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Fixed-Point Theorems for Nonlinear Contraction in Fuzzy-Controlled Bipolar Metric Spaces

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Abstract: In this paper, we introduce the concept of fuzzy-controlled bipolar metric space and prove some fixed-point theorems in this space. Our results generalize and expand some of the literature's well-known results. We also provide some applications of our main results to integral equations.

Keywords: fixed point; fuzzy-controlled bipolar metric space; fuzzy bipolar metric space; fuzzy metric space

MSC: 54H25; 47H10

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1. Introduction

A fuzzy set is a collection of objects with a continuum of grades of membership function that assigns to each object a value ranging between zero and one. In 1960, Schweizer and Sklar [1] introduced the concept of continuous triangular norm. In 1965, fuzzy set theory was scrutinized by Zadeh [2]. In 1975, Kramosil and Michálek [3] provided a basic introduction to the concept of fuzzy metric space, which is an extension of the statistical (probabilistic) metric space. This list provides the best foundation for the development of fixed-point theorem in fuzzy metric spaces. Afterward, in 1988, Grabiec [4] described the completion postulate of fuzzy metric space (now referred to as G-complete fuzzy metric space [5]). The result of the Banach contraction was then extended into G-complete fuzzy metric spaces. George and Veeramani [6] altered the definition of the Cauchy sequence instigated by Grabiec [4] because even R is not complete according to Grabiec's criterion of completion. Mutlu and Gurdal [7] introduced bipolar metric space as a kind of partial distance. We provide bipolar metric spaces, for the most part in the context of completeness, and prove some adjunctions of known fixed-point theorems. Bartwal et al. [8] initiated the definition of fuzzy bipolar metric space and proved some fixed-point theorems. In 2022, Tanusri Senapati, Ankush Chanda, and Vladimir Rakocevic [9] promoted the concept of weak orthogonal metric spaces as a generalization of orthogonal metric spaces.

Recently, Sezen [10] provided an idea regarded controlled fuzzy metric spaces and proved some related fixed-point results. Rakesh Tiwari and Shraddha Rajput [11] introduced the notion of bipolar-controlled fuzzy metric spaces. The above analysis shows that there are several works on fixed-point theory based on the previous two types of complete fuzzy metric space [12–20].

2. Preliminaries

Now, let us recall some basic definitions and lemmas that are used in this article. Schweizer and Sklar [1] introduced the notion of a continuous $\aleph$-norm as:

Definition 1. *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous $\mathfrak{J}$-norm(continuous triangular norm) such that*

1. $*$ *is commutative and associative;*
2. $*$ *is continuous;*
3. $\mathfrak{a}*1 = \mathfrak{a}$ *for every* $\mathfrak{a} \in [0,1]$;
4. $\mathfrak{a}*\mathfrak{b} \leq \mathfrak{c}*\mathfrak{d}$ *whenever* $\mathfrak{a} \leq \mathfrak{c}, \mathfrak{b} \leq \mathfrak{d}$ *and* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in [0,1]$.

Kramosil and Michalek [3] introduced the concept of fuzzy metric space as follows:

Definition 2 ([3])**.** *Let $\Psi \neq \emptyset$. The triplet $(\Psi, \Gamma, *)$ is called a fuzzy metric space (FMS) if a fuzzy set (F set) Γ is on $\Psi^2 \times (0, +\infty)$, and $*$ represents a continuous $\mathfrak{J}$-norm, such that $\forall\ \eta, \sigma, \omega \in \Psi$ and $\aleph, \mathfrak{s} > 0$;*

(i) $\Gamma(\eta, \sigma, \aleph) > 0$;
(ii) $\Gamma(\eta, \sigma, \aleph) = 1$ *iff* $\eta = \sigma$;
(iii) $\Gamma(\eta, \sigma, \aleph) = \Gamma(\sigma, \eta, \aleph)$;
(iv) $\Gamma(\eta, \omega, \aleph + \mathfrak{s}) \geq \Gamma(\eta, \sigma, \aleph)*\Gamma(\sigma, \omega, \mathfrak{s})$;
(v) $\Gamma(\eta, \sigma, .) : (0, +\infty) \longrightarrow (0, 1]$ *is continuous.*

The notion of a fuzzy bipolar metric space was introduced by A. Bartwal, R. C. Dimri and G. Prasad [8] as follows:

Definition 3 ([8])**.** *Let Θ and Ψ be two nonvoid sets. A quadruple $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ is called a fuzzy bipolar metric space (FBMS), where $*$ is a continuous $\mathfrak{J}$-norm and an F set $\Gamma_{\mathfrak{b}}$ is on $\Theta \times \Psi \times (0, +\infty)$, such that $\forall\ \aleph, \mathfrak{s}, \mathfrak{r} > 0$:*

(FB1) $\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) > 0$ *for all* $(\eta, \sigma) \in \Theta \times \Psi$;
(FB2) $\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) = 1$ *iff* $\eta = \sigma$ *for all* $\eta \in \Theta$ *and* $\sigma \in \Psi$;
(FB3) $\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) = \Gamma_{\mathfrak{b}}(\sigma, \eta, \aleph)$ *for all* $\eta, \sigma \in \Theta \cap \Psi$;
(FB4) $\Gamma_{\mathfrak{b}}(\eta_1, \sigma_2, \aleph + \mathfrak{s} + \mathfrak{r}) \geq \Gamma_{\mathfrak{b}}(\eta_1, \sigma_1, \aleph)*\Gamma_{\mathfrak{b}}(\eta_2, \sigma_1, \mathfrak{s})*\Gamma_{\mathfrak{b}}(\eta_2, \sigma_2, \mathfrak{r})$ *for all* $\eta_1, \eta_2 \in \Theta$ *and* $\sigma_1, \sigma_2 \in \Psi$;
(FB5) $\Gamma_{\mathfrak{b}}(\eta, \sigma, .) : \mathbb{R}_+ \to [0,1]$ *is left continuous;*
(FB6) $\Gamma_{\mathfrak{b}}(\eta, \sigma, .)$ *is nondecreasing for all* $\eta \in \Theta$ *and* $\sigma \in \Psi$.

Following this definition is an extended version of Definition 2 from fuzzy bipolar metric space to the fuzzy-controlled bipolar metric space setting.

Definition 4. *Let Θ and Ψ be two nonvoid sets and $\mu : \Theta \times \Psi \to [1, +\infty)$. A quadruplicate $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ is called a fuzzy-controlled bipolar metric space (FCBMS), where $*$ is a continuous $\mathfrak{J}$-norm and an F set $\Gamma_{\mathfrak{b}}$ is on $\Theta \times \Psi \times (0, +\infty)$, such that $\forall\ \aleph, \mathfrak{s}, \mathfrak{r} > 0$:*

(FCB1) $\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) > 0$ *for all* $(\eta, \sigma) \in \Theta \times \Psi$;
(FCB2) $\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) = 1$ *iff* $\eta = \sigma$ *for all* $\eta \in \Theta$ *and* $\sigma \in \Psi$;
(FCB3) $\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) = \Gamma_{\mathfrak{b}}(\sigma, \eta, \aleph)$ *for all* $\eta, \sigma \in \Theta \cap \Psi$;
(FCB4) $\Gamma_{\mathfrak{b}}(\eta_1, \sigma_2, \aleph + \mathfrak{s} + \mathfrak{r}) \geq \Gamma_{\mathfrak{b}}(\eta_1, \sigma_1, \frac{\aleph}{\mu(\eta_1, \sigma_1)})*\Gamma_{\mathfrak{b}}(\eta_2, \sigma_1, \frac{\mathfrak{s}}{\mu(\eta_2, \sigma_1)})*\Gamma_{\mathfrak{b}}(\eta_2, \sigma_2, \frac{\mathfrak{r}}{\mu(\eta_2, \sigma_2)})$ *for all* $\eta_1, \eta_2 \in \Theta$ *and* $\sigma_1, \sigma_2 \in \Psi$;
(FCB5) $\Gamma_{\mathfrak{b}}(\eta, \sigma, .) : \mathbb{R}_+ \to [0,1]$ *is left continuous;*
(FCB6) $\Gamma_{\mathfrak{b}}(\eta, \sigma, .)$ *is nondecreasing for all* $\eta \in \Theta$ *and* $\sigma \in \Psi$.

We present two examples from fuzzy-controlled bipolar metric spaces as follows:

Example 1. *Let* $\Theta = \{1,2,3,4\}$, $\Psi = \{2,4,5,6\}$ *and a mapping* $\mu : \Theta \times \Psi \to [1,+\infty)$ *defined by* $\mu(\eta,\sigma) = \eta + \sigma + 1$. $\Gamma_{\mathfrak{b}} : \Theta \times \Psi \times (0,+\infty) \to [0,1]$ *is defined by*

$$\Gamma_{\mathfrak{b}}(\eta,\sigma,\aleph) = \frac{\min\{\eta,\sigma\} + \aleph}{\max\{\eta,\sigma\} + \aleph},$$

for all $\eta \in \Theta$ *and* $\sigma \in \Psi$. *Then* $(\Theta,\Psi,\Gamma_{\mathfrak{b}},\star)$ *is an FCBMS with the continuous* $\mathfrak{J}$*-norm* $\star$ *such that* $\mathfrak{x} * \mathfrak{b} = \mathfrak{x}\mathfrak{b}$. *Now,* $\mu(1,2) = 4, \mu(1,4) = 6, \mu(1,5) = 7, \mu(1,6) = 8, \mu(2,2) = 5, \mu(2,4) = 7, \mu(2,5) = 8, \mu(2,6) = 9, \mu(3,2) = 6, \mu(3,4) = 8, \mu(3,5) = 9, \mu(3,6) = 10, \mu(4,2) = 7, \mu(4,4) = 9, \mu(4,5) = 10$ *and* $\mu(4,6) = 11$.

Axioms (FCB1) to (FCB3) and (FCB5), (FCB6) are easily verified; now, we prove (FCB4). Let $\eta_1 = 1, \sigma_2 = 4, \sigma_1 = 2$ *and* $\eta_2 = 3$. *Then*

$$\Gamma_{\mathfrak{b}}(1,4,\aleph + \mathfrak{s} + \mathfrak{r}) = \frac{1 + \aleph + \mathfrak{s} + \mathfrak{r}}{4 + \aleph + \mathfrak{s} + \mathfrak{r}}.$$

Then,

$$\frac{1 + \aleph + \mathfrak{s} + \mathfrak{r}}{4 + \aleph + \mathfrak{s} + \mathfrak{r}} \geq \left(\frac{4+\aleph}{8+\aleph}\right)\left(\frac{12+\mathfrak{s}}{18+\mathfrak{s}}\right)\left(\frac{24+\mathfrak{r}}{32+\mathfrak{r}}\right), \ \forall\ \aleph, \mathfrak{s}, \mathfrak{r} > 0.$$

So,

$$\Gamma_{\mathfrak{b}}(1,4,\aleph + \mathfrak{s} + \mathfrak{r}) \geq \Gamma_{\mathfrak{b}}(1,2,\frac{\aleph}{\mu(1,2)}) \star \Gamma_{\mathfrak{b}}(3,2,\frac{\mathfrak{s}}{\mu(3,2)}) \star \Gamma_{\mathfrak{b}}(3,4,\frac{\mathfrak{r}}{\mu(3,4)}).$$

Proceeding this way, $(\Theta,\Psi,\Gamma_{\mathfrak{b}},\star)$ *is an FCBMS.*

Example 2. *If we use the minimal* $\mathfrak{J}$*-norm rather than the product* $\mathfrak{J}$*-norm in Example 1, then* $(\Theta,\Psi,\Gamma_{\mathfrak{b}},\star)$ *is not an FCBMS. For instance, let* $\eta_1 = 1, \sigma_2 = 4, \sigma_1 = 2$, $\eta_2 = 3$ *and* $\aleph = 0.02, \mathfrak{s} = 0.03, \mathfrak{r} = 0.04$ *with* $\mu(\eta,\sigma) = \eta + \sigma + 1$, *then*

$$\Gamma_{\mathfrak{b}}(1,4,0.02 + 0.03 + 0.04) = \frac{1 + 0.09}{4 + 0.09} = 0.2665,$$

and

$$\Gamma_{\mathfrak{b}}(1,2,\frac{0.02}{\mu(1,2)}) = 0.50124, \quad \Gamma_{\mathfrak{b}}(3,2,\frac{0.03}{\mu(3,2)}) = 0.6672, \quad \Gamma_{\mathfrak{b}}(3,4,\frac{0.04}{\mu(3,4)}) = 0.7503.$$

Clearly,

$$\Gamma_{\mathfrak{b}}(1,4,0.02 + 0.03 + 0.04) \ngeq \Gamma_{\mathfrak{b}}(1,2,\frac{0.02}{\mu(1,2)}) \star \Gamma_{\mathfrak{b}}(3,2,\frac{0.03}{\mu(3,2)}) \star \Gamma_{\mathfrak{b}}(3,4,\frac{0.04}{\mu(3,4)}).$$

$(\Theta,\Psi,\Gamma_{\mathfrak{b}},\star)$ *is not an FCBMS with a minimum* $\mathfrak{J}$*-norm.*

Furthermore, let us recall the definitions of a bisequence, Cauchy bisequence (CBS), complete bisequence, and some lemmas in the setting of fuzzy-controlled bipolar metric spaces:

Definition 5. *Let* $(\Theta,\Psi,\Gamma_{\mathfrak{b}},*)$ *be an FCBMS. Then:*

(i) *A sequence* $(\{\eta_\alpha\},\{\sigma_\alpha\}) \in \Theta \times \Psi$ *is named a bisequence on* $(\Theta,\Psi,\Gamma_{\mathfrak{b}},*)$.
(ii) *A bisequence* $(\{\eta_\alpha\},\{\sigma_\alpha\})$ *on FCBMS* $(\Theta,\Psi,\Gamma_{\mathfrak{b}},*)$ *is called a CBS if for each* $\epsilon > 0$, *we can find* $\alpha_0 \in \mathbb{N}$ *satisfying* $\Gamma_{\mathfrak{b}}(\eta_\alpha,\sigma_\beta,\aleph) \to 1$ *as* $\alpha,\beta \to \infty$ *for all* $\alpha,\beta \geq \alpha_0$ $(\alpha,\beta \in \mathbb{N})$, $\aleph > 0$.

Definition 6. *The FCBMS $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ is called complete if every CBS $(\{\eta_\alpha\}, \{\sigma_\alpha\}) \in \Theta \times \Psi$ is convergent.*

Lemma 1 ([8])**.** *Let $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ be an FBMS such that*

$$\Gamma_{\mathfrak{b}}(\eta, \sigma, h\aleph) \geq \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph)$$

for all $\eta \in \Theta, \sigma \in \Psi$ and $h \in (0,1)$. Then $\eta = \sigma$.

Lemma 2. *Let $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ be an FCBMS such that*

$$\Gamma_{\mathfrak{b}}(\eta, \sigma, h\aleph) \geq \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph)$$

for all $\eta \in \Theta, \sigma \in \Psi$ and $h \in (0,1)$. Then $\eta = \sigma$.

Proof. We have

$$\Gamma_{\mathfrak{b}}(\eta, \sigma, h\aleph) \geq \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph). \tag{1}$$

Since $h\aleph < \aleph$ for all $\aleph > 0$ and $h \in (0,1)$, by (FCB-6) we have

$$\Gamma_{\mathfrak{b}}(\eta, \sigma, h\aleph) \leq \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph). \tag{2}$$

From (1) and (2) and definition of FCBMS, we get $\eta = \sigma$. □

Definition 7. *A point $\eta \in \Theta \cap \Psi$ is called a fixed point for the mapping Π on $\eta \in \Theta \cap \Psi$ if $\eta = \Pi\eta$.*

Sezen [10] proved the following fixed-point theorem for fuzzy-controlled metric space:

Theorem 1. *Let $(\Theta, \Gamma_{\mathfrak{b}}, *)$ be a fuzzy-controlled metric space with $\mathfrak{b} : \Theta \times \Theta \to [1, \infty)$ and suppose that*

$$\lim_{t\to\infty} \Gamma_{\mathfrak{b}}(a, c, t) = 1,$$

for all $a \in \Theta$. If $g : \Theta \to \Theta$ satisfies:

$$\Gamma_{\mathfrak{b}}(ga, gc, ht) \geq \Gamma_{\mathfrak{b}}(a, c, t),$$

for all $a, c \in \Theta, t > 0$, where $h \in (0,1)$. Additionally, assume that for every $a \in \Theta$, we obtain $lim_{n\to\infty}\mathfrak{b}(a_n, c)$ and $lim_{n\to\infty}\mathfrak{b}(c, a_n)$, exist and are finite. Then, g has a unique fixed point in Θ.

Mihet [16] introduced the Ψ class of mappings as follows:

Definition 8. *Let Ψ be the class of all maps $\psi : [0,1] \to [0,1]$ such that ψ is non-decreasing, continuous, and $\psi(\xi) > \xi$, $\forall\, \xi \in (0,1)$. If $\psi \in \Psi$, then $\lim_{n\to\infty} \psi^n(\xi) = 1$ and $\psi(1) = 1$, $\forall\, \xi \in (0,1)$.*

Theorem 2. *Let $(\Theta, \Gamma_{\mathfrak{b}}, *)$ be a controlled fuzzy metric space and $g : \Theta \to \Theta$ be a mapping satisfying*

$$\Gamma_{\mathfrak{b}}(a, c, t) > 0 \Rightarrow \Gamma_{\mathfrak{b}}(ga, gc, t) \geq \psi(\Gamma_{\mathfrak{b}}(a, c, t)),$$

for all $a, c \in \Theta$ and $t > 0$. Then, g has a unique fixed point in Θ.

In this study, motivated by the results of Mutlu, A., Gürdal, U. [7], Bartwal, A., Dimri, R. C., Prasad, G. [8] and Sezen [10], we proved a fixed-point theorem for fuzzy-controlled contraction mappings in bipolar metric spaces.

3. Main Results

First, we generalize and improve upon Sezen's [10] Theorem 1 for fuzzy-controlled bipolar metric space.

Theorem 3. *Let* $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ *be a complete FCBMS with* $\mu : \Theta \times \Psi \to [1, +\infty)$ *such that*

$$\lim_{\aleph \to \infty} \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) = 1, \forall \eta \in \Theta, \sigma \in \Psi. \tag{3}$$

Let $\Pi : \Theta \cup \Psi \to \Theta \cup \Psi$ *be a mapping satisfying*

(i) $\Pi(\Theta) \subseteq \Theta$ *and* $\Pi(\Psi) \subseteq \Psi$*;*

(ii) $\Gamma_{\mathfrak{b}}(\Pi(\eta), \Pi(\sigma), h\aleph) \geq \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph), \forall \eta \in \Theta, \sigma \in \Psi$ *and* $\aleph > 0$*, where* $h \in (0, 1)$*.*

Additionally, assume that for every $\eta \in \Theta$*,*

$$\lim_{\alpha \to \infty} \mu(\eta_\alpha, \sigma) \text{ and } \lim_{\alpha \to \infty} \mu(\sigma, \eta_\alpha) \text{ exist and are finite.}$$

Then Π *has a unique fixed point.*

Proof. Let $\eta_0 \in \Theta$ and $\sigma_0 \in \Psi$. Then $\Pi(\eta_\alpha) = \eta_{\alpha+1}$ and $\Pi(\sigma_\alpha) = \sigma_{\alpha+1}$, $\forall \alpha \in \mathbb{N} \cup \{0\}$. Therefore, $(\{\eta_\alpha\}, \{\sigma_\alpha\})$ is a bisequence on FCBMS $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$. Now,

$$\Gamma_{\mathfrak{b}}(\eta_1, \sigma_1, \aleph) = \Gamma_{\mathfrak{b}}(\Pi(\eta_0), \Pi(\sigma_0), \aleph) \geq \Gamma_{\mathfrak{b}}(\eta_0, \sigma_0, \frac{\aleph}{h}),$$

for all $\aleph > 0$ and $\alpha \in \mathbb{N}$. Then,

$$\Gamma_{\mathfrak{b}}(\eta_\alpha, \sigma_\alpha, \aleph) = \Gamma_{\mathfrak{b}}(\Pi(\eta_{\alpha-1}), \Pi(\sigma_{\alpha-1}), \aleph) \geq \Gamma_{\mathfrak{b}}(\eta_0, \sigma_0, \frac{\aleph}{h^\alpha}) \tag{4}$$

and

$$\Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_\alpha, \aleph) = \Gamma_{\mathfrak{b}}(\Pi(\eta_\alpha), \Pi(\sigma_{\mathfrak{n}-1}), \aleph) \geq \Gamma_{\mathfrak{b}}(\eta_1, \sigma_0, \frac{\aleph}{h^\alpha}), \tag{5}$$

for all $\aleph > 0$ and $\alpha \in \mathbb{N}$.

Let $\alpha < \beta \in \mathbb{N}$. Then,

$$\begin{aligned}
\Gamma_{\mathfrak{b}}(\eta_\alpha, \sigma_\beta, \aleph) \geq & \Gamma_{\mathfrak{b}}(\eta_\alpha, \sigma_\alpha, \frac{\aleph}{3\mu(\eta_\alpha, \sigma_\alpha)}) \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_\alpha, \frac{\aleph}{3\mu(\eta_{\alpha+1}, \sigma_\alpha)}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_\beta, \frac{\aleph}{3\mu(\eta_{\alpha+1}, \sigma_\beta)}) \\
& \vdots \\
& \geq \Gamma_{\mathfrak{b}}(\eta_\alpha, \sigma_\alpha, \frac{\aleph}{3\mu(\eta_\alpha, \sigma_\alpha)}) \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_\alpha, \frac{\aleph}{3\mu(\eta_{\alpha+1}, \sigma_\alpha)}) \star \cdots \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\beta-1}, \sigma_{\beta-1}, \frac{\aleph}{3^{\beta-1}\mu(\eta_\alpha, \sigma_\alpha)\mu(\eta_{\alpha+1}, \sigma_{\alpha+1}) \cdots \mu(\eta_{\beta-1}, \sigma_{\beta-1})}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_\beta, \sigma_{\beta-1}, \frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha+1}, \sigma_\alpha)\mu(\eta_{\alpha+2}, \sigma_{\alpha+1}) \cdots \mu(\eta_\beta, \sigma_{\beta-1})}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_\beta, \sigma_\beta, \frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha+1}, \sigma_\beta)\mu(\eta_{\alpha+2}, \sigma_\beta) \cdots \mu(\eta_\beta, \sigma_\beta)}).
\end{aligned}$$

Now applying (4) and (5) on each term of the RHS of the above inequality, we obtain

$$\Gamma_{\mathfrak{b}}(\eta_\alpha, \sigma_\beta, \aleph) \geq \Gamma_{\mathfrak{b}}(\eta_0, \sigma_0, \frac{\aleph}{3h^\alpha \mu(\eta_\alpha, \sigma_\alpha)}) \star \Gamma_{\mathfrak{b}}(\eta_0, \sigma_1, \frac{\aleph}{3h^{\alpha+1}\mu(\eta_{\alpha+1}, \sigma_\alpha)})$$
$$\star \cdots \cdots \star \Gamma_{\mathfrak{b}}(\eta_0, \sigma_0, \frac{\aleph}{3^{\beta-1}h^{\beta+1}\mu(\eta_{\alpha+1}, \sigma_\beta)\mu(\eta_{\alpha+2}, \sigma_\beta)\cdots\mu(\eta_\beta, \sigma_\beta)}).$$

From (3), as $\alpha, \beta \to \infty$, we get

$$\Gamma_{\mathfrak{b}}(\eta_\alpha, \sigma_\beta, \aleph) \geq 1, \text{ for all } \aleph > 0.$$

Therefore, the bisequence $(\{\eta_\alpha\}, \{\sigma_\alpha\})$ is a CBS. Because $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ is a complete space, the bisequence $(\{\eta_\alpha\}, \{\sigma_\alpha\}) \to \varpi$. Then, $\{\eta_\alpha\} \to \varpi$ and $\{\sigma_\alpha\} \to \varpi$, where $\varpi \in \Theta \cap \Psi$. From (FCB4), we derive

$$\Gamma_{\mathfrak{b}}(\Pi(\varpi), \varpi, \aleph) \geq \Gamma_{\mathfrak{b}}(\Pi(\varpi), \Pi(\sigma_\alpha), \frac{\aleph}{3\mu(\varpi, \sigma_\alpha)}) * \Gamma_{\mathfrak{b}}(\Pi(\eta_\alpha), \Pi(\sigma_\alpha), \frac{\aleph}{3\mu(\eta_\alpha, \sigma_\alpha)})$$
$$* \Gamma_{\mathfrak{b}}(\Pi(\eta_\alpha), \varpi, \frac{\aleph}{3\mu(\eta_\alpha, \varpi)}),$$

for all $\alpha \in \mathbb{N}$ and $\aleph > 0$ and as $\alpha \to \infty$,

$$\Gamma_{\mathfrak{b}}(\Pi(\varpi), \varpi, \aleph) \to 1*1*1 = 1.$$

Therefore, $\Pi(\varpi) = \varpi$. Let $\mathfrak{v} \in \Theta \cap \Psi$ is another fixed point of Π. Because

$$\Gamma_{\mathfrak{b}}(\varpi, \mathfrak{v}, \aleph) = \Gamma_{\mathfrak{b}}(\Pi(\varpi), \Pi(\mathfrak{v}), \aleph) \geq \Gamma_{\mathfrak{b}}(\varpi, \mathfrak{v}, \frac{\aleph}{h})$$

for $h \in (0,1)$ and $\forall \aleph > 0$. Hence, $\varpi = \mathfrak{v}$. □

The following example supports Theorem 3.

Example 3. *Let $\Theta = [0,1]$, $\Psi = \{0\} \cup \mathbb{N} - \{1\}$ and $\mu : \Theta \times \Psi \to [1, +\infty)$ be a mapping defined by $\mu(\eta, \sigma) = \eta + \sigma + 1$. Define*

$$\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) = e^{-\frac{(\eta-\sigma)^2}{\aleph}}, \ \forall \eta \in \Theta, \sigma \in \Psi, \aleph > 0.$$

Clearly, $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ is a complete FCBMS, where $*$ is a continuous $\mathfrak{J}$-norm defined as $\mathfrak{x}*\mathfrak{b} = \mathfrak{x}\mathfrak{b}$. Define $\Pi : \Theta \cup \Psi \to \Theta \cup \Psi$ by

$$\Pi(\varpi) = \begin{cases} \frac{\varpi}{2}, & \text{if } \varpi \in [0,1], \\ 0, & \text{if } \varpi \in \mathbb{N} - \{1\}, \end{cases}$$

for all $\varpi \in \Theta \cup \Psi$. Clearly, $\Pi(\Theta) \subseteq \Theta$ and $\Pi(\Psi) \subseteq \Psi$. Let $\eta \in [0,1]$ and $\sigma \in \mathbb{N} - \{1\}$, then

$$\begin{aligned} \Gamma_{\mathfrak{b}}(\Pi(\eta), \Pi(\sigma), h\aleph) &= \Gamma_{\mathfrak{b}}\left(\frac{\eta}{2}, 0, h\aleph\right) \\ &= e^{-\frac{\left(\frac{\eta}{2}\right)^2}{h\aleph}} \\ &\geq e^{-\frac{(\eta-\sigma)^2}{\aleph}} \\ &= \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph). \end{aligned}$$

Now,

$$\lim_{\alpha\to\infty}\mu(\eta_\alpha,\sigma) = \lim_{\alpha\to\infty}(\frac{\eta}{2^\alpha}+\sigma+1) \text{ and } \lim_{\alpha\to\infty}\mu(\sigma,\eta_\alpha) = \lim_{\alpha\to\infty}(\sigma+\frac{\eta}{2^\alpha}+1) \text{ exist and are finite.}$$

Therefore, all the conditions of Theorem 3 are satisfied. Hence, Π has a unique fixed point, i.e., $\varpi = 0$.

We prove the following result to modify the hypothesis (i) of Theorem 3 as follows:

Theorem 4. *Let $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ be a complete FCBMS with $\mu : \Theta \times \Psi \to [1, +\infty)$ such that*

$$\lim_{\aleph\to\infty}\Gamma_{\mathfrak{b}}(\eta,\sigma,\aleph) = 1, \forall \eta \in \Theta, \sigma \in \Psi. \tag{6}$$

Let $\Pi : \Theta \cup \Psi \to \Theta \cup \Psi$ be a mapping satisfying

(i) $\Pi(\Theta) \subseteq \Psi$ *and* $\Pi(\Psi) \subseteq \Theta$;

(ii) $\Gamma_{\mathfrak{b}}(\Pi(\sigma), \Pi(\eta), h\aleph) \geq \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph), \forall \eta \in \Theta, \sigma \in \Psi$ *and* $\aleph > 0$, *here* $h \in (0, 1)$.

Additionally, assume that for every $\eta \in \Theta$,

$$\lim_{\alpha\to\infty}\mu(\eta_\alpha,\sigma) \text{ and } \lim_{\alpha\to\infty}\mu(\sigma,\eta_\alpha) \text{ exist and are finite.}$$

Then Π has a unique fixed point.

Proof. Let $\eta_0 \in \Theta$ and $\sigma_0 \in \Psi$. Then, $\Pi(\eta_\alpha) = \sigma_\alpha$ and $\Pi(\sigma_\alpha) = \eta_{\alpha+1}$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Therefore, $(\{\eta_\alpha\}, \{\sigma_\alpha\})$ is a bisequence on FCBMS $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$. Now,

$$\Gamma_{\mathfrak{b}}(\eta_1,\sigma_0,\aleph) = \Gamma_{\mathfrak{b}}(\Pi(\sigma_0),\Pi(\eta_0),\aleph) \geq \Gamma_{\mathfrak{b}}(\eta_0,\sigma_0,\frac{\aleph}{h}),$$

for all $\aleph > 0$ and $\alpha \in \mathbb{N}$. Then,

$$\Gamma_{\mathfrak{b}}(\eta_\alpha,\sigma_\alpha,\aleph) = \Gamma_{\mathfrak{b}}(\Pi(\sigma_{\alpha-1}),\Pi(\eta_\alpha),\aleph) \geq \Gamma_{\mathfrak{b}}(\eta_0,\sigma_0,\frac{\aleph}{h^{2\alpha}}) \tag{7}$$

and

$$\Gamma_{\mathfrak{b}}(\eta_{\alpha+1},\sigma_\alpha,\aleph) = \Gamma_{\mathfrak{b}}(\Pi(\sigma_\alpha),\Pi(\eta_\alpha),\aleph) \geq \Gamma_{\mathfrak{b}}(\eta_0,\sigma_0,\frac{\aleph}{h^{2\alpha+1}}), \tag{8}$$

for all $\aleph > 0$ and $\alpha \in \mathbb{N}$. Let $\alpha < \beta \in \mathbb{N}$. Then,

$$\begin{aligned}
\Gamma_{\mathfrak{b}}(\eta_\alpha,\sigma_\beta,\aleph) \geq & \Gamma_{\mathfrak{b}}(\eta_\alpha,\sigma_\alpha,\frac{\aleph}{3\mu(\eta_\alpha,\sigma_\alpha)}) \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1},\sigma_\alpha,\frac{\aleph}{3\mu(\eta_{\alpha+1},\sigma_\alpha)}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1},\sigma_\beta,\frac{\aleph}{3\mu(\eta_{\alpha+1},\sigma_\beta)}) \\
& \vdots \\
& \geq \Gamma_{\mathfrak{b}}(\eta_\alpha,\sigma_\alpha,\frac{\aleph}{3\mu(\eta_\alpha,\sigma_\alpha)}) \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1},\sigma_\alpha,\frac{\aleph}{3\mu(\eta_{\alpha+1},\sigma_\alpha)}) \star \cdots \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\beta-1},\sigma_{\beta-1},\frac{\aleph}{3^{\beta-1}\mu(\eta_\alpha,\sigma_\alpha)\mu(\eta_{\alpha+1},\sigma_{\alpha+1})\cdots\mu(\eta_{\beta-1},\sigma_{\beta-1})}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_\beta,\sigma_{\beta-1},\frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha+1},\sigma_\alpha)\mu(\eta_{\alpha+2},\sigma_{\alpha+1})\cdots\mu(\eta_\beta,\sigma_{\beta-1})}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_\beta,\sigma_\beta,\frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha+1},\sigma_\beta)\mu(\eta_{\alpha+2},\sigma_\beta)\cdots\mu(\eta_\beta,\sigma_\beta)}).
\end{aligned}$$

Now, applying (7) and (8) on each term of the RHS of the above inequality, we obtain

$$\varGamma_{\mathfrak{b}}(\eta_\alpha,\sigma_\beta,\aleph) \geq \varGamma_{\mathfrak{b}}(\eta_0,\sigma_0,\frac{\aleph}{3h^{2\alpha}\mu(\eta_\alpha,\sigma_\alpha)}) \star \varGamma_{\mathfrak{b}}(\eta_0,\sigma_0,\frac{\aleph}{3h^{2\alpha+1}\mu(\eta_{\alpha+1},\sigma_\alpha)})$$
$$\star\cdots\cdots\star \varGamma_{\mathfrak{b}}(\eta_0,\sigma_0,\frac{\aleph}{3^{\beta-1}h^{2\beta+1}\mu(\eta_{\alpha+1},\sigma_\beta)\mu(\eta_{\alpha+2},\sigma_\beta)\cdots\mu(\eta_\beta,\sigma_\beta)}).$$

From (6), as $\alpha,\beta\to\infty$, we obtain

$$\varGamma_{\mathfrak{b}}(\eta_\alpha,\sigma_\beta,\aleph) \geq 1 \text{ for all } \aleph > 0.$$

Therefore, the bisequence $(\{\eta_\alpha\},\{\sigma_\alpha\})$ is a CBS. Because $(\Theta,\Psi,\varGamma_{\mathfrak{b}},*)$ is a complete space, the bisequence $(\{\eta_\alpha\},\{\sigma_\alpha\})$ is a convergent bisequence. Then, $\{\eta_\alpha\}\to\varpi$ and $\{\sigma_\alpha\}\to\varpi$, where $\varpi\in\Theta\cap\Psi$. Because

$$\varGamma_{\mathfrak{b}}(\varPi(\varpi),\varpi,\aleph) \geq \varGamma_{\mathfrak{b}}(\varPi(\varpi),\varPi(\eta_\alpha),\frac{\aleph}{3\mu(\varPi(\varpi),\varPi(\eta_\alpha))})$$
$$*\varGamma_{\mathfrak{b}}(\varPi(\sigma_\alpha),\varPi(\eta_\alpha),\frac{\aleph}{3\mu(\varPi(\sigma_\alpha),\varPi(\eta_\alpha))})$$
$$*\varGamma_{\mathfrak{b}}(\varpi,\varPi(\eta_\alpha),\frac{\aleph}{3\mu(\varpi,\varPi(\eta_\alpha))}),$$

for all $\alpha\in\mathbb{N}$ and $\aleph>0$ and as $\alpha\to\infty$,

$$\varGamma_{\mathfrak{b}}(\varPi(\varpi),\varpi,\aleph) \to 1*1*1 = 1.$$

Therefore, $\varPi(\varpi)=\varpi$. Let $\mathfrak{v}\in\Theta\cap\Psi$ is another fixed point of $\varPi$. Because

$$\varGamma_{\mathfrak{b}}(\varpi,\mathfrak{v},\aleph) = \varGamma_{\mathfrak{b}}(\varPi(\mathfrak{v}),\varPi(\varpi),\aleph) \geq \varGamma_{\mathfrak{b}}(\varpi,\mathfrak{v},\frac{\aleph}{h})$$

for $h\in(0,1)$ and $\forall\ \aleph>0$. Hence $\varpi=\mathfrak{v}$. □

We demonstrate our results with an example.

Example 4. *Let* $\Theta=[0,1]$, $\Psi=[1,2]$, *and* $\mu:\Theta\times\Psi\to[1,+\infty)$ *be a mapping defined by* $\mu(\eta,\sigma)=2(\eta+\sigma)+1$. *Define*

$$\varGamma_{\mathfrak{b}}(\eta,\sigma,\aleph) = e^{-\frac{(\eta-\sigma)^2}{\aleph}},\ \forall\ \eta\in\Theta,\sigma\in\Psi,\aleph>0.$$

Then, $(\Theta,\Psi,\varGamma_{\mathfrak{b}},*)$ *is a complete FCBMS with product* $\mathfrak{J}$*-norm. Define* $\varPi:\Theta\cup\Psi\to\Theta\cup\Psi$ *by* $\varPi(\varpi)=\frac{1+\varpi}{2}$ *for all* $\varpi\in\Theta\cup\Psi$. $\varPi(\Theta)\subseteq\Psi$ *and* $\varPi(\Psi)\subseteq\Theta$. *Let* $\eta\in\Theta$ *and* $\sigma\in\Psi$, *then*

$$\begin{aligned}\varGamma_{\mathfrak{b}}(\varPi(\eta),\varPi(\sigma),h\aleph) &= \varGamma_{\mathfrak{b}}\left(\frac{1+\eta}{2},\frac{1+\sigma}{2},h\aleph\right)\\ &= e^{-\frac{(\eta-\sigma)^2}{4h\aleph}}\\ &\geq e^{-\frac{(\eta-\sigma)^2}{\aleph}}\\ &= \varGamma_{\mathfrak{b}}(\eta,\sigma,\aleph).\end{aligned}$$

Now,

$$\lim_{\alpha\to\infty}\mu(\eta_\alpha,\sigma) = \lim_{\alpha\to\infty}(\frac{2^\alpha-1+\eta}{2^\alpha}+\sigma+1) \text{ and } \lim_{\alpha\to\infty}\mu(\sigma,\eta_\alpha) = \lim_{\alpha\to\infty}(\sigma+\frac{2^\alpha-1+\eta}{2^\alpha}+1)$$

exist and are finite. Therefore, all the hypotheses of Theorem 4 are fulfilled. Hence, $\varPi$ *has a unique fixed point, i.e.,* $\varpi=1$.

Here, we prove the following theorem to modify the condition (ii) of Theorem 3 with an increasing function. This theorem is an extension of Theorem 2 of Sezen [10] as follows:

Theorem 5. *Let* $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ *be a complete FCBMS with* $\mu : \Theta \times \Psi \to [1, +\infty)$ *and* $\Pi : \Theta \cup \Psi \to \Theta \cup \Psi$ *a mapping satisfying*

(i) $\Pi(\Theta) \subseteq \Theta$ *and* $\Pi(\Psi) \subseteq \Psi$;

(ii) *For* $\eta \in \Theta, \sigma \in \Psi$ *and* $\aleph > 0, \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph) > 0 \Rightarrow \Gamma_{\mathfrak{b}}(\Pi(\eta), \Pi(\sigma), \aleph) \geq \Psi(\Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph))$, *where* $\Psi : (0,1] \to (0,1]$ *is an increasing mapping such that* $\lim_{\alpha \to \infty} \Psi^{\alpha}(h) = 1$ *and* $\Psi(h) \geq h \; \forall \, h \in (0,1]$.

Additionally, assume that for every $\eta \in \Theta$,

$$\lim_{\alpha \to \infty} \mu(\eta_{\alpha}, \sigma) \text{ and } \lim_{\alpha \to \infty} \mu(\sigma, \eta_{\alpha}) \text{ exist and are finite.}$$

Then Π *has a fixed point.*

Proof. Let $\eta_0 \in \Theta$ and $\sigma_0 \in \Psi$. Then $\Pi(\eta_{\alpha}) = \eta_{\alpha+1}$ and $\Pi(\sigma_{\alpha}) = \sigma_{\alpha+1}$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Therefore, $(\{\eta_{\alpha}\}, \{\sigma_{\alpha}\})$ is a bisequence on FCBMS $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$. From (FCB2) for all $\aleph > 0$ and condition (ii) from Theorem 5, we obtain

$$\Gamma_{\mathfrak{b}}(\eta_{\alpha}, \sigma_{\alpha}, \aleph) \geq \Psi^{\alpha}(\Gamma_{\mathfrak{b}}(\eta_0, \sigma_0, \aleph)) \tag{9}$$

and

$$\Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_{\alpha}, \aleph) \geq \Psi^{\alpha}(\Gamma_{\mathfrak{b}}(\eta_1, \sigma_0, \aleph)). \tag{10}$$

Letting $\alpha < \beta$, for $\alpha, \beta \in \mathbb{N}$, then

$$\begin{aligned}
\Gamma_{\mathfrak{b}}(\eta_{\alpha}, \sigma_{\beta}, \aleph) \geq & \Gamma_{\mathfrak{b}}(\eta_{\alpha}, \sigma_{\alpha}, \frac{\aleph}{3\mu(\eta_{\alpha}, \sigma_{\alpha})}) \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_{\alpha}, \frac{\aleph}{3\mu(\eta_{\alpha+1}, \sigma_{\alpha})}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_{\beta}, \frac{\aleph}{3\mu(\eta_{\alpha+1}, \sigma_{\beta})}) \\
& \vdots \\
& \geq \Gamma_{\mathfrak{b}}(\eta_{\alpha}, \sigma_{\alpha}, \frac{\aleph}{3\mu(\eta_{\alpha}, \sigma_{\alpha})}) \star \Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \sigma_{\alpha}, \frac{\aleph}{3\mu(\eta_{\alpha+1}, \sigma_{\alpha})}) \star \cdots \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\beta-1}, \sigma_{\beta-1}, \frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha}, \sigma_{\alpha})\mu(\eta_{\alpha+1}, \sigma_{\alpha+1}) \cdots \mu(\eta_{\beta-1}, \sigma_{\beta-1})}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\beta}, \sigma_{\beta-1}, \frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha+1}, \sigma_{\alpha})\mu(\eta_{\alpha+2}, \sigma_{\alpha+1}) \cdots \mu(\eta_{\beta}, \sigma_{\beta-1})}) \\
& \star \Gamma_{\mathfrak{b}}(\eta_{\beta}, \sigma_{\beta}, \frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha+1}, \sigma_{\beta})\mu(\eta_{\alpha+2}, \sigma_{\beta}) \cdots \mu(\eta_{\beta}, \sigma_{\beta})}).
\end{aligned}$$

Now, applying (9) and (10) on each term of the RHS of the above inequality, we have

$$\begin{aligned}
\Gamma_{\mathfrak{b}}(\eta_{\alpha}, \sigma_{\beta}, \aleph) \geq & \Psi^{\alpha}(\Gamma_{\mathfrak{b}}(\eta_0, \sigma_0, \frac{\aleph}{3\mu(\eta_{\alpha}, \sigma_{\alpha})})) \star \Psi^{\alpha}(\Gamma_{\mathfrak{b}}(\eta_1, \sigma_0, \frac{\aleph}{3\mu(\eta_{\alpha+1}, \sigma_{\alpha})})) \\
& \star \cdots \cdots \star \Psi^{\alpha}(\Gamma_{\mathfrak{b}}(\eta_0, \sigma_0, \frac{\aleph}{3^{\beta-1}\mu(\eta_{\alpha+1}, \sigma_{\beta})\mu(\eta_{\alpha+2}, \sigma_{\beta}) \cdots \mu(\eta_{\beta}, \sigma_{\beta})})).
\end{aligned}$$

As $\alpha, \beta \to \infty$, $\Gamma_{\mathfrak{b}}(\eta_{\alpha}, \sigma_{\beta}, \aleph) \to 1 \; \forall \; \aleph > 0$. Applying the same lines of the proof of Theorem 3, then ϖ is a fixed point of Π. Because $\Gamma_{\mathfrak{b}}(\eta_{\alpha}, \varpi, \aleph) \to \aleph$, for all $\aleph > 0$ and $\Gamma_{\mathfrak{b}}(\eta_{\alpha+1}, \Pi(\varpi), \aleph) = \Gamma_{\mathfrak{b}}(\Pi(\eta_{\alpha}), \Pi(\varpi), \aleph) \geq \Psi(\Gamma_{\mathfrak{b}}(\eta_{\alpha}, (\varpi), \aleph)) \geq \Gamma_{\mathfrak{b}}(\eta_{\alpha}, (\varpi), \aleph)$. Therefore, $\eta_{\alpha+1} \to \Pi(\varpi)$, which means that $\Pi(\varpi) = \varpi$. □

The following example is provided to demonstrate Theorem 5.

Example 5. *Let $\Theta = \{2,4,5,6\}, \Psi = \{1,2\}, \mathfrak{x} * \mathfrak{b} = \mathfrak{x}\mathfrak{b}$ for all $\mathfrak{x}, \mathfrak{b} \in [0,1]$, and $\mu : \Theta \times \Psi \to [1,+\infty)$ be a mapping defined by $\mu(\eta,\sigma) = \eta + \sigma + 1$. Define*

$$\Gamma_{\mathfrak{b}}(\eta,\sigma,\aleph) = \frac{\min\{\eta,\sigma\} + \aleph}{\max\{\eta,\sigma\} + \aleph} \text{ for all } \eta \in \Theta, \sigma \in \Psi \text{ and for all } \aleph > 0.$$

Then, $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ is a complete FCBMS. A self-map Ψ on $(0,1]$ is defined by $\Psi(h) = \sqrt{h}$. Let $\Pi : \Theta \cup \Psi \to \Theta \cup \Psi$ be a mapping such that $\Pi(2) = \Pi(4) = \Pi(1) = 2, \Pi(5) = \Pi(6) = 4$. Then, all the hypotheses of Theorem 5 are fulfilled. Hence, $\eta = 2$ is a fixed point of Π.

Finally, we prove the following theorem to modify the condition (i) of Theorem 5 as follows:

Theorem 6. *Let $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, *)$ be a complete FCBMS with $\mu : \Theta \times \Psi \to [1,+\infty)$, and $\Pi : \Theta \cup \Psi \to \Theta \cup \Psi$ a mapping satisfying*

(i) $\Pi(\Theta) \subseteq \Psi$ *and* $\Pi(\Psi) \subseteq \Theta$;

(ii) *For* $\eta \in \Theta, \sigma \in \Psi$ *and* $\aleph > 0, \Gamma_{\mathfrak{b}}(\eta,\sigma,\aleph) > 0 \Longrightarrow \Gamma_{\mathfrak{b}}(\Pi(\sigma),\Pi(\eta),\aleph) \geq \Psi(\Gamma_{\mathfrak{b}}(\eta,\sigma,\aleph))$.

Additionally, assume that for every $\eta \in \Theta$,

$$\lim_{\alpha\to\infty} \mu(\eta_\alpha,\sigma) \text{ and } \lim_{\alpha\to\infty} \mu(\sigma,\eta_\alpha) \text{ are exist and finite.}$$

Then, Π has a fixed point.

Proof. The Theorem proof follows from Theorems 4 and 5. □

4. Application

In this section, we prove the existence of solution for the integral equation. In the literature, the solution of fixed-point theorem through integral equation in fuzzy bipolar metric space was initiated by Gunaseelan Mani, Arul Joseph Gnanaprakasam, Haq Absar Ul, Jarad Fahd, and Baloch Imran Abbas [13,15]. Motivated by the above work, we obtained the solution to the integral equation in the fuzzy-controlled bipolar metric space setting by using Theorem 3.

Consider the integral equation

$$\eta(\mathfrak{p}) = \mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{O}_1\cup\mathcal{O}_2} \mathcal{G}(\mathfrak{p},\mathfrak{s},\eta(\mathfrak{s}))d\mathfrak{s},\ \mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2,$$

where $\mathcal{O}_1 \cup \mathcal{O}_2$ is a Lebesgue measurable set, and

(T1) $\mathcal{G} : (\mathcal{O}_1^2 \cup \mathcal{O}_2^2) \times \mathbb{R}_+ \to \mathbb{R}_+$ and $b \in L^\infty(\mathcal{O}_1) \cup L^\infty(\mathcal{O}_2)$,

(T2) There is a continuous function $\theta : \mathcal{O}_1^2 \cup \mathcal{O}_2^2 \to \mathbb{R}_+$ and $h \in (0,1)$ such that

$$|\mathcal{G}(\mathfrak{p},\mathfrak{s},\eta(\mathfrak{s})) - \mathcal{G}(\mathfrak{p},\mathfrak{s},\sigma(\mathfrak{s}))| \leq h\theta(\mathfrak{p},\mathfrak{s})(|\eta(\mathfrak{p}) - \sigma(\mathfrak{p})|),$$

for all $\mathfrak{p},\mathfrak{s} \in \mathcal{O}_1^2 \cup \mathcal{O}_2^2$,

(T3) $\sup_{\mathfrak{p}\in\mathcal{O}_1\cup\mathcal{O}_2} \int_{\mathcal{O}_1\cup\mathcal{O}_2} \theta(\mathfrak{p},\mathfrak{s})d\mathfrak{s} \leq 1$.

Define the mapping $\Gamma_{\mathfrak{b}} : \Theta \times \Psi \times (0,+\infty) \to [0,1]$ by

$$\Gamma_{\mathfrak{b}}(\eta,\sigma,\aleph) = e^{-\frac{\sup_{\mathfrak{p}\in\mathcal{O}_1\cup\mathcal{O}_2} |\eta(\mathfrak{p})-\sigma(\mathfrak{p})|}{\aleph}},$$

for all $\eta \in \Theta, \sigma \in \Psi$. Define $\mu : \Theta \times \Psi \to [1,+\infty)$ as a mapping defined by $\mu(\eta,\sigma) = \eta + \sigma + 1$. Then, $(\Theta, \Psi, \Gamma_{\mathfrak{b}}, \star)$ is a complete FCBMSs.

Theorem 7. *Under assumptions* $(T1)$–$(T3)$*, the integral equation has a unique solution in* $L^{\infty}(\mathcal{O}_1) \cup L^{\infty}(\mathcal{O}_2)$.

Proof. Let $\Theta = L^{\infty}(\mathcal{O}_1)$ and $\Psi = L^{\infty}(\mathcal{O}_2)$ be two normed linear spaces, where $\mathcal{O}_1, \mathcal{O}_2$ are Lebesgue measurable sets, and $m(\mathcal{O}_1 \cup \mathcal{O}_2) < \infty$.

Define the mappings $\Pi : L^{\infty}(\mathcal{O}_1) \cup L^{\infty}(\mathcal{O}_2) \to L^{\infty}(\mathcal{O}_1) \cup L^{\infty}(\mathcal{O}_2)$ by

$$\Pi(\eta(\mathfrak{p})) = \mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{O}_1 \cup \mathcal{O}_2} \mathcal{G}(\mathfrak{p}, \mathfrak{s}, \eta(\mathfrak{s}))d\mathfrak{s}, \ \mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2.$$

Now,

$$\begin{aligned}
\Gamma_{\mathfrak{b}}(\Pi\eta(\mathfrak{p}), \Pi\sigma(\mathfrak{p}), h\aleph) &= e^{-\sup_{\mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2} \frac{|\Pi\eta(\mathfrak{p}) - \Pi\sigma(\mathfrak{p})|}{h\aleph}} \\
&= e^{-\sup_{\mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2} \frac{|\mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{O}_1 \cup \mathcal{O}_2} \mathcal{G}(\mathfrak{p},\mathfrak{s},\eta(\mathfrak{s}))d\mathfrak{s} - \mathfrak{b}(\mathfrak{p}) - \int_{\mathcal{O}_1 \cup \mathcal{O}_2} \mathcal{G}(\mathfrak{p},\mathfrak{s},\sigma(\mathfrak{s}))d\mathfrak{s})|}{h\aleph}} \\
&= e^{-\sup_{\mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2} \frac{\left|\mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{O}_1 \cup \mathcal{O}_2} \mathcal{G}(\aleph,\mathfrak{s},\eta(\mathfrak{s}))d\mathfrak{s} - \left(\mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{O}_1 \cup \mathcal{O}_2} \mathcal{G}(\aleph,\mathfrak{s},\sigma(\mathfrak{s}))d\mathfrak{s}\right)\right|}{h\aleph}} \\
&\geq e^{-\sup_{\mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2} \frac{\int_{\mathcal{O}_1 \cup \mathcal{O}_2} |\mathcal{G}(\mathfrak{p},\mathfrak{s},\eta(\mathfrak{s})) - \mathcal{G}(\mathfrak{p},\mathfrak{s},\sigma(\mathfrak{s}))|d\mathfrak{s}}{h\aleph}} \\
&\geq e^{-\sup_{\mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2} \frac{\int_{\mathcal{O}_1 \cup \mathcal{O}_2} h\theta(\mathfrak{p},\mathfrak{s})(|\eta(\mathfrak{p}) - \sigma(\mathfrak{p})|)d\mathfrak{s}}{h\aleph}} \\
&\geq e^{-\sup_{\mathfrak{p} \in \mathcal{O}_1 \cup \mathcal{O}_2} \frac{|\eta(\mathfrak{p}) - \sigma(\mathfrak{p})|}{\aleph}} \\
&= \Gamma_{\mathfrak{b}}(\eta, \sigma, \aleph).
\end{aligned}$$

Hence, all the hypotheses of Theorem 3 are verified; consequently, the integral equation has a unique solution. □

5. Conclusions

In this study, we introduced a new class of controlled bipolar metric spaces in a fuzzy environment, in which the triple-controlled bipolar metric space was used. On the foundation of this variety of controlled bipolar metric spaces, we additionally proved some fixed-point theorems in FCBMSs. In order to strengthen the main results, an additive example and supportive application was also presented. In [20], fixed-point theorems without continuity were provied by using triangular property in FMSs by Shamas et al. It is an interesting open problem to study the triangular property in FCBMSs and obtain fixed-point results on the triangular property in FCBMSs.

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