_\theta \subseteq \Psi_{\omega_\theta} \cap \Psi_{(\omega_\theta,\theta)}$. To see that $\Psi_{\omega_\theta} \cap \Psi_{(\omega_\theta,\theta)} \subseteq \Psi_\theta$, let $G \in \Psi_{\omega_\theta} \cap \Psi_{(\omega_\theta,\theta)}$. Then $G \in \Psi_{\omega_\theta}$ and $G \in \Psi_{(\omega_\theta,\theta)}$. Since $G \in \Psi_{\omega_\theta}$, then $Int_{\Psi_{\omega_\theta}}(G) = G$. Since $G \in \Psi_{(\omega_\theta,\theta)}$, then $Int_{\Psi_{\omega_\theta}}(G) = Int_{\Psi_\theta}(G)$. Thus, we have $Int_{\Psi_\theta}(G) = G$, and hence $G \in \Psi_\theta$. □

We expect Theorems 25 and 26 to play an important role in specific types of soft continuity that can be defined by the classes of soft sets introduced in this paper. In particular, they will give decomposition theorems for such soft continuity types.

5. Conclusions

This paper belongs to the field of soft topology. The concepts of soft ω_θ-open sets, soft (ω_θ, ω)-sets, and soft (ω_θ, θ)-sets in soft topological spaces are introduced, and their properties are investigated. In particular, the relationships between these classes of soft sets and their analogs in general topology are examined (Theorems 8, 9, 21 and 22, and Corollaries 1, 2, 7 and 8). Also, it is proved that the family of soft ω_θ-open sets form a soft topology that lies between the soft topologies of soft θ-open sets and soft ω^0-open sets (Therems 5 and 6). Moreover, it is proved that the soft topologies of ω_θ-open sets and soft ω^0-open sets are equivalent for soft regular spaces (Theorem 10). Furthermore, it is proved that the soft topology of ω_θ-open sets is a soft discrete space for soft locally countable spaces (Theorem 7). In addition to these, decomposition theorems of soft θ-openness and soft ω_θ-openness in terms of soft (ω_θ, ω)-sets and soft (ω_θ, θ)-sets are introduced (Theorems 25 and 26).

In the upcoming work, we plan: (1) To define soft separation axioms via soft ω_θ-open sets; (2) To investigate the behavior of our new notions under product STSs; (3) To define soft ω_θ-continuity; (4) To extend these concepts to include soft bi-topological spaces.

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