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A Compound Class of Inverse-Power Muth and Power Series Distributions

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Abstract: This paper introduces the inverse-power Muth power series model, which is a composition of the inverse-power Muth and the class of power series distributions. The use of the Bell distribution in this context is emphasized for the first time in the literature. Probability density, survival and hazard functions are studied, as well as their moments. Using the stochastic representation of the model, the maximum-likelihood estimators are implemented by the use of the expectation-maximization algorithm, while standard errors are calculated using Oakes' method. Monte Carlo simulation studies are conducted to show the performance of the maximum-likelihood estimators in finite samples. Two applications to real datasets are shown, where our proposal is compared with some models based on power series compositions.

Keywords: EM algorithm; inverse-power Muth distribution; likelihood; power series distribution

MSC: 62E10; 62F10

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1. Introduction

A brief literature review reveals several distributions related to the power series (PS) distribution models, as shown in [1], which add a parameter to the distribution to provide more flexibility to the proposed models compared to the baseline distribution. The number of parameters depends on the composition of the baseline model, considering the classes of distributions. Some of the most relevant PS distributions are: extended Weibull power series proposed by Silva et al. [2]; Gompertz power series (GPS) by Jafari and Tahmasebi [3]; Burr XII power series studied in Silva and Cordeiro [4]; inverse Weibull power series (IWPS) presented by Shafiei et al. [5]; exponential Pareto power series introduced by Elbatal et al. [6]; Janardan power series by [7]; compound power series with the negative multinomial model introduced by [8]; generalized Burr XII power series performed by Elbatal et al. [9]; Inverse Gamma Power Series (IGPS), of three-parameter by Rivera et al. [10]; three-parameter Inverse Lindley power series (ILPS) proposed by Shakhatreh et al. [11]; four-parameter exponentiated inverse Lomax power series proposed by Hassan et al. [12]; and five-parameter exponentiated power generalized Weibull power series proposed by Aldahlan et al. [13], among others.

In this vein, we propose to study the composition between the recently proposed Inverse-Power Muth (IPM) distribution, introduced by Chesneau and Agiwal [14], which has a positive basis with three parameters, and the class of PS distributions proposed by Noack [15]. This allows us to develop models with more flexibility to fit data and a non-monotone hazard function for different applications, such as cancer and mortality studies, which may not have a monotonic hazard rate function. Furthermore, we highlight for the first time the use of the Bell distribution in this context.

The IPM distribution emerges as a distribution whose hazard function is non-monotonic [14]. This distribution is obtained by calculating the inverse of the Power Muth distribution [16], which in turn comes from the distribution proposed by Muth [17]. Such a distribution, compared to other lifetime distributions, has a less heavy right tail and has enough flexibility to accommodate a large panel of lifetime datasets [14]. Singh et al. [18] indicated that the Muth distribution has been proposed previously by Teissier to model the frequency of mortality related to the aging process. Muth had used it for reliability analysis; however, the same authors believed that they could not have seen this article while working with the distribution at that time. In a brief bibliographic review of the current literature, the article by Muth [17] is taken as a reference when generating new families of distributions, such as those proposed by Abdullah et al. [19] and Almarashi et al. [20]; these articles study the truncated family generation and the T-X generator, respectively. For our research, we will continue with the distribution name of Muth.

A random variable (r.v.) T follows an IPM distribution [14] with parameters $\beta \leq 1$, and $\theta, \gamma > 0$, if its probability density function (PDF) is given by

$$d(t;\beta,\gamma,\theta) = \gamma\theta^{-\gamma}t^{-\gamma-1}(e^{\beta(t\theta)^{-\gamma}} - \beta)e^{[\beta(t\theta)^{-\gamma} - \frac{1}{\beta}(e^{(\beta(t\theta)^{-\gamma})} - 1)]}, \quad t \geq 0.$$

Its cumulative distribution function (CDF) is given by

$$D(t;\beta,\gamma,\theta) = P(T \leq t;\beta,\gamma,\theta) = e^{[\beta(t\theta)^{-\gamma} - \frac{1}{\beta}(e^{(\beta(t\theta)^{-\gamma})} - 1)]}, \quad t \geq 0$$

and its survival function is

$$G(t;\beta,\gamma,\theta) = P(T > t;\beta,\gamma,\theta) = 1 - e^{[\beta(t\theta)^{-\gamma} - \frac{1}{\beta}(e^{(\beta(t\theta)^{-\gamma})} - 1)]}, \quad t \geq 0.$$

The non-central moment for the IPM [14] distribution, i.e., $m(r) = E(T^r)$, exists for $r < \gamma$ and is given by

$$m(r) = \beta \exp\left(\frac{1}{\beta}\right)\theta^{-r}[(-1)^{\xi}\beta]^{s/\gamma}\int_{1/\beta}^{1/\xi-1}[(-1)^{\xi}\log(y\beta)]^{-s/\gamma}(y-1)\exp(-y)dy,$$

where $\theta, \gamma > 0$, $\xi \to 1$ when $\beta < 0$, and $\xi \to 0$ when $\beta \in (0,1]$. An important feature of the distribution is the quantile function (QF), which is given by

$$Q(y;\beta,\gamma,\theta) = \frac{1}{\theta}\left\{\frac{1}{\beta}\log\left[-\beta\Delta\left(-\frac{1}{\beta}y\exp\left(-\frac{1}{\beta}\right)\right)\right]\right\}^{-1/\gamma},$$

where $\Delta(.)$ denotes the Lambert function. For any complex t, the Lambert function is defined as the inverse of the function $g(t) = t\exp(t)$. An implementation in R software is available through the LambertW package (Goerg [21]).

The paper is organized as follows: Section 2 presents the proposed model based on the IPM and the power series (PS) models, infinite linear combination expressions, and further examination of the moments for the model. Section 3 discusses the estimation of parameters using the expectation-maximization (EM) algorithm (Dempster et al. [22]) and the estimation of their respective variances applying Oakes' method [23]. The performance of estimators in finite populations is studied in Section 4. In Section 5, two applications of the IPM-PS model are considered in comparison with other models based also on PS or IPM distributions. Finally, Section 6 presents the main conclusions of this manuscript.

2. The Model

Suppose that in a certain context, the event of interest can be developed by M non-observable concurrent causes. For example, in the context of cancer, those causes are represented by carcinogenic cells. Suppose now that M follows a PS distribution with probability mass function (PMF) given by

$$P(M = m; \sigma) = \frac{a_m \sigma^m}{A(\sigma)}, \quad m = 1, 2, \ldots, \tag{1}$$

where $a_m > 0, \sigma \in \Theta$ is called the power parameter and $A(\sigma) = \sum_{m=1}^{\infty} a_m \sigma^m$ is the power series.

Table 1 shows some distributions that belong to the PS family and that will also be used for the development of this research. The Bell distribution has been used loosely in the literature. Table 1 presents the values of a_m, $A(\sigma)$, and the parameter space Θ for the distributions that we will use here. It should be noted that B_m is the Bell number. Additionally, please note that we are considering $m > 0$, which indicates that we are using the zero-truncated version of the Poisson (ZTP) [24] instead of the traditional Poisson model.

Table 1. Special cases of the $PS(\sigma, A(\sigma))$ distribution.

Distribution	Notation	a_m	$A(\sigma)$	Θ
Geometric	Geo(σ)	1	$\sigma(1-\sigma)^{-1}$	$(0,1)$
Poisson	Po(σ)	$(m!)^{-1}$	$e^{\sigma} - 1$	$(0,+\infty)$
Bell	Be(σ)	$\frac{B_m}{m!}$	$\exp(e^{\sigma} - 1) - 1$	$(0,+\infty)$
Logarithmic	Lo(σ)	$(m)^{-1}$	$-\log(1-\sigma)$	$(0,1)$

$^*B_m = e^{-1} \sum_{k=0}^{\infty} k^m / k!$

2.1. Model Construction

Let W_a be a r.v. denoting the a-th time at which one of the M possible concurrent causes produces the event of interest, for $a = 1, 2, \ldots, M$. Given M, let $W_1, \ldots, W_M$ be independent and identically distributed r.v.'s following the IPM (β, γ, θ) distributions. The IPM-PS is defined as the marginal distribution of $T = \min(W_1, \ldots, W_M)$. The motivation to introduce this distribution is related to a competing risk scenario, where M possible causes can produce the event of interest and it is enough for a single cause to fail for the event of interest to occur, a scheme similar to that used in parallel systems with the particularity that the number of components, M, is random. The CDF for the IPM-PS model can be calculated as

$$F(t; \beta, \gamma, \theta, \sigma) = \sum_{m=1}^{\infty} P(T > t | M = m; \beta, \gamma, \theta) \times P(M = m; \sigma). \tag{2}$$

As the $W_i's$ are conditionally independent, given M, and identically distributed, then we have

$$P(T > t | M = m; \beta, \gamma, \theta) = 1 - G(t; \beta, \gamma, \theta)^m. \tag{3}$$

Therefore, considering Equations (2) and (3), we obtain

$$F(t; \beta, \gamma, \theta, \sigma) = 1 - \frac{\sum_{m=1}^{\infty} a_m (\sigma(G(t; \beta, \gamma, \theta))^m}{A(\sigma)} = 1 - \frac{A(\sigma(G(t; \beta, \gamma, \theta)))}{A(\sigma)}.$$

Thus, the survival function of the IPM-PS model is given by

$$S(t; \beta, \gamma, \theta, \sigma) = \frac{A(\sigma(G(t; \beta, \gamma, \theta)))}{A(\sigma)}$$

and its corresponding density function is given by

$$f(t;\beta,\gamma,\theta,\sigma) = \frac{e^{[\beta(t\theta)^{-\gamma}-\frac{1}{\beta}(e^{(\beta(t\theta)^{-\gamma})}-1)]}\sigma\gamma t^{-\gamma-1}\theta^{-\gamma}(e^{(\beta(t\theta)^{-\gamma})}-\beta))A'(\sigma(G(t;\beta,\gamma,\theta))}{A(\sigma)}. \tag{4}$$

The hazard rate function is

$$\begin{aligned} h(t;\beta,\gamma,\theta,\sigma) &= \frac{f(t;\beta,\gamma,\theta,\sigma)}{S(t;\beta,\gamma,\theta,\sigma)} \\ &= \frac{e^{[\beta(t\theta)^{-\gamma}-\frac{1}{\beta}(e^{(\beta(t\theta)^{-\gamma})}-1)]}\sigma\gamma t^{-\gamma-1}\theta^{-\gamma}(e^{(\beta(t\theta)^{-\gamma})}-\beta))A'(\sigma(G(t;\beta,\gamma,\theta))}{A(\sigma(G(t;\beta,\gamma,\theta))}. \end{aligned}$$

Figures 1 and 2 present the PDF and hazard function of the Inverse-Power Muth Poisson (IPM-P) and Inverse-Power Muth Bell (IPM-B) models, where, for $\theta = 1, \beta = 0.1$, and $\sigma = 0.5$, we can see that there are heavy tails on the right in their PDF, which suggests that the mean does not exist. In Appendix A, we present the same plots for the inverse-power Muth geometric (IPM-G) and inverse-power Muth logarithmic (IPM-L) distributions.

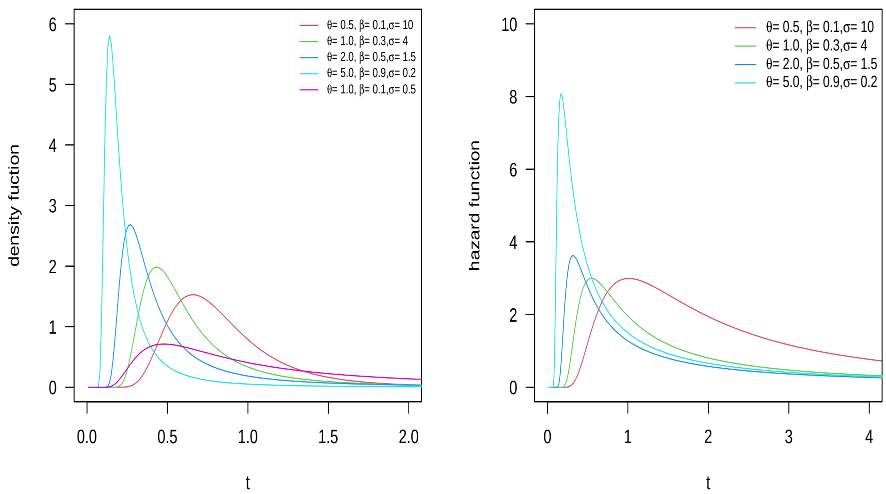


Figure 1. PDF and hazard function for the IPM-P distribution with $\gamma = 1$.

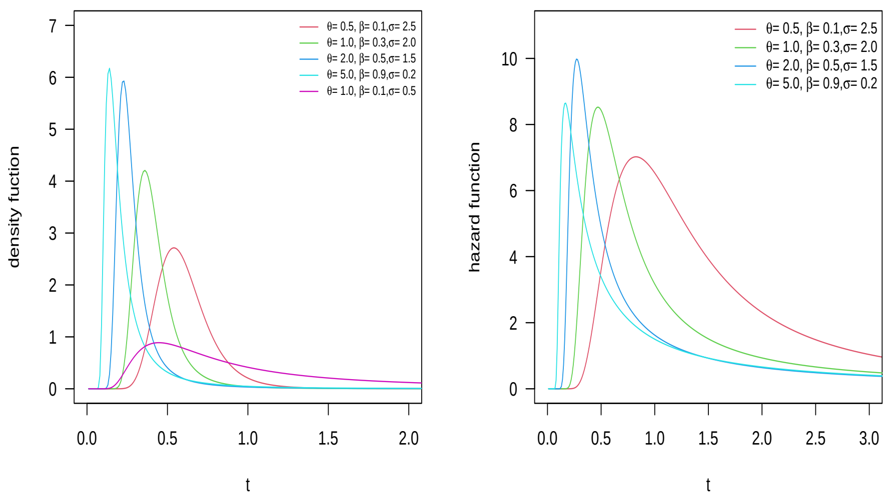


Figure 2. PDF and hazard function for the IPM-B distribution with $\gamma = 1$.

2.2. Model Properties

Proposition 1. *Let $A'(\sigma) = \sum_{m=1}^{\infty} m a_m \sigma^{m-1}$ and considering the PDF for the IPM-PS given in Equation (4), it is determined that*

$$f(t;\beta,\gamma,\theta,\sigma) = \sum_{m=1}^{\infty} P(M = m;\sigma) \cdot f_{T|M}(t;m,\beta,\gamma,\theta),$$

where $f_{T|M}$ is the conditional PDF of T given the value M for the IPM distribution.

Proof. Taking the derivative of Equation (2), we obtain

$$\begin{aligned} f(t;\beta,\gamma,\theta,\sigma) &= \frac{\partial}{\partial t} F(t;\beta,\gamma,\theta,\sigma) \\ &= \sum_{m=1}^{\infty} \frac{\partial P(T > t \mid M = m;\beta,\gamma,\theta,\sigma)}{\partial t} \times P(M = m;\sigma) \\ &= \sum_{m=1}^{\infty} f_{T|M}(t;\beta,\gamma,\theta) \times P(M = m;\sigma). \end{aligned}$$

□

The PDF of the IPM-PS can be expressed as the infinite linear combination of IPM distributions, which will provide a wide use for the application of the EM algorithm, as will be shown in Section 3.

Proposition 2. *The r-th moment of the IPM-PS $(\beta,\gamma,\theta,\sigma)$ distribution, with $r < \gamma$, is given by*

$$E(T^r) = \frac{\sigma}{A(\sigma)} \cdot u(r),$$

where $u(r)$ is defined as

$$u(r) = \beta \exp\left(\frac{1}{\beta}\right) \theta^{-r} [(-1)^{\xi} \beta]^{r/\gamma} \int_{1/\beta}^{1/\xi - 1} A'\Big(\sigma\Big(G\Big(\Big(\frac{log(\beta y)^{-\frac{1}{\gamma}}}{\theta}\Big);\beta,\gamma,\theta\Big)\Big)\Big) [(-1)^{\xi} \log(y\beta)]^{-r/\gamma} (y-1) \exp(-y) dy,$$

where $\xi \to 1$ for $\beta < 0$, and $\xi \to 0$ for $\beta \in (0,1]$.

Proof. By definition of expected value, the *r-th* moment of the r.v. *T* is calculated as

$$\begin{aligned} E(T^r) &= \int_0^{\infty} t^r f(t;\beta,\gamma,\theta,\sigma) dt \\ &= \int_0^{\infty} t^r \frac{A'(\sigma(G(t;\beta,\gamma,\theta,\sigma)))}{A(\sigma)} d(t;\beta,\gamma,\theta,\sigma)\sigma dt \\ &= \frac{\sigma}{A(\sigma)} \int_0^{\infty} A'(\sigma(G(t;\beta,\gamma,\theta,\sigma))) t^r d(t;\beta,\gamma,\theta,\sigma) dt. \end{aligned}$$

By making the change, $y = \frac{\exp(\beta(t\theta)^{-\gamma})}{\beta}$, and following Chesneau and Agiwall [14] in the given expression of moments, we have that

$$E(T^r) = \frac{\sigma}{A(\sigma)} \beta \exp\left(\frac{1}{\beta}\right) \theta^{-r} [(-1)^{\xi} \beta]^{r/\gamma} \int_{1/\beta}^{1/\xi - 1} A'\Big(\sigma\Big(G\Big(\Big(\frac{log(\beta y)^{-\frac{1}{\gamma}}}{\theta}\Big);\beta,\gamma,\theta\Big)\Big)\Big) [(-1)^{\xi} \log(y\beta)]^{-r/\gamma} (y-1) \exp(-y) dy.$$

Therefore,

$$E(T^r) = \frac{\sigma}{A(\sigma)} \cdot u(r).$$

□

In the case that $r \geq \gamma$ the moments tend to infinity. In Appendix B, we present explicit forms for $u(r)$ for each of the PS cases considered.

From Table 2, we observed that the moments and variance are smaller as the parameters β and σ take larger values.

Table 2. First and second moments and variance of the IPM-PS distribution, with $\gamma = 3$ and $\theta = 1$.

				IPM-G					
$(\sigma;\beta)$	(0.1;−1)	(0.2;−1)	(0.5;−1)	(0.1;0.5)	(0.2;0.5)	(0.5;0.5)	(0.1;1)	(0.2;1)	(0.5;1)
$M(1)$	1.4953	1.3291	0.8307	1.0860	0.9654	0.6034	0.9569	0.8505	0.5316
$M(2)$	3.7277	3.3135	2.0710	1.7405	1.5471	0.9670	1.0671	0.9485	0.5928
V	1.4919	1.5470	1.3809	0.5611	0.6152	0.6029	0.1515	0.2251	0.3102
				IPM-L					
$M(1)$	1.5769	1.4891	1.1984	1.1453	1.0815	0.8705	1.0091	0.9529	0.7669
$M(2)$	3.9312	3.7123	2.9878	1.8355	1.7333	1.3950	1.1253	1.0627	0.8553
V	1.4447	1.4950	1.5515	0.5238	0.5636	0.6373	0.1070	0.1546	0.2671
				IPM-P					
$(\sigma;\beta)$	(0.1;−1)	(0.5;−1)	(2;−1)	(0.1;0.5)	(0.5;0.5)	(2;0.5)	(0.1;1)	(0.5;1)	(2;1)
$M(1)$	1.5797	1.2805	0.5201	1.1474	0.9301	0.3777	1.0109	0.8194	0.3328
$M(2)$	3.9383	3.1924	1.2966	1.8388	1.4906	0.6054	1.1273	0.9138	0.3711
V	1.4428	1.5527	1.0261	0.5224	0.6255	0.4627	0.1054	0.2423	0.2604
				IPM-B					
$M(1)$	1.4981	0.9098	0.0056	1.0881	0.6608	0.0041	0.9587	0.5822	0.0036
$M(2)$	3.7348	2.2681	0.0139	1.7438	1.0590	0.0065	1.0691	0.6492	0.0040
V	1.4905	1.4404	0.0139	0.5599	0.6224	0.0065	0.1500	0.3103	0.0040

2.3. Shannon Entropy

The Shannon entropy [25] measures the amount of uncertainty for a random variable. It is defined as $K(T) = -E(\log(f_T(T)))$. For the IPM-PS model, this measure is presented in the following proposition.

Proposition 3. *The Shannon entropy for the IPM-PS model is given by*

$$K(T) = \frac{\sigma\big(A(\sigma) - a_0\big)}{A(\sigma)} \log\left(\frac{A(\sigma)}{\sigma}\right) - \frac{\sigma}{A(\sigma)} \int_0^{\sigma} A'(u) \log A'(u) du$$
$$+ \frac{\sigma}{A(\sigma)}\left[\log(\gamma\sigma) E\left[A'\left(\sigma G\left(\frac{V^{-1/\gamma}}{\theta}\right)\right)\right] + \frac{\gamma+1}{\gamma} E\left[\log(V) A'\left(\sigma G\left(\frac{V^{-1/\gamma}}{\theta}\right)\right)\right]\right.$$
$$\left. + E\left[\log(a(V)) A'\left(\sigma G\left(\frac{V^{-1/\gamma}}{\theta}\right)\right)\right]\right],$$

where $G(t)$ is the survival function of the IPM distribution, $V \sim Muth(\theta)$ and $a(\cdot)$ denotes the PDF of the Muth distribution.

Proof. By definition, $K(T)$ is given by

$$\begin{aligned} K(T) = -E[\log(f_T(T))] = &\frac{\sigma}{A(\sigma)} \log\left(\frac{A(\sigma)}{\sigma}\right) \int_0^{+\infty} d(t) A'(\sigma G(t)) dt \\ &- \frac{\sigma}{A(\sigma)} \int_0^{+\infty} \log(d(t)) d(t) A'(\sigma G(t)) dt \\ &- \frac{\sigma}{A(\sigma)} \int_0^{+\infty} \log(A'(\sigma G(t))) d(t) A'(\sigma G(t)) dt, \end{aligned} \quad (5)$$

where $d(\cdot)$ is the PDF of the IPM distribution. Then, using the change of variable $u = \sigma G(t)$ in the first integral of Equation (5), we obtain

$$\int_0^{+\infty} d(t)A'(\sigma G(t))dt = \int_0^{\sigma} A'(u)du = A(\sigma) - A(0) = A(\sigma) - a_0. \tag{6}$$

On the other hand, by manipulating the second integral in (5), using the change of variable $u = (t\sigma)^{-\gamma}$ and considering $a(\cdot)$ the PDF of the Muth distribution, we obtain

$$\begin{aligned} \int_0^{+\infty} \log(d(t))d(t)A'(\sigma G(t))dt = -\Bigg[&\int_0^{+\infty} \log(\gamma\theta)a(u)A'\Big(\sigma G\Big(\frac{u^{-1/\gamma}}{\theta}\Big)\Big)du \\ &+ \frac{\gamma+1}{\gamma}\int_0^{+\infty} \log(u)a(u)A'\Big(\sigma G\Big(\frac{u^{-1/\gamma}}{\theta}\Big)\Big)du \\ &+ \int_0^{+\infty} \log(a(u))a(u)A'\Big(\sigma G\Big(\frac{u^{-1/\gamma}}{\theta}\Big)\Big)du\Bigg], \end{aligned} \tag{7}$$

then

$$\begin{aligned} \int_0^{+\infty} \log(d(t))d(t)A'(\sigma G(t))dt = -\Bigg[&\log(\gamma\sigma)E\Bigg[A'\Bigg(\sigma G\Bigg(\frac{V^{-1/\gamma}}{\theta}\Bigg)\Bigg)\Bigg] \\ &+ \frac{\gamma+1}{\gamma}E\Bigg[\log(V)A'\Bigg(\sigma G\Bigg(\frac{V^{-1/\gamma}}{\theta}\Bigg)\Bigg)\Bigg] \\ &+ E\Bigg[\log(a(V))A'\Bigg(\sigma G\Bigg(\frac{V^{-1/\gamma}}{\theta}\Bigg)\Bigg)\Bigg]\Bigg]. \end{aligned} \tag{8}$$

Finally, using the change of variable $u = \sigma G(t)$ in the third integral of Equation (5), we obtain

$$\int_0^{+\infty} \log(A'(\sigma G(t)))d(t)A'(\sigma G(t))dt = \int_0^{\sigma} A'(u)\log(A'(u))du. \tag{9}$$

The result is obtained by replacing the results of Equations (6), (8) and (9) in Equation (5). □

In Appendix C, we present explicit forms for the integral in Equation (9) for each of the considered cases of the PS.

2.4. Pseudo-Random Number Generator of the Model

To generate random numbers from the r.v. $T \sim$IPM-PS$(\beta, \gamma, \theta, \sigma)$, we use the inverse transform method, which is summarized as follows (Algorithm 1).

Algorithm 1 Simulating values of the IPM-PS model

(i) Simulate $Y \sim U(0, 1)$,

(ii) Compute $T = Q\left(1 - \frac{A^{-1}\left((1-Y)A(\sigma)\right)}{\sigma}; \beta, \gamma, \theta\right)$,

Where Q is the quantile function of the IPM model and A^{-1} is the inverse function of the power series, which can be seen in Appendix D.

3. Parameter Estimation

In the current section, we discuss the estimation of the parameters of the IPM-PS distribution, using the maximum-likelihood (ML) method. For $\mathbf{t} = t_1, \ldots, t_n$ a random sample of the IPM-PS model, the (observed) log-likelihood function for the parameter vector $\boldsymbol{\zeta} = (\beta, \gamma, \theta, \sigma)$, is given by

$$\begin{aligned}\ell(\boldsymbol{\zeta}|\mathbf{t}) &= n\{\log(\sigma) + \log(\gamma) - \gamma\log(\theta) - \log[A(\sigma)]\} \\ &+ \sum_{i=1}^{n}\Big\{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}} - 1) - (\gamma+1)\log(t_i) + \log\left(e^{\beta(t_i\theta)^{-\gamma}} - \beta\right\} \\ &+ \log[A'(\sigma(G(t_i;\beta,\gamma,\theta)]\Big\}.\end{aligned} \tag{10}$$

The ML estimators are obtained by direct maximization of the log-likelihood function given in (10). As such a procedure requires a four-dimensional maximization problem, we propose an EM-type algorithm (Dempster et al. [22]) as an alternative and more robust process that allows the maximization of the likelihood function, considering some variables of the model as "latent" or unknown.

EM Algorithm

For this particular problem, the vector $\mathbf{M} = (M_1, \ldots, M_n)$ represents the latent variables, the vector $\mathbf{t} = (t_1, \ldots, t_n)$ denotes the observed data and the vector $\mathbf{B}_{comp} = (\mathbf{t}, \mathbf{M})$ represents the complete data. Therefore, the complete log-likelihood function for the parameter vector $\boldsymbol{\zeta}$, is given by

$$\ell_c(\boldsymbol{\zeta} \mid \mathbf{t}) = \ell_{1c}(\sigma \mid \mathbf{t}) + \ell_{2c}(\beta, \gamma, \theta \mid \mathbf{t}),$$

where

$$\begin{aligned}\ell_{1c}(\sigma \mid \mathbf{t}) &= -n\log[A(\sigma)] + \sum_{i=1}^{n} M_i \log(\sigma) \quad \text{and,} \\ \ell_{2c}(\beta, \gamma, \theta \mid \mathbf{t}) &= n[\log(\gamma) - \gamma\log(\theta)] + \sum_{i=1}^{n}\Big\{M_i\log[G(t_i;\beta,\gamma,\theta)] + \beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}} - 1) \\ &- (\gamma+1)\log(t_i) + \log\left(e^{\beta(t_i\theta)^{-\gamma}} - \beta\right) - \log[G(t_i;\beta,\gamma,\theta)]\Big\}.\end{aligned}$$

Let $\boldsymbol{\zeta}^{(k)}$ be the estimate of $\boldsymbol{\zeta}$ at the k-th iteration, and the function $Q(\boldsymbol{\zeta} \mid \mathbf{t}, \boldsymbol{\zeta}^{(k)})$ defined as the conditional expectation of $l_c(\boldsymbol{\zeta} \mid \mathbf{t})$. Given the observed data and the vector of parameters at the k-th iteration, it follows that

$$Q(\boldsymbol{\zeta} \mid \mathbf{t}, \boldsymbol{\zeta}^{(k)}) = Q_1(\sigma \mid \mathbf{t}, \boldsymbol{\zeta}^{(k)}) + Q_2(\beta, \gamma, \theta \mid \mathbf{t}, \boldsymbol{\zeta}^{(k)}),$$

where

$$\begin{aligned}Q_1(\sigma \mid \mathbf{t}, \boldsymbol{\zeta}^{(k)}) &= -n\log[A(\sigma)] + \sum_{i=1}^{n}\widetilde{M_i^{(k)}}\log(\sigma) \quad \text{and,} \\ Q_2(\beta, \gamma, \theta \mid \mathbf{t}, \boldsymbol{\zeta}^{(k)}) &= n[\log(\theta) + \log(\gamma) - \gamma\log(\theta)] + \sum_{i=1}^{n}\Bigg(\widetilde{M_i^{(k)}}\log(G(t_i;\beta,\gamma,\theta)) + \beta(t_i\theta)^{-\gamma} \\ &- \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}} - 1) - (\gamma+1)\log(t_i) + \log\left(e^{\beta(t_i\theta)^{-\gamma}} - \beta\right) - \log(G(t_i;\beta,\gamma,\theta))\Bigg),\end{aligned}$$

where $\widetilde{M_i^{(k)}} = \mathbb{E}[M_i|t_i;\boldsymbol{\zeta}^{(k)}]$. Following Gallardo et al. [26] and considering the function $Q(\boldsymbol{\zeta} \mid \boldsymbol{\zeta}^{(k)}, \mathbf{t})$ with the respective derivatives, the EM algorithm is summarized as following.

- E step: For $i = 1, \ldots, n$, define $v_{ik} = \sigma^{(k-1)}(G(t;\beta^{(k-1)}, \gamma^{(k-1)}, \theta^{(k-1)}))$ and calculate:

$$\widetilde{M_i^{(k)}} = \begin{cases} 1+v_{ik}, & \text{if} \quad M_i \sim \text{Poisson}, \\ \frac{1+v_{ik}}{1-v_{ik}}, & \text{if} \quad M_i \sim \text{geometric}, \\ v_{ik}(e^{v_{ik}}+1)+1, & \text{if} \quad M_i \sim \text{Bell}, \\ \frac{-2(v_{ik})^2\log(1-v_{ik})+3(v_{ik})^2+4(v_{ik})\log(1-v_{ik})-2v_{ik}-2\log(1-v_{ik})}{(1-v_{ik})(v_{ik}+\log(1-v_{ik})(1-v_{ik}))}+1, & \text{if} \quad M_i \sim \text{logarithmic}. \end{cases} \tag{11}$$

- step M-I: $\sigma^{(k)}$ is updated as the solution of the non-linear equation

$$\sum_{i=1}^{n} \frac{\widetilde{M_i^{(k)}}}{\sigma} = \frac{nA'(\sigma)}{A(\sigma)}, \tag{12}$$

  where $\sum_{i=1}^{n} \widetilde{M_i^{(k)}}$ is the sum of the elements of $(\widetilde{M_1^{(k)}}, \ldots, \widetilde{M_n^{(k)}})$.
- step M-II: Given the vector $(\beta^{(k)}, \gamma^{(k)}, \theta^{(k)})$, update (β, γ, θ) by maximizing $Q_2(\beta, \gamma, \theta \mid \mathbf{t}, \boldsymbol{\zeta}^{(k)})$ in relation to each of the parameters.
- If the convergence condition is reached, the algorithm stops. Otherwise, we return to step E for a new iteration.

  Figure 3 shows the scheme for the EM algorithm for clarifying the way to use it.

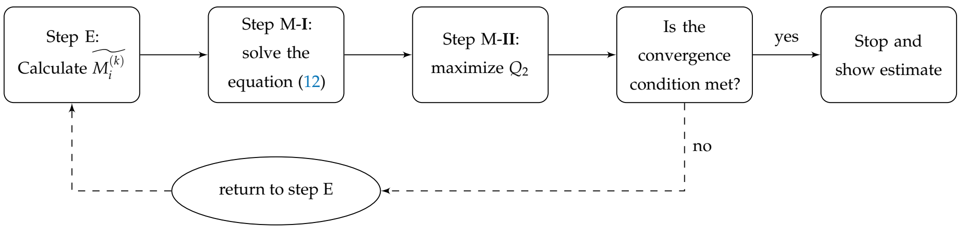


Figure 3. EM algorithm scheme for the IPM-PS distribution.

By taking advantage of the complete log-likelihood function, the variance of the estimator of $\widehat{\boldsymbol{\zeta}} = (\widehat{\beta}, \widehat{\gamma}, \widehat{\theta}, \widehat{\sigma})$ can be computed by Oakes' method presented by Oakes [23]. This method computes the observed information matrix instead of the Fisher information matrix. Details of this method are provided in Appendix E. We highlight that in the literature, variance is usually obtained from numerical methods that approximate the observed information matrix associated with the log-likelihood function in relation to the parameters to be estimated. However, we highlight that the Oakes method provides an exact method for calculating this matrix, also avoiding possible approximation errors.

Asymptotic confidence intervals (ACIs) with confidence level $1-\mu$, $0<\mu<1$, can be constructed based on the asymptotic normality of the MLs, i.e.,

$$ACI(\zeta_i, 1-\mu) = (\widehat{\zeta}_i - z_{1-\mu/2}SE(\widehat{\zeta}_i), \widehat{\zeta}_i + z_{1-\mu/2}SE(\widehat{\zeta}_i)),$$

where $\zeta \in (\beta, \gamma, \theta, \sigma)$.

4. Simulation Study

In this section, a simulation study is carried out to investigate the performance of the ML estimators for the IPM-PS model in finite samples. For this, the EM algorithm was used to calculate the estimates and their corresponding standard errors (SE) and the root of the mean squared error (RMSE). This process was replicated 1000 times with sample sizes of $n=100$, $n=200$ and $n=500$. For the parameters, the following were considered to be true values: $\sigma = 0.5, 1, 2$ in the IPM-P and IPM-B models and $\sigma = 0.5$ in the IPM-G and IPM-L models. For β, $\beta = 0.5, 0.2, 0.1$ and -0.2 were considered to be true values; for $\theta = 1, 2$; and for $\gamma = 1$ in all the cases. The confidence level considered to build the ACIs is $1 - \mu = 0.95$. The IPM-G, IPM-P, IPM-B, and IPM-L models were considered for this study.

From Tables 3–6, as the sample size increases ($n = 100$, $n = 200$, and $n = 500$), the bias of all the estimators decreases and the variances become smaller for each of the estimators. Standard deviations for all cases were also calculated, but are not reported here. These values were also close to the SE terms, which suggests that the variances of the estimators are well estimated. Finally, the CP terms converge to 0.95 as the sample size increases.

Table 3. Simulation study using the EM algorithm for the **IPM-G** with 1000 replicates. For all the cases $\gamma = 1$.

True Value				$n = 100$				$n = 200$				$n = 500$			
σ	θ	β		**bias**	**RMSE**	**SE**	**CP**	**bias**	**RMSE**	**SE**	**CP**	**bias**	**RMSE**	**SE**	**CP**
0.5	1	0.5	$\widehat{\beta}$	−0.0783	0.2152	0.7088	0.9310	−0.0516	0.1879	0.5608	0.9570	−0.0270	0.1543	0.3481	0.9660
			$\widehat{\gamma}$	0.1415	0.3005	0.3507	0.9830	0.0851	0.2266	0.3132	0.9770	0.0366	0.1552	0.2365	0.9600
			$\widehat{\theta}$	0.0542	0.1328	0.3841	0.9990	0.0315	0.0887	0.2806	0.9700	0.0115	0.0490	0.1764	0.9660
			$\widehat{\sigma}$	−0.0336	0.1055	0.5604	0.9990	−0.0156	0.0583	0.4467	0.9870	−0.0012	0.0152	0.2789	0.9700
		0.2	$\widehat{\beta}$	0.0432	0.2111	0.7731	0.9770	0.0404	0.1643	0.5337	0.9710	0.0244	0.1133	0.4813	0.9650
			$\widehat{\gamma}$	0.0291	0.2075	0.3240	0.9870	0.0036	0.1581	0.2447	0.9710	−0.0144	0.1000	0.2726	0.9660
			$\widehat{\theta}$	0.0717	0.1825	0.3119	0.9800	0.0427	0.1368	0.2091	0.9770	0.0049	0.0633	0.1412	0.9660
			$\widehat{\sigma}$	−0.0824	0.1625	0.7174	0.9800	−0.0530	0.1189	0.5045	0.9770	−0.0118	0.0438	0.4565	0.9600
		0.1	$\widehat{\beta}$	0.0318	0.3060	0.7434	0.9910	0.0370	0.1503	0.5364	0.9820	0.0224	0.0873	0.3129	0.9740
			$\widehat{\gamma}$	0.0220	0.1894	0.3427	0.9900	−0.0004	0.1342	0.1942	0.9820	−0.0149	0.0793	0.1091	0.9690
			$\widehat{\theta}$	0.1113	0.2970	0.2888	0.9950	0.0495	0.1596	0.2069	0.9910	0.0049	0.0714	0.1289	0.9790
			$\widehat{\sigma}$	−0.1054	0.1927	0.8066	0.9850	−0.0620	0.1356	0.5439	0.9810	−0.0157	0.0545	0.3312	0.9740
		−0.2	$\widehat{\beta}$	−0.0930	0.5089	0.9152	0.9920	−0.3430	0.3655	0.6773	0.8670	−0.0577	0.1943	0.4614	0.9770
			$\widehat{\gamma}$	−0.0131	0.1356	0.5921	0.9820	−0.0122	0.0952	0.4580	0.9770	−0.0102	0.0600	0.1467	0.9690
			$\widehat{\theta}$	0.1882	0.5432	0.3252	0.9870	0.6742	0.5457	0.2439	0.9790	0.0299	0.1385	0.1670	0.9670
			$\widehat{\sigma}$	−0.0843	0.2032	1.5664	0.9870	−0.0321	0.1230	1.2458	0.9790	−0.0027	0.0490	0.6344	0.9690
	2	0.5	$\widehat{\beta}$	−0.0806	0.2093	0.6795	0.9480	−0.0514	0.1836	0.5117	0.9580	−0.0259	0.1457	0.3659	0.9640
			$\widehat{\gamma}$	0.1295	0.3017	0.3595	0.9480	0.0793	0.2278	0.2954	0.9570	0.0306	0.1502	0.2486	0.9610
			$\widehat{\theta}$	0.0753	0.3383	0.3635	0.9810	0.0493	0.2218	0.2676	0.9770	0.0130	0.1080	0.1768	0.9640
			$\widehat{\sigma}$	−0.0298	0.1464	1.0769	0.9710	−0.0147	0.0859	0.8224	0.9680	−0.0026	0.0273	0.5823	0.9610
		0.2	$\widehat{\beta}$	0.0740	0.2136	0.7317	0.9840	0.0601	0.1733	0.5319	0.9720	0.0357	0.1092	0.3357	0.9620
			$\widehat{\gamma}$	−0.0043	0.2099	0.3093	0.9940	−0.0331	0.1597	0.2420	0.9800	−0.0301	0.0960	0.1545	0.9773
			$\widehat{\theta}$	0.1360	0.4811	0.2876	0.9990	0.0386	0.3577	0.2110	0.9840	0.0117	0.1953	0.1302	0.9730
			$\widehat{\sigma}$	−0.1161	0.2187	1.3044	0.9890	−0.0635	0.1622	1.0014	0.9810	−0.0307	0.0906	0.6472	0.9790
		0.1	$\widehat{\beta}$	0.0607	0.2558	0.7508	0.9820	0.0583	0.1669	0.5352	0.9750	0.0319	0.0834	0.3194	0.9610
			$\widehat{\gamma}$	−0.0236	0.1883	0.2961	0.9880	−0.0389	0.1417	0.1934	0.9850	−0.0369	0.0802	0.1125	0.9770
			$\widehat{\theta}$	0.1271	0.5441	0.2831	0.9900	0.0454	0.3748	0.2013	0.9850	−0.0184	0.2258	0.1271	0.9710
			$\widehat{\sigma}$	−0.1214	0.2400	1.4822	0.9900	−0.0758	0.1800	1.0671	0.9850	−0.0251	0.1043	0.6699	0.9700

Table 4. Simulation study using the EM algorithm for the **IPM-P** with 1000 replicates. For all the cases $\gamma = 1$.

True Value				$n = 100$				$n = 200$				$n = 500$			
σ	θ	β		**bias**	**RMSE**	**SE**	**CP**	**bias**	**RMSE**	**SE**	**CP**	**bias**	**RMSE**	**SE**	**CP**
0.5	1	0.5	$\widehat{\beta}$	−0.0622	0.2006	1.9905	0.9560	−0.0363	0.1847	1.6212	0.9370	−0.0024	0.1493	1.2423	0.9250
			$\widehat{\gamma}$	0.0651	0.2493	0.3158	0.9810	0.0375	0.1969	0.2600	0.9750	0.0091	0.1452	0.2192	0.9640
			$\widehat{\theta}$	−0.0147	0.1516	0.3144	0.9950	−0.0107	0.1141	0.2390	0.9880	−0.0007	0.0612	0.1604	0.9600
			$\widehat{\sigma}$	0.2062	0.6869	0.3978	0.9970	0.1358	0.4748	0.3247	0.9860	0.0308	0.2132	0.2472	0.9790
1	1	0.5	$\widehat{\beta}$	−0.0671	0.2085	1.9345	0.9780	−0.0362	0.1757	1.5427	0.9680	−0.0047	0.1427	1.1113	0.9630
			$\widehat{\gamma}$	0.1230	0.2947	0.3180	0.9720	0.0747	0.2279	0.2599	0.9670	0.0300	0.1518	0.2044	0.9610
			$\widehat{\theta}$	0.0477	0.1797	0.3475	0.9990	0.0318	0.1319	0.2552	0.9890	0.0232	0.0788	0.1712	0.9750
			$\widehat{\sigma}$	−0.0689	0.7308	0.3882	0.9860	−0.0660	0.5294	0.3077	0.9800	−0.0733	0.2831	0.2201	0.9700
2	1	0.5	$\widehat{\beta}$	−0.0498	0.2110	1.9715	0.9890	−0.0269	0.1836	1.4665	0.9790	0.0140	0.1499	0.9608	0.9620
			$\widehat{\gamma}$	0.2355	0.4180	0.3275	0.9850	0.1689	0.3203	0.2734	0.9770	0.0789	0.2077	0.2061	0.9660
			$\widehat{\theta}$	0.1628	0.2579	0.4444	0.9910	0.1314	0.2060	0.3156	0.9800	0.0900	0.1405	0.2010	0.9690
			$\widehat{\sigma}$	−0.6167	1.0444	0.3925	0.9870	−0.5380	0.8661	0.2884	0.9700	−0.4140	0.6041	0.1868	0.9680
		0.2	$\widehat{\beta}$	0.0313	0.2495	2.3864	0.9870	0.0317	0.1588	1.8656	0.9750	0.0166	0.0944	1.1976	0.9690
			$\widehat{\gamma}$	−0.0124	0.1935	0.3168	0.9740	−0.0226	0.1509	0.2235	0.9640	−0.0148	0.0916	0.1374	0.9600
			$\widehat{\theta}$	−0.0050	0.2085	0.2938	0.9920	−0.0132	0.1564	0.2118	0.9840	−0.0053	0.0944	0.1310	0.9710
			$\widehat{\sigma}$	0.0839	0.7085	0.6032	0.9760	0.0707	0.6137	0.4598	0.9690	0.0243	0.3381	0.2975	0.9600
		0.1	$\widehat{\beta}$	−0.0712	2.0289	0.5630	0.9980	0.0211	0.4026	1.8901	0.9860	0.0226	0.0769	1.2122	0.9750
			$\widehat{\gamma}$	−0.0311	0.1827	1.5422	0.9890	−0.0362	0.1445	0.2628	0.9630	−0.0223	0.0838	0.1108	0.9600
			$\widehat{\theta}$	0.1051	2.3623	0.2907	0.9860	−0.0180	0.4423	0.2134	0.9720	−0.0124	0.1098	0.1326	0.9630
			$\widehat{\sigma}$	0.1186	0.8074	2.1261	0.9930	0.1449	0.7578	0.5784	0.9820	0.0339	0.4028	0.3221	0.9720
		−0.2	$\widehat{\beta}$	−0.4583	9.7743	2.7164	0.9830	−0.4739	12.7700	2.1530	0.9730	−0.0805	0.3246	1.5611	0.9640
			$\widehat{\gamma}$	−0.0713	0.1657	0.8047	0.9920	−0.0574	0.1282	0.5628	0.9700	−0.0297	0.0823	0.2070	0.9640
			$\widehat{\theta}$	1.9332	44.2624	0.2751	0.9810	0.6835	19.8681	0.2140	0.9750	0.0041	0.3105	0.1543	0.9670
			$\widehat{\sigma}$	0.4587	1.1872	1.5392	0.9350	0.3772	0.9874	1.1873	0.9480	0.2001	0.6025	0.5607	0.9680
	2	0.5	$\widehat{\beta}$	−0.0590	0.2046	1.9976	0.9990	−0.0304	0.1798	1.6217	0.9970	−0.0080	0.1471	1.2477	0.9780
			$\widehat{\gamma}$	0.0573	0.2482	0.3209	0.9720	0.0264	0.1917	0.2614	0.9660	0.0030	0.1434	0.2185	0.9640
			$\widehat{\theta}$	−0.0344	0.3092	0.3113	0.9710	−0.0322	0.2375	0.2369	0.9630	−0.0191	0.1419	0.1599	0.9600
			$\widehat{\sigma}$	0.1937	0.6670	0.7954	0.9920	0.1345	0.4885	0.6508	0.9910	0.0442	0.2692	0.4960	0.9960
		0.2	$\widehat{\beta}$	0.0340	0.2413	2.4225	0.9990	0.0386	0.1560	1.8048	0.9970	0.0244	0.1040	1.2031	0.9750
			$\widehat{\gamma}$	−0.0328	0.1934	0.3199	0.9900	−0.0439	0.1442	0.2199	0.9800	−0.0426	0.0997	0.1404	0.9750
			$\widehat{\theta}$	−0.0440	0.4214	0.2895	0.9760	−0.0698	0.3155	0.2063	0.9660	−0.0712	0.2201	0.1306	0.9600
			$\widehat{\sigma}$	0.1083	0.7683	1.2248	0.9900	0.0881	0.6356	0.8844	0.9890	0.0750	0.4350	0.5925	0.9770
		0.1	$\widehat{\beta}$	0.0424	0.3420	2.5115	0.9950	0.0432	0.1536	1.8904	0.9840	0.0251	0.0815	1.2635	0.9760
			$\widehat{\gamma}$	−0.0544	0.1891	0.3697	0.9890	−0.0615	0.1495	0.1969	0.9780	−0.0453	0.0904	0.1155	0.9620
			$\widehat{\theta}$	−0.0891	0.5873	0.2878	0.9760	−0.1124	0.3823	0.2082	0.9690	−0.0837	0.2395	0.1376	0.9600
			$\widehat{\sigma}$	0.1957	0.9359	1.4253	0.9770	0.1527	0.7494	0.9803	0.9700	0.0904	0.4559	0.6600	0.9600

Table 5. Simulation study using the EM algorithm for the **IPM-B** with 1000 replicates. For all the cases $\gamma = 1$.

True Value				$n = 100$				$n = 200$				$n = 500$			
σ	θ	β		**bias**	**RMSE**	**SE**	**CP**	**bias**	**RMSE**	**SE**	**CP**	**bias**	**RMSE**	**SE**	**CP**
0.5	1	0.5	$\widehat{\beta}$	−0.0647	0.2052	0.8458	0.9460	−0.0478	0.1802	0.6633	0.9590	−0.0147	0.1416	0.4704	0.9680
			$\widehat{\gamma}$	0.1279	0.2899	0.3174	0.9810	0.0847	0.2194	0.2624	0.9700	0.0290	0.1470	0.2165	0.9660
			$\widehat{\theta}$	0.0497	0.1415	0.3509	0.9860	0.0337	0.1005	0.2645	0.9770	0.0138	0.0524	0.1724	0.9660
			$\widehat{\sigma}$	−0.0459	0.1832	0.4464	0.9800	−0.02568	0.1092	0.3532	0.9700	−0.0096	0.0355	0.2483	0.9660
1	1	0.5	$\widehat{\beta}$	−0.0847	0.2197	0.6442	0.9140	−0.0610	0.1796	0.4419	0.9320	−0.0410	0.1520	0.2637	0.9660
			$\widehat{\gamma}$	0.3063	0.4903	0.3231	0.9410	0.1980	0.3386	0.2662	0.9530	0.1006	0.2180	0.2066	0.9650
			$\widehat{\theta}$	0.1920	0.2799	0.4858	0.9770	0.1303	0.1996	0.3501	0.9840	0.0621	0.1136	0.2256	0.9670
			$\widehat{\sigma}$	−0.2241	0.3527	0.4201	0.9980	−0.1380	0.2337	0.3036	0.9720	−0.0570	0.1193&	0.1887	0.9670

Table 5. *Cont.*

True Value				$n = 100$				$n = 200$				$n = 500$			
σ	θ	β		bias	RMSE	SE	CP	bias	RMSE	SE	CP	bias	RMSE	SE	CP
2	1	0.5	$\widehat{\beta}$	−0.1390	0.2193	0.3709	0.9800	−0.1131	0.2028	0.2551	0.9720	−0.0769	0.1726	0.1665	0.9690
			$\widehat{\gamma}$	0.4175	0.7326	0.7068	0.9720	0.2426	0.3786	0.6131	0.9750	0.1299	0.2173	0.5371	0.9690
			$\widehat{\theta}$	0.2107	0.3243	1.1496	0.9710	0.1259	0.1832	0.8568	0.9860	0.0606	0.0907	0.6019	0.9970
			$\widehat{\sigma}$	−0.1562	0.3359	0.4613	0.9520	−0.0743	0.1487	0.3479	0.9600	−0.0249	0.0592	0.2285	0.9670
		0.2	$\widehat{\beta}$	0.0480	0.2630	0.9211	0.9830	0.0406	0.1583	0.6999	0.9710	0.0302	0.1154	0.4400	0.9620
			$\widehat{\gamma}$	0.0248	0.2076	0.2906	0.9820	0.0048	0.1533	0.2014	0.9710	−0.0150	0.1023	0.1310	0.9640
			$\widehat{\theta}$	0.0713	0.2349	0.3063	0.9820	0.0371	0.1427	0.2312	0.9710	0.0110	0.0792	0.1428	0.9690
			$\widehat{\sigma}$	−0.1069	0.2299	0.6043	0.9830	−0.0648	0.1703	0.45543	0.9800	−0.0273	0.0914	0.2912	0.9770
		0.1	$\widehat{\beta}$	0.0342	0.2980	1.0518	0.9850	0.0365	0.1766	0.7190	0.9830	0.0244	0.1115	0.4275	0.9790
			$\widehat{\gamma}$	−0.0005	0.1861	0.3416	0.9920	−0.0024	0.1333	0.1743	0.9830	−0.0170	0.0828	0.0993	0.9770
			$\widehat{\theta}$	0.0662	0.2706	0.3296	0.9850	0.0390	0.1859	0.2283	0.9790	0.0079	0.1049	0.1405	0.9670
			$\widehat{\sigma}$	−0.0831	0.2456	0.8136	0.9990	−0.0680	0.1914	0.5084	0.9940	−0.0246	0.1092	0.3080	0.9870
		−0.2	$\widehat{\beta}$	−0.0708	0.5232	1.0911	0.9910	−0.0421	0.3010	0.8699	0.9800	−0.0405	0.1763	0.6136	0.9610
			$\widehat{\gamma}$	−0.0469	0.1546	0.5067	0.9940	−0.0306	0.1091	0.2603	0.9740	−0.0168	0.0697	0.1356	0.9630
			$\widehat{\theta}$	0.0746	0.5990	0.3029	0.9890	0.0283	0.2960	0.2443	0.9740	0.0078	0.1507	0.1721	0.9670
			$\widehat{\sigma}$	0.0228	0.3499	1.2728	0.9830	0.0135	0.2384	0.8195	0.9750	0.0152	0.1336	0.5331	0.9610
	2	0.5	$\widehat{\beta}$	−0.0619	0.1945	0.8590	0.9980	−0.0416	0.1753	0.6607	0.9890	−0.0078	0.1412	0.4426	0.9990
			$\widehat{\gamma}$	0.1125	0.2791	0.3363	0.9330	0.0729	0.2173	0.2650	0.9420	0.0159	0.1457	0.2127	0.9620
			$\widehat{\theta}$	0.0729	0.3018	0.3538	0.9800	0.0496	0.2202	0.2627	0.9720	0.0111	0.1150	0.1710	0.9670
			$\widehat{\sigma}$	−0.0283	0.1944	0.9046	0.9700	−0.0194	0.1299	0.7037	0.9890	−0.0050	0.0526	0.4684	0.9620
		0.2	$\widehat{\beta}$	0.0729	0.2117	0.9636	0.9900	0.0547	0.1564	0.6760	0.9850	0.0360	0.1081	0.4496	0.9780
			$\widehat{\gamma}$	−0.0273	0.1958	0.3055	0.9910	−0.0261	0.1517	0.2120	0.9720	−0.0333	0.0975	0.1393	0.9610
			$\widehat{\theta}$	0.0640	0.4303	0.3059	0.9920	0.0357	0.3279	0.2164	0.9830	−0.0037	0.1831	0.1390	0.9680
			$\widehat{\sigma}$	−0.0937	0.2673	1.2401	0.9910	−0.0715	0.2148	0.8714	0.9860	−0.0324	0.1177	0.5893	0.9690
		0.1	$\widehat{\beta}$	0.0749	0.2478	0.9590	0.9990	0.0682	0.1670	0.6936	0.9880	0.0354	0.0876	0.4347	0.9720
			$\widehat{\gamma}$	−0.0373	0.1903	0.3065	0.9910	−0.0583	0.1447	0.1788	0.9840	−0.0433	0.0885	0.1002	0.9730
			$\widehat{\theta}$	0.0770	0.4571	0.2880	0.9780	−0.0244	0.3578	0.2171	0.9660	−0.0335	0.2197	0.1390	0.9600
			$\widehat{\sigma}$	−0.1132	0.2756	1.3742	0.9970	−0.0447	0.2279	0.9692	0.9870	−0.0210	0.1420	0.6188	0.9670

Table 6. Simulation study using the EM algorithm for the **IPM-L** with 1000 replicates. For all the cases $\gamma = 1$.

True Value				$n = 100$				$n = 200$				$n = 500$			
σ	θ	β		bias	RMSE	SE	CP	bias	RMSE	SE	CP	bias	RMSE	SE	CP
0.5	1	0.5	$\widehat{\beta}$	−0.0913	0.2344	1.3674	0.9830	−0.0599	0.1952	1.0293	0.9730	−0.0365	0.1549	0.6736	0.9620
			$\widehat{\gamma}$	0.1068	0.2685	0.4348	0.9830	0.0550	0.2051	0.3611	0.9730	0.0148	0.1424	0.2791	0.9760
			$\widehat{\theta}$	0.0100	0.1186	0.3590	0.9830	−0.0102	0.0794	0.2659	0.9710	−0.0235	0.0514	0.1677	0.9660
			$\widehat{\sigma}$	−0.0437	0.1403	0.6819	0.9730	−0.0199	0.0769	0.5082	0.9710	−0.0102	0.0192	0.3254	0.9620
		0.2	$\widehat{\beta}$	−0.3935	1.6522	1.3146	0.9740	0.0161	0.1571	0.9409	0.9700	−0.0014	0.0963	0.6257	0.9640
			$\widehat{\gamma}$	0.0066	0.1899	0.6875	0.9910	−0.0069	0.1390	0.2749	0.9820	−0.0142	0.0852	0.1846	0.9740
			$\widehat{\theta}$	0.1861	1.6107	0.2626	0.9910	−0.0048	0.1053	0.1808	0.9720	−0.0271	0.0631	0.1135	0.9640
			$\widehat{\sigma}$	−0.1176	0.1962	0.9611	0.9940	−0.0758	0.1442	0.5331	0.9800	−0.0293	0.0677	0.3649	0.9740
		0.1	$\widehat{\beta}$	0.0042	0.4007	1.2466	0.9850	0.0154	0.1490	0.8994	0.9760	0.0025	0.0896	0.5491	0.9690
			$\widehat{\gamma}$	0.0016	0.1708	0.4576	0.9930	−0.0082	0.1162	0.2241	0.9810	−0.0163	0.0703	0.1307	0.9750
			$\widehat{\theta}$	0.0608	0.3558	0.2454	0.9850	0.0043	0.1201	0.1726	0.9760	−0.0269	0.0726	0.1080	0.9650
			$\widehat{\sigma}$	−0.1368	0.2118	0.9204	0.9930	−0.0904	0.1640	0.5495	0.9860	−0.0351	0.0812	0.3475	0.9690
		−0.2	$\widehat{\beta}$	−0.8713	1.3807	1.3926	0.9870	−0.1653	0.5025	1.0040	0.9720	−0.1012	0.2560	0.6365	0.9610
			$\widehat{\gamma}$	−0.0079	0.1100	1.2154	0.9850	−0.0110	0.0795	0.4796	0.9770	−0.0106	0.0525	0.1965	0.9960
			$\widehat{\theta}$	1.6920	3.4927	0.2594	0.9980	0.1181	0.4368	0.1866	0.9870	0.0243	0.1559	0.1196	0.9790
			$\widehat{\sigma}$	−0.1205	0.2113	1.3655	0.9870	−0.0667	0.1424	1.1179	0.9770	−0.0189	0.0593	0.5751	0.9690

Table 6. *Cont.*

True Value				$n=100$				$n=200$				$n=500$			
σ	θ	β		bias	RMSE	SE	CP	bias	RMSE	SE	CP	bias	RMSE	SE	CP
	2	0.5	$\widehat{\beta}$	−0.0965	0.2355	1.2621	0.9820	−0.0541	0.1903	0.9674	0.9760	−0.0320	0.1521	0.6883	0.9670
			$\widehat{\gamma}$	0.1020	0.2691	0.4254	0.9820	0.0378	0.2007	0.3501	0.9760	0.0082	0.1424	0.2801	0.9650
			$\widehat{\theta}$	−0.0087	0.2752	0.3522	0.9720	−0.0554	0.2024	0.2537	0.9690	−0.0565	0.1107	0.1619	0.9650
			$\widehat{\sigma}$	−0.0279	0.1621	1.3228	0.9820	−0.0034	0.1044	0.9964	0.9760	−0.0076	0.0282	0.6670	0.9680
		0.2	$\widehat{\beta}$	0.0260	0.2239	1.2350	0.9890	0.0190	0.1622	0.9383	0.9750	0.1019	0.6279	0.6200	0.9690
			$\widehat{\gamma}$	−0.0117	0.1867	0.3544	0.9870	−0.0280	0.1361	0.2810	0.9750	−0.0241	0.0879	0.1840	0.9670
			$\widehat{\theta}$	−0.0061	0.3345	0.2585	0.9900	−0.0570	0.2333	0.1800	0.9800	−0.0743	0.1438	0.1147	0.9720
			$\widehat{\sigma}$	−0.1002	0.2056	1.4290	0.9870	−0.0636	0.1509	1.0908	0.9700	−0.0263	0.0760	0.7386	0.9690
		0.1	$\widehat{\beta}$	0.0180	0.2640	1.1769	0.9880	0.0133	0.1712	0.8844	0.9790	0.0022	0.0750	0.5468	0.9680
			$\widehat{\gamma}$	−0.0212	0.1646	0.3278	0.9800	−0.0311	0.1141	0.2360	0.9750	−0.0263	0.0693	0.1289	0.9680
			$\widehat{\theta}$	0.0163	0.3927	0.2341	0.9830	−0.0587	0.2823	0.1714	0.9780	−0.0799	0.1674	0.1069	0.9680
			$\widehat{\sigma}$	−0.1189	0.2260	1.4648	0.9830	−0.0674	0.1733	1.1255	0.9790	−0.0305	0.0991	0.6944	0.9680

5. Applications

In this section, we present two applications of the IPM-PS distribution. The parameter estimation was performed based on the EM algorithm, as discussed in Section 3.

5.1. Application 1

This dataset presents body mass index (BMI, weight/height^2) values and is available from the "bsamGP" package (Jo et al. [27]) of the "R" software [28], containing 314 observations. The BMI acts as an indicator to assess the nutritional status of the patients. For comparative purposes, the IPM-B, Gompertz–Poisson (GP), Beta Burr XII (BBXII), Weibull-Poisson (WP), IPM, and the Weibull (W) distributions were also considered.

Based on the Akaike information criterion (Akaike, [29]) and the Bayesian information criterion (Schwarz, [30]), we observe that the IPM-B provides the best fit to the dataset among the models considered, as confirmed in Table 7, where both the AIC and BIC criteria are the lowest values compared to the rest of the distributions. Furthermore, Figure 4 shows the corresponding fit of the different distributions that were compared.

Table 7. Parameter estimation, standard errors (in parentheses), AIC and BIC comparison criteria for IMC values.

Distribution	IPM-B	GP	WP	BBXII	IPM	W
$\widehat{\beta}$	−12.610 (1.5148)	0.0037 (1.1×10^{-6})	5.3277 (0.0043)	15.156 (1.2348)	0.144 (0.253)	–
$\widehat{\gamma}$	6.0506 (0.6685)	0.1198 (2.9×10^{-5})	0.0280 (6.7×10^{-7})	0.9763 (0.6613)	2.484 (0.505)	4.169 (0.1610)
$\widehat{\theta}$	0.0660 (0.0143)	0.0997 (0.4975)	4.2861 (0.3996)	14.812 (0.9717)	1.075 (0.069)	28.565 (0.4119)
$\widehat{\sigma}$	0.0016 (0.5050)	–	–	0.8248 (0.5488)	–	–
$\widehat{c}$	–	–	–	7.4191 (1.4849)	–	–
AIC	1909.79	2174.64	2012.36	1911.41	1913.79	2063.75
BIC	1924.79	2185.89	2023.61	1930.16	1926.82	2071.25

An alternative to check the adequacy of a model is to use the quantile residuals (QR, Dunn, and Smith [31]). If the model is appropriate, these residuals will have a standard normal distribution. Anderson Darling (AD), Cramer–von Mises (CvM), and Kolmogorov–Smirnov (KS) normality tests were performed to verify the normality of the QR. Figure 5 shows the QQ-plot for the QR in the IPM-B model and the p-values for the referred tests, suggesting that the residuals are normally distributed. Finally, Table 8 presents the point and interval estimates (based on the delta method) for the percentage of overweight patients, i.e., patients whose IBM is greater than 30 using the IPM-B, BBXII, IPM, and WP models. Please note that the IPM-B provides the more accurate 95% CI. At first, one might question whether 0.0002 is too small a difference. However, if we think about estimating the percentage of people who are overweight in a small country (say with 10 million inhabitants), this difference translates into a precision of 2000 people in the interval estimate.

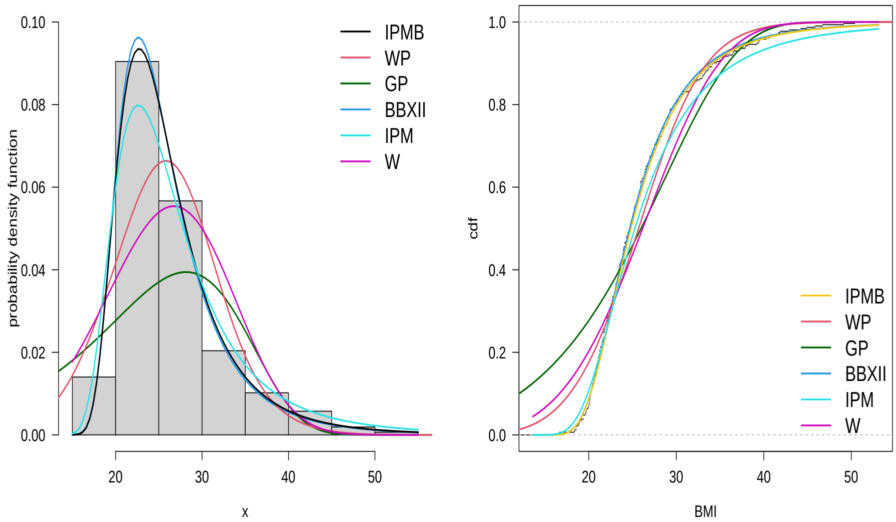
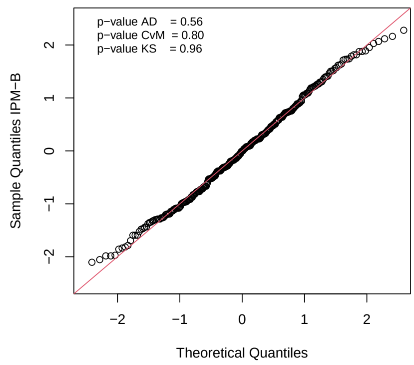
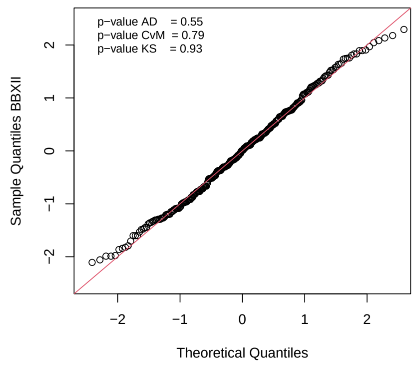


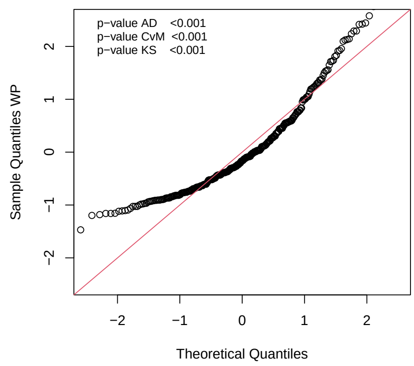
Figure 4. PDF of the IPM-B, WP, GP, and W models on the IMC dataset (**left panel**) and the empirical CDF compared with the estimated CDF for the same models (**right panel**).



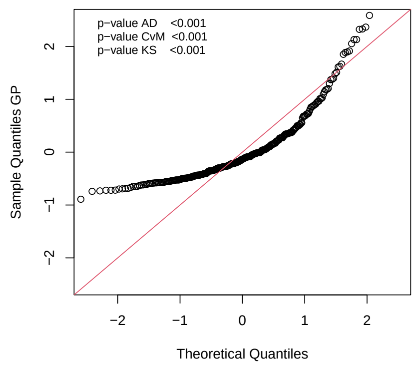
(**a**) IPM-B



(**b**) BBXII

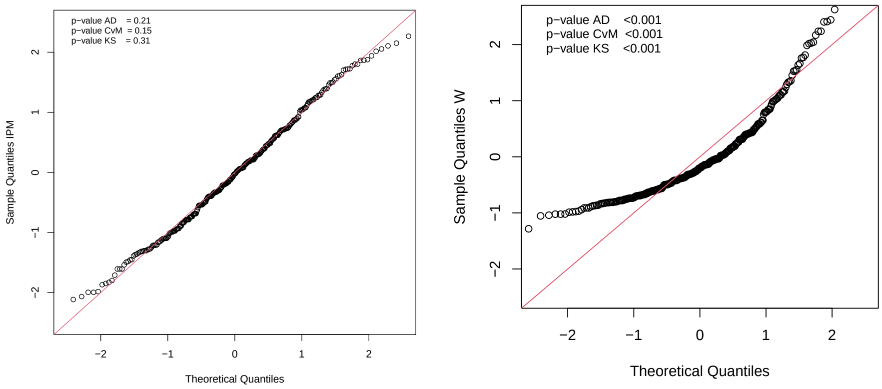


(**c**) WP



(**d**) GP

Figure 5. *Cont.*



(**e**) IPM (**f**) W

Figure 5. QQ plot for the QR for the IPM-B, BBXII, WP, GP, IPM, and W distributions for the beta-carotene data.

Table 8. Point and interval estimation for the percentage of patients with overweight using different distributions.

Distribution	P**(IBM > 30)**	**95% CI**	**Interval Length**
IPMB	0.1910	(0.1836–0.1985)	0.0149
BBXII	0.1922	(0.1847–0.1998)	0.0151
IPM	0.1938	(0.1763–0.2113)	0.0175
WP	0.2364	(0.2264–0.2464)	0.0199

5.2. Application 2

This dataset was taken from Murthy et al. [32], and is related to the failure times of 40 articles. For the current example, the IPM-P, Gompertz–Geometric (GG), Weibull–Geometric (WG), IPM, and W distributions were used for comparative purposes. The IPM-P model provides the best fit for the dataset, as confirmed in Table 9, where the AIC and BIC values are the lowest compared to the rest of the distributions. In addition, Figure 6 compares the histogram and empirical CDF with the corresponding fit provided for the different distributions.

Table 9. Parameter estimation, standard errors (in parentheses), AIC and BIC criteria for failure times of 40 articles.

Distribution	**IPM-P**	**GG**	**WG**	**BBXII**	**IPM**	**W**
$\widehat{\beta}$	0.1398 (0.2946)	0.0728 (0.3357)	0.0078 (6.6 $\times 10^{-5}$)	7.3236 (3.5 $\times 10^{-4}$)	0.019 (0.041)	–
$\widehat{\gamma}$	2.4751 (0.6374)	1.6926 (0.0125)	2.4962 (0.1401)	11.366 (8.4 $\times 10^{-3}$)	5.893 (0.309)	2.3451 (0.2798)
$\widehat{\theta}$	1.0698 (0.2219)	2.2587 (0.0225)	0.9392 (0.0032)	0.3442 (2.2 $\times 10^{-1}$)	0.044 (0.001)	1.3937 (0.0998)
$\widehat{\sigma}$	0.0459 (1.9492)	–	–	0.0673 (6.3 $\times 10^{-1}$)	–	–
$\widehat{c}$	–	–	–	6.3903 (3.8 $\times 10^{-3}$)	–	–
AIC	64.68	72.85	65.62	65.92	68.68	66.58
BIC	71.44	77.92	71.69	74.36	76.13	69.96

Figure 7 presents the QQ-plot for the QR in the IPM-P model and the p-values for the normality tests, suggesting that these residuals satisfy the assumption of normality and therefore, the IPM-P distribution is suitable for this dataset.

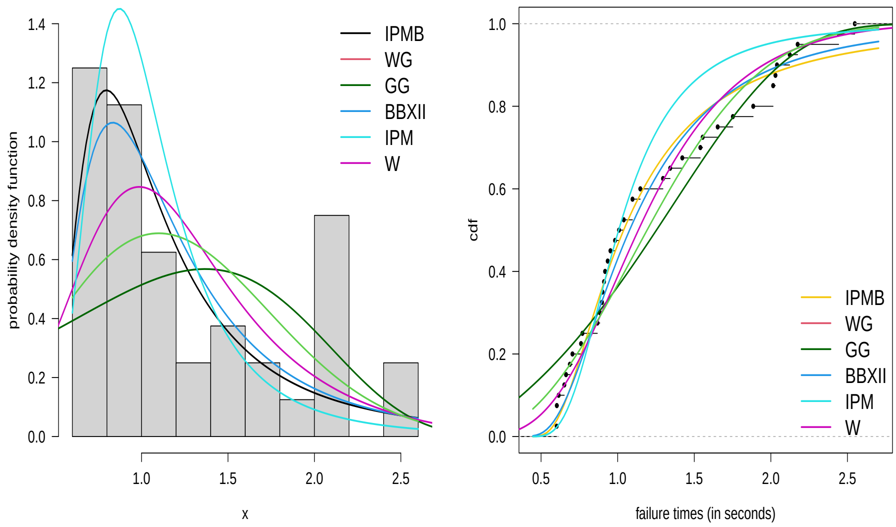
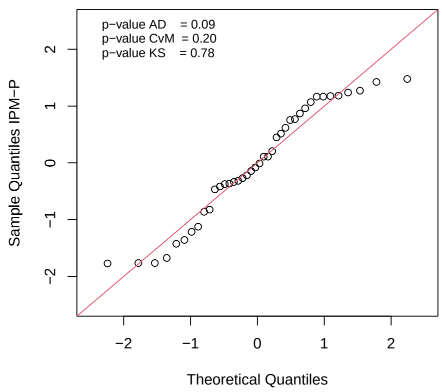
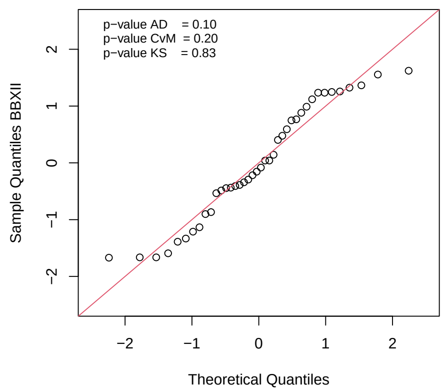


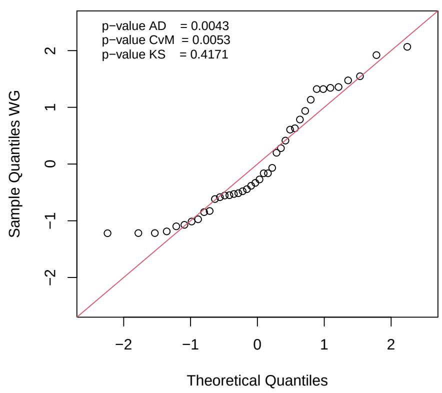
Figure 6. PDF of IPM-P, WG, GP, GG, IPM, and W models for failure time dataset (**left panel**) and the empirical CDF versus the estimated CDF for the same models (**right panel**).



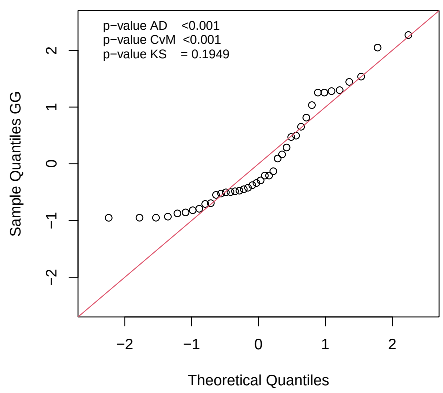
(**a**) IPM-P



(**b**) BBXII

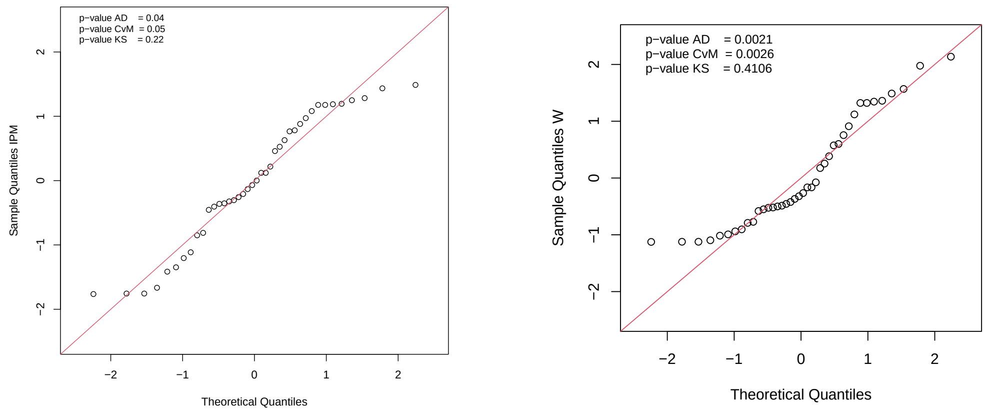


(**c**) WG



(**d**) GG

Figure 7. *Cont*.



(**e**) IPM (**f**) W

Figure 7. QQ plot of the QR for IPM-P, BBXII, WG, GG, IPM, and W distributions for the failure time dataset.

6. Conclusions

This paper has introduced the IPM-PS distribution, which is a composition between the IPM and PS models. We have also considered the Bell distribution, which has not been used as much in the literature as the Poisson, geometric or logarithmic distributions in the same context. Some of its most relevant properties were also studied, such as the PDF, the survival and hazard functions, and the moments. Parameter estimation was performed by the ML method using the EM algorithm. We highlight the implementation of the EM algorithm to perform the parameter estimation for the model, because it provides robustness in obtaining the maximum-likelihood estimators. We also emphasize the use of Oakes' method to compute the hessian matrix of the model, because it provides an exact method and not an approximation of said matrix, as it is usually used. In addition, a simulation study was carried out showing the good statistical properties of these estimators in finite samples. As applications, two real datasets on the failure of an article and the measurement of BMI were considered. Several common models that are PS-based, such as the WPS and GPS, were also considered. Applying traditional information criteria, we have shown that our proposal provides a suitable alternative to model positive data. In addition to the possible interpretability of the model in terms of a competitive risk scheme, the model proved to be more useful in terms of precision compared to other usual alternatives in the literature. Therefore, it is of interest to disseminate the use and applications of this model. Further research could take into account the competitive risk structure of the IPM-PS model. For instance, by adding the case $M = 0$ in Equation (1), a new cure rate model could be proposed similar to the recent work of Vásquez et al. [33] and Azimi et al. [34]. In such a case, the distribution of susceptible individuals is given by the IPM-PS distribution. An alternative way to extend this proposal would be to modify the PS distribution for M in Equation (1), e.g., using the COM-Poisson [35] or the modified PS model [36], which is a generalization of the PS class of distributions.

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Informed Consent Statement: Not applicable.

Data Availability Statement: The data used in Section 5 were duly referenced.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Additional Plots

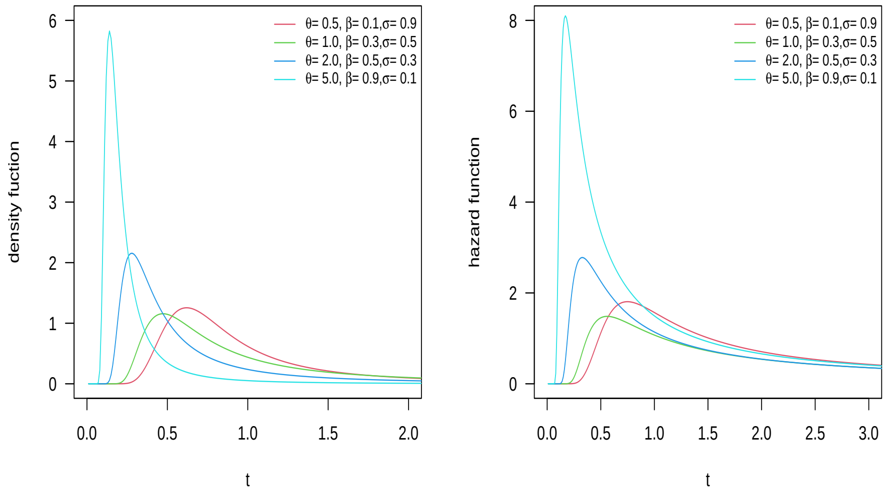

Figure A1. PDF and hazard function for the IPM-G distribution, with $\gamma = 1$.

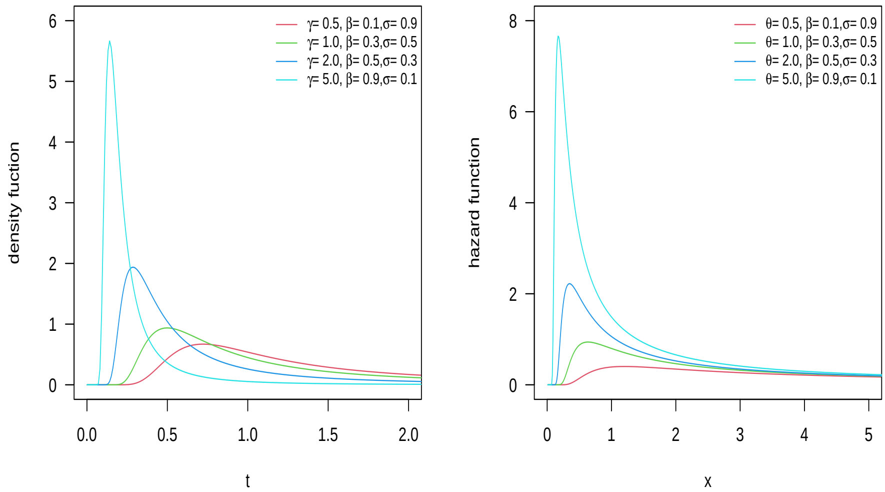

Figure A2. PDF and hazard function for the IPM-L distribution, with $\theta = 1$.

Appendix B. Explicit form for the Integral in $u(r)$ of *r-th Moment* of the IPM- PS

$$u(r)=\begin{cases}\int_{1/\beta}^{1/\xi-1}\exp\left(\sigma\left(G\left(\left(\frac{log(\beta y)^{-\frac{1}{\gamma}}}{\theta}\right);\beta,\gamma,\theta\right)\right)\right)[(-1)^{\xi}\log(y\beta)]^{-r/\gamma}(y-1)\exp(-y)dy, & \text{for} \quad M\sim \text{Poisson},\\ \int_{1/\beta}^{1/\xi-1}\frac{1}{\left(-(\sigma(G((\frac{log(\beta y)^{-\frac{1}{\gamma}}}{\theta});\beta,\gamma,\theta)))+1\right)^2}[(-1)^{\xi}\log(y\beta)]^{-r/\gamma}(y-1)\exp(-y)dy, & \text{for} \quad M\sim \text{geometric},\\ \int_{1/\beta}^{1/\xi-1}\left[\left(e^{\left(\sigma\left(G\left(\left(\frac{log(\beta y)^{-\frac{1}{\gamma}}}{\theta}\right);\beta,\gamma,\theta\right)\right)\right)+\exp\left(\sigma\left(G\left(\left(\frac{log(\beta y)^{-\frac{1}{\gamma}}}{\theta}\right);\beta,\gamma,\theta\right)\right)\right)-1}\right)\right. & \\ \left.[(-1)^{\xi}\log(y\beta)]^{-r/\gamma}(y-1)\exp(-y)\right]dy, & \text{for} \quad M\sim \text{Bell},\\ \beta\exp\left(\frac{1}{\beta}\right)\theta^{-r}[(-1)^{\xi}\beta]^{r/\gamma} & \\ \cdot\int_{1/\beta}^{1/\xi-1}\frac{1}{1-\left(\sigma\left(G\left(\left(\frac{log(\beta y)^{-\frac{1}{\gamma}}}{\theta}\right);\beta,\gamma,\theta\right)\right)\right)}[(-1)^{\xi}\log(y\beta)]^{-r/\gamma}(y-1)\exp(-y)dy, & \text{for} \quad M\sim \text{logarithmic}.\end{cases}$$

Such integrals should be computed numerically. For instance, in R [28] using the `integrate` function.

Appendix C. Explicit form for the Integral in Equation (9) for the Shannon Entropy

$$\int_0^{\sigma}A'(u)\log(A'(u))du=\begin{cases}\sigma e^{\sigma}-e^{\sigma}+1, & \text{for} \quad M\sim \text{Poisson},\\ \frac{-2\log(1-\sigma)-2}{1-\sigma}+2, & \text{for} \quad M\sim \text{geometric},\\ -\frac{1}{e}\left[Ei(\sigma)-Ei(0)\right]+e^{e^{\sigma}-1}\left(e^{\sigma}+\sigma-2\right)-1 & \text{for} \quad M\sim \text{Bell},\\ \frac{\log^2(-\sigma+1)}{2}, & \text{for} \quad M\sim \text{logarithmic}.\end{cases}$$

where $Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$ is exponential integral function.

Appendix D. Inverse and Derivatives of $A(\cdot)$

$$A'(\sigma)=\begin{cases}e^{\sigma}, & \text{if} \quad M_i\sim \text{Poisson},\\ \frac{1}{(-\sigma+1)^2}, & \text{if} \quad M_i\sim \text{geometric},\\ e^{\sigma+e^{\sigma}-1}, & \text{if} \quad M_i\sim \text{Bell},\\ \frac{1}{1-\sigma}, & \text{if} \quad M_i\sim \text{logarithmic}.\end{cases}$$

$$A''(\sigma)=\begin{cases} e^{\sigma}, & \text{if} \quad M_i \sim \text{Poisson}, \\ \frac{2}{(-\sigma+1)^3}, & \text{if} \quad M_i \sim \text{geometric}, \\ e^{\sigma+e^{\sigma}-1}(e^{\sigma}+1), & \text{if} \quad M_i \sim \text{Bell}, \\ \frac{1}{(1-\sigma)^2}, & \text{if} \quad M_i \sim \text{logarithmic}. \end{cases}$$

$$A^{-1}(\sigma)=\begin{cases} \log(\sigma+1), & \text{if} \quad M_i \sim \text{Poisson}, \\ \sigma(1+\sigma)^{-1}, & \text{if} \quad M_i \sim \text{geometric}, \\ \log(\log(\sigma+1)+1)-1, & \text{if} \quad M_i \sim \text{Bell}, \\ 1-e^{-\sigma}, & \text{if} \quad M_i \sim \text{logarithmic}. \end{cases}$$

Appendix E. Oakes' Method for the IMP-PS Distribution

The standard errors can be calculated as the square root of the diagonal elements of the inverse of the Fisher information matrix, i.e.,

$$SE(\hat{\zeta})=\sqrt{diag(I(\hat{\zeta})^{-1})},$$

where the Fisher information matrix is given by $I(\zeta)^{-1}=-\frac{\partial^2 l(\zeta)}{\partial\zeta\partial\zeta^T}$, l is the observed log-likelihood function and $\zeta=(\beta,\gamma,\theta,\sigma)$.

Then, the formula for calculating the standard errors by the Oakes method [23] is defined as

$$-\frac{\partial^2 l(\zeta)}{\partial\zeta\partial\zeta^T}\bigg|_{\zeta=\hat{\zeta}}=-\left[\frac{\partial^2 Q(\zeta;\zeta^k)}{\partial\zeta\partial\zeta^T}+\frac{\partial^2 Q(\zeta;\zeta^k)}{\partial\zeta\partial\zeta^{(k)T}}\right]\bigg|_{\zeta^{(k)}=\hat{\zeta}},$$

where Q is the Q-function.

In this way, with the estimation of the parameters, they are substituted in the respective derivatives to obtain the Fisher information matrix to obtain the standard errors. The following are the calculations of the derivatives necessary to find this matrix.

Appendix E.1. Double Derivatives of the Function Q in Relation to the Parameters ζ

$$\frac{\partial^2 Q}{\partial^2\sigma}=\frac{n(A'(\sigma))^2}{A^2(\sigma)}-\frac{n(A''(\sigma))}{A(\sigma))}-\sum_{i=1}^{n}\frac{M_i}{\sigma^2},$$

$$\frac{\partial^2 Q}{\partial\sigma\partial\beta}=\frac{\partial^2 Q}{\partial\sigma\partial\gamma}=\frac{\partial^2 Q}{\partial\sigma\partial\theta}=0,$$

$$\frac{\partial^2 Q}{\partial\beta\partial\sigma}=\frac{\partial^2 Q}{\partial\gamma\partial\sigma}=\frac{\partial^2 Q}{\partial\theta\partial\sigma}=0,$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial^2\beta} =& \sum_{i=1}^{n}\Bigg(\frac{e^{\beta(t_i\theta)^{-\gamma}}\beta t_i^{-\gamma}\theta^{-\gamma} - 2\Big(e^{\beta(t_i\theta)^{-\gamma}}-1\Big)}{\beta^3} - \frac{\Big(e^{\beta(\beta t_i)^{-\gamma}}\beta(\theta t_i)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}}\Big)(\theta t_i)^{-\gamma}}{\beta^2} + \\
&\frac{-e^{\beta(t_i\theta)^{-\gamma}}\beta t_i^{-2\gamma}\theta^{-2\gamma} + 2e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}-1}{\Big(e^{\beta(t_i\theta)^{-\gamma}}-\beta\Big)^2} + \\
&\Bigg(\frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2}\Bigg)\Bigg(\Bigg(e^{\beta(\theta t_i)^{-\gamma}-\frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}}\Bigg((\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}\beta(\theta t_i)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}}+1}{\beta^2}\Bigg)\Bigg) \\
&\Bigg((t_i\theta)^{-\gamma} + \frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta^2} - \frac{e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}}{\beta}\Bigg) + e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)} \\
&\Bigg(\frac{e^{\beta(\theta t_i)^{-\gamma}}\beta\theta^{-\gamma}t_i^{-\gamma} - 2\Big(e^{\beta(\theta t_i)^{-\gamma}}-1\Big)}{\beta^3} - \frac{\Big(e^{\beta(\theta t_i)^{-\gamma}}\beta(\theta t_i)^{-\gamma}-e^{\beta(\theta t_i)^{-\gamma}}\Big)(\theta t_i)^{-\gamma}}{\beta^2}\Bigg)\Bigg)(1-G(t_i,\beta,\gamma,\theta)) \\
&-\Bigg(e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\Big((t_i\theta)^{-\gamma} + \frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta^2} - \frac{e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}}{\beta}\Big)\Bigg)\Bigg(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\beta}\Bigg)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial\beta\partial\gamma} =& \sum_{i=1}^{n}\Bigg(-\log(\theta t_i)(\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)(\theta t_i)^{-\gamma}}{\beta} \\
&-\frac{-e^{\beta(\theta t_i)^{-\gamma}}\beta\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)(\theta t_i)^{-\gamma}}{\beta} \\
&+\frac{e^{\beta(\theta t_i)^{-\gamma}}\beta^2\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} + e^{2\beta(\theta t_i)^{-\gamma}}\beta\theta^{-2\gamma}\log(\theta t_i)t_i^{-2\gamma} - e^{2\beta(\theta t_i)^{-\gamma}}\beta\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} - e^{2\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)(\theta t_i)^{-\gamma}}{\Big(e^{\beta(\theta t_i)^{-\gamma}}-\beta\Big)^2} \\
&+\frac{(1+M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2}\Big(\Big(e^{\beta(\theta t_i)^{-\gamma}-\frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}}\Big(-\beta\log(\theta t_i)(\theta t_i)^{-\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)(\theta t_i)^{-\gamma}\Big)\Big) \\
&\Big((t_i\theta)^{-\gamma} + \frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta^2} - \frac{e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}}{\beta}\Big) \\
&+\big(e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\big(-\log(\theta t_i)(\theta t_i)^{-\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma}\big)\big)\Big)(1-G(t_i,\beta,\gamma,\theta)) \\
&-\Bigg(e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\Big((t_i\theta)^{-\gamma} + \frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta^2} - \frac{e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}}{\beta}\Big)\Bigg)\Big(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\gamma}\Big)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial\beta\partial\theta} =& \sum_{i=1}^{n}\Bigg(-\gamma\theta^{-\gamma-1}t_i^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}\gamma\theta^{-\gamma-1}t_i^{-\gamma}}{\beta} - \frac{-e^{\beta(\theta t_i)^{-\gamma}}\beta\gamma\theta^{-2\gamma-1}t_i^{-2\gamma} - e^{\beta(\theta t_i)^{-\gamma}}\gamma\theta^{-\gamma-1}t_i^{-\gamma}}{\beta} \\
&+\frac{e^{\beta(\theta t_i)^{-\gamma}}\beta^2\gamma\theta^{-2\gamma-1}t_i^{-2\gamma} - e^{2\beta(\theta t_i)^{-\gamma}}\gamma\theta^{-\gamma-1}t_i^{-\gamma}}{\Big(e^{\beta(\theta t_i)^{-\gamma}}-\beta\Big)^2} \\
&+\frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2}\Big(\big(\big(e^{\beta(\theta t_i)^{-\gamma}-\frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}}\Big(-\beta\gamma\theta^{-\gamma-1}t_i^{-\gamma} + \gamma e^{\beta(\theta t_i)^{-\gamma}}\theta^{-\gamma-1}t_i^{-\gamma}\Big)\big) \\
&\big((t_i\theta)^{-\gamma} + \frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta^2} - \frac{e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}}{\beta}\big) + \big(e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\big)\big(-\gamma\theta^{-\gamma-1}t_i^{-\gamma} \\
&+e^{\beta(\theta t_i)^{-\gamma}}\gamma\theta^{-2\gamma-1}t_i^{-2\gamma}\big)\big)(1-G(t_i,\beta,\gamma,\theta)) \\
&-\big(e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}(t_i\theta)^{-\gamma} + \frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta^2} - \frac{e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}}{\beta}\big)\big(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\gamma}\big)\big)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial\gamma\partial\beta} = \sum_{i=1}^{n} \Bigg(\Big(& -\log(\theta t_i)(\theta t_i)^{-\gamma} \Big) + e^{\beta(\theta t_i)^{-\gamma}} \log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} \\
& + \Big(\frac{e^{2\beta(\theta t_i)^{-\gamma}}\big(\theta^2 t_i^2\big)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}}\beta\big(\theta^2 t_i^2\big)^{-\gamma} - e^{2\beta(\theta t_i)^{-\gamma}}\theta^{-2\gamma} t_i^{-2\gamma} + 2e^{\beta(\theta t_i)^{-\gamma}}(\theta t_i)^{-\gamma} - 1}{\Big(e^{\beta(\theta t_i)^{-\gamma}} - \beta\Big)^2} \\
& + \frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2} \Bigg(\Big(e^{\beta(\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}} \Bigg((\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}\beta(\theta t_i)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}} + 1}{\beta^2} \Bigg) \Big) \\
& \big(\log(t_i\theta)(t_i\theta)^{-\gamma}(e^{\beta(t_i\theta)^{-\gamma}} - \beta)\big) + e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)} \big(\log(t_i\theta)(t_i\theta)^{-\gamma} \Big(e^{\beta(t_i\theta)^{-\gamma}}(\theta t_i)^{-\gamma} - 1 \Big)\big) \Bigg) \\
& (1-G(t_i,\beta,\gamma,\theta)) - e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)} \big(\log(t_i\theta)(t_i\theta)^{-\gamma}(e^{\beta(t_i\theta)^{-\gamma}} - \beta)\big)\Big(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\beta} \Big) \Bigg) \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial^2\gamma} = & -\frac{n}{\gamma^2} + \sum_{i=1}^{n} \Bigg(\beta\log(\theta t_i)\log(\theta t_i)(\theta t_i)^{-\gamma} + \\
& \log(\theta t_i)\Big(-e^{\beta(\theta t_i)^{-\gamma}}\beta\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}}\log(\gamma t_i)(\theta t_i)^{-\gamma} \Big) \\
& + \frac{e^{\beta(\theta t_i)^{-\gamma}}\beta^2\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} + e^{2\beta(\theta t_i)^{-\gamma}}\beta\theta^{-2\gamma}\log(\theta t_i)t_i^{-2\gamma} - e^{2\beta(\theta t_i)^{-\gamma}}\beta\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} - e^{2\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)(\theta t_i)^{-\gamma}}{\Big(e^{\beta(\theta t_i)^{-\gamma}} - \beta\Big)^2} \\
& + \frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2} \Bigg(\Bigg(\Bigg(e^{\beta(\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}} \Big(-\beta\log(\theta t_i)(\theta t_i)^{-\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)(\theta t_i)^{-\gamma} \Big) \Bigg) \\
& \Big(\log(t_i\theta)(t_i\theta)^{-\gamma}(e^{\beta(t_i\theta)^{-\gamma}} - \beta)\Big) + \Big(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)} \Big) \\
& \Bigg(\log(\theta t_i)\Bigg(-\log(\theta t_i)\Big(e^{\beta(\theta t_i)^{-\gamma}} - \beta \Big)(\theta t_i)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}}\beta\log(\theta t_i)\big(\theta^2 t_i^2\big)^{-\gamma} \Bigg) \Bigg) \Bigg) (1-G(t_i,\beta,\gamma,\theta)) \\
& - \Big(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)} \Big) \Big(\big(\log(t_i\theta)(t_i\theta)^{-\gamma}(e^{\beta(t_i\theta)^{-\gamma}} - \beta)\big)\Big(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\gamma} \Big) \Big) \Bigg) \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial\gamma\partial\theta} = & -\frac{1}{\theta} + \sum_{i=1}^{n} \Bigg(-\beta\Big(\theta^{-\gamma-1}t_i^{-\gamma} - \gamma\theta^{-\gamma-1}\log(\theta t_i)t_i^{-\gamma} \Big) \\
& - e^{\beta(\theta t_i)^{-\gamma}}\beta\gamma\theta^{-2\gamma-1}\log(\theta t_i)t_i^{-2\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\Big(\theta^{-\gamma-1}t_i^{-\gamma} - \gamma\theta^{-\gamma-1}\log(\theta t_i)t_i^{-\gamma} \Big) \\
& + \frac{e^{\beta(\theta t_i)^{-\gamma}}\beta^2\gamma\theta^{-2\gamma-1}t_i^{-2\gamma} - e^{2\beta(\theta t_i)^{-\gamma}}\gamma\theta^{-\gamma-1}t_i^{-\gamma}}{\Big(e^{\beta(\theta t_i)^{-\gamma}} - \beta\Big)^2} \\
& + \frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2} \Bigg(\Big(\Big(e^{\beta(\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}} \Big(-\beta\gamma\theta^{-\gamma-1}t_i^{-\gamma} + \gamma e^{\beta(\theta t_i)^{-\gamma}}\theta^{-\gamma-1}t_i^{-\gamma} \Big) \Big) \\
& \big(\log(t_i\theta)(t_i\theta)^{-\gamma}(e^{\beta(t_i\theta)^{-\gamma}} - \beta)\big) + \big(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)} \big)\big(\theta^{-\gamma-1}t_i^{-\gamma}\Big(e^{\beta(\theta t_i)^{-\gamma}} - \beta \Big) + \\
& \log(\theta t_i)\Big(-\gamma\theta^{-\gamma-1}t_i^{-\gamma}\Big(e^{\beta(\theta t_i)^{-\gamma}} - \beta \Big) - e^{\beta(\theta t_i)^{-\gamma}}\gamma\beta\theta^{-2\gamma-1}t_i^{-2\gamma} \Big) \big) \Big) \\
& (1-G(t_i,\beta,\gamma,\theta) - \big(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)} \big(\log(t_i\theta)(t_i\theta)^{-\gamma}(e^{\beta(t_i\theta)^{-\gamma}} - \beta)\big)\big)\Big(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\theta} \Big) \Bigg) \Bigg),
\end{aligned}
$$

$$\frac{\partial^2 Q}{\partial\theta\partial\beta} = \sum_{i=1}^{n}\Bigg(-\gamma\theta^{-\gamma-1}t_i^{-\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\gamma\theta^{-2\gamma-1}t_i^{-2\gamma} - \frac{\gamma\theta^{-\gamma-1}t_i^{-\gamma}\left(-e^{\beta(\theta t_i)^{-\gamma}}\beta(\theta t_i)^{-\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\right)}{\left(e^{\beta(\theta t_i)^{-\gamma}} - \beta\right)^2}$$

$$+ \frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2}\Bigg(\Bigg(\Bigg(e^{\beta(\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}}\left((\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}\beta(\theta t_i)^{-\gamma} - e^{\beta(\theta t_i)^{-\gamma}} + 1}{\beta^2}\right)\Bigg)$$

$$\left(\gamma(t_i\theta)^{-\gamma-1}(\beta + e^{\beta(t_i\theta)^{-\gamma}})\right) + \left(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\right)\left(\gamma(t_i\theta)^{-\gamma-1}\left(1 + e^{\beta(t_i\theta)^{-\gamma}}(\theta t_i)^{-\gamma}\right)\right)\Bigg)$$

$$\left(1 - G(t_i,\beta,\gamma,\theta)\right) - \left(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\gamma(t_i\theta)^{-\gamma-1}(\beta + e^{\beta(t_i\theta)^{-\gamma}})\right)\left(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\beta}\right)\Bigg)\Bigg),$$

$$\frac{\partial^2 Q}{\partial\theta\partial\gamma} = -\frac{n}{\theta} + \sum_{i=1}^{n}\Bigg(-\beta\left(\theta^{-\gamma-1}t_i^{-\gamma} + \gamma\left(-\theta^{-\gamma-1}\log(\theta)t_i^{-\gamma} - \theta^{-\gamma-1}\log(t_i)t_i^{-\gamma}\right)\right)$$

$$- e^{\beta(\theta t_i)^{-\gamma}}\beta\gamma\theta^{-2\gamma-1}\log(\theta t_i)t_i^{-2\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\left(\theta^{-\gamma-1}t_i^{-\gamma} + \gamma\left(-\theta^{-\gamma-1}\log(\theta)t_i^{-\gamma} - \theta^{-\gamma-1}\log(t_i)t_i^{-\gamma}\right)\right)$$

$$- \frac{\beta\left(\left(\theta^{-\gamma-1}t_i^{-\gamma} + \gamma\left(-\theta^{-\gamma-1}\log(\theta)t_i^{-\gamma} - \theta^{-\gamma-1}\log(t_i)t_i^{-\gamma}\right)\right)\left(e^{\beta(\theta t_i)^{-\gamma}} - \beta\right) + e^{\beta(\theta t_i)^{-\gamma}}\gamma\beta\theta^{-2\gamma-1}\log(\theta t_i)t_i^{-2\gamma}\right)}{\left(e^{\beta(\theta t_i)^{-\gamma}} - \beta\right)^2}$$

$$+ \frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2}\Bigg(\Bigg(\Bigg(e^{\beta(\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}}\left(-\beta\log(\theta t_i)(\theta t_i)^{-\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\log(\theta t_i)(\theta t_i)^{-\gamma}\right)\Bigg)$$

$$\left(\gamma(t_i\theta)^{-\gamma-1}(\beta + e^{\beta(t_i\theta)^{-\gamma}})\right)$$

$$- \left(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\right)$$

$$\left(\left(\beta + e^{\beta(\theta t_i)^{-\gamma}}\right)(\theta t_i)^{-\gamma-1} + \gamma\left(-\log(\theta t_i)\left(\beta + e^{\beta(\theta t_i)^{-\gamma}}\right)(\theta t_i)^{-\gamma-1} - e^{\beta(\theta t_i)^{-\gamma}}\beta\log(\theta t_i)(\theta t_i)^{-2\gamma-1}\right)\right)\Bigg)$$

$$(1 - G(t_i,\beta,\gamma,\theta)) - \left(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\gamma(t_i\theta)^{-\gamma-1}(\beta + e^{\beta(t_i\theta)^{-\gamma}})\right)\left(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\gamma}\right)\Bigg)\Bigg),$$

$$\frac{\partial^2 Q}{\partial^2\theta} = -\frac{n}{\theta^2} + \frac{n\gamma}{\theta^2} + \sum_{i=1}^{n}\Bigg(\gamma\beta\theta^{-\gamma-2}t_i^{-\gamma}(-\gamma-1) + \gamma t_i^{-\gamma}\left(-e^{\beta(\theta t_i)^{-\gamma}}\beta\gamma\theta^{-2\gamma-2}t_i^{-\gamma} + e^{\beta(\theta t_i)^{-\gamma}}\theta^{-\gamma-2}(-\gamma-1)\right)$$

$$- \frac{\gamma\beta t_i^{-\gamma}\left(\theta^{-\gamma-2}(-\gamma-1)\left(e^{\beta(\theta t_i)^{-\gamma}} - \beta\right) + e^{\beta(\theta t_i)^{-\gamma}}\gamma\beta\theta^{-2\gamma-2}t_i^{-\gamma}\right)}{\left(e^{\beta(\theta t_i)^{-\gamma}} - \beta\right)^2}$$

$$+ \frac{(1-M_i)}{(1-G(t_i,\beta,\gamma,\theta))^2}\Bigg(\Bigg(\Bigg(e^{\beta(\theta t_i)^{-\gamma} - \frac{e^{\beta(\theta t_i)^{-\gamma}}-1}{\beta}}\left(-\beta\gamma\theta^{-\gamma-1}t_i^{-\gamma} + \gamma e^{\beta(\theta t_i)^{-\gamma}}\theta^{-\gamma-1}t_i^{-\gamma}\right)\Bigg)$$

$$\left(\gamma(t_i\theta)^{-\gamma-1}(\beta + e^{\beta(t_i\theta)^{-\gamma}})\right) + \left(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\right)$$

$$\left(\gamma\left(\theta^{-\gamma-2}t_i^{-\gamma-1}(-\gamma-1)\left(\beta + e^{\beta(\theta t_i)^{-\gamma}}\right) - e^{\beta(\theta t_i)^{-\gamma}}\gamma\beta\theta^{2(-\gamma-1)}t_i^{-2\gamma-1}\right)\right)\Bigg)$$

$$(1 - G(t_i,\beta,\gamma,\theta)) - \left(e^{\beta(t_i\theta)^{-\gamma} - \frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\gamma(t_i\theta)^{-\gamma-1}(\beta + e^{\beta(t_i\theta)^{-\gamma}})\right)\left(-\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\theta}\right)\Bigg)\Bigg),$$

where

$$\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\beta} = e^{\beta(t_i\theta)^{-\gamma} - \frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta}}\left((t_i\theta)^{-\gamma} - \frac{e^{\beta(t_i\theta)^{-\gamma}}\beta(t_i\theta)^{-\gamma} - e^{\beta(t_i\theta)^{-\gamma}} + 1}{\beta^2}\right),$$

$$\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\gamma} = e^{\beta(t\theta)^{-\gamma} - \frac{e^{\beta(t\theta)^{-\gamma}}-1}{\beta}}\left(-\beta\log(t\theta)(t\theta)^{-\gamma} + e^{\beta(t\theta)^{-\gamma}}\log(t\theta)(t\theta)^{-\gamma}\right),$$

$$\frac{\partial G(t_i,\beta,\gamma,\theta)}{\partial\theta} = e^{\beta(t\theta)^{-\gamma}-\frac{e^{\beta(t\theta)^{-\gamma}}-1}{\beta}}\left(-\beta\gamma t^{-\gamma}\theta^{-\gamma-1} + e^{\beta(t\theta)^{-\gamma}}\gamma t^{-\gamma}\theta^{-\gamma-1}\right).$$

Appendix E.2. Double Derivatives of Function Q in Relation to Parameters and Expectation Mi

$$\frac{\partial^2 Q}{\partial\sigma\partial\psi^{(k)}} = \sum_{i=1}^{n}\frac{\frac{\partial M_i}{\partial\psi^{(k)}}}{\sigma},$$

$$\frac{\partial^2 Q}{\partial\beta\partial\psi^{(k)}} = \sum_{i=1}^{n}(-\frac{\partial M_i}{\partial\psi^{(k)}})\left(\frac{e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\left((t_i\theta)^{-\gamma}+\frac{e^{\beta(t_i\theta)^{-\gamma}}-1}{\beta^2}-\frac{e^{\beta(t_i\theta)^{-\gamma}}(t_i\theta)^{-\gamma}}{\beta}\right)}{1-G(t_i,\beta,\gamma,\theta)}\right),$$

$$\frac{\partial^2 Q}{\partial\gamma\partial\psi^{(k)}} = \sum_{i=1}^{n}(-\frac{\partial M_i}{\partial\psi^{(k)}})\left(\frac{e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\left(\log(t_i\theta)(t_i\theta)^{-\gamma}(e^{\beta(t_i\theta)^{-\gamma}}-\beta)\right)}{1-G(t_i,\beta,\gamma,\theta)}\right),$$

$$\frac{\partial^2 Q}{\partial\theta\partial\psi^{(k)}} = \sum_{i=1}^{n}(-\frac{\partial M_i}{\partial\psi^{(k)}})\left(\frac{e^{\beta(t_i\theta)^{-\gamma}-\frac{1}{\beta}(e^{\beta(t_i\theta)^{-\gamma}}-1)}\gamma(t_i\theta)^{-\gamma-1}(\beta+e^{\beta(t_i\theta)^{-\gamma}})}{1-G(t_i,\beta,\gamma,\theta)}\right),$$

where

$$\frac{\partial M_i}{\partial\psi^{(k)}} = \frac{\partial M_i}{\partial v_{ik}}\frac{\partial v_{ik}}{\partial\psi^{(k)}},$$

with $\psi = \{\beta,\gamma\ \ o\ \ \theta\}$. Such derivatives also will be specified.

(Geometric)

$$\frac{\partial M_i}{\partial\sigma^{(k)}} = \frac{2}{(1-\sigma P(t_i;\beta,\gamma,\theta))^2}P(t_i;\beta,\gamma,\theta),$$
$$\frac{\partial M_i}{\partial\beta^{(k)}} = \frac{2}{(1-\sigma P(t_i;\beta,\gamma,\theta))^2}\sigma P'_\beta(t;\beta,\gamma,\theta),$$
$$\frac{\partial M_i}{\partial\gamma^{(k)}} = \frac{2}{(1-\sigma P(t_i;\beta,\gamma,\theta))^2}\sigma P'_\gamma(t;\beta,\gamma,\theta),$$
$$\frac{\partial M_i}{\partial\theta^{(k)}} = \frac{2}{(1-\sigma P(t_i;\beta,\gamma,\theta))^2}\sigma P'_\theta(t;\beta,\gamma,\theta).$$

(Poisson)

$$\frac{\partial M_i}{\partial\sigma^{(k)}} = \frac{-e^{-\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta)-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1}{\left(-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1\right)^2}P(t_i;\beta,\gamma,\theta),$$
$$\frac{\partial M_i}{\partial\beta^{(k)}} = \frac{-e^{-\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta)-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1}{\left(-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1\right)^2}\sigma P'_\beta(t;\beta,\gamma,\theta),$$
$$\frac{\partial M_i}{\partial\gamma^{(k)}} = \frac{-e^{-\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta)-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1}{\left(-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1\right)^2}\sigma P'_\gamma(t;\beta,\gamma,\theta),$$
$$\frac{\partial M_i}{\partial\theta^{(k)}} = \frac{-e^{-\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta)-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1}{\left(-e^{-\sigma P(t_i;\beta,\gamma,\theta)}+1\right)^2}\sigma P'_\theta(t;\beta,\gamma,\theta).$$

(Bell)

$$\frac{\partial M_i}{\partial \sigma^{(k)}} = \left(e^{\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta) + e^{\sigma P(t_i;\beta,\gamma,\theta)} + 1\right)P(t_i;\beta,\gamma,\theta),$$

$$\frac{\partial M_i}{\partial \beta^{(k)}} = \left(e^{\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta) + e^{\sigma P(t_i;\beta,\gamma,\theta)} + 1\right)\sigma P'_\beta(t;\beta,\gamma,\theta),$$

$$\frac{\partial M_i}{\partial \gamma^{(k)}} = \left(e^{\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta) + e^{\sigma P(t_i;\beta,\gamma,\theta)} + 1\right)\sigma P'_\gamma(t;\beta,\gamma,\theta),$$

$$\frac{\partial M_i}{\partial \theta^{(k)}} = \left(e^{\sigma P(t_i;\beta,\gamma,\theta)}\sigma P(t_i;\beta,\gamma,\theta) + e^{\sigma P(t_i;\beta,\gamma,\theta)} + 1\right)\sigma P'_\theta(t;\beta,\gamma,\theta).$$

(Logarithmic)

$$\frac{\partial M_i}{\partial \sigma^{(k)}} = P(t_i;\beta,\gamma,\theta),$$

$$\frac{\partial M_i}{\partial \beta^{(k)}} = \sigma P'_\beta(t;\beta,\gamma,\theta),$$

$$\frac{\partial M_i}{\partial \gamma^{(k)}} = \sigma P'_\gamma(t;\beta,\gamma,\theta),$$

$$\frac{\partial M_i}{\partial \theta^{(k)}} = \sigma P'_\theta(t;\beta,\gamma,\theta),$$

where

$$P'_\beta(t;\beta,\gamma,\theta) = -e^{\beta(t\theta)^{-\gamma} - \frac{e^{\beta(t\theta)^{-\gamma}}-1}{\beta}}\left((t\theta)^{-\gamma} - \frac{e^{\beta(t\theta)^{-\gamma}}\beta(t\theta)^{-\gamma} - e^{\beta(t\theta)^{-\gamma}} + 1}{\beta^2}\right),$$

$$P'_\gamma(t;\beta,\gamma,\theta) = -e^{\beta(t\theta)^{-\gamma} - \frac{e^{\beta(t\theta)^{-\gamma}}-1}{\beta}}\left(-\beta\log(t\theta)(t\theta)^{-\gamma} + e^{\beta(t\theta)^{-\gamma}}\log(t\theta)(t\theta)^{-\gamma}\right),$$

$$P'_\theta(t;\beta,\gamma,\theta) = -e^{\beta(t\theta)^{-\gamma} - \frac{e^{\beta(t\theta)^{-\gamma}}-1}{\beta}}\left(e^{\beta(t\theta)^{-\gamma}}\gamma t^{-\gamma}\theta^{-\gamma-1} - \beta\gamma t^{-\gamma}\theta^{-\gamma-1}\right).$$

Appendix F. PDF of Distributions Used in Applications

The PDF of the Gompertz power series distribution is given by

$$f(x;\beta,\gamma,\theta) = \theta\beta e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\frac{A'(\theta e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})}{A(\theta)}$$

where $x > 0$, $\beta > 0$ and $\gamma > 0$.

The PDF of the Weibull power series distribution is given by

$$f(x;\beta,\alpha,\theta) = \alpha\theta\beta^\alpha x^{\alpha-1}e^{-(\beta x)^\alpha}\frac{A'(\theta e^{-(\beta x)^\alpha})}{A(\theta)},$$

where $x > 0$, $\beta > 0$ and $\alpha > 0$.

The PDF of the Beta Burr XII distribution is given by

$$f(x) = \frac{ckx^{c-1}}{s^c B(a,b)}\left[1 + (x/s)^c\right]^{-(kb+1)}\left\{1 - \left[1 + (x/s)^c\right]^{-k}\right\}^{a-1}, \quad \text{(A1)}$$

where $B(a,b)$ is the beta function and $x > 0$, $a > 0$, $b > 0$, $k > 0$, $c > 0$ and $s > 0$.

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