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A Note on Finite Coarse Shape Groups

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Abstract: In this paper, we investigate properties concerning some recently introduced finite coarse shape invariants—the k-th finite coarse shape group of a pointed topological space and the k-th relative finite coarse shape group of a pointed topological pair. We define the notion of finite coarse shape group sequence of a pointed topological pair (X, X_0, x_0) as an analogue of homotopy and (coarse) shape group sequences and show that for any pointed topological pair, the corresponding finite coarse shape group sequence is a chain. On the other hand, we construct an example of a pointed pair of metric continua whose finite coarse shape group sequence fails to be exact. Finally, using the aforementioned pair of metric continua together with a pointed dyadic solenoid, we show that finite coarse-shape groups, in general, differ from both shape and coarse-shape groups.

Keywords: topological space; inverse system; pro-category; pro*-category; shape; finite coarse shape; coarse shape; shape group; polyhedron; exactness

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1. Introduction

The shape theory of metric compacta was founded by K. Borsuk in 1967 [1,2]. Later on, the shape theory was extended to the class of all topological spaces by S. Mardešić [3] and K. Morita [4]. Further generalizations were made by N. Koceić-Bilan and N. Uglešić. They have founded the coarse shape theory for all topological spaces using the inverse systems approach. The coarse shape classification of topological spaces is generally coarser than the homotopy type classification and shape classification, although they all coincide on the class of polyhedra.

The authors have recently introduced a new classification of topological spaces based on *the finite coarse shape theory* [5]. This new theory is quite abstract and defined for an arbitrary pair (C, D) consisting of a category C, and it is a full and pro-reflective (dense) subcategory D. In the special case when $C = HTop$ (the homotopy category of all topological spaces) and $D = HPol$ (the category of all topological spaces having the homotopy type of a polyhedron), one speaks of *the (topological) finite coarse shape category* $Sh^{\circledast}$. The standard shape category Sh of topological spaces can be considered as a proper subcategory of the finite coarse shape category $Sh^{\circledast}$ and $Sh^{\circledast}$ is a proper subcategory of the coarse shape category Sh^{*}. Following the general construction, for $C = HTop_{\star}$ ($C = HTop^2_{\star}$) (the homotopy category of all pointed topological spaces (pairs)) and $D = HPol_{\star}$ ($D = HPol^2_{\star}$) (the category of all pointed topological spaces (pairs) having the homotopy type of a pointed polyhedron (polyhedral pair)), one obtains *the pointed topological finite coarse shape category (of pairs)* $Sh^{\circledast}_{\star}$ ($Sh^{\circledast 2}_{\star}$).

In reference [6] the authors introduced the notion of *the k-th finite coarse shape group* $\check{\pi}^{\circledast}_k(X, x_0)$ *of a pointed topological space* (X, x_0). For each $k \in \mathbb{N}$, $\check{\pi}^{\circledast}_k(X, x_0)$ is a group (for $k = 0$ a pointed set) having $Sh^{\circledast}_{\star}\Big(\big(\mathbb{S}^k, s_0\big), (X, x_0)\Big)$ as underlying set with a group operation defined in a certain way (more on this in Section 2.1). Analogously, one can define the notion of *the k-th relative finite coarse shape group* $\check{\pi}_k^{\circledast}(X, X_0, x_0)$ *of a pointed pair of topological spaces* (X, X_0, x_0). For each $k \in \mathbb{N} \setminus \{1\}$, $\check{\pi}_k^{\circledast}(X, X_0, x_0)$ is a group (for $k = 1$ a pointed set) having $Sh_{\star}^{\circledast 2}\left(\left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right), (X, X_0, x_0)\right)$ as underlying set. Furthermore, the existence of a certain *k-th (relative) finite coarse shape group functor* $\check{\pi}_k^{\circledast}$ from the corresponding finite coarse shape category to the category Grp ($Set_\star$) implies that the (relative) finite coarse shape groups are invariants of the finite coarse shape theory.

In this paper, we introduce the notion of *finite coarse shape group sequence of a pointed topological pair* (X, X_0, x_0). Since homotopy group sequences [7] and coarse shape group sequences are exact [8], while shape group sequences are only semiexact at each term ("chains") [3], it makes sense to investigate whether finite coarse shape group sequences have these properties. We will show that finite coarse-shape group sequences are chains and construct an example of a pointed pair of metric compacta such that the corresponding finite coarse-shape group sequence fails to be exact. Moreover, the aforementioned pointed pair of metric compacta will facilitate us to show that finite coarse-shape groups, in general, differ from both shape and coarse shape groups.

2. Preliminaries

2.1. The Finite Coarse Shape

We will now recall some elementary notions and properties of the finite coarse shape theory. *A finite $*$-morphism* (shorter $\circledast$*-morphism*) of inverse systems $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = \left(Y_\mu, q_{\mu\mu'}, M\right)$ in a category C is an ordered pair $\left(f, f_\mu^m\right)$ consisting of a function $f : M \to \Lambda$ (called an index function) and, for every $\mu \in M$, of a sequence of morphisms $f_\mu^m : X_{f(\mu)} \to Y_\mu$, $m \in \mathbb{N}$, in C such that:

(1) for every pair of comparable indices $\mu, \mu' \in M$, $\mu \le \mu'$, there exist $\lambda \in \Lambda$, $\lambda \ge f(\mu), f(\mu')$, and $m_{\mu\mu'} \in \mathbb{N}$ such that, for every $m \ge m_{\mu\mu'}$,

$$f_\mu^m p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^m p_{f(\mu')\lambda};$$

(2) for every $\mu \in M$

$$\operatorname{card}\left\{f_\mu^m : m \in \mathbb{N}\right\} < \aleph_0.$$

By composition of $\circledast$-morphisms $\left(f, f_\mu^m\right) : \mathbf{X} \to \mathbf{Y}$ and $(g, g_\nu^m) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ we mean a $\circledast$-morphism $(h, h_\nu^m) : \mathbf{X} \to \mathbf{Z}$ such that

$$h = fg \text{ and } h_\nu^m = g_\nu^m f_{g(\nu)}^m, \text{ for all } m \in \mathbb{N}, \nu \in N.$$

Given a category C, by $inv^{\circledast}$-C we denote the category having all inverse systems in C as objects and, for any pair of object $\mathbf{X}$ and $\mathbf{Y}$, having all $\circledast$-morphisms between $\mathbf{X}$ and $\mathbf{Y}$ as morphisms with the composition mentioned above as the categorial composition.

A $\circledast$-morphism $\left(f, f_\mu^m\right) : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to a $\circledast$-morphism $\left(f', f_\mu'^m\right) : \mathbf{X} \to \mathbf{Y}$, and one writes $\left(f, f_\mu^m\right) \sim \left(f', f_\mu'^m\right)$, if every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \ge f(\mu), f'(\mu)$, and $m_\mu \in M$ such that, for every $m \ge m_\mu$,

$$f_\mu^m p_{f(\mu)\lambda} = f_\mu'^m p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation on every set of $\circledast$-morphisms between inverse systems in C. The equivalence class $\left[\left(f, f_\mu^m\right)\right]$ of a $\circledast$-morphism $\left(f, f_\mu^m\right) : \mathbf{X} \to \mathbf{Y}$ is denoted by $\mathbf{f}^{\circledast}$.

All inverse systems in C as objects and all equivalence classes $\mathbf{f}^{\circledast}$ as morphisms form a category denoted by $pro^{\circledast}$-C. The composition in $pro^{\circledast}$-C is defined by the representatives, i.e,

$$\mathbf{g}^{\circledast} \circ \mathbf{f}^{\circledast} = \mathbf{h}^{\circledast} = [(h, h_\nu^m)],$$

where $(h, h_\nu^m) = \left(fg, g_\nu^m f_{g(\nu)}^m\right)$.

The joining which keeps the inverse systems in C fixed and associates to every morphism $\mathbf{f} = [(f, f_\mu)] : \mathbf{X} \to \mathbf{Y}$ of *pro*-C the *pro*$^\circledast$-C morphism $\mathbf{f}^\circledast = \left[\left(f, f_\mu^m\right)\right] : \mathbf{X} \to \mathbf{Y}$ such that $f_\mu^m = f_\mu$, for all $\mu \in M$, $m \in \mathbb{N}$, determines a faithful functor $\mathbf{J}_C^\circledast : pro\text{-}C \to pro^\circledast\text{-}C$. Hence, *pro*-$C$ can be considered as a subcategory of *pro*$^\circledast$-C.

Analogously, the joining which keeps the inverse systems in C fixed and associates to every morphism $\mathbf{f}^\circledast = \left[\left(f, f_\mu^m\right)\right] : \mathbf{X} \to \mathbf{Y}$ of *pro*$^\circledast$-C the same morphism as morphism $\mathbf{f}^* = \left[\left(f, f_\mu^m\right)\right] : \mathbf{X} \to \mathbf{Y}$ of *pro**-C determines a faithful functor $\mathbf{J}_C^* : pro^\circledast\text{-}C \to pro^*\text{-}C$. Hence, *pro*$^\circledast$-$C$ is a subcategory of *pro**-C.

Let C be an arbitrary category and $D \subseteq C$ a dense and full subcategory. Let $\mathbf{p} : (X) \to \mathbf{X}$ and $\mathbf{p}' : (X) \to \mathbf{X}'$ be D-expansions of the same object $X \in Ob(C)$ and let $\mathbf{q} : (Y) \to \mathbf{Y}$ and $\mathbf{q}' : (Y) \to \mathbf{Y}'$ be D-expansions of the same object $Y \in Ob(C)$. A morphism $\mathbf{f}^\circledast : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to a morphism $\mathbf{f}'^\circledast : \mathbf{X}' \to \mathbf{Y}'$ in *pro*$^\circledast$-D, denoted by $\mathbf{f}^\circledast \sim \mathbf{f}'^\circledast$, provided

$$\mathbf{J}_D^\circledast(\mathbf{j}) \circ \mathbf{f}^\circledast = \mathbf{f}'^\circledast \circ \mathbf{J}_D^\circledast(\mathbf{i}),$$

where $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ and $\mathbf{j} : \mathbf{Y} \to \mathbf{Y}'$ are canonical isomorphisms between different expansions of objects X and Y, respectively. The relation $\sim$ is an equivalence relation such that $\mathbf{f}^\circledast \sim \mathbf{f}'^\circledast$ and $\mathbf{g}^\circledast \sim \mathbf{g}'^\circledast$ imply $\mathbf{g}^\circledast \mathbf{f}^\circledast \sim \mathbf{g}'^\circledast \mathbf{f}'^\circledast$ whenever the compositions $\mathbf{g}^\circledast \mathbf{f}^\circledast$ and $\mathbf{g}'^\circledast \mathbf{f}'^\circledast$ exist. The equivalence class of a morphism $\mathbf{f}^\circledast$ is denoted by $\langle \mathbf{f}^\circledast \rangle$.

Based on the relation $\sim$ of *pro*$^\circledast$-D, to every pair (C, D) (where D is a dense and full subcategory of C) we associate a category $Sh_{(C,D)}^\circledast$ such that:

- $Ob\left(Sh_{(C,D)}^\circledast\right) = Ob(C)$;
- for any pair X, Y of objects in $Sh_{(C,D)}^\circledast$, the set $Sh_{(C,D)}^\circledast(X, Y)$ consists of equivalence classes $\langle \mathbf{f}^\circledast \rangle$ of all morphisms $\mathbf{f}^\circledast : \mathbf{X} \to \mathbf{Y}$ in *pro*$^\circledast$-D, where $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ are any D-expansions of objects X and Y respectively;
- the composition of $\langle \mathbf{f}^\circledast \rangle : X \to Y$ and $\langle \mathbf{g}^\circledast \rangle : Y \to Z$ is defined by the representatives, i.e.,
$$\left\langle \mathbf{g}^\circledast \right\rangle \circ \left\langle \mathbf{f}^\circledast \right\rangle := \left\langle \mathbf{g}^\circledast \circ \mathbf{f}^\circledast \right\rangle : X \to Z.$$

The category $Sh_{(C,D)}^\circledast$ is called *the abstract finite coarse shape category of a pair* (C, D), while the morphisms $\langle \mathbf{f}^\circledast \rangle : X \to Y$ in $Sh_{(C,D)}^\circledast$ are called *finite coarse shape morphisms* and denoted by $F^\circledast : X \to Y$. A finite coarse shape morphism $F^\circledast : X \to Y$ can be described by a diagram

$$\begin{tikzcd} X \arrow[r, dashed, "F^\circledast"] \arrow[d, "\mathbf{p}"'] & Y \arrow[d, "\mathbf{q}"] \\ \mathbf{X} \arrow[r, "\mathbf{f}^\circledast"] & \mathbf{Y} \end{tikzcd}$$

It is important to emphasize that the set $Sh_{(C,D)}^\circledast(X, Y)$ is in a one-to-one correspondence with the set *pro*$^\circledast$-$D(\mathbf{X}, \mathbf{Y})$, for any D-expansions $\mathbf{X}$ and $\mathbf{Y}$ of objects X and Y respectively.

Isomorphic objects X and Y in the category $Sh_{(C,D)}^\circledast$ are said to *have the same finite coarse shape type*. This is denoted by $sh^\circledast(X) = sh^\circledast(Y)$.

Functors $\mathbf{J}_C^\circledast : pro\text{-}C \to pro^\circledast\text{-}C$ and $\mathbf{J}_C^* : pro^\circledast\text{-}C \to pro^*\text{-}C$ induce faithful functors $J_{(C,D)}^\circledast : Sh_{(C,D)} \to Sh_{(C,D)}^\circledast$ and $J_{(C,D)}^* : Sh_{(C,D)}^\circledast \to Sh_{(C,D)}^*$, respectively, by putting:

- $J_{(C,D)}^\circledast(X) = J_{(C,D)}^*(X) = X$, for every object $X \in C$,
- $J_{(C,D)}^\circledast(F) = F^\circledast = \langle \mathbf{J}_D^\circledast(\mathbf{f}) \rangle$, for every shape morphism $F = \langle \mathbf{f} \rangle$,
- $J_{(C,D)}^*(F^\circledast) = F^* = \langle \mathbf{J}_D^*(\mathbf{f}^\circledast) \rangle$, for every finite coarse shape morphism $F^\circledast = \langle \mathbf{f}^\circledast \rangle$.

Hence, the abstract shape category $Sh_{(C,D)}$ (see [3]) can be considered as a subcategory of the abstract finite coarse shape category $Sh^{\circledast}_{(C,D)}$ and $Sh^{\circledast}_{(C,D)}$ is a subcategory of the abstract coarse shape category $Sh^{*}_{(C,D)}$.

The composition of functors $S_{(C,D)} : C \to Sh_{(C,D)}$ (the shape functor) and $J^{\circledast}_{(C,D)}$ is called *the abstract finite coarse shape functor*, denoted by $S^{\circledast}_{(C,D)} : C \to Sh^{\circledast}_{(C,D)}$.

Throughout this paper, we will restrict to $C = HTop_\star$ ($C = HTop^2_\star$) and $D = HPol_\star$ ($D = HPol^2_\star$). In this case, one speaks of *the pointed topological finite coarse shape category (of pairs)*, briefly denoted by $Sh^{\circledast}_\star$ ($Sh^{\circledast 2}_\star$), and *the finite coarse shape functor* $S^{\circledast}$.

Recall that the objects of $HTop_\star$ are all the pointed topological spaces (X, x_0), $x_0 \in X$, and morphisms are all the homotopy classes $[f]$ of mappings of pointed spaces $f : (X, x_0) \to (Y, y_0)$, i.e., homotopy classes of functions $f : X \to Y$ satisfying $f(x_0) = y_0$. Analogously, objects of $HTop^2_\star$ are all the pointed pairs of topological spaces (X, X_0, x_0), $x_0 \in X_0 \subseteq X$, and morphisms are all the homotopy classes $[f]$ of mappings of pointed pairs $f : (X, X_0, x_0) \to (Y, Y_0, y_0)$, i.e., homotopy classes of functions $f : X \to Y$ satisfying $f(X_0) \subseteq Y_0$ and $f(x_0) = y_0$. We will usually denote an H-map $[f]$ by omitting the brackets unless we need to especially distinct some mapping f and the corresponding homotopy class $[f]$. By reducing the object classes to all the pointed pairs having homotopy type of some pointed polyhedral pair, and to all pointed spaces having homotopy type of some pointed polyhedron, one gets full subcategories $HPol^2_\star \subseteq HTop^2_\star$ and $HPol_\star \subseteq HTop_\star$, respectively.

It is well known (Theorem 1.4.7, Theorem 1.4.8, [3]) that every pointed pair of topological spaces (X, X_0, x_0) admits an $HPol^2_\star$-expansion

$$\mathbf{p} : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

and that every pointed topological space (X, x_0) admits an $HPol_\star$-expansion

$$\mathbf{q} : (X, x_0) \to \Big((X_\mu, x_{0\mu}), q_{\mu\mu'}, M \Big).$$

To end this section, let us recall the notion of normally embedded subspace. Let (X, x_0) be a pair of topological spaces. We say that X_0 is *normally embedded* in X provided for every normal covering $\mathcal{U}_0$ of X_0 there is a normal covering $\mathcal{U}$ of X such that

$$\mathcal{U}|_{X_0} = (U \cap X_0 : U \in \mathcal{U})$$

refines $\mathcal{U}_0$. An important property of normally embedded subspace is given in Corollary 1.6.7, [3]: for a pointed topological pair (X, X_0, x_0), where the subspace X_0 is normally embedded in X, there exists an $HPol^2_\star$-expansion

$$\mathbf{p} = [(p_\lambda)] : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

such that

$$\mathbf{p} = [(p_\lambda)] : (X, x_0) \to ((X_\lambda, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda) \text{ and}$$

$$\mathbf{p}_{|X_0} = \Big[\Big(p_{\lambda|X_0} \Big) \Big] : (X_0, x_0) \to \Big((X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'|X_{0\lambda'}}, \Lambda \Big)$$

are $HPol_\star$-expansions of the pointed spaces (X, x_0) and (X_0, x_0) respectively. Such an expansion $\mathbf{p}$ of a pointed pair (X, X_0, x_0) is called *a normal $HPol^2_\star$-expansion*.

2.2. (Relative) Homotopy Groups and (Relative) Finite Coarse Shape Groups

In this section, let us recall some basic notions and properties concerning (relative) homotopy groups [7] and (relative) finite coarse shape groups [6]. For every pointed topological pair (X, X_0, x_0) and every $k \in \mathbb{N}$ *the relative k-dimensional homotopy group* $\pi_k(X, X_0, x_0)$ consists of all the homotopy classes of maps from $\left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right)$ to (X, X_0, x_0), where $\mathbb{B}^k$

is the unit k-dimensional disk with boundary $\partial\mathbb{B}^k = \mathbb{S}^{k-1}$. For $k \geq 2$, $\pi_k(X, X_0, x_0)$ has a group structure with the operation being commutative for every $k \geq 3$.

Similarly, for every pointed topological space (X, x_0) and for every $k \in \mathbb{N}_0$, *the k-th homotopy group* $\pi_k(X, x_0)$ consists of all the homotopy classes of maps from $\left(S^k, s_0\right)$ to (X, x_0). For every $k \in \mathbb{N}$, $\pi_k(X, x_0)$ is a group with the operation being commutative for every $k \geq 2$. Since every map from $\left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right)$ to (X, X_0, x_0) can be identified with a map from $\mathbb{B}^k/\mathbb{S}^{k-1} = \mathbb{S}^k$ to X mapping the base point $s_0 = \mathbb{S}^{k-1}/\mathbb{S}^{k-1}$ to x_0, the k-th homotopy group $\pi_k(X, x_0)$ coincides with the relative k-dimensional homotopy group $\pi_k(X, \{x_0\}, x_0)$, for every $k \geq 2$.

For every pointed topological pair (X, X_0, x_0) and every $k \in \mathbb{N}$ *a homotopy boundary homomorphism*

$$\partial_k : \pi_k(X, X_0, x_0) \to \pi_{k-1}(X_0, x_0)$$

is defined by the rule

$$\partial_k(f) = f|_{\left(\mathbb{S}^{k-1}, s_0\right)} : \left(\mathbb{S}^{k-1}, s_0\right) \to (X_0, x_0),$$

for any $f \in \pi_k(X, X_0, x_0)$. In other words, the image $\partial_k(f)$ of any H-map $f : \left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right) \to (X, X_0, x_0)$ is the restriction of f to the boundary $\mathbb{S}^{k-1}$ of $\mathbb{B}^k$, i.e., an element of the $(k-1)$-th homotopy group of the pointed subspace (X_0, x_0) of X. A homotopy boundary homomorphism ∂_k is a homomorphism of groups for every $k \geq 2$, while ∂_1 is a base point preserving function. Now, the homotopy group sequence is defined as a sequence

$$\begin{aligned} \ldots \xrightarrow{\partial_{k+1}} \pi_k(X_0, x_0) \xrightarrow{\pi_k(i)} \pi_k(X, x_0) \xrightarrow{\pi_k(j)} \pi_k(X, X_0, x_0) \xrightarrow{\partial_k} \cdots \\ \cdots \xrightarrow{\partial_1} \pi_0(X_0, x_0) \xrightarrow{\pi_0(i)} \pi_0(X, x_0), \end{aligned} \tag{1}$$

where $i : (X_0, x_0) \to (X, x_0)$ and $j : (X, \{x_0\}, x_0) \to (X, X_0, x_0)$ are homotopy classes of the appropriate inclusions.

For every $k \in \mathbb{N}_0$ and every pointed space (X, x_0) *the k-th finite coarse shape group* $\check{\pi}_k^{\circledast}(X, x_0)$ can be defined as follows. For every $k \in \mathbb{N}$, $\check{\pi}_k^{\circledast}(X, x_0)$ is a group (for $k \geq 2$ an abelian group) having $Sh_\star^{\circledast}\left(\left(\mathbb{S}^k, s_0\right), (X, x_0)\right)$ as underlying set with a group operation given by the formula

$$A^{\circledast} + B^{\circledast} = \left\langle \mathbf{a}^{\circledast} \right\rangle + \left\langle \mathbf{b}^{\circledast} \right\rangle = \left\langle \mathbf{a}^{\circledast} + \mathbf{b}^{\circledast} \right\rangle = \langle [(a_\lambda^n)] + [(b_\lambda^n)] \rangle = \langle [(a_\lambda^n + b_\lambda^n)] \rangle. \tag{2}$$

The finite coarse shape morphisms $A^{\circledast}$ and $B^{\circledast}$ are represented by *pro*$^{\circledast}$-*HPol*$_\star$ morphisms $\mathbf{a}^{\circledast} = \left[(a_\lambda^n)\right]$ and $\mathbf{b}^{\circledast} = \left[(b_\lambda^n)\right] : \left(\mathbb{S}^k, s_0\right) \to ((X_\lambda, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$ respectively, where

$$\mathbf{p} : (X, x_0) \to ((X_\lambda, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

is an *HPol*$_\star$-expansion of a pointed space (X, x_0). The sum $a_\lambda^n + b_\lambda^n$ denotes the sum in the group $\pi_k(X_\lambda, x_{0\lambda})$. Especially, $\check{\pi}_0^{\circledast}(X, x_0)$ is a pointed set of all finite coarse shape morphisms from $(\mathbb{S}^0, s_0)$ to (X, x_0), i.e., the set $Sh_\star^{\circledast}((\mathbb{S}^0, s_0), (X, x_0))$.

For every $k \in \mathbb{N}$ and every pointed topological pair (X, X_0, x_0) *the k-th relative finite coarse shape group* $\check{\pi}_k^{\circledast}(X, X_0, x_0)$ can be defined as follows. For every $k \geq 2$, $\check{\pi}_k^{\circledast}(X, X_0, x_0)$ is a group (for $k \geq 3$ an abelian group) having the set $Sh_\star^{\circledast}\left(\left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right), (X, X_0, x_0)\right)$ as underlying set with a group operation given by the Formula (2).

The finite coarse shape morphisms $A^{\circledast}$ and $B^{\circledast}$ are hereby represented by *pro*$^{\circledast}$-*HPol*$_\star^2$ morphisms $\mathbf{a}^{\circledast} = \left[(a_\lambda^n)\right]$ and $\mathbf{b}^{\circledast} = \left[(b_\lambda^n)\right] : \left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$, respectively, where

$$\mathbf{p} : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda).$$

is an $HPol^2_\star$-expansion of a pointed pair (X, X_0, x_0). The sum $a^n_\lambda + b^n_\lambda$ in this case denotes the sum in the group $\pi_k(X_\lambda, X_{0\lambda}, x_{0\lambda})$. For $k = 1$, $\check{\pi}^{\circledast}_1(X, X_0, x_0)$ is a pointed set consisting of all finite coarse shape morphisms from $(\mathbb{B}^1, \mathbb{S}^0, s_0)$ to (X, X_0, x_0), i.e., the set $Sh^{\circledast 2}_\star\big((\mathbb{B}^1, \mathbb{S}^0, s_0), (X, X_0, x_0)\big)$.

It is obvious that, for $X_0 = \{x_0\}$, an $HPol^2_\star$-expansion of a pointed pair (X, X_0, x_0) is

$$\mathbf{p} : (X, \{x_0\}, x_0) \to ((X_\lambda, \{x_{0\lambda}\}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

and that every homotopy class from $\left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right)$ to $(X, \{x_0\}, x_0)$ can be identified with a homotopy class from $\left(\mathbb{S}^k, s_0\right)$ to (X, x_0), so the sets $Sh^{\circledast 2}_\star\left(\left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right), (X, \{x_0\}, x_0)\right)$ and $Sh^{\circledast}_\star\left(\left(\mathbb{S}^k, s_0\right), (X, x_0)\right)$ can be identified for every $k \in \mathbb{N}$. This, together with $\pi_k(X, x_0) = \pi_k(X, \{x_0\}, x_0)$ for every $k \geq 2$, implies that the k-th finite coarse shape group $\check{\pi}^{\circledast}_k(X, x_0)$ can be considered as the k-th relative finite coarse shape group $\check{\pi}^{\circledast}_k(X, \{x_0\}, x_0)$, for every $k \geq 2$.

For every $k \in \mathbb{N}_0$ and for every finite coarse shape morphism $F^{\circledast} : (X, x_0) \to (Y, y_0)$, a homomorphism of finite coarse shape groups (for $k = 0$ a base point preserving function)

$$\check{\pi}^{\circledast}_k\left(F^{\circledast}\right) : \check{\pi}^{\circledast}_k(X, x_0) \to \check{\pi}^{\circledast}_k(Y, y_0)$$

is defined by the rule

$$\check{\pi}^{\circledast}_k\left(F^{\circledast}\right)\left(A^{\circledast}\right) = F^{\circledast} \circ A^{\circledast},$$

for any finite coarse shape morphism $A^{\circledast} \in \check{\pi}^{\circledast}_k(X, x_0)$. Analogously, for every $k \in \mathbb{N}$ and for every finite coarse shape morphism $F^{\circledast} : (X, X_0, x_0) \to (Y, Y_0, y_0)$, a homomorphism of relative finite coarse shape groups (for $k = 1$ a base point preserving function)

$$\check{\pi}^{\circledast}_k\left(F^{\circledast}\right) : \check{\pi}^{\circledast}_k(X, X_0, x_0) \to \check{\pi}^{\circledast}_k(Y, Y_0, y_0)$$

is defined by the same rule. These two joinings induce functors $\check{\pi}^{\circledast}_k : Sh^{\circledast}_\star \to Grp$ ($\check{\pi}^{\circledast}_0 : Sh^{\circledast}_\star \to Set_\star$) and $\check{\pi}^{\circledast}_k : Sh^{\circledast 2}_\star \to Grp$ ($\check{\pi}^{\circledast}_1 : Sh^{\circledast 2}_\star \to Set_\star$) respectively, associating with every pointed topological space (X, x_0) the k-th finite coarse shape group $\check{\pi}^{\circledast}_k(X, x_0)$ and with every pointed topological pair (X, X_0, x_0) the k-th relative finite coarse shape group $\check{\pi}^{\circledast}_k(X, X_0, x_0)$ respectively. The functors $\check{\pi}^{\circledast}_k : Sh^{\circledast}_\star \to Grp$ and $\check{\pi}^{\circledast}_k : Sh^{\circledast 2}_\star \to Grp$ are called *the k-th (relative) finite coarse shape group functors*.

To obtain our main goals, the following result from [6] will be of a great significance.

Theorem 1. *Let (X, x_0) be a pointed space and let $\mathbf{p} = [(p_\lambda)] : (X, x_0) \to ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ be an $HPol_\star$-expansion of (X, x_0). Then, for every $k \in \mathbb{N}$,*

$$\check{\pi}^{\circledast}_k(X, x_0) \cong \varprojlim\left(\check{\pi}^{\circledast}_k(X_\lambda, x_\lambda), \check{\pi}^{\circledast}_k(p_{\lambda\lambda'}), \Lambda\right).$$

In reference [6], Example 5.1, the authors have derived an explicit formula for the k-th finite coarse shape group of a pointed stable space, i.e., a space having the shape type of some pointed polyhedron. Namely, if (P, p_0) is a pointed stable space and $k \in \mathbb{N}$, then the k-th finite coarse shape group of (P, p_0) is given by

$$\check{\pi}^{\circledast}_k(P, p_0) = \prod_{n \in \mathbb{N}}{}^{f} \pi_k(P, p_0) \Big/ \bigoplus_{n \in \mathbb{N}} \pi_k(P, p_0),$$

where $\prod_{n \in \mathbb{N}}{}^{f} \pi_k(P, p_0)$ denotes a subgroup of $\prod_{n \in \mathbb{N}} \pi_k(P, p_0)$ consisting only of the elements of $\prod_{n \in \mathbb{N}} \pi_k(P, p_0)$ having (at most) finitely many mutually different coordinate values. In the same way one obtains the following result.

Proposition 1. *Let (P, P_0, p_0) be a pointed pair of stable spaces and let $k \in \mathbb{N}$ be an arbitrary integer.*

(i) *If $k \geq 2$, then the k-th relative finite coarse shape group of (P, P_0, p_0) is a group given by*

$$\check{\pi}_k^{\circledast}(P, P_0, p_0) = \prod_{n\in\mathbb{N}}^{f} \pi_k(P, P_0, p_0) \Big/ \bigoplus_{n\in\mathbb{N}} \pi_k(P, P_0, p_0).$$

(ii) *If $k = 1$, then*

$$\check{\pi}_1^{\circledast}(P, P_0, p_0) = \prod_{n\in\mathbb{N}}^{f} \pi_k(P, P_0, p_0)/\sim,$$

where $\sim$ is an equivalence relation on the direct product of pointed sets $\prod_{n\in\mathbb{N}}^{f} \pi_k(P, P_0, p_0)$ identifying elements having almost all equal coordinates.

3. The Finite Coarse Shape Group Sequence of a Pointed Pair (X, X_0, x_0)

In this section, we introduce the notion of *finite coarse shape group sequence of a pointed topological pair* as the finite coarse shape theory analogue of the homotopy group sequences and the (coarse) shape group sequences. Recall that a sequence

$$\cdots \to G' \stackrel{f'}{\to} G \stackrel{f}{\to} G'' \to \cdots$$

of group homomorphisms is said to be *exact* (*semiexact*) at term G if $\operatorname{Im} f' = \operatorname{Ker} f$ ($\operatorname{Im} f' \subseteq \operatorname{Ker} f$). Even if G, G' and G'' are no more than pointed sets and f is only a base point preserving function, the property of exactness (semiexactness) at G still makes sense provided $\operatorname{Ker} f$ is considered as the preimage $f^{-1}(o)$ of the base point (or neutral element) o. A sequence of homomorphisms is said to be *exact* (*a chain*) if it is exact (semiexact) at each term. It is known that homotopy group sequences and coarse shape group sequences are exact, while shape group sequences are, generally, no more than chains. Therefore, it is interesting to investigate whether finite coarse-shape group sequences have these properties.

Definition 1. *Let (X, X_0, x_0) be a pointed topological pair such that X_0 is normally embedded in X and let $k \in \mathbb{N}$ be an arbitrary integer. The joining*

$$\partial_k^{\circledast} : \check{\pi}_k^{\circledast}(X, X_0, x_0) \to \check{\pi}_{k-1}^{\circledast}(X_0, x_0)$$

given by the rule

$$\partial_k^{\circledast}(A^{\circledast}) = A^{\circledast}|_{(\mathbb{S}^{k-1}, s_0)} : \left(\mathbb{S}^{k-1}, s_0\right) \to (X_0, x_0),$$

for every finite coarse shape morphism $A^{\circledast} \in \check{\pi}_k^{\circledast}(X, X_0, x_0)$, is called the boundary homomorphism of finite coarse shape groups.

Notice that $A^{\circledast}|_{(\mathbb{S}^{k-1}, s_0)}$ is a well defined finite coarse shape morphism. Indeed, since X_0 is normally embedded in X, there exists a normal $HPol_\star^2$-expansion

$$\mathbf{p} : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

of a pointed pair (X, X_0, x_0). If $A^{\circledast}$ in $Sh_\star^{\circledast 2}$ is represented by a *pro*$^{\circledast}$-$HPol_\star^2$ morphism

$$\mathbf{a}^{\circledast} = [(a_\lambda^n)] : \left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda),$$

then $\partial_k^{\circledast}(A^{\circledast})$ in $Sh_\star^{\circledast}$ is represented by a *pro*$^{\circledast}$-$HPol_\star$ morphism

$$\mathbf{a}^{\circledast}|_{(\mathbb{S}^{k-1}, s_0)} := \left[\left(a_\lambda^n|_{(\mathbb{S}^{k-1}, s_0)}\right)\right] : \left(\mathbb{S}^{k-1}, s_0\right) \to ((X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda),$$

i.e.,

$$A^{\circledast}|_{\left(\mathbb{S}^{k-1},s_0\right)} \in Sh_{\star}^{\circledast}\left(\left(\mathbb{S}^{k-1},s_0\right),(X_0,x_0)\right).$$

Theorem 2. *For every $k \geq 2$, the boundary homomorphism of the finite coarse shape group $\partial_k^{\circledast}$ is a group homomorphism.*

Proof. Let

$$\mathbf{p} : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda} x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

be a normal $HPol_{\star}^2$-expansion of a pointed pair (X, X_0, x_0) and let $A^{\circledast} \in \check{\pi}^{\circledast}(X, X_0, x_0)$ be an arbitrary finite coarse shape morphism. Then $A^{\circledast}$ is represented by a $pro^{\circledast}$-$HPol_{\star}^2$ morphism

$$\mathbf{a}^{\circledast} = [(a_\lambda^n)] : \left(\mathbb{B}^k, \mathbb{S}^{k-1}, s_0\right) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda),$$

and $\partial_k^{\circledast}(A^{\circledast})$ is represented by a $pro^{\circledast}$-$HPol_{\star}$ morphism

$$\mathbf{a}^{\circledast}|_{\left(\mathbb{S}^{k-1},s_0\right)} = [(a_\lambda^n|_{\mathbb{S}^{k-1}})] : \left(\mathbb{S}^{k-1}, s_0\right) \to ((X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda).$$

Notice that, for coordinate functions a_λ^n, the relation

$$a_\lambda^n|_{\left(\mathbb{S}^{k-1},s_0\right)} = \partial_k(a_\lambda^n)$$

holds. Since the homomotopy boundary homomorphism ∂_k is a group homomorphism, we infer that for every two finite coarse shape morphisms $A^{\circledast}, B^{\circledast} \in \check{\pi}_k^{\circledast}(X, X_0, x_0)$ the following equalities

$$\begin{aligned}
\partial_k^{\circledast}(A^{\circledast} + B^{\circledast}) &= \partial_k^{\circledast}(\langle \mathbf{a}^{\circledast}\rangle + \langle \mathbf{b}^{\circledast}\rangle) \overset{(2)}{=} \partial_k^{\circledast}(\langle[(a_\lambda^n + b_\lambda^n)]\rangle) = \\
&= \left\langle \left[((a_\lambda^n + b_\lambda^n))|_{\left(\mathbb{S}^{k-1},s_0\right)}\right]\right\rangle = \langle[(\partial_k(a_\lambda^n + b_\lambda^n))]\rangle = \langle[(\partial_k(a_\lambda^n) + \partial_k(b_\lambda^n))]\rangle = \\
&= \langle[(\partial_k(a_\lambda^n))]\rangle + \langle[(\partial_k(b_\lambda^n))]\rangle = \left\langle\left[\left(a_\lambda^n|_{\left(\mathbb{S}^{k-1},s_0\right)}\right)\right]\right\rangle + \left\langle\left[\left(b_\lambda^n|_{\left(\mathbb{S}^{k-1},s_0\right)}\right)\right]\right\rangle = \\
&= \left\langle \mathbf{a}^{\circledast}|_{\left(\mathbb{S}^{k-1},s_0\right)}\right\rangle + \left\langle \mathbf{b}^{\circledast}|_{\left(\mathbb{S}^{k-1},s_0\right)}\right\rangle = A^{\circledast}|_{\left(\mathbb{S}^{k-1},s_0\right)} + B^{\circledast}|_{\left(\mathbb{S}^{k-1},s_0\right)} = \partial_k^{\circledast}(A^{\circledast}) + \partial_k^{\circledast}(B^{\circledast})
\end{aligned}$$

hold. □

Remark 1. *For $k = 1$ the boundary homomorphism of finite coarse shape groups $\partial_1^{\circledast} : \check{\pi}_1^{\circledast}(X, X_0, x_0) \to \check{\pi}_0^{\circledast}(X_0, x_0)$ is a base point preserving function.*

Definition 2. *Let (X, X_0, x_0) be a pointed topological pair such that X_0 is normally embedded in X and let*

$$\check{I}_k^{\circledast} := \check{\pi}_k^{\circledast}(S^{\circledast}(i)) : \check{\pi}_k^{\circledast}(X_0, x_0) \to \check{\pi}_k^{\circledast}(X, x_0), \textit{ for every } k \in \mathbb{N}_0,$$

$$\check{J}_k^{\circledast} := \check{\pi}_k^{\circledast}(S^{\circledast}(j)) : \check{\pi}_k^{\circledast}(X, \{x_0\}, x_0) \to \check{\pi}_k^{\circledast}(X, X_0, x_0), \textit{ for every } k \in \mathbb{N}.$$

The sequence

$$\begin{gathered}
\cdots \xrightarrow{\partial_{k+1}^{\circledast}} \check{\pi}_k^{\circledast}(X_0, x_0) \xrightarrow{\check{I}_k^{\circledast}} \check{\pi}_k^{\circledast}(X, x_0) \xrightarrow{\check{J}_k^{\circledast}} \check{\pi}_k^{\circledast}(X, X_0, x_0) \xrightarrow{\partial_k^{\circledast}} \cdots \\
\cdots \xrightarrow{\check{I}_1^{\circledast}} \check{\pi}_1^{\circledast}(X, x_0) \xrightarrow{\check{J}_1^{\circledast}} \check{\pi}_1^{\circledast}(X, X_0, x_0) \xrightarrow{\partial_1^{\circledast}} \check{\pi}_0^{\circledast}(X_0, x_0) \xrightarrow{\check{I}_0^{\circledast}} \check{\pi}_0^{\circledast}(X, x_0),
\end{gathered} \tag{3}$$

is called the finite coarse shape group sequence of the pointed pair (X, X_0, x_0).

If X_0 is normally embedded in X and if $\mathbf{p} : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$ is a normal $HPol^2_\star$-expansion of the pointed pair (X, X_0, x_0), then it is readily seen that the finite coarse shape morphisms $S^{\circledast}(i)$ and $S^{\circledast}(j)$ are represented by $pro^{\circledast}$-$HPol_\star$ morphisms

$$[(1_\Lambda, i^n_\lambda)] : \left((X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}|_{X_{0\lambda'}}, \Lambda\right) \to ((X_\lambda, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda) \tag{4}$$

and

$$[(1_\Lambda, j^n_\lambda)] : ((X_\lambda, \{x_{0\lambda}\}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda), \tag{5}$$

respectively, where $i^n_\lambda = i_\lambda : (X_{0\lambda}, x_{0\lambda}) \to (X_\lambda, x_{0\lambda})$ and $j^n_\lambda = j_\lambda : (X_\lambda, \{x_{0\lambda}\}, x_{0\lambda}) \to (X_\lambda, X_{0\lambda}, x_{0\lambda})$ are homotopy classes of the corresponding inclusions, for every $\lambda \in \Lambda$ and $n \in \mathbb{N}$. Let us now clarify the properties of finite coarse shape group sequences.

Theorem 3. *If (X, X_0, x_0) is a pointed topological pair such that X_0 is normally embedded in X, then the finite coarse shape group sequence of (X, X_0, x_0) is a chain.*

Proof. Let

$$\mathbf{p} : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda}, x_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$$

be a normal $HPol^2_\star$-expansion of the pointed pair (X, X_0, x_0). Given an arbitrary $k \in \mathbb{N}$, we will prove semiexactness of the finite coarse shape group sequence of (X, X_0, x_0) at the term $\check{\pi}^{\circledast}_k(X, x_0)$ (the proof can be done analogously at any other term).

For an arbitrary finite coarse shape morphism $F^{\circledast} = \left\langle \left[\left(f^n_\lambda \right) \right] \right\rangle \in \check{\pi}^{\circledast}_k(X_0, x_0)$, because Equations (4) and (5), it holds that

$$\check{J}^{\circledast}_k \circ \check{I}^{\circledast}_k(F^{\circledast}) = \check{J}^{\circledast}_k(\langle [((1_\Lambda, i^n_\lambda))] \rangle \circ \langle [(f^n_\lambda)] \rangle) = \langle [((1_\Lambda, j^n_\lambda))] \rangle \circ \langle [(i^n_\lambda \circ f^n_\lambda)] \rangle = \langle [(j^n_\lambda \circ i^n_\lambda \circ f^n_\lambda)] \rangle$$

Notice that, for every $\lambda \in \Lambda$ and $n \in \mathbb{N}$, the H-map $j^n_\lambda \circ i^n_\lambda \circ f^n_\lambda$ is an element of the k-dimensional relative homotopy group $\pi_k(X, X_0, x_0)$ and

$$i^n_\lambda \circ f^n_\lambda \in \mathrm{Im}\, \pi_k(i^n_\lambda) = \mathrm{Im}\, \pi_k(i_\lambda).$$

Since a homotopy group sequence is exact at any term, i.e., $\mathrm{Ker}\, \pi_k\left(j^n_\lambda\right) = \mathrm{Im}\, \pi_k\left(i^n_\lambda\right)$, we infer that

$$\pi_k(j^n_\lambda)(i^n_\lambda \circ f^n_\lambda) = j^n_\lambda \circ i^n_\lambda \circ f^n_\lambda = o,$$

where o is the neutral element of $\pi_k(X, X_0, x_0)$. This means that

$$\check{J}^{\circledast}_k \circ \check{I}^{\circledast}_k(F^{\circledast}) = O^{\circledast},$$

that is, $\mathrm{Im}\, \check{I}^{\circledast}_k \subseteq \mathrm{Ker}\, \check{J}^{\circledast}_k$ and the statement is proved. □

The following example shows that finite coarse shape group sequences, in general, fail to be exact.

Example 1. *Let $\left(P^2, P^1, \star\right)$ be the pointed pair which consists of the real projective plane P^2, the real projective line P^1 and of an arbitrary base point $\star$. Let $p : \left(P^2, P^1, \star\right) \to \left(P^2, P^1, \star\right)$ be a mapping defined by the commutative diagram*

$$\begin{tikzcd}
\left(\mathbb{B}^2, \mathbb{S}^1, \star\right) \arrow[d, "\varphi"'] & \left(\mathbb{B}^2, \mathbb{S}^1, \star\right) \arrow[l, "f"'] \arrow[d, "\varphi"] \\
\left(P^2, P^1, \star\right) & \left(P^2, P^1, \star\right) \arrow[l, "p"]
\end{tikzcd},$$

where $\mathbb{B}^2$ is the unit disc in $\mathbb{C}$, $\mathbb{S}^1 = \partial\mathbb{B}^2$ is the unit circle, $f(z) = z^3$ and $\varphi : \mathbb{B}^2 \to P^2$ is the quotient map which identifies pairs of antipodal points of $\mathbb{S}^1$.

Let $(X, X_0, \star)$ denote the pointed pair of metric continua, which is the inverse limit of the inverse sequence $\left((P^2, P^1, \star), p_{ii+1}, \mathbb{N}\right)$, where $p_{ii+1} = p$, for every $i \in \mathbb{N}$. By Theorem 9, I.5.3 of [3], this inverse limit induces an $HTop^2_\star$-expansion $\mathbf{p} = [(p_i)] : (X, X_0, \star) \to \left((P^2, P^1, \star), [p_{ii+1}], \mathbb{N}\right)$ of the pointed pair $(X, X_0, \star)$, where $[p_{ii+1}] = [p]$, for every $i \in \mathbb{N}$. Moreover, since both P^2 and P^1 are compacts having the homotopy type of a polyhedron, $\mathbf{p}$ is an $HPol^2_\star$-expansion of $(X, X_0, \star)$. It is readily seen that $\mathbf{p}$ is a normal $HPol^2_\star$-expansion of the pointed pair $(X, X_0, \star)$.

Recall that $\pi_1(P^1, \star) \cong \mathbb{Z}$, $\pi_1(P^2, \star) \cong \mathbb{Z}_2$ and $\pi_1(P^2, P^1, \star) = 0$. Let us now describe the (relative) finite coarse shape groups $\check{\pi}_1^{\circledast}(X_0, \star)$, $\check{\pi}_1^{\circledast}(X, \star)$ and $\check{\pi}_1^{\circledast}(X, X_0, \star)$. From Theorem 1 and Example 5.1 [6], we infer that

$$\check{\pi}_1^{\circledast}(X_0, \star) \cong \lim_{\leftarrow}\left(\check{\pi}_1^{\circledast}\left(P^1, \star\right), \check{\pi}_1^{\circledast}([p_{ii+1}]), \mathbb{N}\right) \cong$$

$$\cong \lim_{\leftarrow}\left(\prod_{n\in\mathbb{N}}^{f} \pi_1\left(P^1, \star\right) \Big/ \bigoplus_{n\in\mathbb{N}} \pi_1\left(P^1, \star\right), \check{\pi}_1^{\circledast}([p_{ii+1}]), \mathbb{N}\right) \cong \lim_{\leftarrow}\left(\prod_{n\in\mathbb{N}}^{f} \mathbb{Z} \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}, p'_{ii+1}, \mathbb{N}\right),$$

where p'_{ii+1} denotes the multiplication by 3. Hence, the 1st finite coarse shape group of $(X_0, \star)$ is a subgroup of the direct product of groups $\prod_{n\in\mathbb{N}}^{f} \mathbb{Z} \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$, i.e.,

$$\check{\pi}_1^{\circledast}(X_0, \star) \leqslant \prod_{p\in\mathbb{N}}\left(\prod_{n\in\mathbb{N}}^{f} \mathbb{Z} \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}\right),$$

but $\check{\pi}_1^{\circledast}(X_0, \star)$ is actually trivial. Suppose the contrary, i.e., that there exists an element

$$0 \neq x = (x_i | i \in \mathbb{N}) \in \lim_{\leftarrow}\left(\prod_{n\in\mathbb{N}}^{f} \mathbb{Z} \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}, p'_{ii+1}, \mathbb{N}\right).$$

Notice that such an element x has the following properties:

(a) For every $i \in \mathbb{N}$, the i-th coordinate x_i of x is a sequence $\left(x_i^m\right)_m$ of integers such that $\operatorname{card}\left\{x_i^m : m \in \mathbb{N}\right\} < \aleph_0$.

(b) For every pair $i, i' \in \mathbb{N}$, $i \leq i'$, there exists $m_{ii'} \in \mathbb{N}$ such that

$$x_i^m = p_{ii'}^m \circ x_{i'}^m = p_{ii'} \circ x_{i'}^m = 3^{i'-i} x_{i'}^m, \text{ for every } m \geq m_{ii'}.$$

(c) Since $x \neq 0$, there exists $i_0 \in \mathbb{N}$ such that

$$x_{i_0} \neq 0 \in \prod_{n\in\mathbb{N}}^{f} \mathbb{Z} \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}.$$

The property (c), by the definition of the group $\prod_{n\in\mathbb{N}}^{f} \mathbb{Z} \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$, means that for every $m \in \mathbb{N}$ there exists $m' \in \mathbb{N}$, $m' > m$, such that $0 \neq x_{i_0}^{m'} \in Z$. In other words, there exists a subsequence $\left(x_{i_0}^{m_k}\right)_k$ of $\left(x_{i_0}^m\right)$ such that $x_{i_0}^{m_k} \neq 0$, for every $k \in \mathbb{N}$. Let us now, for every $k \in \mathbb{N}$, denote

$$r_k = \max\left\{r \in \mathbb{Z}_0^+ : 3^r | x_{i_0}^{m_k}\right\}.$$

Since

$$\operatorname{card}\left\{x_{i_0}^{m_k} : k \in \mathbb{N}\right\} \leq \operatorname{card}\left\{x_{i_0}^m : m \in \mathbb{N}\right\} < \aleph_0,$$

the following maximum

$$q = \max\{3^{r_k} : k \in \mathbb{N}\} \in \mathbb{Z}_0^+ \tag{6}$$

surely exists. Furthermore, by the property (b), for every $i > i_0$ there exists $m_{i_0 i} \in \mathbb{N}$ such that

$$x_{i_0}^m = 3^{i-i_0} x_i^m, \text{ for every } m \geq m_{i_0 i}.$$

Hence, for every $i > i_0$ there exist $k_i \in \mathbb{N}$ such that

$$0 \neq x_{i_0}^{m_{k_i}} = 3^{i-i_0} x_i^{m_{k_i}}.$$

Let $r_0 \in \mathbb{N}$ such that $3^{r_0} > q$. Then for $i = i_0 + r_0 > i_0$ there exists $k_{i_0+r_0} \in \mathbb{N}$ such that

$$0 \neq x_{i_0}^{m_{k_{i_0+r_0}}} = 3^{r_0} x_{i_0+r_0}^{m_{k_{i_0+r_0}}}.$$

Hence, $3^{r_0} | x_{i_0}^{m_{k_{i_0+r_0}}}$ and this is a contradiction to the equality (6). Thus, the 1st finite coarse shape group $\check{\pi}_1^{\circledast}(X_0, \star)$ is trivial.

Let us now describe $\check{\pi}_1^{\circledast}(X, \star)$. Since $\pi_1(P^2, \star) \cong \mathbb{Z}_2$, from Theorem 1 and Example 5.1 [6] we infer that

$$\check{\pi}_1^{\circledast}(X, \star) \cong \lim_{\leftarrow}\left(\check{\pi}_1^{\circledast}\left(P^2, \star\right), \check{\pi}_1^{\circledast}([p_{ii+1}]), \mathbb{N}\right) \cong$$

$$\cong \lim_{\leftarrow}\left(\prod_{n\in\mathbb{N}}^{f} \pi_1\left(P^2, \star\right) \Big/ \bigoplus_{n\in\mathbb{N}} \pi_1\left(P^2, \star\right), \check{\pi}_1^{\circledast}([p_{ii+1}]), \mathbb{N}\right) \cong \lim_{\leftarrow}\left(\prod_{n\in\mathbb{N}}^{f} \mathbb{Z}_2 \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2, p'_{ii+1}, \mathbb{N}\right),$$

where p'_{ii+1} denotes the multiplication by $3 (\bmod 2) \equiv 1$. Hence, the 1st finite coarse shape group of $(X, \star)$ is a subgroup of the direct product of groups $\prod_{n\in\mathbb{N}} \mathbb{Z}_2 \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2$, i.e.,

$$\check{\pi}_1^{\circledast}(X, \star) \leqslant \prod_{p\in\mathbb{N}}\left(\prod_{n\in\mathbb{N}} \mathbb{Z}_2 \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2\right).$$

Every sequence (α_n) of zeros and ones determines an element

$$x = (x_i | i \in \mathbb{N}) \in \lim_{\leftarrow}\left(\prod_{n\in\mathbb{N}} \mathbb{Z}_2 \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2, p'_{ii+1}, \mathbb{N}\right)$$

which, due to the fact that p'_{ii+1} is the multiplication by 1, must be of the form

$$x_i^n = \alpha_n, \quad \text{for all } i, n \in \mathbb{N}.$$

Thus,

$$x = x' \iff \alpha_n = \alpha'_n, \text{ for almost all } n \in \mathbb{N},$$

i.e.,

$$\check{\pi}_1^{\circledast}(X, \star) \cong \left(\prod_{n\in\mathbb{N}} \mathbb{Z}_2 \Big/ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2\right).$$

Finally, since $\pi_1(P^2, P^1, \star) = 0$, it is obvious that $\check{\pi}_1^{\circledast}(X, X_0, \star)$ is trivial. This means that the finite coarse shape group sequence of the pointed pair $(X, X_0, \star)$ at the term $\check{\pi}_1^{\circledast}(X, \star)$ is of the form

$$0 \longrightarrow \neq 0 \longrightarrow 0$$

and so cannot be exact.

Another consequence of Example 1 is explained in the following remark.

Remark 2. *Example 1 and Example 5.2, [6] together prove that finite coarse shape groups, in general, differ from both shape and coarse shape groups. In fact, the 1st finite coarse shape group*

$\check{\pi}_1^{\circledast}(X,\star)$, where X is the metric continuum defined in Example 1, is uncountable, while the corresponding shape group $\check{\pi}_1(X,\star)$ is isomorphic to the finite group $\mathbb{Z}_2$ (Example 2.3.3, [3]). On the other hand, in Example 5.2, [6] the authors have proved that the 1st finite coarse shape group $\check{\pi}_1^{\circledast}(D,\star)$ of the pointed dyadic solenoid $(D,\star) = \lim\limits_{\leftarrow}((X_i,\star), p_{ii+1}, \mathbb{N})$, where $X_i = \mathbb{S}^1$ and $p_{ii+1}(z) = z^2$, for every $i \in \mathbb{N}$, is trivial, while the corresponding coarse shape group $\check{\pi}_1^{*}(D,\star)$ is uncountable.

4. Conclusions

The finite coarse shape theory is a recently introduced theory that provides a categorical framework for the classification of topological spaces. The standard shape category *Sh* of topological spaces can be considered as a proper subcategory of the finite coarse shape category $Sh^{\circledast}$, while $Sh^{\circledast}$ is a proper subcategory of the coarse shape category Sh^{*}. In reference [6] the authors introduced the notions of *the k-th finite coarse shape group* $\check{\pi}_k^{\circledast}(X,x_0)$ of a pointed topological space (X,x_0) and, analogously, *the k-th relative finite coarse shape group* $\check{\pi}_k^{\circledast}(X,X_0,x_0)$ of a pointed topological pair (X,X_0,x_0). The existence of a *k-th (relative) finite coarse shape group functor* $\check{\pi}_k^{\circledast}$ from the corresponding finite coarse shape category to the category Grp ($Set_{\star}$) implies that the (relative) finite coarse shape groups are invariants of the finite coarse shape theory.

In this paper, the authors introduced the notion of *finite coarse shape group sequence of a pointed topological pair* (X,X_0,x_0) and investigated its properties. We have shown that such a sequence forms a chain (Theorem 3) that is not exact at each term (Example 1). Furthermore, (Example 1), together with Example 5.2, [6], shows that finite coarse shape groups, in general, differ from both shape and coarse shape groups. This result is very important because it shows that finite coarse shape groups are not only technically but also essentially completely new objects that can be a useful tool for comparing the shape properties of various topological spaces, especially stable spaces.

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