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Comparative Study with Applications for Gompertz Models under Competing Risks and Generalized Hybrid Censoring Schemes

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Abstract: In reliability and survival analysis, the time-to-failure data play an important role in the development of the reliability and life characteristics of the products. In some cases, these kinds of data are modeled using a competing risks model. The problem of conducting comparative life testing under a competing risks model when the units come from different lines of production has recently been addressed. In this paper, we address this problem when the life of the unit is distributed using the Gompertz distribution, noting that the units come from two lines of production and two independent causes of failure are activated. The data are collected under a joint generalized type-II hybrid censoring scheme. Maximum likelihood estimators of the unknown parameters are derived, along with the corresponding asymptotic confidence intervals. We also adopt two bootstrap confidence intervals. Using independent gamma priors, the Bayes estimators relative to squared error loss function are obtained with credible intervals. The properties and quality of estimators are measured by performing a Monte Carlo simulation study. Finally, a real-life data set is analyzed to discuss the applicability of the proposed methods to real phenomena. The optimal plan with respect to comments on the numerical results is discussed in the conclusion.

Keywords: gompertz distribution; competing risks model; joint censoring schemes; generalized type-II hybrid censoring scheme; maximum likelihood; parametric bootstrap; Bayes methods

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1. Introduction

There are many situations in life-testing and reliability experiments in which units are lost or removed from the test before failure. The data observed from such experiments are called censored data. To save time and costs, censored data are used. Type-I and Type-II censoring schemes are the two most frequently used censoring schemes. In Type-I censoring, failures are observed until the pre-determined time τ (time censoring), while in Type-II censoring (failure censoring), when the time of r failures is reached, the experiment is terminated, where r is specified before experimenting with n items on the test: $0 < r < n$. Various modified censoring schemes such as progressive censoring and multiply censoring are also available and are used to analyze the lifetime data. In different situations, it is more common to provide the optimal test period and the corresponding number of failures needed for statistical inference. A mixture of Type-I and Type-II censoring schemes is known as a hybrid censoring scheme (HCS). This type of scheme has received considerable attention among practitioners. Several HCSs have been introduced in the literature. For example, Childs et al. [1] introduced the generalized Type-I and Type-II HCSs, Kundu and Joarder [2] introduced the progressively Type-II HCSs, and Balakrishnan et al. [3] and Lone and Panahi [4] introduced unified HCSs.

In real-life experiments, a product can fail for a variety of reasons, and these reasons are referred to as competing risks since we can only observe the product failing for one reason but not the others. In reliability and survival analysis, this kind of observations are modeled by a competing risks model. When using the competing risks models, our goal is to assess the risk of a particular cause in relation to other potential causes for failure. This model has been used earlier by different authors; for example, Cox [5] discussed the competing risks model using the exponential populations.

Several properties of a competing risks model have been presented by Crowder [6], Balakrishnan and Han [7], Modhesh and Abd-Elmougod [8], Bakoban and Abd-Elmougod [9], Debnath and Mohiuddine [10] and Alghamdi [11]. Recently, the characteristics of the competing risks model under the accelerated life test model were discussed by many authors; for example, Ganguly and Kundu [12] and Hanaa and Neveen [13]. A joint censoring scheme (JCS) may occur while conducting comparative life tests on products from different lines of production under the same conditions. This type of censoring scheme has been discussed by different authors. For example, Rao et al. [14] developed the rank order theory under JCS, while Johnson and Mehrotra, [15] presented the most locally powerful rank tests under JCS. Mehrotra and Bhattacharyya [16] used JCS to explore the problem of measuring the equality of two exponential distributions. The confidence intervals using JCS regarding the exponential distribution were developed by Mehrotra and Bhattacharyya [17]. Balakrishnan and Rasouli [18] and Rasouli and Balakrishnan [19] developed the exact likelihood inferences for the exponential distributions under JSCs and progressive JSCs. The estimation and prediction of two exponential distributions are discussed in the work of Shafay et al. [20]. Recently, this problem has been handled by Algarni et al. [21], Mondal-Kundu [22], Mondal-Kundu [23], Almarashi et al. [24], Tahani el al. [25] and Abdulaziz et al. [26]. To describe human mortality and provide actuarial tables, the Gompertz distribution was developed. This distribution is widely used as a life time distribution in demography, actuarial, biology, and medical research and plays a vital role in modelling survival times. Many product 's life times are modelled in reliability and survival studies by an increasing hazard rate or a Gompertz distribution. Assuming skewness and kurtosis of this distribution are fixed constants and independent of the distribution parameters, the Gompertz distribution has been used to obtain age-specific fertility rates. Comparative life tests are adopted for products deriving from different lines of production under the same conditions in the presence of the competing risks model. The problem of inference of unknown quantities in the population is formulated using the population characteristics and censoring methodologies. Here, we discuss these problems when the failure time of population units has a Gompertz lifetime distribution with a CDF given by

$$F(t) = 1 - \exp\left(-\frac{\theta}{\beta}(\exp(\beta t) - 1)\right),\ t > 0, \theta, \beta > 0. \tag{1}$$

The Gompertz distribution has density function that is in zero mode when $0 < \beta \leq \theta$ and hence monotonically decreases at $(0, \infty)$. However, if $\beta > \theta$, then take the mode $t_{mod} = (\frac{1}{\beta})\log(\frac{\beta}{\theta})$; hence, it increases in $(0, t_{mod})$ and decreases in (t_{mod}, ∞). For more details, see Soliman et al. [27,28]. The statistical inference of Gompertz distribution for independent competing risks model was developed by Lodhi et al. [29] and for dependent competing risks model was developed by Wang et al. [30]. Gompertz distribution reduces to exponential distribution when $\beta \to 0$. As far as we know, no works were observed under joint Type-II GHCS in the case of Gompertz distribution. In this paper, we adopted the joint Type-II GHCS in comparative Gompertz populations in the presence of a competing risks model. We used different methods of estimation: the ML, bootstrap and Bayes methods. The model parameters and reliability of the system were estimated using point and interval estimates. Different tolls such as MSE and coverage percentage were used to assess and compare the results through Monte Carlo simulation studys. Finally, we analysed a real data set to demonstrate our goals.

The rest of the article is organized as follows: A description of a generalized hybrid censoring scheme is presented in Section 2. The model and its assumptions are formulated in Section 3. In Section 4, using joint Type-II GHC competing risks data, we discuss the maximum likelihood estimation MLE of the parameters, as well as the reliability and failure rate functions. Based on the asymptotic normality of the MLEs, the approximate confidence intervals were obtained in the same section. Two bootstrap confidence intervals (based on bootstrap-p and bootstrap-t methods) are discussed in Section 5. The Bayes estimations under squared error loss function and gamma priors are obtained in Section 6. An assessment and comparison of the results, using a Monte Carlo simulation study, are reported in Section 7. Section 8 deals with a real-life data set for illustration purposes. Finally, conclusions and concluding remarks are discussed in Section 9.

2. Generalized Hybrid Censoring Scheme

For HCS, suppose that (τ, m) are the ideal test time and the corresponding number of failures. Hence, in Type-I HCS the test is terminated at $\min(\tau, T_m)$, where T_m is the failure time of m-th failure. The test is terminated at $\max(\tau, T_m)$ in Type-II HCS. For an extensive review of HCSs, see Childs et al. [1], Gupta and Kundu [31], Zhang et al. [32], Kundu and Pradhan [33] and Algarn et al. [34]. The problems of the low expected number of failures and long test time are still present in Type-I and Type-II HCSs. To solve these problems, Chanrasekar et al. [35] established the generalized hybrid censoring scheme (GHCS), which can be described as follows:

Type-I GHCS: Consider a life testing experiment with n units, two fixed positive integers (m_1, m_2) and the ideal test time τ that was previously proposed, such that $1 < m_1 < m_2 \leq n$. When the test is running, the failure time T_i, $i \geq 1$ is recorded until the failure T_{m_1} is observed. Hence, if $T_{m_1} < \tau$, then the test is terminated at ω, where $\omega = \min(T_{m_2}, \tau)$. If $T_{m_1} > \tau$, the test is terminated at $\omega = T_{m_1}$. Therefore, the data under Type-I GHCS are $\underline{\mathbf{t}} = (t_1, t_2, \ldots, t_k)$, where the number of failed units k and the corresponding test termination time ω are defined by $(k, \omega) = (m_1, T_{m_1})$ if $T_{m_1} \geq \tau$, $(k, \omega) = (m_2, T_{m_2})$, if $T_{m_1} < T_{m_2} \leq \tau$, $(k, \omega) = (m_1 \leq k \leq m_2, \tau)$, if $T_{m_1} < \tau < T_{m_2}$. For more details, see Chakrabarty et al. [36].

Type-II GHCS: Assume that n units are involved in the experiment. The two times (τ_1, τ_2), $0 < \tau_1 < \tau_2 \leq \infty$ and the integer number m have been proposed previously. When the test is running the failure time T_i, $i \geq 1$ is recorded until the time τ_1 is observed. If $T_m < \tau_1$, then the test is terminated at $\omega = \tau_1$. However, if $\tau_1 < T_m < \tau_2$, the test is terminated at $\omega = T_m$ and if $\tau_1 < \tau_2 < T_m$, the test is terminated at $\omega = \tau_2$. Therefore, the data under Type-II GHCS: $\underline{\mathbf{t}} = \{t_1, t_2, \ldots, t_k\}$, where k is the number of failed units. The integer number k and the corresponding test terminated time ω are defined by $(k, \omega) = (k > m, \tau_1)$, if $T_m < \tau_1$, $(k, \omega) = (m, T_m)$ if $\tau_1 < T_m < \tau_2$ and $(k, \omega) = (k < m, \tau_2)$, if $\tau_1 < \tau_2 < T_m$. In this paper, we adopted Type-II GHCS, which guarantees to terminate the experiment at a pre-fixed time $\tau_2 > \tau_1$, with τ_1 and τ_2 as the shortest and longest test times, respectively. The time τ_2 is the absolute longest time for which the experiment is allowed to continue, which is suitable for many applications. Hence, experiments using Type-II GHCS are guaranteed to be completed by time τ_2, which is the suitable time for which the researcher is willing to continue the experiment. The possibility of removing units from the test, other than the last point, is not available in two GHCS schemes (Type-I and Type-II). However, the possibility of removing survival units from the test is available in generalized progressive censoring schemes (GPCSs); see Balakrishnan [37], Balakrishnan and Cramer [38] and Elsherpieny et al. [39].

3. Modeling

Suppose that, from a population consisting of two lines Ω_1 and Ω_2, the joint random sample of size $N = n_1 + n_2$ is randomly selected as (n_1 from Ω_1 and n_2 from Ω_2). We considered only two potential causes of failure, and we adopted Type-II GHCS with two times (τ_1, τ_2), $0 < \tau_1 < \tau_2 \leq \infty$ and the integer number m. During the experiment, the failure time T_i, unit type $\eta_i = \{1, 0\}$ (where 1 means the unit from the line Ω_1 and 0 the unit from the line Ω_2) and the cause of failure $\rho_i = \{1, 2\}$ (failure under causes one or two) were recorded. When the first failure was observed, we recorded (t_1, η_1, ρ_1); when the second failure was observed, (t_2, η_2, ρ_2) was recorded. Under Type-II GHCS, the number of failure units and the corresponding test termination time were denoted by (k, ω), respectively. The experiment was continued until the time τ_1. If, $T_m < \tau_1$ then the test was terminated at $\omega = \tau_1$. However, if $\tau_1 < T_m < \tau_2$, the test was terminated at $\omega = T_m$ and if $\tau_1 < \tau_2 < T_m$, the test was terminated at $\omega = \tau_2$. Therefore, the observed joint Type-II GHC competing risks data were defined by: $\mathbf{t} = \{(t_1, \eta_1, \rho_1), (t_2, \eta_2, \rho_2), \ldots, (t_k, \eta_k, \rho_k)\}$, where $(k, \omega) = (k > m, \tau_1)$, if $T_m < \tau_1$, $(k, \omega) = (m, T_m)$ if $\tau_1 < T_m < \tau_2$ and $(k, \omega) = (k < m, \tau_2)$, if $\tau_1 < \tau_2 < T_m$. The proposed model under joint Type-II GHC competing risks data $\mathbf{t}$ included the following assumptions

1. The number of failures taken from line Ω_1 is given by $k_1 = \sum_{i=1}^{k} \eta_i$ and those from line Ω_2 are given by $k_2 = \sum_{i=1}^{k} (1 - \eta_i)$.
2. The number of failures taken from line Ω_1 under cause j is given by $m_{1j} = \sum_{i=1}^{k} \eta_i * \delta(\rho = j)$ and those from line Ω_2 are given by $m_{2j} = \sum_{i=1}^{k} (1 - \eta_i) * \delta(\rho = j)$.
3. The latent failure time T_i is defined by $T_i = \min(T_{is1}, T_{is2})$, and s is used to define the unit type, $i = 1, 2, \ldots, k$.
4. The i-th failure time T_{isj} of the line Ω_s and cause j, $i = 1, 2, \ldots, k$ has the Gompertz lifetime distribution with CDF given by

$$F_{sj}(t) = 1 - \exp\left(-\frac{\theta_{sj}}{\beta_s}(\exp(\beta_s t) - 1)\right), \; t > 0, \; \theta_{sj}, \beta_s > 0, \; s, \; j = 1, 2. \tag{2}$$

5. The latent failure time $T_i = \min(T_{is1}, T_{is2})$ has a Gompertz lifetime distribution with a CDF given by

$$F(t) = 1 - \exp\left(-\frac{(\theta_{s1} + \theta_{s2})}{\beta_s}(\exp(\beta_s t) - 1)\right), \; t > 0, \; \theta_{sj}, \beta_s > 0. \tag{3}$$

6. The integer number of failure m_{sj} is obtained from the line Ω_s under j; $s, j = 1, 2$ have the binomial distribution $B\left(k_s, \frac{\theta_{sj}}{\theta_{s1} + \theta_{s2}}\right)$.
7. The likelihood function of the joint Type-II GHC competing risks data $\underline{\mathbf{t}} = \{(t_1, \eta_1, \rho_1), (t_2, \eta_2, \rho_2), \ldots, (t_k, \eta_k, \rho_k)\}$, see Abdulaziz et al. [26] is given by

$$\begin{aligned} L(\underline{\mathbf{t}}|\mathbf{\Theta}) \propto \prod_{i=1}^{k} \Big\{ & \left[[f_{11}(t_i)S_{12}(t_i)]^{\delta(\rho_i=1)}[f_{12}(t_i)S_{11}(t_i)]^{\delta(\rho_i=2)}\right]^{\eta_i} \\ & \times \left[[f_{21}(t_i)S_{22}(t_i)]^{\delta(\rho_i=1)}[f_{22}(t_i)S_{21}(t_i)]^{\delta(\rho_i=2)}\right]^{1-\eta_i} \Big\} \\ & \times [S_{11}(\omega)S_{12}(\omega)]^{n_1-k_1}[S_{21}(\omega)S_{22}(\omega)]^{n_2-k_2}, \end{aligned} \tag{4}$$

where $f_{sj}(.)$ and $S_{sj}(.)$ are the density and reliability functions of type s and cause j, where $s, j = 1, 2$ and $\delta(\rho_i = j)$ are defined by

$$\delta(\rho = j) = \begin{cases} 1, \; \rho = j \\ 0, \; \rho \neq j \end{cases}, \; j = 1, \; 2, \tag{5}$$

4. Maximum Likelihood Estimation

The likelihood function (4), under joint Type-II GHC competing risks data $\underline{\mathbf{t}}=\{(t_1, \eta_1, \rho_1), (t_2, \eta_2, \rho_2), \ldots, (t_k, \eta_k, \rho_k)\}$ of Gompertz distribution (2) is reduced to

$$L(\mathbf{\Theta}|\underline{\mathbf{t}}) \propto \theta_{11}^{m_{11}}\theta_{12}^{m_{12}}\theta_{21}^{m_{21}}\theta_{22}^{m_{22}} \exp\Bigg\{\beta_1\sum_{i=1}^{k}\eta_i t_i + \beta_2\sum_{i=1}^{k}(1-\eta_i)t_i - \frac{\theta_{11}+\theta_{12}}{\beta_1}\sum_{i=1}^{k}\eta_i\exp(\beta_1 t_i) - \frac{\theta_{21}+\theta_{22}}{\beta_2}\sum_{i=1}^{k}(1-\eta_i)\exp(\beta_2 t_i) - \frac{(n_1-k_1)(\theta_{11}+\theta_{12})}{\beta_1}\exp(\beta_1\omega) - \frac{(n_2-k_2)(\theta_{21}+\theta_{22})}{\beta_2}\exp(\beta_2\omega) + \frac{n_1(\theta_{11}+\theta_{12})}{\beta_1} + \frac{n_2(\theta_{21}+\theta_{22})}{\beta_2}\Bigg\}, \tag{6}$$

where $\mathbf{\Theta}$ is the model parameter vector, $\mathbf{\Theta} = \{\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \beta_1, \beta_2\}$. The natural logarithms of the function (6) are given by

$$\ell(\mathbf{\Theta}|\underline{\mathbf{t}}) \propto m_{11}\log\theta_{11} + m_{12}\log\theta_{12} + m_{21}\log\theta_{21} + m_{22}\log\theta_{22} + \beta_1\sum_{i=1}^{k}\eta_i t_i + \beta_2\sum_{i=1}^{k}(1-\eta_i)t_i - \frac{\theta_{11}+\theta_{12}}{\beta_1}\sum_{i=1}^{k}\eta_i\exp(\beta_1 t_i) - \frac{\theta_{21}+\theta_{22}}{\beta_2}\sum_{i=1}^{k}(1-\eta_i)\exp(\beta_2 t_i) - \frac{(n_1-k_1)(\theta_{11}+\theta_{12})}{\beta_1}\exp(\beta_1\omega) - \frac{(n_2-k_2)(\theta_{21}+\theta_{22})}{\beta_2}\exp(\beta_2\omega) + \frac{n_1(\theta_{11}+\theta_{12})}{\beta_1} + \frac{n_2(\theta_{21}+\theta_{22})}{\beta_2}. \tag{7}$$

4.1. Point Estimations

The MLE depends on the form of the likelihood equations, which can be obtained from the log-likelihood function (7) after taking the partial derivatives with respect to the parameters and equating these to zero. The first partial derivatives of (7) with respected to θ_{sj}, $s, j = 1, 2$ are given by

$$\frac{\partial\ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial\theta_{1j}} = \frac{m_{1j}}{\theta_{1j}} - \frac{1}{\beta_1}\sum_{i=1}^{k}\eta_i\exp(\beta_1 t_i) - \frac{n_1-k_1}{\beta_1}\exp(\beta_1\omega) + \frac{n_1}{\beta_1} = 0, \tag{8}$$

and

$$\frac{\partial\ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial\theta_{2j}} = \frac{m_{2j}}{\theta_{2j}} - \frac{1}{\beta_2}\sum_{i=1}^{k}(1-\eta_i)\exp(\beta_2 t_i) - \frac{n_2-k_2}{\beta_2}\exp(\beta_2\omega) + \frac{n_2}{\beta_2} = 0. \tag{9}$$

Hence, the MLEs of the parameters θ_{sj} given β_s are given by

$$\hat{\theta}_{sj}(\beta_1) = \frac{m_{sj}\beta_s}{D_s},\ s,\ j = 1, 2 \tag{10}$$

where

$$D_1 = \sum_{i=1}^{k}\eta_i\exp(\beta_1 t_i) + (n_1-k_1)\exp(\beta_1\omega) - n_1, \tag{11}$$

and

$$D_2 = \sum_{i=1}^{k}(1-\eta_i)\exp(\beta_2 t_i) + (n_2-k_2)\exp(\beta_2\omega) - n_2. \tag{12}$$

The likelihood equations with respected to β_s, $s = 1, 2$ are given by

$$\frac{\partial \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \beta_1} = \frac{\theta_{11}+\theta_{12}}{\beta_1^2}\left\{\sum_{i=1}^{k}\eta_i(1-\beta_1 t_i)\exp(\beta_1 t_i) + (n_1-k_1)(1-\beta_1 w)\exp(\beta_1\omega) - n_1\right\} + \sum_{i=1}^{k}\eta_i t_i = 0 \tag{13}$$

and

$$\frac{\partial \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \beta_2} = \frac{\theta_{21}+\theta_{22}}{\beta_2^2}\left\{\sum_{i=1}^{k}(1-\eta_i)(1-\beta_2 t_i)\exp(\beta_2 t_i) + (n_2-k_2)(1-\beta_2 w)\exp(\beta_2\omega) - n_2\right\} + \sum_{i=1}^{k}(1-\eta_i)t_i = 0, \tag{14}$$

The likelihood Equations (13) and (14) have shown that the ML estimators of the model parameters $\beta_s, s = 1, 2$ are reduced to two non-linear equations, which require an iteration method to solve.

Theorem 1. *For a given $m_{sk} > 0, s, k = 1, 2$, the conditional ML estimators of parameters β_s are presented by*

$$\beta_s^{i+1} = h(\beta_s^i), \tag{15}$$

where

$$h(\beta_s^i) = \begin{cases} \frac{(m_{11}+m_{12})}{D_1\sum_{i=1}^{k}\eta_i t_i}\left\{\sum_{i=1}^{k}\eta_i(\beta_1 t_i - 1)\exp(\beta_1 t_i) + n_1(\beta_1\omega - 1)\exp(\beta_1\omega) + n_1\right\}, & s = 1 \\ \frac{(m_{21}+m_{22})}{D_2\sum_{i=1}^{k}(1-\eta_i)t_i}\left\{\sum_{i=1}^{k}(1-\eta_i)(\beta_1 t_i - 1)\exp(\beta_1 t_i) + n_1(\beta_1\omega - 1)\exp(\beta_1\omega) + n_2\right\}, & s = 2 \end{cases} \tag{16}$$

Proof. From the iteration relation in (15) and fixed point method, the iteration is stopped after $|\beta_s^{i+1} - \beta_s^i|$ is sufficiently small. For fixed point theorem and its applications one can refer to Abdul Mannan et al. [40]. By substituting from (10) into (13) and (14). Using the properties of the operator-function in (16), we can immediately obtain the proof. □

Remark 1. *The iteration procedure needs a suitable initial value, which can be obtained using the profile likelihood function given by*

$$g(\beta_1, \beta_2|\underline{\mathbf{t}}) = m_{11}\log\frac{m_{11}\beta_1}{D_1} + m_{12}\log\frac{m_{12}\beta_1}{D_1} + m_{21}\log\frac{m_{21}\beta_2}{D_2} + m_{22}\log\frac{m_{22}\beta_2}{D_2} + \beta_1\sum_{i=1}^{k}\eta_i t_i + \beta_2\sum_{i=1}^{k}(1-\eta_i)t_i - (m_{11}+m_{12}+m_{21}+m_{22}). \tag{17}$$

The ML estimate of parameters θ_{sj} can be obtained from (10) by substituting the values of β_s by $\hat{\beta}_s$. Using the invariance property of MLEs, the ML estimators of reliability function and the corresponding failure rate function can be obtained from

$$\hat{S}_{sj}(t) = \exp\left(-\frac{\hat{\theta}_{sj}}{\hat{\beta}_s}(\exp(\hat{\beta}_s t) - 1)\right), \tag{18}$$

and

$$\hat{h}_{sj}(t) = \hat{\theta}_{sj}\exp(\hat{\beta}_s t), \tag{19}$$

in which we replace θ_{sj} and β_s with their MLEs

Remark 2. *It should be noted that it is more difficult to formulate the exact distributions of $\hat{\theta}_{1j}$ and $\hat{\theta}_{2j}$, which are specified as a combination of discrete and continuous distributions; see Kundu and Joarder [2]. The estimators of the parameters, reliability and failure rate functions are formulated with respect to the value of integers m_{sj}. Therefore, when using the value of $m_{1j} = (0 \text{ or } k_1)$, and $m_{2j} = (0 \text{ or } k_2)$, the estimates $\hat{\theta}_{1j}$ and $\hat{\theta}_{2j}$, respectively, are not exist.*

4.2. Interval Estimate

In many cases, providing an interval of values that may contain the parameter's true value with some degree of certainty is preferable to only reporting a point estimate of the unknown parameter. As the exact distributions of the MLEs are difficult to determine, in this subsection, we investigate the asymptotic confidence intervals of ACIs based on the asymptotic normality of MLEs. The definition of the Fisher information matrix in the literature provides the negative expectation of the second partial derivatives of the log-likelihood function. The asymptotic confidence intervals are formulated with respect to the Fisher information matrix of the model parameters. However, in different cases, the problem of obtaining the expectation of second partial derivatives is more serious, especially in models with high-dimensional cases. Therefore, the observed information matrix was adopted as the natural alternative to the Fisher information matrix. The observed information matrix of the model parameters $\mathbf{\Theta} = \{\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \beta_1, \beta_2\}$ is defined by the formula

$$i(\mathbf{\Theta}|\underline{\mathbf{t}}) - \left(\frac{\partial^2 \ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial \mathbf{\Theta}_i \partial \mathbf{\Theta} l}\right),\ i,l = 1,2,\ldots,6, \tag{20}$$

where the second derivatives are given by

$$\begin{aligned}\frac{\partial^2 \ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial \beta_1^2} = \frac{\theta_{11}+\theta_{12}}{\beta_1^3}\Bigg\{&\sum_{i=1}^{k}\eta_i\Big[-(\beta_1 t_i)^2 + 2\beta_1 t_i - 2\Big]\exp(\beta_1 t_i) \\ &+(n_1-k_1)\Big[-(\beta_1 w)^2 + 2\beta_1 w - 2\Big]\exp(\beta_1 w) - n_1\Bigg\},\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial^2 \ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial \beta_2^2} = \frac{\theta_{21}+\theta_{22}}{\beta_2^3}\Bigg\{&\sum_{i=1}^{k}(1-\eta_i)\Big[-(\beta_2 t_i)^2 + 2\beta_2 t_i - 2\Big]\exp(\beta_2 t_i) \\ &+(n_2-k_2)\Big[-(\beta_2 w)^2 + 2\beta_2 w - 2\Big]\exp(\beta_2 w) - n_2\Bigg\},\end{aligned} \tag{22}$$

$$\frac{\partial^2 \ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial \theta_{sj}^2} = \left.\frac{-m_{sj}}{\theta_{sj}^2}\right|_{s,\, j=1,2}, \tag{23}$$

$$\frac{\partial^2 \ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial \theta_{sj}\partial \theta_{il}} = 0,\ \text{For each } sj \neq il, \tag{24}$$

$$\begin{aligned}\frac{\partial^2 \ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial \beta_1 \partial \theta_{1j}} = \frac{\partial^2 \ell(\mathbf{\Theta}|\underline{\mathbf{t}})}{\partial \theta_{1j}\partial \beta_1} = \frac{1}{\beta_1^2}\Bigg\{&\sum_{i=1}^{k}\eta_i(1-\beta_1 t_i)\exp(\beta_1 t_i) + (n_1-k_1)(1-\beta_1\omega)\exp(\beta_1\omega) \\ &-\frac{n_1}{\beta_1^2}\Bigg\},\ j=1,2,\end{aligned} \tag{25}$$

$$\frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \beta_2 \partial \theta_{2j}} = \frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \theta_{2j} \partial \beta_2} = \frac{1}{\beta_2^2} \left\{ \sum_{i=1}^{k} (1-\eta_i)(1-\beta_2 t_i) \exp(\beta_2 t_i) + \frac{n_2}{\beta_2^2}(1-\beta_2 \omega) \exp(\beta_2 \omega) - \frac{n_2}{\beta_2^2} \right\},\ j = 1, 2, \tag{26}$$

and

$$\frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \beta_1 \partial \theta_{2j}} = \frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \beta_2 \partial \theta_{1j}} = \frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \theta_{2j} \partial \beta_1} = \frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \theta_{1j} \partial \beta_2} = \frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \beta_2 \partial \beta_1} = 0. \tag{27}$$

The observed information matrix at the estimate value of model parameters $\hat{\boldsymbol{\Theta}} = \{\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\beta}_1, \hat{\beta}_2\}$ is denoted by $i_0(\boldsymbol{\Theta}|\underline{\mathbf{t}})$

$$i_0(\boldsymbol{\Theta}|\underline{\mathbf{t}}) = -\left(\frac{\partial^2 \ell(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\partial \boldsymbol{\Theta}_i \partial \boldsymbol{\Theta} l} \right)_{\hat{\boldsymbol{\Theta}} = \{\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\beta}_1, \hat{\beta}_2\}},\ i, l = 1, 2, \ldots, 6, \tag{28}$$

For the model parameter $\boldsymbol{\Theta} = \{\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \beta_1, \beta_2\}$, ,the approximate distribution of MLE $\hat{\boldsymbol{\Theta}}$ is such that $(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})$ has a normal $N(\boldsymbol{\Theta}, i_0(\boldsymbol{\Theta}|\underline{\mathbf{t}})$ distribution, where $(i_0^{-1}(\boldsymbol{\Theta}|\underline{\mathbf{t}})$ denotes the inverse of the observed Fisher information matrix. The approximate $(1 - 2\alpha)\%$ confidence intervals of the model parameters are given by

$$\left\{ \begin{array}{c} \theta_{11} \mp z_\alpha e_{11},\ \theta_{12} \mp z_\alpha e_{23} \\ \theta_{21} \mp z_\alpha e_{33},\ \theta_{22} \mp z_\alpha e_{44} \\ \beta_1 \mp z_\alpha e_{55},\ \beta_2 \mp z_\alpha e_{66} \end{array} \right., \tag{29}$$

where e_{ii}, $i = 1, \ldots, 6$ are non-zero values of the elements of diagonal of $i_0^{-1}(\boldsymbol{\Theta}|\underline{\mathbf{t}})$ and the value of z_α is a standard normal value computed under the significance level α.

Equation (29) has shown that the lower bound of interval estimate can be of a negative value. Hence, the asymptotic distribution of $\log\boldsymbol{\Theta}_i$, $i = 1, 2, \ldots, 6$ can be described by the delta method of the logarithmic transformation; see [41,42].

The pivotal $Z = \frac{\log\boldsymbol{\Theta}_i - \log\hat{\boldsymbol{\Theta}}_i}{\mathrm{Var}(\log\hat{\boldsymbol{\Theta}}_i)}$ has normal properties, with mean 0 and variance 1. Therefore, $100(1 - 2\alpha)\%$ approximate interval estimators of $\Theta = \{\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \beta_1, \beta_2\}$ can be defined by

$$\left(\frac{\hat{\boldsymbol{\Theta}}_i}{\exp\left(\gamma_\alpha \sqrt{\mathrm{Var}(\log \hat{\boldsymbol{\Theta}}_i)}\right)}, \hat{\boldsymbol{\Theta}}_i \exp\left(\gamma_\alpha \sqrt{\mathrm{Var}(\log \hat{\boldsymbol{\Theta}}_i)}\right) \right), \tag{30}$$

where $\mathrm{Var}(\log \hat{\boldsymbol{\Theta}}_i) = \frac{\mathrm{Var}(\hat{\boldsymbol{\Theta}}_i)}{\hat{\boldsymbol{\Theta}}_i}$ and $i = 1, 2, \ldots, 6$; for more details, see Shih and Emura [43].

5. Bootstrap Confidence Intervals

The bootstrap method is a resampling technique for statistical inference that can be used to construct confidence intervals (CIs) for the model parameters. In the literature, the bootstrap technique is frequently used to gauge an estimator's bias and variance. This technique is widely used in calibrate hypothesis tests. There are two types of bootstrap techniques, parametric and nonparametric techniques; see Davison and Hinkley [44] and Efron and Tibshirani [45]. In the parametric bootstrap technique, the percentile bootstrap-p and bootstrap-t techniques are applied; see Efron [46] and Hall [47]. In this section, we adopted the percentile bootstrap-p and bootstrap-t techniques to formulate the confidence intervals of the model parameters, which can be implemented with the following algorithm (Algorithm 1).

Algorithm 1 Percentile bootstrap-p and bootstrap-t confidence interval.

Step 1: For given the original joint competing risks Type-II GHC data $\underline{\mathbf{t}}=\{(t_1,\eta_1,\rho_1),(t_2,\eta_2,\rho_2),\ldots,(t_k,\eta_k,\rho_k)\}$, compute the ML estimates of the model parameters $\hat{\mathbf{\Theta}}=\{\hat{\theta}_{11},\hat{\theta}_{12},\hat{\theta}_{21},\hat{\theta}_{22},\hat{\beta}_1,\hat{\beta}_2\}$.
Step 2: Generate two samples of size n_1 from Gompertz($\hat{\beta}_1$, $\hat{\theta}_{11}+\hat{\theta}_{12}$) and sample of size n_2 from Gompertz($\hat{\beta}_2$, $\hat{\theta}_{21}+\hat{\theta}_{22}$).
Step 3: For a given (τ_1,τ_2) and m, generate the joint Type-II GHC competing risks data defined by $\underline{\mathbf{t}}^*=\{(t_1^*,\eta_1,\rho_1),(t_2^*,\eta_2,\rho_2),\ldots,(t_k^*,\eta_k,\rho_k)\}$.
Step 4: Using the bootstrap sample $\underline{\mathbf{t}}^*$, compute the integers k, k_1, k_2 and determine the termination time ω.
Step 5: The numbers of failure m_{sj} (obtained from the line Ω_s under the obtained j, where $s,j=1,2$) are generated from the binomial distribution with parameters k_s and $\frac{\theta_{sj}}{\theta_{s1}+\theta_{s2}}$.
Step 6: The bootstrap estimates $\hat{\mathbf{\Theta}}^*=\{\hat{\theta}^*_{11},\hat{\theta}^*_{12},\hat{\theta}^*_{21},\hat{\theta}^*_{22},\hat{\beta}^*_1,\hat{\beta}^*_2\}$ are computed using (10) and (15).
Step 7: Repeat Steps (2–6) **N** times.
Step 8: The resulting bootstrap estimates are arranged in ascending order, $(\hat{\mathbf{\Theta}}_i^{*(1)},\hat{\mathbf{\Theta}}_i^{*(2)},\ldots,\hat{\mathbf{\Theta}}_i^{*(\mathbf{N})})$, $i=1,2,\ldots,6$.

Percentile bootstrap confidence interval (PBCI)

Let $F(z)=P(\hat{\mathbf{\Theta}}_i^*\leqslant z)$, $i=1,2,\ldots,6$ be the empirical cumulative distribution function of $\hat{\mathbf{\Theta}}_i^*$; then, the point bootstrap estimate of $\mathbf{\Theta}_i$ is given by

$$\hat{\mathbf{\Theta}}_{i(\text{boot})}=\frac{1}{\mathbf{N}}\sum_{i=1}^{\mathbf{N}}\hat{\mathbf{\Theta}}_i^{*(\mathbf{i})}. \tag{31}$$

The corresponding $100(1-2\alpha)\%$ PBCIs are given by

$$(\hat{\mathbf{\Theta}}_i^{*(\mathbf{N}\alpha)},\ \hat{\mathbf{\Theta}}_i^{*(\mathbf{N}(1-\alpha))}), \tag{32}$$

where $\hat{\mathbf{\Theta}}_i^*=F^{-1}(z)$.

Bootstrap-t confidence interval (BTCI)

From the ascending order sample $(\hat{\mathbf{\Theta}}_i^{*(1)},\hat{\mathbf{\Theta}}_i^{*(2)},\ldots,\hat{\mathbf{\Theta}}_i^{*(\mathbf{N})})$, $i=1,2,\ldots,6$, we built the order statistics values $\Delta_i^{*(1)}<\Delta_i^{*(2)}<\cdots<\Delta_i^{*(\mathbf{N})}$, where

$$\Delta_i^{*[l]}=\frac{\hat{\mathbf{\Theta}}_i^{*(i)}-\hat{\mathbf{\Theta}}_i}{\sqrt{\operatorname{var}\left(\hat{\mathbf{\Theta}}_i^{*(i)}\right)}},\ l=1,2,\ldots,\mathbf{N},\ i=1,2,3,4,5,6. \tag{33}$$

Hence, $100(1-2\alpha)\%$ BTCIs are given by

$$\left(\tilde{\Delta}^*_{i\text{boot-t}(\alpha)},\ \tilde{\Delta}^*_{i\text{boot-t}(1-\alpha)}\right), \tag{34}$$

where $\tilde{\Delta}^*_{i\text{boot-t}}$ is given by

$$\tilde{\Delta}^*_{i\text{boot-t}} = \hat{\Delta}^*_i + \sqrt{\text{Var}(\hat{\Delta}^*_i)}F^{-1}(z), \tag{35}$$

and $F^{-1}(z) = P(\hat{\Delta}^*_i \leqslant z)$ is the cumulative distribution function of $\hat{\Delta}^*_i$.

6. Bayesian Approach

In this section, to obtain the joint Type-II GHC competing risks data $\underline{\mathbf{t}} = \{(t_1, \eta_1, \rho_1), (t_2, \eta_2, \rho_2), \ldots, (t_k, \eta_k, \rho_k)\}$, we consider the problem of the Bayesian estimation of model parameters. We assume that the prior distributions for the unknown parameters are independent gamma priors. Therefore, the prior information formulated for the parameter vector $\boldsymbol{\Theta} = \{\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \beta_1, \beta_2\}$ as

$$\boldsymbol{\Theta}_i \propto \boldsymbol{\Theta}_i^{a_i-1} \exp(-b_i \boldsymbol{\Theta}_i),\ \boldsymbol{\Theta}_i > 0,\ a_i,\ b_i > 0,\ i = 1,2,3,4,5,6. \tag{36}$$

Hence, the joint prior density function of the model parameters is given by

$$\Pi^*(\boldsymbol{\Theta}) = \prod_{i=1}^{6} \frac{b_i^{a_i}}{\Gamma(a_i)} \boldsymbol{\Theta}_i^{a_i-1} \exp(-b_i \boldsymbol{\Theta}_i). \tag{37}$$

The joint posterior density function of the model parameters is given by

$$\Pi(\boldsymbol{\Theta}|\underline{\mathbf{t}}) = \frac{\Pi^*(\boldsymbol{\Theta}) \times L(\boldsymbol{\Theta}|\underline{\mathbf{t}})}{\int_{\boldsymbol{\Theta}} \Pi^*(\boldsymbol{\Theta}) \times L(\boldsymbol{\Theta}|\underline{\mathbf{t}}) \times d\boldsymbol{\Theta}} \propto \Pi^*(\boldsymbol{\Theta}) \times L(\boldsymbol{\Theta}|\underline{\mathbf{t}}). \tag{38}$$

Inserting (6) and (37) in (38) and ignoring the additive constant, the joint posterior density can be expressed as

$$\begin{aligned}
\Pi(\boldsymbol{\Theta}|\underline{\mathbf{t}}) \propto\, & \theta_{11}^{m_{11}+a_1-1} \theta_{12}^{m_{12}+a_2-1} \theta_{21}^{m_{21}+a_3-1} \theta_{22}^{m_{22}+a_4-1} \beta_1^{a_5-1} \beta_2^{a_6-1} \exp\Bigg\{ \beta_1 \sum_{i=1}^{k} \eta_i t_i \\
& + \beta_2 \sum_{i=1}^{k} (1-\eta_i) t_i - \frac{\theta_{11}+\theta_{12}}{\beta_1} \sum_{i=1}^{k} \eta_i \exp(\beta_1 t_i) - \frac{\theta_{21}+\theta_{22}}{\beta_2} \sum_{i=1}^{k} (1-\eta_i) \exp(\beta_2 t_i) \\
& - \frac{(n_1-k_1)(\theta_{11}+\theta_{12})}{\beta_1} \exp(\beta_1 \omega) - \frac{(n_2-k_2)(\theta_{21}+\theta_{22})}{\beta_2} \exp(\beta_2 \omega) + \frac{n_1(\theta_{11}+\theta_{12})}{\beta_1} \\
& + \frac{n_2(\theta_{21}+\theta_{22})}{\beta_2} - b_1\theta_{11} - b_2\theta_{12} - b_3\theta_{21} - b_4\theta_{22} - b_5\beta_1 - b_6\beta_2 \Bigg\}.
\end{aligned} \tag{39}$$

Under the squared error loss (SEL) function, the Bayes estimate of the parameter is the posterior mean. Then, the Bayes estimate of the parameters or any function of the parameters, such as reliability or failure rate functions, say $\Psi(\boldsymbol{\Theta})$, is given by

$$\hat{\Psi}_B(\boldsymbol{\Theta}) = \int_{\boldsymbol{\Theta}} \Psi(\boldsymbol{\Theta}) \Pi(\boldsymbol{\Theta}|\underline{\mathbf{t}}) d\boldsymbol{\Theta}. \tag{40}$$

Equation (40) shows that the Bayes estimate of $\Psi(\mathbf{\Theta})$ needs to compute a high-dimensional integral. Appropriate numerical methods could be used to approximate Bayesian estimation.

One of the most common methods applied in this paper is the Markov Chain Monte Carlo method (MCMC method). Compared with traditional methods, the MCMC method is more flexible and provides an alternative approach to parameter estimation. The key to the MCMC technique is obtaining posterior distribution in the empirical form and generating MCMC samples from the posterior distribution, and then computing Bayes estimators and constructing the associated credible intervals. Therefore, we describe this technique as follows.

From Equation (39), the posterior full conditional density functions of the parameters and data can be obtained as

$$\Pi_j(\theta_{1j}|\mathbf{\Theta}_{-\theta_{1j}},\underline{\mathbf{t}}) \propto \theta_{1j}^{m_{1j}+a_j-1} \exp\left\{-b_j\theta_{1j} - \frac{\theta_{1j}}{\beta_1}\sum_{i=1}^{k}\eta_i \exp(\beta_1 t_i) - \frac{(n_1-k_1)\theta_{1j}}{\beta_1}\exp(\beta_1\omega) + \frac{n_1\theta_{1j}}{\beta_1}\right\}, \tag{41}$$

$$\Pi_{j+2}(\theta_{2j}|\mathbf{\Theta}_{-\theta_{2j}},\underline{\mathbf{t}}) \propto \theta_{2j}^{m_{1j}+a_{j+2}-1} \exp\left\{-b_{j+2}\theta_{2j} - \frac{\theta_{2j}}{\beta_2}\sum_{i=1}^{k}(1-\eta_i) \exp(\beta_2 t_i) - \frac{(n_2-k_2)\theta_{2j}}{\beta_2}\exp(\beta_2\omega) + \frac{n_2\theta_{2j}}{\beta_2}\right\}, \tag{42}$$

$$\Pi_5(\beta_1|\mathbf{\Theta}_{-\beta_1},\underline{\mathbf{t}}) \propto \beta_1^{a_5-1} \exp\left\{\beta_1\sum_{i=1}^{k}\eta_i t_i - \frac{\theta_{11}+\theta_{12}}{\beta_1}\sum_{i=1}^{k}\eta_i \exp(\beta_1 t_i) - \frac{(n_1-k_1)(\theta_{11}+\theta_{12})}{\beta_1}\exp(\beta_1\omega) - b_5\beta_1 + \frac{n_1(\theta_{11}+\theta_{12})}{\beta_1}\right\}, \tag{43}$$

and

$$\Pi_6(\beta_2|\mathbf{\Theta}_{-\beta_2},\underline{\mathbf{t}}) \propto \beta_2^{a_6-1} \exp\left\{\beta_2\sum_{i=1}^{k}(1-\eta_i) t_i - \frac{\theta_{21}+\theta_{22}}{\beta_2}\sum_{i=1}^{k}(1-\eta_i) \exp(\beta_2 t_i) - \frac{n_2(\theta_{21}+\theta_{22})}{\beta_2}\exp(\beta_2\omega) - b_6\beta_2 + \frac{n_2(\theta_{21}+\theta_{22})}{\beta_2}\right\}, \tag{44}$$

where $j = 1,2$ and, for example, $(\theta_{11}|\mathbf{\Theta}_{-\theta_{11}},\underline{\mathbf{t}})$ mean $(\theta_{11}|\theta_{12},\theta_{21},\theta_{22},\beta_1,\beta_2,\underline{\mathbf{t}})$. The full conditional posterior distributions show that the posterior distribution is reduced to four gamma distributions, for which any conventional methods of generating random numbers can be used. And two general unknown functions make it impossible to generate random samples directly from the conditional posterior distributions. Therefore, to generate random samples from the two unknown distributions, the Metropolis–Hastings (M–H) algorithm with normal proposal distribution can be used; see [48]. The following steps describe the algorithm used to generate from the posterior distribution (Algorithm 2).

Algorithm 2 Gibbs with M-H sampler algorithms.

Step 1: Begin with the indicated number $J = 1$ and the initial parameter values $\boldsymbol{\Theta}^{(0)} = \{\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\beta}_1, \hat{\beta}_2\}$.

Step 2: The values $\theta_{1j}^{(J)}$ and $\theta_{2j}^{(J)}$ are generated from gamma distributions given by (40) and (41), respectively, $j = 1, 2$.

Step 3: The values $\beta_j^{(J)}$ generated under M–H algorithms with a normal proposal distribution with a mean $\beta_j^{(J-1)}$ and variance $e_{j+4\,j+4}$, obtained from an approximate information matrix, $j = 1, 2$, as follows

(I) For the index $j = 1, 2$, begin with starting points $\beta_j^{(J-1)}$, where $\beta_j^{(0)} = \hat{\beta}_j$.

(II) Generate a candidate sample points $\beta_j^{(*)}$, from $N(\beta_j^{(J-1)}, e_{j+4\,j+4})$, as proposal distributions.

(III) Compute the probability (the acceptance probability) from (43) and (44)

$$P_j\left(\beta_j^{(J-1)}, \beta_j^{(*)}\right) = \min\left[1, \frac{\Pi_{j+4}(\beta_j^{(*)}|\boldsymbol{\Theta}_{-1}, \underline{\mathbf{t}})}{\Pi_{j+4}(\beta_j^{(J-1)}|\boldsymbol{\Theta}_{-1}, \underline{\mathbf{t}})}\right]. \quad (45)$$

(IV) Generate U_j from uniform (0, 1).

(V) If $U_j \leq P_j\left(\beta_j^{(J-1)}, \beta_j^{(*)}\right)$, we accept the candidate sample points $\beta_j^{(*)}$ as $\beta_j^{(J)}$. Otherwise, the values $\beta_j^{(*)}$ are rejected and $\beta_j^{(J)} = \beta_j^{(J-1)}$ is set.

Step 4: Put $J = J + 1$

Step 5: Repeat steps (2–4) N times.

Step 6: Put the generated parameter vector $\boldsymbol{\Theta}_i^{(J)}$ in ascending order; for example, $\boldsymbol{\Theta}_i^{[J]}$, $i = 1, 2, \ldots, 6$.

6.1. MCMC Bayesian Point Estimations

The initial simulated variants of the algorithm are often discarded at the start of the analysis (burn-in time) to eliminate the bias caused by the initially selected value. Suppose that the number of iterations needed to reach the stationary distribution is $\mathbf{N}^*$ (burn-in). In all computations, we take the number $\mathbf{N}^* = 1000$ iteration. Hence, the Bayes point estimator when using the MCMC method is given by

$$\hat{\Psi}_B(\boldsymbol{\Theta}) = \mathrm{E}_\Pi(\Psi(\boldsymbol{\Theta})|\underline{\mathbf{t}}) = \frac{1}{\mathbf{N} - \mathbf{N}^*} \sum_{l=\mathbf{N}^*+1}^{N} \Psi\left(\boldsymbol{\Theta}_i^{(l)}\right),\ i = 1, 2, \ldots, 6. \quad (46)$$

The corresponding variance in the Bayes estimate is given by

$$\widehat{V}(\Psi(\boldsymbol{\Theta})|\underline{\mathbf{t}}) = \frac{1}{\mathbf{N} - \mathbf{N}^*} \sum_{l=\mathbf{N}^*+1}^{N} \left(\Psi\left(\boldsymbol{\Theta}_i^{(l)}\right) - \hat{\Psi}_B(\boldsymbol{\Theta})\right)^2. \quad (47)$$

6.2. MCMC Bayesian Interval Estimations

To establish the two-sided credible intervals of $\Psi(\boldsymbol{\Theta})$; sort $\Psi\left(\boldsymbol{\Theta}_i^{(l)}\right)$, $i = 1, 2, 3, 4, 5, 6$; $j = \mathbf{N}^* + 1, \mathbf{N}^* + 2, \ldots, N$. in ascending order. Hence, $100(1 - 2\alpha)\%$ credible intervals of $\Psi(\boldsymbol{\Theta})$ can be constructed as:

$$\left(\Psi(\boldsymbol{\Theta})_{\alpha(\mathbf{N}-\mathbf{N}^*)}, \Psi(\boldsymbol{\Theta})_{(1-\alpha)(\mathbf{N}-\mathbf{N}^*)}\right). \quad (48)$$

7. Simulation Studies

In this section, the estimation results obtained and developed in this paper are assessed and compared using the Monte Carlo simulation study. In our study, we assessed the effect of changing sample size $N = n_1 + n_2$, and affected sample size m, two times (τ_1, τ_2) and parameters vector $\boldsymbol{\Theta} = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \beta_1, \beta_2)$. Therefore, we adopted two sets of parameter values $\boldsymbol{\Theta}_1 = \{0.05, 0.1, 0.07, 0.12, 0.4, 0.5\}$ and $\boldsymbol{\Theta}_2 = \{0.2, 0.3, 0.4, 0.2, 1.0, 1.0\}$. For the censoring schemes, different combinations were adopted and are shown in Tables 1–4. The prior information was selected using the relation (prior mean $\simeq \frac{a_i}{b_i}$), where a_i and b_i are hyper-parameters of gamma prior. The point estimate were tested by computing the mean squared error (MSE). The interval estimates were evaluated using average length (AL) criterion, as well as the coverage probabilities (CPs). Using the Bayesian approach, we adopted.

Table 1. MSEs of the parameter estimates for the choose $\boldsymbol{\Theta}_1 = \{0.05, 0.1, 0.07, 0.12, 0.4, 0.5\}$.

(n_1, n_2, m)	(τ_1, τ_2)		θ_{11}	θ_{12}	θ_{21}	θ_{22}	β_1	β_2
(25, 25, 30)	(2.0, 6.0)	ML	0.0345	0.0728	0.0421	0.0854	0.2310	0.3523
		Boot	0.0488	0.0897	0.0564	0.1324	0.3380	0.5201
		BayesP^1	0.0311	0.0704	0.0398	0.0822	0.2289	0.3510
		BayesP^2	0.0241	0.0589	0.0255	0.0645	0.2009	0.3324
(25, 25, 40)	(2.0, 6.0)	ML	0.0321	0.0707	0.0390	0.0831	0.2287	0.3504
		Boot	0.0459	0.0871	0.0535	0.1301	0.3347	0.5172
		BayesP^1	0.0291	0.0684	0.0375	0.0800	0.2251	0.3500
		BayesP^2	0.0219	0.0554	0.0221	0.0619	0.1880	0.3303
(25, 25, 30)	(2.0, 8.0)	ML	0.0327	0.0713	0.0392	0.0841	0.2279	0.3511
		Boot	0.0448	0.0869	0.0525	0.1307	0.3351	0.5168
		BayesP^1	0.0294	0.0682	0.0366	0.0805	0.2251	0.3503
		BayesP^2	0.0224	0.0551	0.0221	0.0615	0.1883	0.3311
(40, 40, 50)	(2.0, 8.0)	ML	0.0285	0.0677	0.0352	0.0805	0.2244	0.3451
		Boot	0.0418	0.0850	0.0501	0.1275	0.3315	0.5128
		BayesP^1	0.0262	0.0651	0.0345	0.0762	0.2214	0.3462
		BayesP^2	0.0184	0.0511	0.0181	0.0576	0.1842	0.3259
(40, 40, 65)	(2.0, 8.0)	ML	0.0241	0.0628	0.0309	0.0765	0.2205	0.3411
		Boot	0.0379	0.0801	0.0428	0.1225	0.3282	0.5100
		BayesP^1	0.0219	0.0609	0.0311	0.0721	0.2171	0.3429
		BayesP^2	0.0142	0.0471	0.0155	0.0527	0.1811	0.3215

Table 2. AL and CP of the parameter estimates for the chosen $\boldsymbol{\Theta}_1 = \{0.05, 0.1, 0.07, 0.12, 0.4, 0.5\}$.

(n_1, n_2, m)	(τ_1, τ_2)		θ_{11}		θ_{12}		θ_{21}		θ_{22}		β_1		β_2	
			AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
(25, 25, 30)	(2.0, 6.0)	ML	0.145	0.88	0.321	0.89	0.185	0.89	0.374	0.87	1.254	0.89	1.452	0.88
		Boot-p	0.170	0.89	0.355	0.90	0.221	0.89	0.399	0.86	1.280	0.89	1.772	0.89
		Boot-t	0.119	0.89	0.300	0.90	0.154	0.91	0.342	0.89	1.228	0.90	1.418	0.89
		BayesP^1	0.137	0.88	0.318	0.88	0.172	0.89	0.366	0.89	1.241	0.90	1.438	0.90
		BayesP^2	0.082	0.91	0.274	0.90	0.119	0.91	0.315	0.90	1.202	0.90	1.379	0.91
(25, 25, 40)	(2.0, 6.0)	ML	0.122	0.90	0.302	0.91	0.164	0.90	0.351	0.89	1.229	0.89	1.425	0.90
		Boot-p	0.148	0.89	0.325	0.90	0.188	0.90	0.365	0.89	1.251	0.89	1.741	0.90
		Boot-t	0.089	0.92	0.269	0.92	0.117	0.93	0.311	0.92	1.200	0.91	1.379	0.91
		BayesP^1	0.112	0.89	0.289	0.91	0.142	0.960	0.331	0.91	1.219	0.93	1.405	0.92
		BayesP^2	0.066	0.92	0.249	0.92	0.089	0.91	0.287	0.93	1.181	0.92	1.351	0.91

Table 2. *Cont.*

(n_1, n_2, m)	(τ_1, τ_2)		θ_{11}		θ_{12}		θ_{21}		θ_{22}		β_1		β_2	
(25, 25, 30)	(2.0, 8.0)	ML	0.129	0.91	0.311	0.91	0.158	0.91	0.355	0.89	1.233	0.88	1.417	0.91
		Boot-p	0.151	0.88	0.321	0.91	0.192	0.89	0.361	0.88	1.244	0.90	1.729	0.91
		Boot-t	0.085	0.91	0.271	0.92	0.112	0.91	0.315	0.91	1.197	0.92	1.375	0.93
		BayesP^1	0.115	0.89	0.287	0.90	0.133	0.92	0.318	0.91	1.211	0.91	1.399	0.93
		BayesP^2	0.062	0.91	0.241	0.91	0.085	0.92	0.291	0.92	1.178	0.91	1.348	0.92
(40, 40, 50)	(2.0, 6.0)	ML	0.091	0.92	0.281	0.93	0.142	0.92	0.325	0.91	1.211	0.92	1.401	0.94
		Boot-p	0.124	0.90	0.311	0.91	0.162	0.92	0.343	0.91	1.232	0.94	1.715	0.93
		Boot-t	0.071	0.93	0.251	0.91	0.100	0.91	0.292	0.95	1.178	0.94	1.362	0.93
		BayesP^1	0.100	0.90	0.277	0.91	0.121	0.92	0.317	0.92	1.202	0.92	1.381	0.93
		BayesP^2	0.047	0.95	0.228	0.91	0.055	0.93	0.262	0.94	1.157	0.91	1.332	0.94
(40, 40, 65)	(2.0, 8.0)	ML	0.075	0.91	0.263	0.94	0.118	0.92	0.303	0.93	1.191	0.94	1.382	0.92
		Boot-p	0.101	0.91	0.301	0.92	0.151	0.90	0.324	0.92	1.214	0.91	1.700	0.91
		Boot-t	0.054	0.94	0.236	0.92	0.084	0.93	0.275	0.92	1.151	0.92	1.338	0.94
		BayesP^1	0.089	0.92	0.255	0.93	0.101	0.92	0.300	0.94	1.187	0.91	1.359	0.94
		BayesP^2	0.025	0.91	0.207	0.92	0.035	0.94	0.241	0.94	1.135	0.92	1.309	0.96

Non-informative prior (P^1) and informative prior (P^2), where $\mathrm{P}^1 \equiv (a_i, b_i) = (0.0001, 0.0001)$, and $P^2 = \{(0.5, 5), (0.5, 4), (1, 6), (1, 4), (1, 3), (2, 4)\}$ **for** θ_1 and $P^2 = \{$**(1, 3), (2, 5), (2, 4), (1, 3), (2, 2), (3, 2)**$\}$ **for** θ_2 are selected. For the MCMC method, we reported 11,000 iterations and the first 1000 iterations were discarded. The simulation results were formulated according to the following algorithm (Algorithm 3).

Algorithm 3 Monte Carlo simulation study.

Step 1: From Gompertz distribution with two parameters $\theta_{s1} + \theta_{s2}$ and β_s generate samples of size n_1 and n_2, $s = 1, 2,$, respectively.
Step 2: From the joint sample of size $n = n_1 + n_2$ and for given censoring parameters m, τ_1, τ_2. If, $T_m < \tau_1$; then, $k \geq m$ and the test is terminated at $\omega = \tau_1$. However, if $\tau_1 < T_m < \tau_2$, $k = m$ and the test is terminated at $\omega = T_m$ **and if** $\tau_1 < \tau_2 < T_m$, $k \leq m$ and the test is terminated at $\omega = \tau_2$.
Step 3: From step 2, the number of failures **k**, test termination time ω and failure times are generated. Hence, the observed joint Type-II GHC competing risks data are obtained.
Step 4: The two values k_1 and k_2 (number of units from the first and second line in joint Type-II GHC competing risks data) are observed.
Step 5: The integer numbers m_{sj}, $s, j = 1, 2$ are generated from binomial distributions.
Step 6: We obtain various estimates by considering 1000 replications of samples. Steps (1–4) are repeated 1000 times.
Step 7: For each sample, the MLE, bootstrap and Bayes estimate are computed.
Step 8: The values of each MSE, AL and CP are computed, and the results are reported in Tables 1–4.

Discussion: Recently, the problem of obtaining adequate information about the competing lifetime distributions and their parameters it has been of interest to many authors. Therefore, the reliability experimenter may resort to censoring techniques. In this paper, we proposed joint Type-II GHCS. The behavior of different estimation methods under different censoring schemes can be obtained from a simulation study. The numerical results presented in Tables 1–4 show that the proposed model and the methods of estimation work well. The quality of the proposed model did not change for different model parameters. We summarize some points that describe the capabilities and the behavior of estimators as follows.

1. The values of MSEs decrease when sample size $n_1 + n_2$ or effected sample size m increases.
2. The model quality improves at increasing τ_1 and τ_2.
3. The results under classical ML and non-informative Bayes estimation are both closed.
4. Informative prior Bayes estimates present the best estimation.
5. Estimation results under two Gompertz distribution parameters are more acceptable.
6. Interval estimations are more acceptable using bootstrap-t and informative Bayes estimation.

Table 3. MSEs of the parameter estimates for the $\boldsymbol{\Theta}_2 = \{0.2, 0.3, 0.4, 0.2, 1.0, 1.0\}$.

(n_1, n_2, m)	(τ_1, τ_2)		θ_{11}	θ_{12}	θ_{21}	θ_{22}	β_1	β_2
(25, 25, 30)	(0.3, 6.0)	ML	0.1262	0.1895	0.2541	0.0678	0.4521	0.5742
		Boot	0.1310	0.1952	0.2598	0.0751	0.4660	0.5789
		BayesP[1]	0.1254	0.1882	0.2537	0.0661	0.4509	0.5729
		BayesP[2]	0.1142	0.1751	0.2410	0.0556	0.4390	0.5642
(25, 25, 40)	(0.3, 6.0)	ML	0.1228	0.1866	0.2505	0.0645	0.4482	0.5715
		Boot	0.1284	0.1919	0.2571	0.0715	0.4628	0.5761
		BayesP[1]	0.1219	0.1861	0.2511	0.0633	0.4481	0.5700
		BayesP[2]	0.1114	0.1725	0.2379	0.0526	0.4354	0.5609
(25, 25, 30)	(0.3, 0.9)	ML	0.1231	0.1872	0.2511	0.0641	0.4491	0.5709
		Boot	0.1287	0.1924	0.2568	0.0708	0.4622	0.5767
		BayesP[1]	0.1225	0.1866	0.2515	0.0639	0.4487	0.5705
		BayesP[2]	0.1117	0.1719	0.2366	0.0522	0.4358	0.5613
(40, 40, 50)	(0.3, 0.9)	ML	0.1135	0.1461	0.2415	0.0608	0.4451	0.5671
		Boot	0.1239	0.1885	0.2540	0.0677	0.4591	0.5719
		BayesP[1]	0.1181	0.1822	0.2471	0.0600	0.4449	0.5662
		BayesP[2]	0.1075	0.1681	0.2329	0.0488	0.4315	0.5571
(40, 40, 65)	(0.3, 0.9)	ML	0.1102	0.1427	0.2385	0.0587	0.4422	0.5645
		Boot	0.1211	0.1861	0.2515	0.0641	0.4565	0.5691
		BayesP[1]	0.1155	0.1800	0.2442	0.0569	0.4422	0.5636
		BayesP[2]	0.1041	0.1652	0.2303	0.0454	0.4287	0.5552

Table 4. AL and CP of the parameter estimates for $\boldsymbol{\Theta}_2 = \{0.2, 0.3, 0.4, 0.2, 1.0, 1.0\}$

(n_1, n_2, m)	(τ_1, τ_2)		θ_{11}		θ_{12}		θ_{21}		θ_{22}		β_1		β_2	
			AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
(25, 25, 30)	(0.3, 0.6)	ML	0.542	0.88	0.741	0.89	1.254	0.89	0.547	0.87	3.245	0.89	2.989	0.89
		Boot-p	0.665	0.86	0.854	0.89	1.452	0.89	0.645	0.88	3.345	0.90	3.214	0.89
		Boot-t	0.490	0.90	0.694	0.91	1.201	0.90	0.500	0.91	3.191	0.90	2.914	0.90
		BayesP[1]	0.511	0.89	0.707	0.90	1.222	0.89	0.519	0.89	3.215	0.89	2.951	0.90
		BayesP[2]	0.425	0.91	0.624	0.91	1.110	0.89	0.421	0.90	3.100	0.90	2.798	0.91
(25, 25, 40)	(0.3, 0.6)	ML	0.502	0.90	0.708	0.90	1.211	0.89	0.502	0.90	3.211	0.89	2.941	0.91
		Boot-p	0.628	0.90	0.817	0.89	1.422	0.89	0.619	0.89	3.312	0.91	3.187	0.89
		Boot-t	0.462	0.90	0.671	0.92	1.175	0.91	0.475	0.91	3.167	0.92	2.887	0.92
		BayesP[1]	0.477	0.89	0.682	0.91	1.200	0.89	0.491	0.90	3.182	0.90	2.912	0.91
		BayesP[2]	0.392	0.93	0.589	0.93	1.081	0.92	0.387	0.92	3.69	0.92	2.764	0.96
(25, 25, 30)	(0.3, 0.9)	ML	0.511	0.92	0.704	0.91	1.217	0.89	0.508	0.91	3.215	0.90	2.947	0.92
		Boot-p	0.624	0.89	0.821	0.91	1.428	0.90	0.612	0.90	3.308	0.92	3.191	0.85
		Boot-t	0.457	0.90	0.670	0.93	1.182	0.90	0.469	0.95	3.162	0.90	2.891	0.93
		BayesP[1]	0.472	0.92	0.687	0.92	1.205	0.90	0.488	0.91	3.188	0.93	2.917	0.94
		BayesP[2]	0.387	0.94	0.580	0.91	1.088	0.91	0.381	0.93	3.64	0.91	2.755	0.93

Table 4. *Cont.*

(n_1, n_2, m)	(τ_1, τ_2)		θ_{11}		θ_{12}		θ_{21}		θ_{22}		β_1		β_2	
(40, 40, 50)	(0.3, 0.9)	ML	0.461	0.92	0.674	0.93	1.174	0.92	0.469	0.93	3.170	0.91	2.912	0.93
		Boot-p	0.591	0.92	0.800	0.91	1.389	0.90	0.594	0.92	3.281	0.92	3.141	0.91
		Boot-t	0.422	0.93	0.625	0.94	1.131	0.92	0.448	0.94	3.129	0.96	2.851	0.94
		BayesP^1	0.439	0.91	0.648	0.93	1.164	0.92	0.435	0.92	3.144	0.93	2.874	0.95
		BayesP^2	0.354	0.92	0.562	0.94	1.047	0.91	0.355	0.95	3.514	0.94	2.724	0.92
(40, 40, 65)	(0.3, 0.9)	ML	0.418	0.95	0.634	0.92	1.141	0.95	0.432	0.93	3.144	0.94	2.888	0.92
		Boot-p	0.554	0.94	0.771	0.93	1.351	0.92	0.571	0.93	3.248	0.96	3.109	0.92
		Boot-t	0.392	0.93	0.600	0.91	1.092	0.95	0.415	0.95	3.094	0.92	2.815	0.91
		BayesP^1	0.414	0.92	0.614	0.93	1.127	0.91	0.411	0.94	3.119	0.94	2.848	0.92
		BayesP^2	0.317	0.93	0.527	0.94	1.001	0.92	0.325	0.94	3.489	0.93	2.687	0.95

8. Real Data Analysis

Real datasets obtained from laboratory experiments were used to discuss the results of this paper. This data presented by Hoel [49] describe the survival time of male mice under a conventional laboratory environment. The test time considered an age of 5–6 weeks and male mice were exposed to radiation dose of 300 roentgens. These data were analyzed by Pareek et al. [50], Sarhan et al. [51] and Cramer and Schmiedt [52]. Data obtained under progressive first failure of compertz population were analyzed by Soliman et al. [27,28]. In this section, we considered two groups of radiated male mice, as shown in Table 5. For causes of failure, we considered Thymine Lymphoma as the first cause and the other causes were considered the second cause of failure. The data were divided by 1000 for simplicity of computation. To generate the joint Type-II GHC competing risks sample, the following algorithms were used (Algorithm 4).

Table 5. Two groups of failure for the laboratory radiation male mice Ω_1 and Ω_2.

							Thymic Lymphoma							
Ω_1	159	189	191	198	200	207	220	235	245	250	256	261	265	266
	280	343	356	383	403	414	428	432						
							Other causes							
Ω_1	40	42	51	62	163	179	206	222	228	252	249	282	324	333
	341	366	385	407	420	431	441	461	462	482	517	517	524	564
	567	586	619	620	621	622	647	651	686	761	763			
							Thymic Lymphoma							
Ω_2	158	192	193	194	195	202	212	215	229	230	237	240	244	247
	259	300	301	321	337	415	434	444	485	496	529	537	624	707
	800													
							Other causes							
Ω_2	136	246	255	376	421	565	616	617	652	655	658	660	662	675
	681	734	736	737	757	769	777	800	807	825	855	857	864	868
	870	870	873	882	895	910	934	942	1015	1019				

Algorithm 4 Generate joint Type-II GHC competing risks data.

Step 1: Suppose that the censoring scheme has m = 70, $\tau_1 = 0.2$, $\tau_2 = 0.4$ and $(n_1, n_2) = (61, 67)$.
Step 2: For the joint sample of size $n = n_1 + n_2$ given in Tables 5 and 6 and the corresponding censoring scheme, we observed that, $\tau_1 < \tau_2 < T_m$.
Step 3: Hence, the value of $k = 58 < m$ and the test was terminated at $\omega = \tau_2 = 0.4$.
Step 4: For the joint Type-II GHC data of zise 58 given in Table 7, we obtained $k_1 = 35$ from the first line and k_2 = 23 from the second line, and $(m_{11}, m_{12}, m_{21}, m_{22})$ = (18, 17, 19, 4).

Using the joint Type-II GHCS presented by Table 6, we plotted the profile log-likelihood function (16) as in Figure 1. The maximum values need to begin with initial values of the parameters β_1 and β_2, showing that the iteration can be run with initial values that are almost in the neighborhood of the maximum values in Figure 1; therefore, the initial values were taken to be (β_1, β_2) = (5, 6). For Bayes estimation, we adopted non-informative prior with $a_i = b_i = 0.0001, i = 1, 2, \ldots, 6$. For the MCMC approach in Bayes method, we ran the chain 11,000 with the first 1000 values as burn-in. The MCMC approach that describes the empirical posterior distribution is shown in Figures 2–7. Hence, the results of the ML point and interval estimates and different Bayes estimates were computed and the results are presented in Tables 7 and 8.

Table 6. Jointly type-II GHCS competing risks sample from Hoal data with $m = 50$.

t_i	0.04	0.042	0.051	0.062	0.136	0.158	0.159	0.163	0.179	0.189	0.191	0.192	0.193	0.194
η_i	1	1	1	1	0	0	1	1	1	1	1	0	0	0
ρ_i	2	2	2	2	2	1	1	2	2	1	1	1	1	1
t_i	0.195	0.198	0.2	0.202	0.206	0.207	0.212	0.215	0.22	0.222	0.228	0.229	0.23	0.235
η_i	0	1	1	0	1	1	0	0	1	1	1	0	0	1
ρ_i	1	1	1	1	2	1	1	1	1	2	2	1	1	1
t_i	0.237	0.24	0.244	0.245	0.246	0.247	0.249	0.25	0.252	0.255	0.256	0.259	0.261	0.265
η_i	0	0	0	1	0	0	1	1	1	0	1	0	1	1
ρ_i	1	1	1	1	2	1	2	1	2	2	1	1	1	1
t_i	0.266	0.28	0.282	0.3	0.301	0.321	0.324	0.333	0.337	0.341	0.343	0.356	0.366	0.376
η_i	1	1	1	0	0	0	1	1	0	1	1	1	1	0
ρ_i	1	1	2	1	1	1	2	2	1	2	1	1	2	2
t_i	0.383	0.385												
η_i	1	1												
ρ_i	1	2												

Table 7. Point estimates with 95% CIs of the parameters.

Pa.	$(.)_{ML}$	$(.)_{Boot}$	$(.)_{B\text{-}MCMC}$	ACI	Boot-p	Boot-t	CI
θ_{11}	0.3365	0.5412	0.4675	(0.0501, 0.6228)	(0.0472, 1.3214)	(0.1784, 0.9115)	(0.1903, 0.9029)
θ_{12}	0.3178	0.4578	0.4442	(0.0450, 0.5905)	(0.1472, 0.8897)	(0.1954, 0.8874)	(0.1782, 0.8789)
θ_{21}	0.3156	0.4652	0.4636	(0.0006, 0.6306)	(0.0015, 0.9541)	(0.1924, 0.8556)	(0.1778, 0.8789)
θ_{22}	0.0664	0.1243	0.1184	(-0.0216, 0.1545)	(0.0824, 0.4123)	(0.0336, 0.2911)	(0.0298, 0.2824)
β_1	5.1907	5.3254	4.1962	(2.1508, 8.2306)	(2.3652, 8.4562)	(1.4215, 7.1921)	(1.3725, 7.0651)
β_2	4.5269	4.7771	3.3093	(0.8129, 8.2408)	(0.9112, 8.7214)	(0.741, 6.4007)	(0.6402, 6.4894)

Table 8. Point estimates of the reliability and failure rates at $t = 0.1$.

Method	R_{11}	R_{12}	R_{21}	R_{22}	h_{11}	h_{12}	h_{21}	h_{22}
$(.)_{ML}$	0.9569	0.9592	0.9609	0.9916	0.5654	0.5340	0.4963	0.1045
$(.)_{Bayes}$	0.9448	0.9475	0.9478	0.9864	0.6869	0.6523	0.6213	0.1588

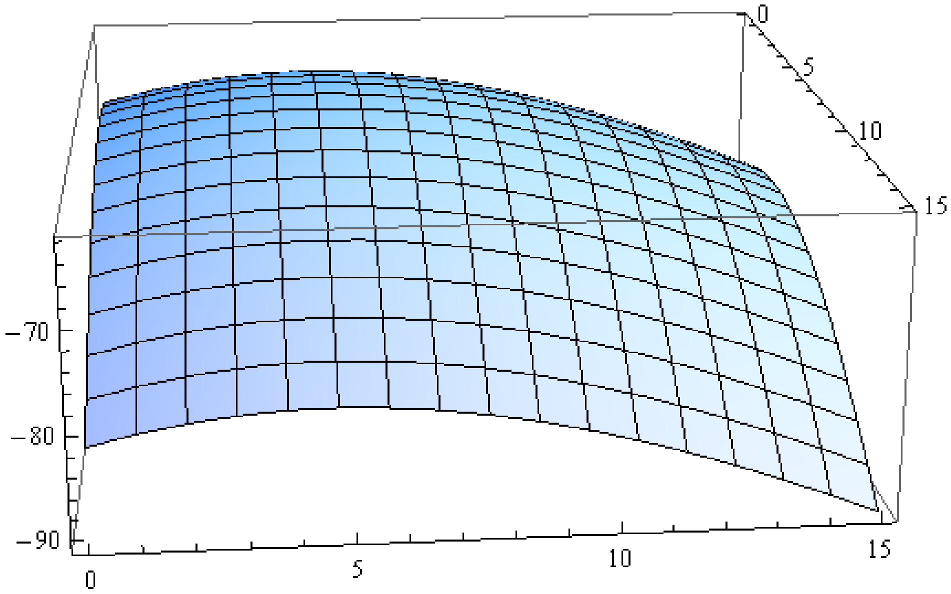

Figure 1. Profile log-likelihood function (y-axis) of (β_1, x-axis) and (β_2, z-axis).

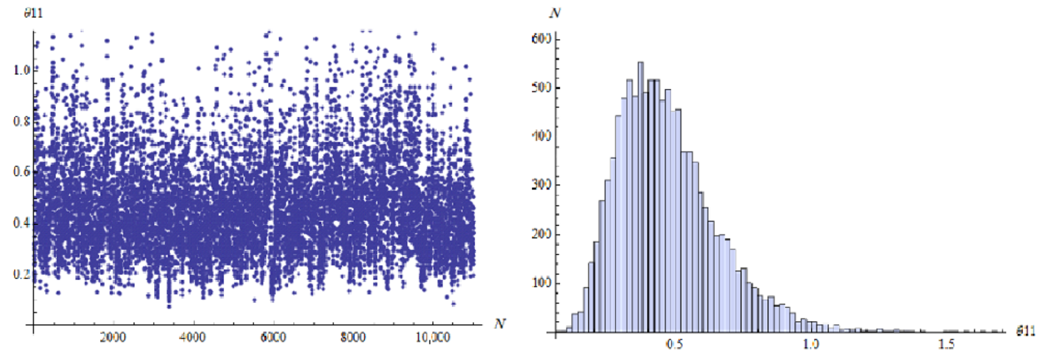

Figure 2. Trace (**Left**) and histogram (**Right**) plots of the parameter θ_{11}.

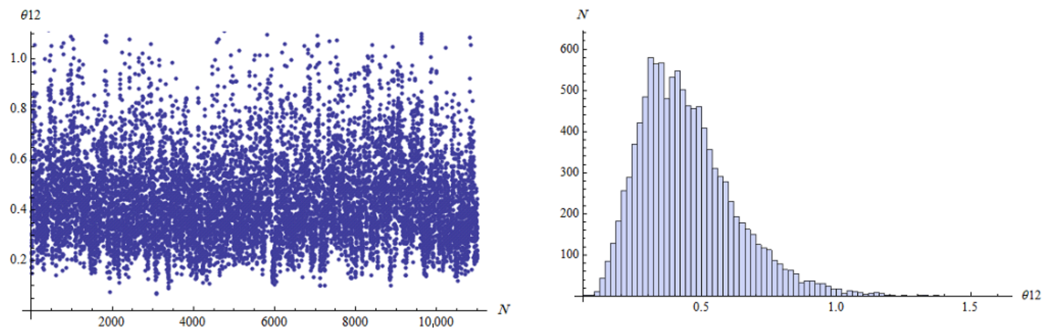

Figure 3. Trace (**Left**) and histogram (**Right**) plots of the parameter θ_{12}.

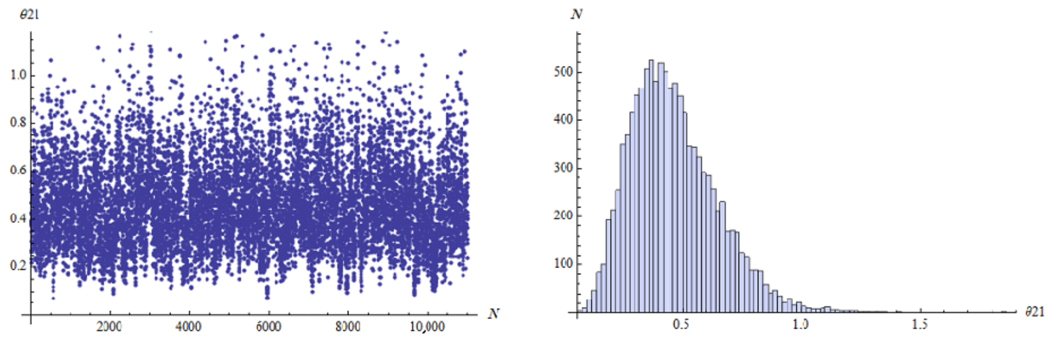

Figure 4. Trace (**Left**) and histogram (**Right**) plots of the parameter θ_{21}.

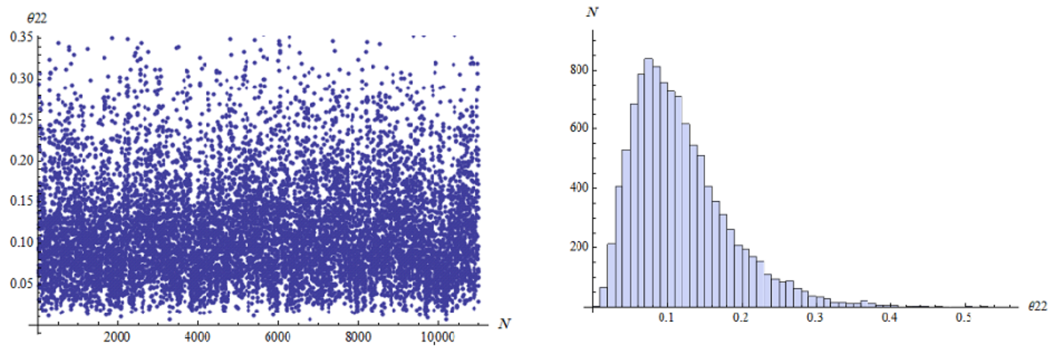

Figure 5. Trace (**Left**) and histogram (**Right**) plots of the parameter θ_{22}.

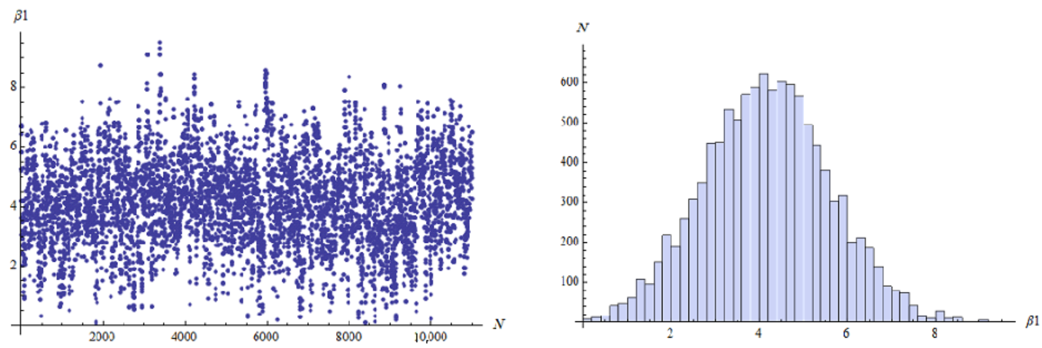

Figure 6. Trace (**Left**) and histogram (**Right**) plots of the parameter β_1.

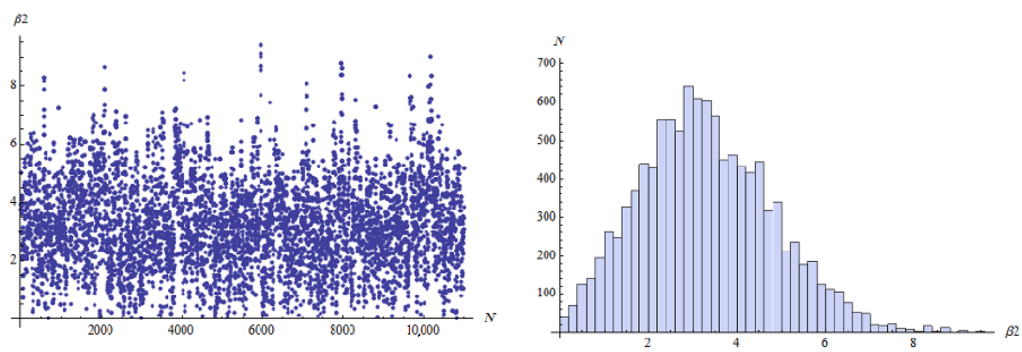

Figure 7. Trace (**Left**) and histogram (**Right**) plots of the parameter β_2.

9. Conclusions

Inference under various censoring techniques is crucial for life testing. Here, the problem of statistical inferences under a joint censoring scheme for Gompertz distribution is considered. Various inferences for unknown parameters of the proposed model were obtained from classical and Bayesian methods. We proposed that Gompertz units have two independent causes of failure, which can be determined using a competing risks model. Classical ML and bootstrap methods were used. Additionally, by using the Bayes technique and the MCMC method, the point and interval estimates were computed based on informative and non-informative priors. The asymptotic confidence intervals and Bayes credible intervals were also discussed. We used real data analysis and Monte Carlo simulation studies to assess and discuss the results. From the numerical result, we observed that the MLEs and non-informative Bayes estimations were closed. The Bayes method and bootstrap-t under informative prior P^1 worked better than other methods. The numerical results generally showed that using an informative prior distribution in Bayes computations produces superior results to likelihood estimates. The estimates obtained under MCMC method also performed well for all sample sizes and affected sample size in terms of MSEs and interval average widths. The study demonstrated that the comparative Gompertz distributions has good flexibility for modeling joint samples of survival times of male mice under a conventional laboratory environment. Finally, we can say that the proposed model and proposed method of estimation work well. Therefore, our results are very important in the field of comparative life testing, especially when units fail due to several causes of failure.

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