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Coefficient Results concerning a New Class of Functions Associated with Gegenbauer Polynomials and Convolution in Terms of Subordination

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Abstract: Gegenbauer polynomials constitute a numerical tool that has attracted the interest of many function theorists in recent times mainly due to their real-life applications in many areas of the sciences and engineering. Their applications in geometric function theory (GFT) have also been considered by many researchers. In this paper, this powerful tool is associated with the prolific concepts of convolution and subordination. The main purpose of the research contained in this paper is to introduce and study a new subclass of analytic functions. This subclass is presented using an operator defined as the convolution of the generalized distribution and the error function and applying the principle of subordination. Investigations into this subclass are considered in connection to Carathéodory functions, the modified sigmoid function and Bell numbers to obtain coefficient estimates for the contained functions.

Keywords: analytic function; starlike function; convex function; univalent function; Gegenbauer polynomials; Bell numbers; sigmoid function

MSC: 30C45; 30C50

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1. Introduction and Preliminaries

The beginning of univalent function theory is largely credited to P. Koebe's article published in 1907 [1]. Problems pertaining to the full class of univalent functions were the primary focus at first. Bieberbach, who published numerous significant papers on the theory of univalent functions in the early 1920s, was a key figure in the early development of geometric function theory. He conjectured his well-known bounds for a normalized univalent function's coefficients in 1916 [2] and established the bound for the second coefficient. It was not until 1984 [3] that the hypothesis was generally proven.

In a paper published in 1915 [4], Alexander intended to obtain sufficient conditions for a function to map the interior of the unit disc in a one-to-one manner. As a result, Alexander developed a number of classes of univalent functions as well as several tests that ensured the univalence of those classes, initiating new lines of research in GFT. Alexander first proposed the concepts of starlike functions, close-to-convex functions and functions of bounded turning, along with other ideas and theorems that were later rediscovered, often without awareness of Alexander's pioneering work. In a nice review paper [5], the authors analyze the content of Alexander's paper emphasizing his intuitive arguments and how those arguments were used by other researchers for further developments. Alexander describes [4] a star-shaped region as a set whose every point may be connected to point a via a linear segment made up only of points contained in the region. The center is designated as point a. The region is said to be convex when any point inside the region may be picked as the center. In an effort to guarantee the univalence of the mapping by controlling the shape of the boundary image, he proposed the idea of mapping the unit disc onto a starlike or convex region. By mandating that the boundary image is a starlike or a convex domain, univalence is achieved since overlapping or looping is avoided. Geometric characterization states that a mapping $w = w(z)$ is star-shaped if $\arg w(z)$ is a never-decreasing function of $\theta = \arg w(z)$ when z describes the unit circle in the counterclockwise direction, and it is a convex function if the argument of the normal vector of the image curve is a non-decreasing function of an increasing θ.

There are numerous intriguing uses for star-shaped bodies in various fields. For instance, star-shaped bodies were explored in the context of compressible fluid penetration in mechanics [6]. Computer simulations were extensively used in statistical mechanics to study models of fluids, liquid crystals, plastic crystals and other solid-phase systems made of hard convex bodies [7]. On the other hand, it has been demonstrated in [8] that hard star-shaped bodies can replace hard convex bodies in computer simulations of constant volume and constant pressure. As with other applications for star-shaped bodies, it has been determined in elasticity theory that the stress field is uniform when m is an odd integer for an m-pointed polygonal inclusion exposed to a uniform eigenstrain [9].

Many univalent function subclasses have captured the interest of GFT researchers. Such subclasses are defined using functions f belonging to the class $\mathcal{A}$ of holomorphic functions that have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ z \in E, \tag{1}$$

where $E = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. The class of starlike functions is comprised functions $f \in \mathcal{A}$ with the geometric representation $Re\frac{zf'(z)}{f(z)} > 0$, the class of convex functions contains functions $f \in \mathcal{A}$ with the geometric characterization given by $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ and the class of close-to-convex functions is characterized by $Re\frac{zf'(z)}{g(z)} > 0$, with g representing a starlike function.

Recently, Babalola [10] improved on a subclass of $\mathcal{A}$ called the class of starlike functions by introducing the class $\mathcal{L}_\lambda(\beta)$, which is defined as the class of functions f belonging to $\mathcal{A}$ that satisfies

$$Re\frac{z(f'(z))^\lambda(z)}{f(z)} > \beta, \tag{2}$$

where $\beta \in [0, 1)$ and $\lambda \geq 1 \in \mathbb{R}$. Since then, many authors have used different approaches to study the class of functions introduced in [10].

Using an analytic function $F(z)$, the starlike and convex functions were investigated by authors such as [11–14] and extended to the class of $F-$starlike and $F-$convex functions denoted by FS^* and FK, respectively, which are represented by

$$Re\frac{F(z)f'(z)}{f(z)} > 0 \tag{3}$$

and

$$Re\left(1 + \frac{F(z)f''(z)}{f'(z)}\right) > 0, \tag{4}$$

respectively, with the condition $F(0) = 0$. By setting $F(z) = z$ in (3) and (4), the well-known starlike and convex functions are obtained [11,14].

Let

$$g(z) = z + \sum_{n=2}^{\infty} b_k z^n. \tag{5}$$

Then, the convolution of (1) and (5) gives

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{6}$$

For details, see [15,16].

A normalized Gegenbauer polynomial has the form

$$\mathcal{G}(z,m) = z + \sum_{n=2}^{\infty} c_{n-1}^{\beta}(m) z^n, \tag{7}$$

where $\beta > -\frac{1}{2}$. The first three coefficients of the forms are

$$c_0^{\beta}(m) = 1, \tag{8}$$

$$c_1^{\beta}(m) = 2\beta m, \tag{9}$$

$$c_2^{\beta}(m) = 2\beta(\beta+1)m^2 - \beta, \tag{10}$$

and the next coefficient is given by

$$c_3^{\beta}(m) = \frac{4\beta(\beta+1)(\beta+2)m^2}{3} - 2\beta(\beta+1)m. \tag{11}$$

In general, the *n-th* coefficient is defined by

$$c_n^{\beta}(m) = \frac{2m(n+\beta-1)c_{n-1}^{\beta}(m) - (n+2\beta-2)c_{n-2}^{\beta}(m)}{n}. \tag{12}$$

It originates from

$$q(z) = \int_{-1}^{1} \mathcal{G}(z,m) d\mu(m),$$

where

$$\mathcal{G}(z,m) = \frac{z}{(1-2mz+z^2)^{\beta}} \tag{13}$$

and μ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[s,t]$ is denoted by $P_{[s,t]}$.

By substituting $\beta = \frac{1}{2}$ in $\mathcal{G}(z,m)$, the Legendre polynomial will be obtained, while by setting $\beta = 1$ in $\mathcal{G}(z,m)$, the famous Chebyshev polynomial will be obtained, which are both tools in the field. These recent results can be seen in [17,18].

Gegenbauer polynomials have been studied intensely and have proved to provide interesting results, as seen in early studies such as [19,20]. They have wide applications in queueing theory, as can be seen in [21], signal analysis, automatic control, scattering theory and many others. Applications in GFT include defining the subclasses of univalent functions [22] and bi-univalent functions [23]. Coefficient studies on the subclasses of bi-univalent functions can be seen in very recent papers, such as [24–27].

Let D denote the sum of the convergent series of the form

$$D = \sum_{n=0}^{\infty} a_n, \tag{14}$$

where $a_n \geq 0$ for all $n \in \mathbb{N}$. The probability mass function of the generalized discrete probability distribution defined using (14) is given by $p(n) = \frac{a_n}{D}, n = 0,1,2,3,\ldots$. Function $p(n)$ is the probability mass function because $p(n) \geq 0$ and $\sum_n p(n) = 1$.

Additionally, let $\psi(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, since $D = \sum_{n=0}^{\infty} a_n$ is convergent, the series ψ is convergent for $|x| < 1$ and $x = 1$. The interest of the present investigation is the power series whose coefficients are probabilities of generalized distributions of the form

$$\mathcal{H}_{\psi}(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{D} z^n. \tag{15}$$

Details can be found in [22,28,29].

The error function is a special function that occurs in probability, statistics, material science, partial differential equation, physics, chemistry, biology, mass flow and diffusion for transportation phenomena. It also occurs in quantum mechanics to eliminate the probability of observing a particle in a particular region. Barton et al. [30] introduced the function of the form

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+1}}{n!(2n+1)}. \tag{16}$$

The properties and inequalities of error functions have also been considered by Alzer [31], Cartilz [32], Coman [33], Elbert [34] and many other researchers in the field.

Ramachandran et al. [35,36] modified (16) to

$$\operatorname{Erf}(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} z^n}{(n-1)!(2n-1)}, \tag{17}$$

which is analytic in the unit disk $U = z : |z| < 1$ and normalized by $\operatorname{Erf}(0) = 0$ and $\operatorname{Erf}'(0) = 1$.

The convolution of (15) and (17) generates the following function, which will be used to define the new subclass of functions that is investigated in this paper:

$$\mathcal{F}_{\leftarrow}(z) = \left(\mathcal{H}_{\psi} * \operatorname{Erf}\right)(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} \frac{a_{n-1}}{D} z^n, \tag{18}$$

as a power series. See [16] for details.

Let $\mathcal{P}$ denote the class of the Carathéodory functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{19}$$

with the conditions $Re\, p(z) > 0$ and $p(0) = 1$.

The functions of the form

$$G(z) = \frac{1}{1+e^{-z}} = \frac{1}{2} + \frac{z}{4} - \frac{z^3}{48} + \frac{z^5}{480} - \frac{17z^7}{80640} + \ldots,$$

referred to as sigmoid functions, are defined in [37] by the following modified form:

$$\gamma(z) = \frac{2}{1+e^{-z}} = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{17z^7}{40320} + \ldots. \tag{20}$$

The sigmoid function has been repeatedly studied by many researchers because it has the following properties: it outputs real numbers between 0 and 1, maps a very large input domain to a small range of outputs, never loses information because it is a one-to-one function, increases monotonically and is also differentiable. The sigmoid function has useful applications in fields such as functional analysis, real analysis, algebra, topology, differential equations and many others. It has numerous methods of evaluation but, here, only the truncated series expansion is considered. See [38–44].

The function of the form

$$\mathcal{Q}(z) = e^{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \dots, \qquad z \in \mathbb{E}, \tag{21}$$

was investigated by Kumar et al. [45]. This function is starlike with respect to one, and its coefficients generate the Bell numbers where $B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15$, $B_5 = 52$ and $B_6 = 203$ are the coefficients generated through binomial expansion. In recent times, some applications of the Beta function were considered in [46–48], while Olatunji and Altinkaya [49] used (21) to investigate the generalized distribution for the analytic function classes associated with error functions and Bell numbers. Further information can also be found in [49–51].

In the present work, the authors draw motivation from prior research in [15,17,29,49]. The aim of this paper is to consider applications of certain special functions in GFT. In particular, certain results are obtained in terms of subordination associated with Carathéodory functions, the modified sigmoid function and Bell numbers for the specific class of functions defined here and given in the next definition. The early coefficient bounds obtained are used to establish the famous Fekete–Szegö inequalities.

The following class is defined and studied in this paper.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{G}S^*_{\mathcal{F}\psi}(m, \beta)$, where $m \in [-1, 1]$ and $\beta \geq 1$, if the following subordination is satisfied*

$$Re \frac{\mathcal{G}\left(\mathcal{F}'_{\psi}\right)(z)}{\mathcal{F}_{\psi}(z)} \prec \sqrt{1+z} \tag{22}$$

with the condition $\mathcal{G}_{\psi}(0) = 0$. The function $\mathcal{F}_{\psi}(z)$, defined by (18), is a convolution of (7) and (17).

The class of functions $\mathcal{G}S^*_{\mathcal{F}_{\leftarrow}}(m, \beta)$, defined above, is investigated in the next section in relation to the Carathéodory function $p(z)$, the modified sigmoid function and Bell numbers by means of the subordination principle, and initial coefficient estimates are obtained. Furthermore, those results are used for investigating the Fekete–Szegö problem.

2. Main Results

The main aim of this work is to investigate the coefficient problems for the class of functions $\mathcal{G}S^*_{\mathcal{F}\psi}(m, \beta)$ defined in this study. The coefficient estimates are obtained using the Carathéodory function $p(z)$ defined by (19), the modified sigmoid function given by (20) and Bell numbers generated by the function given in (21) involving functions associated with Gegenbauer polynomials. The applications of Gegenbauer polynomials (7), the error function (17), a generalized distribution function (15), Carathéodory functions (19), the modified sigmoid function (20), Bell numbers (21) and some other functions in GFT have been considered by several authors in the field. In this study, the authors use combinations of all the functions mentioned above with the purpose of investigating the coefficients of the class of F-starlike functions $\mathcal{G}S^*_{\mathcal{F}\psi}(m, \beta)$ such that every function in the class satisfies the condition seen in (22). The Gegenbauer polynomials used in this work can be found to have some applications in queueing theory [21], signal analysis, automatic control, scattering theory and many other areas. Gengenbauer polynomials, also known as ultraspherical polynomials $C_n^{\alpha}(x)$, are orthogonal polynomials defined on the closed interval $[-1, 1]$. These polynomials are obtained as solutions of the Gengenbauer differential equation, which reduces to the Chebyshev differential equation for $\alpha = 1$.

First, we consider obtaining the coefficient bounds for the class of functions $\mathcal{G}S^*_{\mathcal{F}\psi}(m, \beta)$ associated with the Carathéodory functions given by (19).

Theorem 1. *Let $f(z)$ be defined by (1) and $p(z)$ by (19). Then, $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$, if*

$$\left|\frac{a_1}{s}\right| \leq \left|\frac{3(4c_1^{\beta}(m) - p_1)}{4}\right| \tag{23}$$

and

$$\left|\frac{a_2}{s}\right| \leq \left|\frac{5\left[8p_2 - 7p_1^2 + 16c_1^{\beta}(m)p_1 + 32\left((c_1^{\beta}(m))^2 - c_2^{\beta}(m)\right)\right]}{32}\right|. \tag{24}$$

Proof. Let $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$, then

$$\frac{\mathcal{G}\left(\mathcal{F}'_{\psi}\right)(z)}{\mathcal{F}_{\psi}(z)} = \sqrt{1+\omega(z)}. \tag{25}$$

The left-hand side of (25) gives

$$\frac{\mathcal{G}\left(\mathcal{F}'_{\psi}\right)(z)}{\mathcal{F}_{\psi}(z)} = 1 + \left(c_1^{\beta}(m) - \frac{a_1}{3s}\right)z + \left(c_2^{\beta}(m) - \frac{c_1^{\beta}(m)a_1}{3s} + \frac{a_2}{5s} - \frac{a_1^2}{9s^2}\right)z^2 + \ldots, \tag{26}$$

while the right-hand side gives

$$\sqrt{1 + \frac{p(z)-1}{p(z)+1}} = 1 + \frac{p_1}{4}z + \left(\frac{p_2}{4} - \frac{5p_1^2}{32}\right)z^2 + \ldots. \tag{27}$$

Comparing the coefficients of z and z^2 in (26) and (27), we obtain

$$\frac{a_1}{s} = \frac{3(4c_1^{\beta}(m) - p_1)}{4}$$

and

$$\frac{a_2}{s} = \frac{5\left[8p_2 - 7p_1^2 + 16c_1^{\beta}(m)p_1 + 32\left((c_1^{\beta}(m))^2 - c_2^{\beta}(m)\right)\right]}{32},$$

which completes the proof. □

The next two theorems are concerned with the investigation of certain coefficient problems for the class of functions $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ involving the sigmoid function given by (20).

Theorem 2. *Let $f(z)$ be defined by (1) and $\gamma(z)$ by (20). Then, $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$ if the following condition holds true*

$$\left|\frac{a_2}{s} - \mu\frac{a_1^2}{s^2}\right| \leq \left|\frac{9(4c_1^{\beta}(m) - p_1)^2}{16}\left[\frac{5[8p_2 - 7p_1^2 + 16c_1^{\beta}(m)p_1 + 32((c_1^{\beta}(m))^2 - c_2^{\beta}(m))]}{18(4c_1^{\beta}(m) - p_1)^2} - \mu\right]\right|. \tag{28}$$

Proof. Let $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ $\beta \geq 1$ and $\mu \in \mathbb{R}$. Then

$$\frac{a_2}{2} - \mu\frac{a_1^2}{s^2} = \left[\frac{5[8p_2 - 7p_1^2 + 16c_1^{\beta}(m)p_1 + 32((c_1^{\beta}(m))^2 - c_2^{\beta}(m))]}{32} - \mu\left(\frac{3(4c_1^{\beta}(m) - p_1)}{4}\right)^2\right],$$

$$\frac{a_2}{2} - \mu\frac{a_1^2}{s^2} = \frac{9(4c_1^{\beta}(m) - p_1)^2}{16}\left[\frac{5[8p_2 - 7p_1^2 + 16c_1^{\beta}(m)p_1 + 32((c_1^{\beta}(m))^2 - c_2^{\beta}(m))]}{18(4c_1^{\beta}(m) - p_1)^2} - \mu\right],$$

which finally gives

$$\left|\frac{a_2}{2}-\mu\frac{a_1^2}{s^2}\right| \leq \left|\frac{9(4c_1^{\beta}(m)-p_1)^2}{16}\left[\frac{5[8p_2-7p_1^2+16c_1^{\beta}(m)p_1+32((c_1^{\beta}(m))^2-c_2^{\beta}(m))]}{18(4c_1^{\beta}(m)-p_1)^2}-\mu\right]\right|.$$

□

Theorem 3. *Let $f(z)$ be defined by (1) and $\gamma(z)$ by (20). Then, $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$, if*

$$\left|\frac{a_1}{s}\right| \leq \left|\frac{3(8c_1^{\beta}(m)-1)}{8}\right| \tag{29}$$

and

$$\left|\frac{a_2}{s}\right| \leq \left|\frac{5\left[256(c_1^{\beta}(m))^2-128c_2^{\beta}(m)-48c_1^{\beta}(m)-3\right]}{128}\right|. \tag{30}$$

Proof. Let $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$, where $m \in [-1,1]$ and $\beta \geq 1$. Then

$$\frac{\mathcal{G}\left(\mathcal{F}'_{\psi}\right)(z)}{\mathcal{F}_{\psi}(z)} = \sqrt{1+\omega(z)}.$$

The left-hand side of (25) gives

$$\frac{\mathcal{G}\left(\mathcal{F}'_{\psi}\right)(z)}{\mathcal{F}_{\psi}(z)} = 1+\left(c_1^{\beta}(m)-\frac{a_1}{3s}\right)z+\left(c_2^{\beta}(m)-\frac{c_1^{\beta}(m)a_1}{3s}+\frac{a_2}{5s}-\frac{a_1^2}{9s^2}\right)z^2+\ldots, \tag{31}$$

while the right-hand side gives

$$\sqrt{1+\frac{\gamma(z)-1}{\gamma(z)+1}} = 1+\frac{1}{8}z-\frac{5}{128}z^2+\ldots \tag{32}$$

Comparing the coefficients of z and z^2 in (26) and (32), we obtain

$$\frac{a_1}{s} = \frac{3(8c_1^{\beta}(m)-1)}{8}$$

and

$$\frac{a_2}{s} = \frac{5\left[256(c_1^{\beta}(m))^2-128c_2^{\beta}(m)-48c_1^{\beta}(m)-3\right]}{128},$$

which completes the proof. □

Theorems 4–6 involve the investigation of certain coefficient problems of the class $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ with respect to the Bell numbers (21).

Theorem 4. *Let $f(z)$ be defined by (1) and $\mathcal{Q}(z)$ by (21). Then, $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$ if the following condition holds true*

$$\left|\frac{a_2}{s}-\mu\frac{a_1^2}{s^2}\right| \leq \left|\frac{5\left[256(c_1^{\beta}(m))^2-128c_2^{\beta}(m)-48c_1^{\beta}(m)-3\right]}{36(8c_1^{\beta}(m)-1)^2}-\mu\right|. \tag{33}$$

Proof. Let $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$ and $\mu \in \mathbb{R}$. Then

$$\frac{a_2}{s}-\mu\frac{a_1^2}{s^2} = \frac{5\left[256(c_1^{\beta}(m))^2-128c_2^{\beta}(m)-48c_1^{\beta}(m)-3\right]}{128}-\mu\left(\frac{3(8c_1^{\beta}(m)-1)}{8}\right)^2,$$

$$\frac{a_2}{2} - \mu \frac{a_1^2}{s^2} = \frac{9(8c_1^{\beta}(m)-1)^2}{64}\left(\frac{5\left[256(c_1^{\beta}(m))^2 - 128c_2^{\beta}(m) - 48c_1^{\beta}(m) - 3\right]}{36(8c_1^{\beta}(m)-1)^2} - \mu\right),$$

which finally gives:

$$\left|\frac{a_2}{s} - \mu \frac{a_1^2}{s^2}\right| \leq \left|\frac{5\left[256(c_1^{\beta}(m))^2 - 128c_2^{\beta}(m) - 48c_1^{\beta}(m) - 3\right]}{36(8c_1^{\beta}(m)-1)^2} - \mu\right|.$$

□

Theorem 5. *Let $f(z)$ be defined by (1) and $\mathcal{Q}(z)$ by (21). Then, $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$, if*

$$\left|\frac{a_1}{s}\right| \leq \left|\frac{3(4c_1^{\beta}(m)-1)}{4}\right| \tag{34}$$

and

$$\left|\frac{a_2}{s}\right| \leq \left|\frac{5\left[5 + 16c_1^{\beta}(m) + 32\left(2(c_1^{\beta}(m))^2 - c_2^{\beta}(m)\right)\right]}{32}\right|. \tag{35}$$

Proof. Let $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$. Then

$$\frac{\mathcal{G}\left(\mathcal{F}'_{\psi}\right)(z)}{\mathcal{F}_{\psi}(z)} = \sqrt{1+\omega(z)}.$$

The left-hand side of (25) gives

$$\frac{\mathcal{G}\left(\mathcal{F}'_{\psi}\right)(z)}{\mathcal{F}_{\psi}(z)} = 1 + \left(c_1^{\beta}(m) - \frac{a_1}{3s}\right)z + \left(c_2^{\beta}(m) - \frac{c_1^{\beta}(m)a_1}{3s} + \frac{a_2}{5s} - \frac{a_1^2}{9s^2}\right)z^2 + \dots,$$

while the right-hand side gives

$$\sqrt{1 + \frac{\mathcal{Q}(z)-1}{\mathcal{Q}(z)+1}} = 1 + \frac{1}{4}z + \frac{3}{32}z^2 + \dots. \tag{36}$$

When the coefficients of z and z^2 in (26) and (36), are compared, the following values are obtained:

$$\frac{a_1}{s} = \frac{3(4c_1^{\beta}(m)-1)}{4}$$

and

$$\frac{a_2}{s} = \frac{5\left[5 + 16c_1^{\beta}(m) + 32\left(2(c_1^{\beta}(m))^2 - c_2^{\beta}(m)\right)\right]}{32}.$$

Hence, the proof is completed. □

In Theorem 2, the coefficient estimates were established using the Carathéodory function $p(z)$ given by (19). We now consider in the next theorem the coefficient estimates for the class of functions $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ using the Bell numbers $\mathcal{Q}(z)$ as given by (21).

Theorem 6. *Let $f(z)$ be defined by (1) and $\mathcal{Q}(z)$ by (21). Then, $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$ and $\beta \geq 1$ if the following condition holds true*

$$\left|\frac{a_2}{s} - \mu\frac{a_1^2}{s^2}\right| \leq \left|\frac{9(4c_1^\beta(m)-1)^2}{16}\left[\frac{5\left[5+16c_1^\beta(m)+32\left(2(c_1^\beta(m))^2-c_2^\beta(m)\right)\right]}{18(4c_1^\beta(m)-1)^2}-\mu\right]\right|. \quad (37)$$

Proof. Let $f \in \mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ where $m \in [-1,1]$, $\beta \geq 1$ and $\mu \in \mathbb{R}$. Then

$$\frac{a_2}{2} - \mu\frac{a_1^2}{s^2} = \left[\frac{5\left[5+16c_1^\beta(m)+32\left(2(c_1^\beta(m))^2-c_2^\beta(m)\right)\right]}{32} - \mu\left(\frac{3(4c_1^\beta(m)-1)}{4}\right)^2\right],$$

$$\frac{a_2}{2} - \mu\frac{a_1^2}{s^2} = \frac{9(4c_1^\beta(m)-1)^2}{16}\left[\frac{5\left[5+16c_1^\beta(m)+32\left(2(c_1^\beta(m))^2-c_2^\beta(m)\right)\right]}{18(4c_1^\beta(m)-p_1)^2}-\mu\right],$$

which finally gives

$$\left|\frac{a_2}{2} - \mu\frac{a_1^2}{s^2}\right| \leq \left|\frac{9(4c_1^\beta(m)-1)^2}{16}\left[\frac{5\left[5+16c_1^\beta(m)+32\left(2(c_1^\beta(m))^2-c_2^\beta(m)\right)\right]}{18(4c_1^\beta(m)-p_1)^2}-\mu\right]\right|.$$

□

3. Conclusions

The investigation presented in the paper concerns a new subclass of functions denoted by $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ introduced in Definition 1 by using an operator defined in (18) as the convolution of the generalized distribution and the error function using the concept of subordination. The new class is interesting due to the powerful tools in geometric function theory used for introducing it, namely convolution and subordination. The main aim of the research presented in this paper targets a topic of interest at this moment in GFT: coefficient-related studies. The first theorem proved in Section 2, Theorem 1, provides the coefficient estimates for functions that are part of the class $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$ by involving the Carathéodory function $p(z)$ defined in (19). The next results, proved in Theorem 2 and Theorem 3, use the sigmoid function given by (20) for establishing further coefficient estimates regarding the class $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$. Finally, the Bell numbers given by (21) are used in Theorems 4–6 to provide other forms of coefficient estimates concerning functions from the new class $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$.

The initial results regarding the coefficient estimates obtained here can be used for further specific investigations regarding coefficients of the functions from class $\mathcal{GS}^*_{\mathcal{F}\psi}(m,\beta)$, such as estimations for Hankel determinants of different orders, Toeplitz determinants or the Fekete–Szegö problem.

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