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Some Subordination Results Defined by Using the Symmetric q-Differential Operator for Multivalent Functions

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Abstract: In this article, we use the concept of symmetric q-calculus and convolution in order to define a symmetric q-differential operator for multivalent functions. This operator is an extension of the classical Ruscheweyh differential operator. By using the technique of differential subordination, we derive several interesting applications of the newly defined operator for multivalent functions.

Keywords: analytic functions; symmetric q-calculus; symmetric q-differential operator; multivalent functions; convolution; subordination

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1. Introduction

The study of q-calculus has motivated scholars due to its wide range of applications in different areas of mathematics and physics. Jackson [1,2] was the first to consider the q-calculus theory in order to define the q-derivative (∂_q) and the q-integral operator. Meanwhile, in [3], Ismail et al. used ∂_q and defined q-starlike functions in the field of Geometric function theory and investigated some interesting applications. Later on, Srivastava [4], used the q-calculus in the context of univalent functions theory and he developed many important results. The q-analogue of Ruscheweyh differential operator was introduced by Kanas and Raducanu [5] while in [6], Srivastava et al. introduced the q-Noor integral operator and studied some of its applications for bi-univalent functions. In particular, Srivastava [7,8] pointed out many applications and mathematical explanations of q-derivatives in Geometric function theory. In recent years, many researchers have defined a number of q-differential and integral operators and have published many important results associated with q-starlike and the Janowski functions (for details, see [9–14]).

Let $q \in (0,1)$ and $[l]_q = \frac{1-q^l}{1-q}$ be the q-number for $l \in \mathbb{N}$ and $[l]_q! = \prod\limits_{k=1}^{l} [k]_q$ be the factorial and $[0]_q! = 1$.

The q-Gamma function is defined as:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \text{ and } \Gamma_q(1) = 1.$$

Jackson [1] defined the q-difference operator for analytic functions in the following form:

$$\partial_q \xi(\nu) = \frac{\xi(q\nu) - \xi(\nu)}{\nu(q-1)}, \quad q \in (0,1).$$

Additionally, we have

$$\partial_q \nu^l = [l]_q \nu^{l-1}, \quad \partial_q \left\{ \sum_{l=1}^{\infty} a_l \nu^l \right\} = \sum_{l=1}^{\infty} [l]_q a_l \nu^{l-1}.$$

It can be observed that

$$\lim_{q \to 1^-} \partial_q \xi(\nu) = \xi'(\nu).$$

The symmetric q-calculus has been found to be very useful in different areas, such as fractional calculus and quantum mechanics. The applications of quantum mechanics are discussed in q-symmetric variational calculus in [15] while in [16], Lavagno discussed the symmetric q-calculus in the field of basic-deformed quantum mechanics. More recently, Kanas et al. [17] investigated some new applications of the symmetric q-derivative related to the conic domain and studied a new subclass of analytic functions in the open unit disk U. Khan et al. [18] investigated the new version of generalized symmetric conic domains using the basic concepts of symmetric q-calculus and studied a new subclass of q-symmetric starlike functions. Recently, a number of authors used the q-symmetric operator and studied some new subclasses of analytic functions, (see [19–21]). Here, we present the basic concepts of symmetric q-calculus, which will be useful for our subsequent work.

The symmetric q-number $\widetilde{[l]}_q$ can be defined as:

$$\widetilde{[l]}_q = \frac{q^{-l} - q^l}{q^{-1} - q}, \tag{1}$$

and the symmetric q-number shift factorial is given by

$$\widetilde{[l]}_q! = \widetilde{[l]}_q \widetilde{[l-1]}_q \widetilde{[l-2]}_q \dots \widetilde{[2]}_q \widetilde{[1]}_q, \quad l \geq 1.$$

It can be noted that

$$\widetilde{[0]}_q = 0, \quad \widetilde{[0]}_q! = 1.$$

It is worth mentioning that the symmetric q-number cannot reduce to q-number.

Kamel and Yosr [22] defined the symmetric q-derivative operator for the analytic function, which can be written as follows:

$$\begin{aligned} \widetilde{\partial}_q h(\nu) &= \frac{1}{\nu}\left(\frac{\xi(q\nu) - \xi(q^{-1}\nu)}{q - q^{-1}}\right), \quad \nu \in U, \\ &= 1 + \sum_{l=1}^{\infty} \widetilde{[l]}_q a_l \nu^{l-1}, \quad (\nu \neq 0,\ q \neq 1), \end{aligned} \tag{2}$$

and

$$\widetilde{\partial}_q \nu^l = \widetilde{[l]}_q \nu^{l-1}, \quad \widetilde{\partial}_q \left\{ \sum_{l=1}^{\infty} a_l \nu^l \right\} = \sum_{l=1}^{\infty} \widetilde{[l]}_q a_l \nu^{l-1}.$$

It can be observed that

$$\lim_{q \to 1^-} \widetilde{\partial}_q \xi(\nu) = \xi'(\nu).$$

Let $\mathcal{P}$ represents the set of all functions p, which satisfies the conditions $p(0) = 1$ and $\Re(p(\nu)) > 0$. Let ξ and ρ be analytic in U. If there exists a Schwarz function u, such that $\xi(\nu) = \rho(u(\nu))$, then we will say that ξ is subordinate to ρ (written as $(\xi(\nu) \prec \rho(\nu))$ for $\nu \in \mathcal{U}$.

Assume that $\mathcal{A}_\psi$ denote the class of multivalent functions of the form:

$$\xi(\nu) = \nu^\psi + \sum_{l=2}^{\infty} a_{l+\psi-1} \nu^{l+\psi-1}, \quad (\psi \in \mathbb{N}).$$

The convolution of two functions $\xi_i(\nu) \in \mathcal{A}_\psi$, for $i = 1, 2.$ is defined as:

$$(\xi_1 * \xi_2)(\nu) = \nu^\psi + \sum_{l=2}^{\infty} a_{l+\psi-1,1} a_{l+\psi-1,2} \nu^{l+\psi-1} = (\xi_2 * \xi_1)(\nu),$$

where

$$\xi_i(\nu) = \nu^\psi + \sum_{l=2}^{\infty} a_{l+\psi-1,i} \nu^{l+\psi-1} \ \ (\psi \in \mathbb{N}).$$

Janowski [23] defined the function $h(\mathfrak{Q}_1, \mathfrak{Q}_2; \nu) = \frac{1+\mathfrak{Q}_1\nu}{1+\mathfrak{Q}_2\nu}, \ -1 \leq \mathfrak{Q}_2 < \mathfrak{Q}_1 \leq 1, \ (\nu \in \mathcal{U})$. The image of the unit disc under the mapping $h(\mathfrak{Q}_1, \mathfrak{Q}_2; \nu)$ is the disk symmetrical with respect to the real axis, with its center at $\left(\frac{1-\mathfrak{Q}_1\mathfrak{Q}_2}{1-\mathfrak{Q}_2^2}, 0\right)$ for $\mathfrak{Q}_2 \neq \pm 1$, and the end points of the diameter are

$$A\left(\frac{1-\mathfrak{Q}_1}{1-\mathfrak{Q}_2}, 0\right) \text{ and } B\left(\frac{1+\mathfrak{Q}_1}{1+\mathfrak{Q}_2}, 0\right).$$

In our current investigation, we aim to use basic concepts of symmetric q-calculus and the convolution theory to define a new operator for multivalent analytic functions.

Definition 1. *Let $\xi \in \mathcal{A}_\psi$. Then, the symmetric q-differential operator $\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}} : \mathcal{A}_\psi \to \mathcal{A}_\psi$ is defined as:*

$$\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}} \xi(\nu) = Q_{\lambda,q}^\psi(\nu) * \xi(\nu), \quad \lambda > -1,$$

where

$$Q_{\lambda,q}^\psi(\nu) = \nu^\psi + \sum_{l=2}^{\infty} \frac{\widetilde{[\lambda+\psi]}_{l-\psi,q}}{\widetilde{[l-\psi]}_q!} \nu^{l+\psi-1}.$$

By using the definition of convolution, it can be noted that

$$\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}} \xi(\nu) = \nu^\psi + \sum_{l=2}^{\infty} \frac{\widetilde{[\lambda+\psi]}_{l-\psi,q}}{\widetilde{[l-\psi]}_q!} a_{l+\psi-1} \nu^{l+\psi-1}. \tag{3}$$

From (3), the following identity can be easily verified:

$$q^\lambda \nu \widetilde{\partial}_q \widetilde{\mathfrak{F}_q^{\lambda+\psi-1}} \xi(\nu) = \widetilde{[\lambda+\psi]}_q - \widetilde{[\lambda]}_q \widetilde{\mathfrak{F}_q^{\lambda+\psi-1}} \xi(\nu). \tag{4}$$

For $\psi = 1$, and $q \to 1^-$, Identity (4), implies that

$$\nu \widetilde{\partial}_q \mathfrak{F}_q^\lambda \xi(\nu) = (\lambda+1) \mathfrak{F}_q^{\lambda+1} \xi(\nu) - (\lambda) \mathfrak{F}_q^\lambda \xi(\nu),$$

which is the well-known relation studied by Ruscheweyh in [24].

Remark 1. *If $\psi = 1$ and $q \to 1^-$, the q-differential operator $\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}} \xi(\nu)$ reduces to the Ruscheweyh differential operator introduced by Ruscheweyh in [24].*

2. Lemmas

To prove our main results, we need the following Lemmas:

Lemma 1 ([11])**.** *If an analytic function $\varphi(\nu) \in \mathcal{P}(\beta)$ and*

$$\varphi(\nu) = 1 + c_1\nu + c_2\nu^2 + \cdots, \tag{5}$$

then

$$\Re(\varphi(\nu)) > \beta, \ \ (0 \le \beta < 1).$$

Lemma 2 ([25])**.** *If $\varphi_i \in \mathcal{P}(\beta_j)$ be given by (5), $(0 \le \beta_i < 1, i = 1, 2)$. Then,*

$$\varphi_1 * \varphi_2 \in \mathcal{P}(\beta_3), \ \ \textit{where} \ \ \beta_3 = 1 - (1-\beta_1)(1-\beta_2).$$

Lemma 3 ([26])**.** *If the function φ of the form (5) is in the class $\mathcal{P}(\beta)$, then*

$$\Re(\varphi(\nu)) > 2\beta - 1 + \frac{2(1-\beta)}{1+|\nu|}, \quad (0 \le \beta < 1).$$

Lemma 4 ([27])**.** *The function*

$$(1-\nu)^{\gamma} = e^{\gamma \log(1-\nu)}, \gamma \neq 0,$$

is univalent in $\mathcal{U}$ if and only if γ is either in closed disk

$$|\gamma - 1| \le 1 \ \textit{or} \ |\gamma + 1| \le 1.$$

We can prove Lemmas 5 and 6 using the similar method of Lemmas proved in [11].

Lemma 5. *Let $h(\nu)$ be analytic and convex univalent in $\mathcal{U}$ with $h(0) = 1$. Additionally, let $\rho(\nu) = 1 + b_1\nu + b_2\nu^2 + \cdots$ be analytic in $\mathcal{U}$. If*

$$\rho(\nu) + \frac{\nu\widetilde{\partial_q}\rho(\nu)}{c} \prec h(\nu), \quad (\nu \in \mathcal{U}, \ c \neq 0), \Re(c) \ge 0, \tag{6}$$

then

$$\rho(\nu) \prec \frac{c}{\nu^c}\int_0^{\nu} t^{c-1}h(t)dt.$$

Proof. Suppose that $\mathfrak{h}$ is analytic and convex univalent in $\mathcal{U}$ and ρ is analytic in $\mathcal{U}$. Letting $q \to 1^-$, in (6)

$$\rho(\nu) + \frac{\nu\rho^{'}(\nu)}{c} \prec h(\nu), \qquad (\nu \in \mathcal{U}, \ c \neq 0).$$

Then, from Lemma in [28], we obtain

$$\rho(\nu) \prec \frac{c}{\nu^c}\int_0^{\nu} t^{c-1}h(t)dt.$$

□

Lemma 6. *Let $q(\nu)$ be univalent in $\mathcal{U}$ and let $\theta(w)$ and $\varphi(w)$ be analytic in domain D containing $u(\mathcal{U})$ with $\varphi(w) \neq 0$ when $w \in u(\mathcal{U})$. Set*

$$Q(\nu) = \nu\widetilde{\partial_q}(u(\nu))\varphi(u(\nu)), \ h(\nu) = \theta(u(\nu) + Q(\nu)) \tag{7}$$

and suppose that

(i) $Q(\nu)$ *is starlike univalent in* $\Im$.

(ii) $\Re\left(\frac{\nu\widetilde{\partial_q}(h(\nu)}{Q(\nu)}\right) = \Re\left(\frac{\succeq\widetilde{\partial_q}\theta(u(\nu))}{(\varphi(u(\nu))}\right) + \left(\frac{\nu\widetilde{\partial}qQ(\nu)}{Q(\nu)}\right) > 0.$

If $p(\nu)$ is analytic in $\mathcal{U}$, with $p(\nu) = u(0)$, $p(\mathcal{U}) \subset D$, and

$$\theta(p(\nu)) + \nu\widetilde{\partial}_q(p(\nu))\varphi(p(\nu)) \prec \theta(u(\nu)) + \nu\widetilde{\partial}_q(u(\nu))\varphi(u(\nu)) = h(\nu), \tag{8}$$

then $p(\nu) \prec u(\nu)$ and $u(\nu)$ is the best dominant.

Proof. We can prove Lemma 6 using a method similar to the one shown in Lemma 5. □

3. Main Results

Theorem 1. *Let $\lambda > 0$, $\alpha > 0$, and $-1 \leq \mathfrak{Q}_2 < \mathfrak{Q}_1 \leq 1$. If $\xi \in \mathcal{A}_\psi$ satisfies*

$$(1-\alpha)\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi} + \alpha\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi(\nu)}{\nu^\psi} \prec h(\mathfrak{Q}_1, \mathfrak{Q}_2, \nu),$$

then

$$\Re\left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi}\right)^{\frac{1}{l}}$$

$$> \left(l(\psi)\int_0^1 u^{\left(\widetilde{[\lambda+\psi]}_q/\alpha\widetilde{[\psi]}_q q^\lambda\right)-1}\left(\frac{1-\mathfrak{Q}_1 u}{1-\mathfrak{Q}_2 u}\right)du\right)^{\frac{1}{l}}, \quad l \geq 1, \tag{9}$$

where

$$l(\psi) = \frac{\widetilde{[\lambda+\psi]}_q}{\alpha\widetilde{[\psi]}_q q^\lambda} \text{ and } w(\lambda) = \widetilde{[\lambda+\psi]}_q.$$

Proof. Let

$$\rho(\nu) = \frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi}. \tag{10}$$

For $\xi \in \mathcal{A}_\psi$, and by taking the logarithmic differentiation of (10), we get

$$\frac{\nu\widetilde{\partial}_q(\rho(\nu))}{\rho(\nu)} = \frac{\nu\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)} - \widetilde{[\psi]}_q.$$

Using the identity (4), we get

$$\frac{\nu\widetilde{\partial}_q(\rho(\nu))}{\rho(\nu)} = \frac{w(\lambda)}{q^\lambda}\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi(\nu)}{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)} - \frac{\widetilde{[\lambda]}_q}{q^\lambda} - \widetilde{[\psi]}_q.$$

Let $w(\lambda) = \widetilde{[\lambda]}_q + \widetilde{[\psi]}_q q^\lambda$. Then,

$$\frac{q^\lambda}{w(\lambda)}\nu\widetilde{\partial}_q\rho(\nu) + \rho(\nu) = \frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi(\nu)}{\nu^\psi}. \tag{11}$$

From (4), (10) and (11), we have

$$\rho(\nu) + \frac{1}{l(\psi)}\nu\widetilde{\partial}_q\rho(\nu) \prec h(\mathfrak{Q}_1, \mathfrak{Q}_2, \nu). \tag{12}$$

Now, by applying the Lemma 5, we have

$$\rho(\nu) \prec l(\psi)\nu^{-w(\lambda)/\alpha[\widetilde{\psi}]_q q^\lambda} \int_0^1 u^{\left(w(\lambda)/\alpha[\widetilde{\psi}]_q q^\lambda\right)^{-1}} \left(\frac{1+\mathfrak{Q}_1 t}{1+\mathfrak{Q}_2 t}\right) dt.$$

Using the definition of subordination, we get

$$\left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi}\right) = l(\psi) \int_0^1 u^{\left(w(\lambda)/\alpha[\widetilde{\psi}]_q q^\lambda\right)^{-1}} \left(\frac{1+\mathfrak{Q}_1 uw(\nu)}{1+\mathfrak{Q}_2 uw(\nu)}\right) du. \tag{13}$$

In view of $-1 \le \mathfrak{Q}_2 < \mathfrak{Q}_1 \le 1$ and $\lambda > 0$, it follows from (13) that

$$\Re\left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi}\right) > l(\psi) \int_0^1 u^{\left(w(\lambda)/\alpha[\widetilde{\psi}]_q q^\lambda\right)^{-1}} \left(\frac{1-\mathfrak{Q}_1 u}{1-\mathfrak{Q}_2 u}\right) du. \tag{14}$$

Since

$$\Re(w^{\frac{1}{l}}) \ge (\Re w)^{1/l} \text{ for } \Re w > 0 \text{ and } l \ge 1.$$

Therefore, the inequality (9) is proved.
To show the sharpness of (9), we define $\xi \in \mathcal{A}_\psi$ as:

$$\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi} = l(\psi) \int_0^1 u^{\left(w(\lambda)/\alpha[\widetilde{\psi}]_q q^\lambda\right)^{-1}} \left(\frac{1+\mathfrak{Q}_1 u\nu}{1+\mathfrak{Q}_2 u\nu}\right) du.$$

For this function, we find that

$$(1-\alpha)\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi} + \alpha\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi(\nu)}{\nu^\psi} = \frac{1+\mathfrak{Q}_1\nu}{1+\mathfrak{Q}_2\nu}.$$

So,

$$\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^\psi} \to l(\psi) \int_0^1 u^{\left(w(\lambda)/\alpha[\widetilde{\psi}]_q q^\lambda\right)^{-1}} \left(\frac{1-\mathfrak{Q}_1 u}{1-\mathfrak{Q}_2 u}\right) du, \text{ as } \nu \to -1.$$

This completes the proof. □

Corollary 1. *Let $q \to 1^-$, $\lambda > 0$, $\alpha > 0$, and $-1 \le \mathfrak{Q}_2 < \mathfrak{Q}_1 \le 1$. If $\xi \in \mathcal{A}_\psi$ satisfies*

$$(1-\alpha)\frac{\mathfrak{F}^{\lambda+\psi-1}\xi(\nu)}{\nu^\psi} + \alpha\frac{\mathfrak{F}^{\lambda+\psi}\xi(\nu)}{\nu^\psi} \prec h(\mathfrak{Q}_1, \mathfrak{Q}_2, \nu),$$

then

$$\Re\left(\frac{\mathfrak{F}^{\lambda+\psi-1}\xi(\nu)}{\nu^\psi}\right)^{\frac{1}{l}}$$

$$> \left(\frac{(\lambda+\psi)}{\alpha\psi} \int_0^1 u^{((\lambda+\psi)/\alpha)-1} \left(\frac{1-\mathfrak{Q}_1 u}{1-\mathfrak{Q}_2 u}\right) du\right)^{\frac{1}{l}}, \quad l \ge 1.$$

Corollary 2. *Let $\psi = 1$, $q \to 1^-$, $\lambda > 0$, $\alpha > 0$, and $-1 \le \mathfrak{Q}_2 < \mathfrak{Q}_1 \le 1$. If $\xi \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{\mathfrak{F}^{\lambda}\xi(\nu)}{\nu} + \alpha\frac{\mathfrak{F}^{\lambda+1}\xi(\nu)}{\nu} \prec h(\mathfrak{Q}_1, \mathfrak{Q}_2, \nu),$$

then

$$\Re\left(\frac{\mathfrak{F}^{\lambda}\xi(\nu)}{\nu}\right)^{\frac{1}{l}}$$

$$> \left(\frac{(\lambda+1)}{\alpha}\int_0^1 u^{((\lambda+1)/\alpha)-1}\left(\frac{1-\mathfrak{Q}_1 u}{1-\mathfrak{Q}_2 u}\right)du\right)^{\frac{1}{l}}, \ \ l \ge 1.$$

Corollary 3. *Let $\mathfrak{Q}_1 = 1-2\alpha$, $\mathfrak{Q}_2 = -1$, $\alpha, \lambda > 1$, $l \ge 1$ and $0 \le \beta < 1$. If $\xi \in \mathcal{A}(\psi)$ satisfies*

$$(1-\alpha)\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^{\psi}} + \alpha\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi(\nu)}{\nu^{\psi}} \prec h(1-2\alpha, -1, \nu),$$

then,

$$\Re\left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^{\psi}}\right)^{\frac{1}{l}}$$

$$> \left(\left(\frac{2(1-\beta)\mathfrak{w}(\lambda)}{\alpha\widetilde{[\psi]}_q q^{\lambda}}\int_0^1 \frac{u^{\left(w(\lambda)/\alpha\widetilde{[\psi]}_q q^{\lambda}\right)-1}}{1+u}du\right) + (2\beta-1)u^{\frac{w(\lambda)}{\alpha\widetilde{[\psi]}_q q^{\lambda}}}\right)^{\frac{1}{l}}.$$

Proof. Following the same steps detailed in the proof of Theorem 1 and by considering

$$\rho(\nu) = \frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^{\psi}},$$

the differential subordination (12) becomes

$$\rho(\nu) + \frac{1}{l(\psi)}\nu\widetilde{\partial}_q\rho(\nu) \prec \frac{1+(2\beta-1)\nu}{1+\nu}.$$

Therefore,

$$\Re\left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^{\psi}}\right)^{\frac{1}{l}}$$

$$> \left(l(\psi)\int_0^1 u^{\left(w(\lambda)/\alpha\widetilde{[\psi]}_q q^{\lambda}\right)-1}\left(\frac{1+(2\beta-1)u}{1+u}\right)du\right)^{\frac{1}{l}}$$

$$= \left(l(\psi)\int_0^1 u^{\left(w(\lambda)/\alpha\widetilde{[\psi]}_q q^{\lambda}\right)-1} \times \left((2\beta-1) + \frac{2(1-\beta)u}{1+u}\right)du\right)^{\frac{1}{l}}$$

$$= \left(\left(\frac{2(1-\beta)l(\psi)}{}\int_0^1 \frac{u^{\left(w(\lambda)/\alpha\widetilde{[\psi]}_q q^{\lambda}\right)-1}}{1+u}du\right) + (2\beta-1)u^{\left(w(\lambda)/\alpha\widetilde{[\psi]}_q q^{\lambda}\right)-1}\right)^{\frac{1}{l}}.$$

□

Theorem 2. *Let $\lambda > 0$ and $0 \leq \mu < 1$ and γ be a complex number with $\gamma \neq 0$ satisfying either*

$$|2\gamma(1-\mu)(w(\lambda)/\alpha q^{\lambda}\widetilde{[\psi]}_q - 1| \leq 1$$

or

$$|2\gamma(1-\mu)(w(\lambda)/\alpha q^{\lambda}\widetilde{[\psi]}_q + 1| \leq 1.$$

If $\xi \in \mathcal{A}_{\psi}$ satisfies the condition

$$\Re\left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi(\nu)}{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}\right) > \mu, \tag{15}$$

then

$$\Re\left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^{\psi}}\right)^{\gamma} \prec \frac{1}{(1-\nu)^{2\gamma(1-\mu)w(\lambda)/q^{\lambda}[\psi]_q}}. \tag{16}$$

Proof. Let

$$\psi(\nu) = \left(\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi(\nu)}{\nu^{\psi}}\right)^{\gamma}.$$

Using the (4), (15) and (16), we obtain

$$1 + \frac{1}{l(\psi)}\left(\frac{\nu\widetilde{\partial}_q(\rho(\nu))}{\rho(\nu)}\right) \prec \frac{1+(1-2\mu)\nu}{1-\nu}. \tag{17}$$

Assume that

$$h(\nu) = \frac{1}{(1-\nu)_q^{2\gamma(1-\mu)w(\lambda)/q^{\lambda}\widetilde{[\psi]}}}, \quad \theta(w) = 1, \quad \varphi(w) = \frac{1}{l(\psi)w}.$$

Then, $h(\nu)$ is univalent and we will show that $h(\nu)$, $\theta(w)$, and $\varphi(w)$ satisfy the conditions of Lemma 6. Note that the function

$$Q(\nu) = \nu\widetilde{\partial}_q(h(\nu))\varphi(h(\nu)) = \frac{2(1-\mu)\nu}{1-\nu},$$

is univalent starlike in U and

$$h(\nu) = \theta(h(\nu) + Q(\nu)) = \frac{1+(1-2\mu)\nu}{1-\nu}.$$

By using the Lemma 6, we obtain the required result. □

Corollary 4. *Let $q \to 1^{-}$, $\lambda > 0$ and $0 \leq \mu < 1$. Let γ be a complex number with $\gamma \neq 0$ that satisfies either*

$$|2\gamma(1-\mu)(\lambda+\psi)/\alpha\psi - 1| \leq 1$$

or

$$|2\gamma(1-\mu)(\lambda+\psi)/\alpha\psi + 1| \leq 1.$$

If $\xi \in \mathcal{A}_\psi$ satisfies the condition

$$\Re\left(\frac{\mathfrak{F}^{\lambda+\psi}\xi(\nu)}{\mathfrak{F}^{\lambda+\psi-1}\xi(\nu)}\right) > \mu.$$

Then,

$$\Re\left(\frac{\mathfrak{F}^{\lambda+\psi-1}\xi(\nu)}{\nu^{\psi}}\right)^{\gamma} \prec \frac{1}{(1-\nu)^{2\gamma(1-\mu)(\lambda+\psi)/\psi}},$$

Theorem 3. *Let $\lambda > 0$, $\alpha < 1$, and $-1 \leq \mathfrak{Q}_{2i} < \mathfrak{Q}_{1i} \leq 1$. If each of $\xi_i \in \mathcal{A}_\psi$ satisfy*

$$(1-\alpha)\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi_i(\nu)}{\nu^{\psi}} + \alpha\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi_i(\nu)}{\nu^{\psi}} \prec h(\mathfrak{Q}_{1i}, \mathfrak{Q}_{2i}, \nu),$$

then

$$(1-\alpha)\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\Theta(\nu)}{\nu^{\psi}} + \alpha\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\Theta(\nu)}{\nu^{\psi}} \prec h(1-2\gamma, -1, \nu),$$

where

$$\Theta(\nu) = \widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}(\xi_1 * \xi_2)(\nu), \tag{18}$$

and

$$\gamma = \left\{1 - l(\psi)\int_0^1 \frac{u^{\left(w(\lambda)/q^{\lambda}[\widetilde{\psi}]_q\right)-1}}{1+u}du\right\} \times \left(1 - \frac{4(\mathfrak{Q}_1-\mathfrak{Q}_2)(\mathfrak{Q}_1-\mathfrak{Q}_2)}{(1-\mathfrak{Q}_2)(1-\mathfrak{Q}_2)}\right).$$

Proof. We define the function h_i by

$$h_i = (1-\alpha)\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi_i(\nu)}{\nu^{\psi}} + \alpha\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\xi_i(\nu)}{\nu^{\psi}}, \tag{19}$$

where $\xi_i \in \mathcal{A}_\psi$, $i = 1, 2$. We have $h_i \in P(\beta_i)$, where $\beta_i = \frac{1-\mathfrak{Q}_{1i}}{1-\mathfrak{Q}_{2i}}$, $(i = 1, 2)$.

By making use of (4) and (19), we obtain

$$\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\xi_i(\nu) = l(\psi)\nu^{1-\left(w(\lambda)/q^{\lambda}[\widetilde{\psi}]_q\right)}\int_0^1 t^{\left(w(\lambda)/q^{\lambda}[\widetilde{\psi}]_q\right)-1}h_i(t)dt.$$

In the light of (18), we show that

$$\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\Theta(\nu) = l(\psi)\nu^{1-\left(w(\lambda)/q^{\lambda}[\widetilde{\psi}]_q\right)}\int_0^1 t^{\left(w(\lambda)/q^{\lambda}[\widetilde{\psi}]_q\right)-1}h_0(t)dt,$$

where, for convenience

$$\begin{aligned} h_0 &= (1-\alpha)\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi-1}}\Theta(\nu)}{\nu^{\psi}} + \alpha\frac{\widetilde{\mathfrak{F}_q^{\lambda+\psi}}\Theta(\nu)}{\nu^{\psi}} \\ &= l(\psi)\nu^{1-\left(w(\lambda)/q^{\lambda}[\widetilde{\psi}]_q\right)}\int_0^1 t^{\left(w(\lambda)/q^{\lambda}[\widetilde{\psi}]_q\right)-1}(h_1 * h_2)(t)dt. \end{aligned}$$

By using the Lemma 2, we have $(h_1 * h_2) \in \mathcal{P}(\beta_3)$, where

$$\beta_3 = 1 - 2(1-\beta_1)(1-\beta_2).$$

With an application of Lemma 3, we have

$$
\begin{aligned}
\Re h_0(\nu) &= l(\psi)\int_0^1 u^{\left(w(\lambda)/q^{\lambda}\widetilde{[\psi]}_q\right)-1}\Re(h_1 * h_2)(u\nu)du \\
&\geq l(\psi)\int_0^1 u^{\left(w(\lambda)/q^{\lambda}\widetilde{[\psi]}_q\right)-1}\left((2\beta_3-1)+\frac{2(1-\beta_3)}{1+u|\nu|}\right)du \\
&> l(\psi)\int_0^1 u^{\left(w(\lambda)/q^{\lambda}\widetilde{\psi}]_q\right)-1}\left((2\beta_3-1)+\frac{2(1-\beta_3)}{1+u}\right)du \\
&= 1-\frac{4(\mathfrak{Q}_1-\mathfrak{Q}_2)(\mathfrak{Q}_1-\mathfrak{Q}_2)}{(1-\mathfrak{Q}_2)(1-\mathfrak{Q}_2)}\times\left(1-\frac{[\lambda+\psi]_q}{\alpha\widetilde{[\psi]}_q q^{\lambda}}\int_0^1\frac{u^{\left(w(\lambda)/q^{\lambda}\widetilde{[\psi]}_q\right)-1}}{1+u}du\right) \\
&= \gamma.
\end{aligned}
$$

Hence, Theorem 3 is proved. □

4. Conclusions

By taking inspiration from recent studies on q-calculus and convolution operators for univalent functions, we have defined a new convolution operator for multivalent analytic functions. This newly defined operator for multivalent functions is an extension of the classical Ruscheweyh derivative operator. In this paper, we have successfully derived several properties for a class of multivalent analytic functions connected with a new operator by using the subordination theory. We also highlighted some consequences of our main results, which are stated in the form of corollaries.

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