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On the Existence of Solutions to a Boundary Value Problem via New Weakly Contractive Operator

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Abstract: In this paper, the notion of generalized quasi-weakly contractive operators in metric-like spaces is introduced, and new conditions for the existence of fixed points for such mappings are investigated. A non-trivial example which highlights the novelty of our principal idea is constructed. It is observed comparatively that the proposed concepts herein subsume some important results in the corresponding literature. As an application, one of our obtained findings is utilized to setup novel criteria for the existence of solutions to two-point boundary value problems of a second order differential equation. To attract new researchers in the directions examined in this article, a significant number of corollaries are pointed out and discussed.

Keywords: fixed point; metric-like spaces; partial metric spaces; weakly contractive

MSC: 47H10; 54H25

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1. Introduction

The Banach fixed point theorem (also known as the contraction mapping principle) is an essential tool in the theory of metric spaces. It ensures the existence and uniqueness of fixed points of specific mappings of metric spaces (MSs) and provides a useful search method to find these fixed points. In efforts to explore more fixed point results, several researchers have established generalizations of MSs. The idea of a contraction mapping principle in quasi-metric spaces was introduced by Bakhtin [1]. The latter notion was extended to b-metric spaces by Czerwik [2]. As an improvement of MSs and the corresponding fixed point results, the concept of cone MSs was initiated by Huang and Zhang [3]. In a related development, Mustafa and Sims [4] recently coined a novel approach to generalized MSs. One of the earliest generalizations is the quasi-MS defined by Wilson [5]. In a similar approach, Matthews [6] introduced the concept of partial MS as a part of the investigation into denotational semantics of data flow networks. The main contribution in [6] is the establishment of the fact that self-distance in the partial metric space is not necessarily zero. As a refinement of the partial MS, Amini-Harandi [7] proposed the notion of metric-like space (MlS) by relaxing the axiom of non-negativity and small self-distances in partial MS. In another direction, Alber et al. [8] introduced the idea of weak contraction mappings in the context of Hilbert space by defining additional algebraic structure on the space. Following this, Cho [9] established some fixed point results for weakly contractive mappings in MS which extended some known results. A general remark on invariant point results for weakly contractive operators was made by Aguirre and Reich [10], which formed one of the good reference notes in the literature.

It is noted from the review of the existing literature that little or no work has been conducted on the quasi-weakly contractive operator as a result of MlS. Hence, motivated by the idea in [9], in this manuscript we introduce a new concept of a generalized quasi-weakly contractive operator in MlS and investigate the existence and uniqueness of fixed points of such operators. The idea proposed in this manuscript generalizes several well-known findings in the corresponding literature. Substantial examples are presented to verify our proposed idea and compare it to other corresponding results. A few corollaries which compare our new concepts to other well-known ideas in the literature are presented and analyzed. As an application, in order to investigate new existence conditions for the solution of a class of boundary value problems, one of our obtained corollaries is used. Our proposed ideas herein extend the results of [9,11] and some references therein from complete MS to σ-complete MlS.

The paper is organized as follows: Section 1 presents the introduction and review of the related literature. In Section 2, the fundamental concepts needed in the sequel are collated. The main findings of the paper are discussed in Section 3. Some consequences of our obtained invariant point results in partial metric spaces are established in Section 4. In Section 5, one of the results obtained herein is applied to investigate new conditions for the existence of a solution to a boundary value problem of the second order.

2. Preliminaries

In this section, we record basic ideas needed in later sections.

Definition 1 ([6])**.** *Let Ω be a nonempty set. A function $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}^+$ is called a partial metric on Ω if, for all $l, m, z \in \Omega$, the following conditions are satisfied:*

(1) $\rho(l,l) = \rho(m,m) \Leftrightarrow l = m$;
(2) $\rho(l,l) \leq \rho(l,m)$;
(3) $\rho(l,m) = \rho(m,l)$;
(4) $\rho(l,z) \leq \rho(l,m) + \rho(m,z) - \rho(m,m)$.

The pair (Ω, ρ) is called a partial MS. Note that if $\rho(l,m) = 0$, then $l = m$. An example of a partial metric defined on $\mathbb{R}^+$, is $\rho(l,m) = \max\{l,m\}$, $l,m \geq 0$. For more examples of partial metrics, see [9]. Let the sequence in Ω be $\{l_y\}_{y\in\mathbb{N}}$. Then,

(1) $\{l_y\}_{y\in\mathbb{N}}$ is convergent to l if $\lim\limits_{y\to\infty} \rho(l,l_y) = \rho(l,l)$;
(2) $\{l_y\}_{y\in\mathbb{N}}$ is said to be a Cauchy sequence if $\lim\limits_{y,i\to\infty} \rho(l_y,l_i)$ exists and is finite;
(3) If each Cauchy sequence in Ω converges to a point $l \in \Omega$, then Ω is complete. such that

$$\lim_{y,i\to\infty} \rho(l_y,l_i) = \rho(l,l).$$

Remark 1. *A partial MS Ω is complete if and only if there exists $l \in \Omega$ such that for every Cauchy sequence $\{l_y\}_{y\in\mathbb{N}}$ in Ω,*

$$\lim_{y,i\to\infty} \rho(l_y,l_i) = \rho(l,l).$$

Definition 2 ([7])**.** *A mapping $\sigma : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ is said to be an Ml on Ω if for any $l, m, z \in \Omega$, the following hold:*

(σ_1) $\sigma(l,m) = 0 \Rightarrow l = m$;
(σ_2) $\sigma(l,m) = \sigma(m,l)$;
(σ_3) $\sigma(l,z) \leq \sigma(l,m) + \sigma(m,z)$.

The pair (Ω, σ) is called an MlS.

Definition 3 ([7])**.** *A sequence* $\{l_y\}_{y\in\mathbb{N}}$ *in an MlS* (Ω,σ) *converges to a point* $l\in\Omega$ *if* $\sigma(l,l)=\lim_{y\to\infty}\sigma(l_y,l)$.

Definition 4 ([7])**.** *A sequence* $\{l_y\}_{y\in\mathbb{N}}$ *in an MlS* (Ω,σ) *is called a* σ*-Cauchy sequence if the limit* $\lim_{y,i\to\infty}\sigma(l_y,l_i)$ *exists and is finite. If there is any* $l\in\Omega$ *such that for each* σ*-Cauchy sequence* $\{l_y\}_{y=0}^{\infty}$,

$$\lim_{y\to\infty}\sigma(l_y,l)=\lim_{y,i\to\infty}\sigma(l_y,l_i),$$

then, the MlS (Ω,σ) *is said to be complete.*

Remark 2 ([7])**.** *Every partial MS is an MlS, but the converse is not always true. The example given here recognizes this observation.*

Example 1 ([7])**.** *Let* $\Omega=\{0,1\}$*, and let*

$$\sigma(l,m)=\begin{cases}2, & \text{if } l=m=0;\\ 1, & \text{otherwise.}\end{cases}$$

Then, (Ω,σ) is an MlS, but since $\sigma(0,0)\nleq\sigma(0,1)$, (Ω,σ) is not a partial MS.

Remark 3 ([7])**.** *An Ml on* Ω *satisfies all the conditions of a metric except that* $\sigma(l,l)$ *may be positive for* $l\in\Omega$.

Definition 5 ([12])**.** *Let* (Ω,d) *be an MS. A self-mapping* $\Upsilon:\Omega\longrightarrow\Omega$ *is said to be a quasi-contraction if there exists* $\lambda\in\left[0,\frac{1}{2}\right)$ *such that for all* $l,m\in\Omega$,

$$d(\Upsilon l,\Upsilon m)\le\max\{d(l,m),d(l,\Upsilon l),d(m,\Upsilon m),d(l,\Upsilon m),d(m,\Upsilon l)\}.$$

Definition 6 ([8])**.** *Let* (Ω,d) *be an MS. A mapping* $\Upsilon:\Omega\longrightarrow\Omega$ *is said to be weakly contractive, if for all* $l,m\in\Omega$,

$$d(\Upsilon l,\Upsilon m)\le d(l,m)-\lambda(d(l,m)),$$

where $\lambda:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ *is a continuous and non-decreasing function such that* $\lambda(0)=0$ *and* $\lim_{t\to+\infty}\lambda(t)=+\infty$.

Definition 7. *A function* $f:\Omega\longrightarrow[0,\infty]$*, where* Ω *is an MS, is called lower semi-continuous if, for all* $l\in\Omega$ *and* $\{l_y\}_{y\in\mathbb{N}}\subset\Omega$ *with* $\lim_{y\to\infty}l_y=l$*, we have*

$$f(l)\le\liminf_{y\to\infty}l_y.$$

Let $\Psi=\{\psi:[0,\infty)\to[0,\infty)|\psi$ *be continuous and* $\psi(t)=0\Leftrightarrow t=0\}$*. In addition, let* $\Phi=\{\phi:[0,\infty)\longrightarrow[0,\infty)|\phi$ *be lower semi-continuous and* $\phi(t)=0\Leftrightarrow t=0\}$.

Cho [9] obtained the following result in the context of MS.

Definition 8 ([9])**.** *Let* Ω *be an MS with metric* d*,* $\Upsilon:\Omega\longrightarrow\Omega$ *be a mapping, and let* $\varphi:X\to[0,\infty)$ *be a lower semi-continuous function. Then,* Υ *is called a generalized weakly contractive mapping if it satisfies the following condition:*

$$\begin{aligned}&\psi(d(\Upsilon l,\Upsilon m)+\varphi(\Upsilon l)+\varphi(\Upsilon m)\\ &\quad\le\ \psi(\mathcal{M}(l,m,\varphi))-\phi(\mathcal{N}(l,m,\varphi))\end{aligned}\tag{1}$$

for all $l, m \in \Omega$, where $\psi \in \Psi$, $\phi \in \Phi$ and

$$\mathcal{M}(l,m,\varphi) = \max\left\{ \begin{array}{c} d(l,m) + \varphi(l) + \varphi(m) + d(l,\Upsilon l) + \varphi(l) + \varphi(\Upsilon l), \\ d(l,\Upsilon m) + \varphi(m) + \varphi(\Upsilon m), \\ \frac{1}{2}[d(l,\Upsilon m) + \varphi(l) + \varphi(\Upsilon m) + d(m,\Upsilon l) + \varphi(m) + \varphi(\Upsilon l) \end{array} \right\},$$

$$\mathcal{N}(l,m,\varphi) = \max\{d(l,m) + \varphi(l) + \varphi(m), d(m,\Upsilon m) + \varphi(m) + \varphi(\Upsilon m)\}.$$

The main result of [9] is as follows.

Theorem 1 ([9]). *Let Ω be a complete MS. If Υ is a generalized weakly contractive mapping, then there exists a unique $z \in \Omega$ such that $z = \Upsilon z$ and $\varphi(z) = 0$.*

Lemma 1. *Let (Ω, σ) be an MlS, and let $\{l_y\}_{y\in\mathbb{N}}$ be a sequence in Ω such that if $\{l_y\}_{y\in\mathbb{N}}$ is not a σ-Cauchy sequence in (Ω, σ). Then, there exist $\epsilon^+ > 0$ and two subsequences $\{l_{y(k)}\}_{k\in\mathbb{N}}$ and $\{l_{i(k)}\}_{y\in\mathbb{N}}$ of $\{l_y\}_{k\in\mathbb{N}}$, where y, i are positive integers with $y(k) > i(k) > k$ such that*

$$\sigma(l_{i(k)}, l_{y(k)}) \geq \epsilon^+ \tag{2}$$

and

$$\sigma(l_{i(k)-1}, l_{y(k)}) < \epsilon^+. \tag{3}$$

Moreover, suppose that

$$\lim_{y\to\infty} \sigma(l_y, l_{y+1}) = 0. \tag{4}$$

Then, the following hold:

(1) $\lim_{k\to\infty} \sigma(l_{i(k)}, l_{y(k)}) = \epsilon^+$;
(2) $\lim_{k\to\infty} \sigma(l_{i(k)}, l_{y(k)+1}) = \epsilon^+$;
(3) $\lim_{k\to\infty} \sigma(l_{i(k-1)}, l_{y(k)}) = \epsilon^+$;
(4) $\lim_{k\to\infty} \sigma(l_{i(k)-1}, l_{y(k+1)}) = \epsilon^+$.

Proof. Suppose that $\{l_y\}_{y\in\mathbb{N}}$ is not a σ-Cauchy sequence in (Ω, σ). Then, there exist $\epsilon^+ > 0$ and sequences $\{l_{y(k)}\}_{k\in\mathbb{N}}$ and $\{l_{i(k)}\}_{k\in\mathbb{N}}$ of positive integers y, i with $y(k) > i(k) > k$, such that

$$\sigma(l_{i(k)},\ l_{y(k-1)}) < \epsilon^+,\ \sigma(l_{i(k)}, l_{y(k)}) \geq \epsilon^+,$$

for all $k \in \mathbb{N}$. Then,

$$\begin{aligned} \epsilon^+ \leq \sigma(l_{i(k)}, l_{y(k)}) \leq&\ \sigma(l_{y(k)}, l_{y(k-1)}) + \sigma(l_{y(k-1)}, l_{i(k)}) \\ <&\ \epsilon^+ + \sigma(l_{y(k)}, l_{y(k-1)}). \end{aligned} \tag{5}$$

Applying (4), we deduce from (5) that

$$\lim_{k\to\infty} \sigma(l_{i(k)}, l_{y(k)}) = \epsilon^+. \tag{6}$$

Moreover,

$$\sigma(l_{i(k)}, l_{y(k)}) \leq \sigma(l_{i(k)}, l_{y(k+1)}) + \sigma(l_{y(k+1)}, l_{y(k)})$$

and

$$\sigma(l_{i(k)}, l_{y(k+1)}) \leq \sigma(l_{i(k)}, l_{y(k)}) + \sigma(l_{y(k)}, l_{y(k+1)}).$$

Letting $k \to \infty$ in the above two expressions and employing (4) and (6), we have

$$\lim_{k\to\infty} \sigma(l_{i(k)}, l_{y(k+1)}) = \epsilon^{+}.$$

In similar steps, we can show that the sequences in (3) and (4) tend to ϵ^{+}. □

3. Main Results

In this section, we introduce the concept of a generalized quasi-weakly contractive operator in the framework of MlS and examine the conditions for the existence of a fixed point of such an operator.

Definition 9. *Let (Ω, σ) be an MlS. A self-mapping $\Upsilon : \Omega \longrightarrow \Omega$ is called a generalized quasi-weakly contractive operator, if it satisfies the following condition:*

$$\psi(\sigma(\Upsilon l, \Upsilon l) + \sigma(\Upsilon l, \Upsilon m) + \varphi(\Upsilon l) + \varphi(\Upsilon m)) \leq \psi(\mathcal{C}(l, m, \varphi)) - \phi(\mathcal{L}(l, m, \varphi)), \tag{7}$$

for all $l, m \in \Omega$, where $\psi \in \Psi$, $\phi, \varphi \in \Phi$ and

$$\mathcal{C}(l, m, \varphi) = \max \left\{ \begin{array}{c} \sigma(l, l) + \sigma(l, m) + \varphi(l) + \varphi(m), \sigma(l, \Upsilon l) + \varphi(l) + \varphi(\Upsilon l), \\ +\sigma(m, m)\sigma(m, \Upsilon m) + \varphi(m) + \varphi(\Upsilon m), \\ \frac{1}{2}[\sigma(l, \Upsilon m) + \varphi(l) + \varphi(\Upsilon m) \\ +\sigma(m, \Upsilon l) + \varphi(m) + \varphi(\Upsilon l)] \end{array} \right\} \tag{8}$$

$$\mathcal{L}(l, m, \varphi) = \max \left\{ \begin{array}{c} \sigma(l, l) + \sigma(l, m) + \varphi(l) + \varphi(m), \\ \sigma(m, m) + \sigma(m, \Upsilon m) + \varphi(m) + \varphi(\Upsilon m) \end{array} \right\}. \tag{9}$$

The following is the main result of this paper.

Theorem 2. *Let (Ω, σ) be a σ-complete MlS. If Υ is a generalized quasi-weakly contractive operator, then there exists a unique $u \in \Omega$ such that $u = \Upsilon u$ and $\varphi(u) = 0$.*

Proof. Starting from an arbitrary point $l := l_0 \in \Omega$, we will construct a recursive sequence $\{l_y\}_{y\in\mathbb{N}}$ in the following manner:

$$l_0 := l \text{ and } l_y = \Upsilon l_{y-1}, \text{ for all } n \in \mathbb{N}.$$

We presume that $l_y \neq l_{y-1}$ for all $y \in \mathbb{N}$. In fact, if for some $y \in \mathbb{N}$, it is observed that the expression $l_y = l_{y-1} = \Upsilon l_{y-1}$, then the proof is finished.

By replacing $l = l_{y-1}$ and $m = l_y$ in (8), we obtain

$$\begin{aligned} \mathcal{C}(l_{y-1}, l_y, \varphi) &= \max \left\{ \begin{array}{c} \sigma(l_{y-1}, l_{y-1}) + \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y), \\ \sigma(l_{y-1}, \Upsilon l_{y-1}) + \varphi(l_{y-1}) + \varphi(\Upsilon l_{y-1}), \\ \sigma(l_y, l_y) + \sigma(l_y, \Upsilon l_y) + \varphi(l_y) + \varphi(\Upsilon l_y), \\ \frac{1}{2}[\sigma(l_{y-1}, \Upsilon l_y) + \varphi(l_{y-1}) + \varphi(\Upsilon l_y) \\ +\sigma(l_y, \Upsilon l_{y-1}) + \varphi(l_y) + \varphi(\Upsilon l_{y-1})] \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \sigma(l_{y-1}, l_{y-1}) + \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y), \\ \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y), \\ \sigma(l_y, l_y) + \sigma(l_y, l_{y+1}) + \varphi(l_y) + \varphi(l_{y+1}), \\ \frac{1}{2}[\sigma(l_{y-1}, l_{y+1}) + \varphi(l_{y-1}) + \varphi(l_{y+1}) \\ +\sigma(l_y, l_y) + \varphi(l_y) + \varphi(l_y)] \end{array} \right\}. \end{aligned} \tag{10}$$

We observe that

$$\begin{aligned}
&\frac{1}{2}[\sigma(l_{y-1},l_{y+1})+\varphi(l_{y-1})+\varphi(l_{y+1})+\sigma(l_y,l_y)+\varphi(l_y)+\varphi(l_y)]\\
&\quad\le\frac{1}{2}[\sigma(l_{y-1},l_y)+\sigma(l_y,l_{y+1})+\varphi(l_{y-1})+\varphi(l_{y+1})+\sigma(l_y,l_y)+\varphi(l_y)+\varphi(l_y)]\\
&\quad\le\max\{\sigma(l_{y-1},l_y)+\varphi(l_{y-1})+\varphi(l_y),\\
&\ \sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1})\}.
\end{aligned}$$

Hence, (10) becomes

$$\begin{aligned}
&\mathcal{C}(l_{y-1},l_y,\varphi)=\max\{\sigma(l_{y-1},l_{y-1})+\sigma(l_{y-1},l_y)+\varphi(l_{y-1})+\varphi(l_y),\\
&\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1})\}).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\mathcal{L}(l_{y-1},l_y,\varphi)&=\max\left\{\begin{array}{c}\sigma(l_{y-1},l_{y-1})+\sigma(l_{y-1},l_y)+\varphi(l_{y-1})+\varphi(l_y),\\ \sigma(l_y,\Upsilon l_y)+\varphi(\Upsilon l_y)+\sigma(l_y,l_y)+\varphi(l_y)\})\end{array}\right\}\\
&=\max\left\{\begin{array}{c}\sigma(l_{y-1},l_{y-1})+\sigma(l_{y-1},l_y)+\varphi(l_{y-1})+\varphi(l_y),\\ \sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_{y+1})+\varphi(l_y)\end{array}\right\}.
\end{aligned}$$

Consequently, (7) gives

$$\begin{aligned}
&\psi(\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1}))\\
&=\psi(\sigma(\Upsilon l_{y-1},\Upsilon l_{y-1})+\sigma(\Upsilon l_{y-1},\Upsilon l_y)+\varphi(\Upsilon l_{y-1})+\varphi(\Upsilon l_y)\\
&\le\psi(\mathcal{C}(l_{y-1},l_y,\varphi))-\phi(\mathcal{L}(l_{y-1},l_y,\varphi)).
\end{aligned}\tag{11}$$

If

$$\begin{aligned}
&\sigma(l_{y-1},l_{y-1})+\sigma(l_{y-1},l_y)+\varphi(l_{y-1})+\varphi(l_y)\\
&\quad<\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1})
\end{aligned}$$

for some positive integer y, then it follows from (11) that

$$\begin{aligned}
&\psi(\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1}))\\
&\quad\le\psi(\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1}))\\
&\quad-\phi(\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1}))
\end{aligned}$$

which implies that

$$\phi(\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1}))=0.$$

Hence,

$$\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1})=0,$$

from which we notice that

$$l_y=l_{y+1},\quad \sigma(l_y,l_y)=0\quad\text{and}\quad\varphi(l_y)=\varphi(l_{y+1})=0$$

which is a contradiction. Therefore,

$$\begin{aligned}
&\sigma(l_y,l_y)+\sigma(l_y,l_{y+1})+\varphi(l_y)+\varphi(l_{y+1})\\
&\le\ \sigma(l_{y-1},l_{y-1})+\sigma(l_{y-1},l_y)+\varphi(l_{y-1})+\varphi(l_y)
\end{aligned}\tag{12}$$

for all $y = 1, 2, 3, \ldots$

Hence,

$$\mathcal{C}(l_{y-1}, l_y, \varphi) = \sigma(l_{y-1}, l_{y-1}) + \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y)$$

and

$$\mathcal{L}(l_{y-1}, l_y, \varphi) = \sigma(l_{y-1}, l_{y-1}) + \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y).$$

From (11), we have

$$\begin{aligned} &\psi(\sigma(l_y, l_y) + \sigma(l_y, l_{y+1}) + \varphi(l_y) + \varphi(l_{y+1})) \\ &\leq \ \psi(\sigma(l_{y-1}, l_{y-1}) + \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y)) \\ &- \phi(\sigma(l_{y-1}, l_{y-1}) + \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y)). \end{aligned} \tag{13}$$

It follows from (12) that the sequence $\{\sigma(l_y, l_y) + \sigma(l_y, l_{y+1}) + \varphi(l_y) + \varphi(l_{y+1})\}$ is bounded below and non-increasing.

Therefore,

$$\sigma(l_y, l_y) + \sigma(l_y, l_{y+1}) + \varphi(l_y) + \varphi(l_{y+1}) \to \eta \ as \ n \to \infty,$$

for some $\eta \geq 0$.

Suppose that $\eta > 0$. Taking limit in (13) as $y \to \infty$, using the continuity of ψ and the lower semi-continuity of ϕ, lead to

$$\begin{aligned} \psi(\eta) &\leq \psi(\eta) - \liminf_{y\to\infty} \phi(\sigma(l_{y-1}, l_{y-1}) + \sigma(l_{y-1}, l_y) + \varphi(l_{y-1}) + \varphi(l_y)) \\ &\leq \ \psi(\eta) - \phi(\eta) < \psi(\eta), \end{aligned}$$

which is a contradiction. Thus, $\lim_{y\to\infty}(\sigma(l_y, l_y) + \sigma(l_y, l_{y+1}) + \varphi(l_y) + \varphi(l_{y+1})) = 0$, from which we have

$$\lim_{y\to\infty} \sigma(l_y, l_y) = \lim_{y\to\infty} \sigma(l_y, l_{y+1}) = 0 \tag{14}$$

and

$$\lim_{y\to\infty} \varphi(l_y) = \lim_{y\to\infty} \varphi(l_{y+1}) = 0. \tag{15}$$

Now, we prove that the sequence $\{l_y\}_{y\in\mathbb{N}}$ is Cauchy. Assume that $\{l_y\}_{y\in\mathbb{N}}$ is not Cauchy. Then, by Lemma 1, there exist $\epsilon^+ > 0$ and subsequences $\{l_{y(k)}\}_{k\in\mathbb{N}}$ and $\{l_{i(k)}\}_{k\in\mathbb{N}}$ of $\{l_y\}_{y\in\mathbb{N}}$ such that (2) and (3) hold.

From (8), we have

$$\mathcal{C}(l_{y(k)}, l_{i(k)}, \varphi) = \max\left\{\begin{array}{c} \sigma(l_{y(k)}, l_{y(k)}) + \sigma(l_{y(k)}, l_{i(k)}) + \varphi(l_{y(k)}) + \varphi(l_{i(k)}), \\ \sigma(l_{y(k)}, \Upsilon l_{y(k)}) + \varphi(l_{y(k)}) + \varphi(\Upsilon l_{y(k)}), \\ \sigma(l_{y(k)}, l_{y(k)}) + \sigma(l_{i(k)}, \Upsilon l_{i(k)}) + \varphi(l_{i(k)}) + \varphi(\Upsilon l_{i(k)}), \\ \frac{1}{2}[\sigma(l_{y(k)}, \Upsilon l_{i(k)}) + \varphi(l_{y(k)}) + \varphi(\Upsilon l_{i(k)}) + \sigma(l_{i(k)}, \Upsilon l_{y(k)}) \\ +\varphi(l_{i(k)}) + \varphi(\Upsilon l_{y(k)})]. \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \sigma(l_{y(k)}, l_{y(k)}) + \sigma(l_{y(k)}, l_{i(k)}) + \varphi(l_{y(k)}) + \varphi(l_{i(k)}), \\ \sigma(l_{y(k)}, l_{y(k)+1}) + \varphi(l_{y(k)}) + \varphi(l_{y(k)+1}), \\ \sigma(l_{i(k)}, l_{i(k)}) + \sigma(l_{i(k)}, l_{i(k)+1}) + \varphi(l_{i(k)}) + \varphi(l_{i(k)+1}) \\ \frac{1}{2}[\sigma(l_{y(k)}, l_{i(k)+1}) + \varphi(l_{y(k)}) + \varphi(l_{i(k)+1}) + \sigma(l_{i(k)}, l_{y(k)+1}) \\ +\varphi(l_{i(k)}) + \varphi(l_{y(k)+1})]\} \end{array}\right\}. \tag{16}$$

As $k \to \infty$ in (16), applying Lemma 1 and using Equations (14) and (15) yield

$$\lim_{k\to\infty} \mathcal{C}(l_{y(k)}, l_{i(k)}, \varphi) = \epsilon^{+}. \tag{17}$$

On similar steps, it follows from (9) that

$$\begin{aligned}
\mathcal{L}(l_{y(k)}, l_{i(k)}, \varphi) &= \max\left\{ \begin{array}{c} \sigma(l_{y(k)}, l_{y(k)}) + \sigma(l_{y(k)}, l_{i(k)}) + \varphi(l_{y(k)}) + \varphi(l_{i(k)}), \\ \sigma(l_{i(k)}, l_{i(k)}) + \sigma(l_{i(k)}, \Upsilon l_{i(k)}) + \varphi(l_{i(k)}) + \varphi(\Upsilon l_{i(k)}) \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} \sigma(l_{y(k)}, l_{y(k)}) + \sigma(l_{y(k)}, l_{i(k)}) + \varphi(l_{y(k)}) + \varphi(l_{i(k)}), \\ \sigma(l_{i(k)}, l_{i(k)}) + \sigma(l_{i(k)}, l_{i(k)+1}) + \varphi(l_{i(k)}) + \varphi(l_{i(k)+1}) \end{array} \right\}.
\end{aligned}$$

Thus,

$$\lim_{k\to\infty} \mathcal{L}(l_{y(k)}, l_{i(k)}, \varphi) = \epsilon^{+}. \tag{18}$$

From (7), we have

$$\begin{aligned}
&\psi(\sigma(l_{y(k)+1}, l_{y(k)+1}) + \sigma(l_{y(k)+1}, l_{i(k)+1}) + \varphi(l_{y(k)+1}) + \varphi(l_{i(k)+1}) \\
&\leq \psi(\mathcal{C}(l_{y(k)}, l_{i(k)}, \varphi)) - \phi(\mathcal{L}((l_{y(k)}, l_{i(k)}, \varphi)).
\end{aligned} \tag{19}$$

Letting $k \to \infty$ in (19), and using Lemma 1, the continuity of ψ, the lower semi-continuity of ϕ and by using Equations (15), (17) and (18), we obtain $\psi(\epsilon^{+}) \leq \psi(\epsilon^{+}) - \phi(\epsilon^{+})$, which is a contradiction because $\phi(\epsilon^{+}) > 0$. Therefore, $\{l_y\}_{y\in\mathbb{N}}$ is a Cauchy sequence. The completeness of Ω implies that there exists $u \in \Omega$ such that $\lim_{y\to\infty} l_y = u$. Given that ϕ is lower semi-continuous, $\varphi(u) \leq \liminf_{y\to\infty} \varphi(l_y) \leq \lim_{y\to\infty} \varphi(l_y) = 0$, from which it follows that $\varphi(u) = 0$.

Now, from (8), we obtain

$$\begin{aligned}
\mathcal{C}(l_{y(k)}, u, \varphi) &= \max\left\{ \begin{array}{c} \sigma(l_y, l_y) + \sigma(l_y, u) + \varphi(l_y) + \varphi(u), \\ \sigma(l_y, \Upsilon l_y) + \varphi(l_y) + \varphi(\Upsilon l_y), \\ \sigma(u, u) + \sigma(u, \Upsilon u) + \varphi(u) + \varphi(\Upsilon u), \\ \frac{1}{2}[\sigma(l_y, \Upsilon u) + \varphi(l_y) + \varphi(\Upsilon u) + \sigma(u, \Upsilon l_y) \\ +\varphi(u) + \varphi(\Upsilon l_y)] \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} \sigma(l_y, l_y) + \sigma(l_y, u) + \varphi(l_y) + \varphi(u), \\ \sigma(l_y, l_{y+1}) + \varphi(l_y) + \varphi(l_{y+1}), \\ \sigma(u, u) + \sigma(u, \Upsilon u) + \varphi(u) + \varphi(\Upsilon u), \\ \frac{1}{2}[\sigma(l_y, \Upsilon u) + \varphi(l_y) + \varphi(\Upsilon u) + \sigma(u, l_{y+1}) \\ +\varphi(u) + \varphi(l_{y+1})] \end{array} \right\},
\end{aligned}$$

from which we have

$$\begin{aligned}
\lim_{y\to\infty} \mathcal{C}(l_y, u, \varphi) &= \lim_{y\to\infty} \max\{\sigma(u, u), \sigma(u, u) + \sigma(u, \Upsilon u) + \varphi(\Upsilon u), \\
&\quad \frac{1}{2}[\sigma(u, u) + \sigma(u, \Upsilon u) + \varphi(\Upsilon u)]\} \\
&= \sigma(u, u) + \sigma(u, \Upsilon u) + \varphi(\Upsilon u).
\end{aligned} \tag{20}$$

In like manner, we have

$$\begin{aligned}\lim_{y\to\infty}\mathcal{L}(l_y,u,\varphi)&=\lim_{y\to\infty}\max\{\sigma(l_y,l_y)+\sigma(l_y,u)+\varphi(l_y)+\varphi(u),\\&\qquad\sigma(u,u)+\sigma(u,\Upsilon u)+\varphi(u)+\varphi(\Upsilon u)\}\\&=\max\{\sigma(u,u),\sigma(u,u)+\sigma(u,\Upsilon u)+\varphi(\Upsilon u)\}\\&=\sigma(u,u)+\sigma(u,\Upsilon u)+\varphi(\Upsilon u).\end{aligned}\tag{21}$$

Therefore, from (7), we have

$$\begin{aligned}&\psi(\sigma(l_{y+1},l_{y+1})+\sigma(l_{y+1},\Upsilon u)+\varphi(l_{y+1})+\varphi(\Upsilon u))\\&=\psi(\sigma(\Upsilon l_y,\Upsilon l_y)+\sigma(\Upsilon l_y,\Upsilon u)+\varphi(\Upsilon l_y)+\varphi(\Upsilon u))\\&\leq\ \psi(C(l_y,u,\varphi))-\phi(L(l_y,u,\varphi)).\end{aligned}\tag{22}$$

Letting $y\to\infty$ in (22) and employing the continuity of ψ, the lower continuity of ϕ and using Equations (20) and (19), we have

$$\begin{aligned}&\psi(\sigma(u,u)+\sigma(u,\Upsilon u)+\varphi(\Upsilon u))\\&\leq\ \psi(\sigma(u,u)+\sigma(u,\Upsilon u)+\varphi(\Upsilon u))-\phi(\sigma(u,u)+\sigma(u,\Upsilon u)+\varphi(\Upsilon u)).\end{aligned}\tag{23}$$

The expression (23) implies that

$$\begin{aligned}\phi(\sigma(u,\Upsilon u)+\varphi(\Upsilon u)+\sigma(u,u))&=0\text{ and hence}\\\sigma(u,\Upsilon u)+\varphi(\Upsilon u)+\sigma(u,u)&=0.\end{aligned}$$

Therefore, $u=\Upsilon u,\ \varphi(\Upsilon u)=0$ and $\sigma(u,u)=0$.

To see uniqueness, suppose that p is another fixed point of Υ with $l=u$ and $m=p$. Then, $p=\Upsilon p$ and $\varphi(p)=0$. Now, using (7), we have

$$\begin{aligned}\psi(\sigma(u,u)+\sigma(p,u))&=\psi(\sigma(\Upsilon u,\Upsilon u)+\sigma(\Upsilon p,\Upsilon u))\\&=\psi(\sigma(\Upsilon u,\Upsilon u)+\sigma(\Upsilon u,\Upsilon p)+\varphi(\Upsilon u)+\varphi(\Upsilon p)\\&\leq\ \psi(C(u,p,\varphi))-\phi(L(u,p,\varphi))\\&=\psi(\sigma(u,u)+\sigma(p,u))-\phi(\sigma(u,u)+\sigma(p,u)).\end{aligned}$$

Consequently, $u=p$. □

We construct the following example to verify the hypotheses of Theorem 2.

Example 2. *Let $\Omega=\{0,1,2\}$ together with the metric $\sigma:\Omega\times\Omega\in\mathbb{R}^+$ defined by $\sigma(0,0)=\sigma(1,1)=0$, $\sigma(2,2)=\frac{9}{20}$, $\sigma(0,2)=\sigma(2,0)=\frac{2}{5}$, $\sigma(1,2)=\sigma(2,1)=\frac{3}{5}$, $\sigma(0,1)=\sigma(1,0)=\frac{1}{2}$. Then, (Ω,σ) is a σ-complete MlS. Notice that $\sigma(2,2)\neq 0$. Hence, σ is not a metric. In addition, $\sigma(2,2)>\sigma(2,0)$, implying that σ is not a partial metric. Define a self-mapping $\Upsilon:\Omega\longrightarrow\Omega$ by $\Upsilon(0)=\Upsilon(1)=0$ and $\Upsilon(2)=1$.*
To see that Υ is a generalized quasi-weakly contractive operator, let $\psi(t)=\frac{4t}{5}$, $\phi(t)=\frac{2t}{5}$ and $\varphi(t)=\frac{3t}{5}$. We then consider the following cases:
Case 1: *$l,m\in\Omega$, $l=m$;*
Case 2: *$l,m\in\Omega$, $l\neq m$.*

We demonstrate using the following Table 1 that inequality (7) is satisfied for each of the above cases.

Table 1. Table of values for Cases 1 and 2.

Cases	l	m	LHS of (7)	RHS of (7)
Case 1	0	0	0	0
	1	1	0	0.48
	2	2	0.96	1.32
Case 2	0	1	0	0.44
	0	2	0.88	1.14
	1	0	0	0.44
	1	2	0.88	1.14
	2	0	0.88	1.1
	2	1	0.88	1.14

In the following Figure 1, we illustrate the validity of contractive inequality (7) using Example 2.

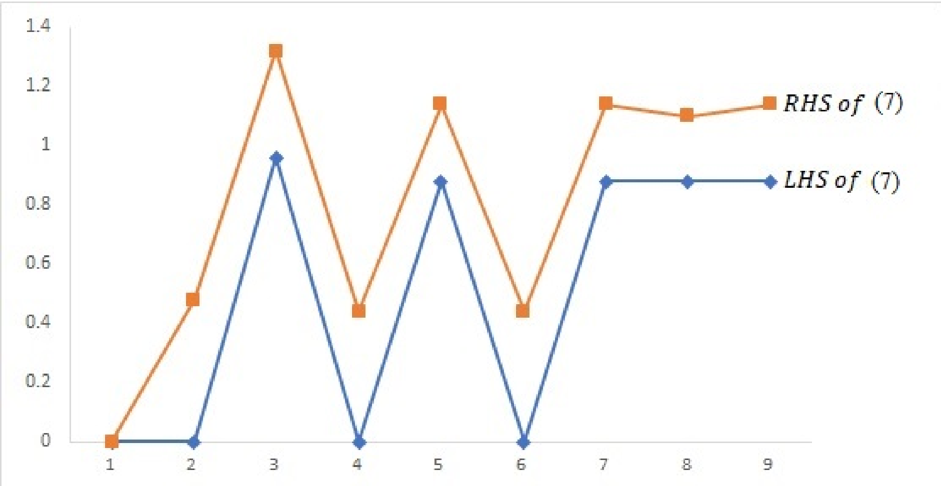

Figure 1. Illustration of contractive inequality (7) using Example 2.

Therefore, all the hypotheses of Theorem 2 are satisfied, and Υ *has a fixed point,* $l = 0$*. Consequently,* Υ *is a generalized quasi-weakly contractive operator.*

To see that the generalized quasi-weakly contractive operator introduced in this manuscript is not the generalized weakly contractive operator introduced by Cho [9], let Ω *be equipped with the Euclidean metric* d*. Then,* (Ω, d) *is a complete MS. However, taking any points* $l, m \in \{1,2\} \subset \Omega$*, we see that*

$$\begin{aligned} \psi(\sigma(\Upsilon l, \Upsilon m) + \varphi(\Upsilon l) + \varphi(\Upsilon m) &= \frac{4}{5}\left(\frac{8}{5}\right) = \frac{32}{25} \\ &> \frac{28}{25} = \frac{4}{5}\left(\frac{14}{5}\right) - \frac{2}{5}\left(\frac{14}{5}\right) \\ &= \frac{4}{5}\left(\max\left(\frac{14}{5}, \frac{14}{5}, \frac{8}{5}, \frac{11}{5}\right)\right) - \frac{2}{5}\left(\max\left(\frac{14}{5}, \frac{8}{5}\right)\right) \\ &= \psi(\mathcal{M}(l, m, \varphi)) - \phi(\mathcal{N}(l, m, \varphi)). \end{aligned}$$

Therefore, the generalized quasi-weakly contractive operator is not the generalized weakly contractive mapping defined by Cho [9], and so Theorem 1 due to Cho [9] is not applicable to this example.

In what follows, we present some consequences of Theorem 2.

Corollary 1. *Let* (Ω, σ) *be a* σ*-complete MIS. Suppose that the self-mapping* Υ *satisfies the following condition:*

$$\begin{aligned}&\psi(\sigma(\Upsilon l,\Upsilon l)+\sigma(\Upsilon l,\Upsilon m)+\varphi(\Upsilon l)+\varphi(\Upsilon m))\\&\leq\ \psi(\mathcal{C}(l,m,\varphi))-\phi(\mathcal{C}(l,m,\varphi)),\end{aligned}\tag{24}$$

for all $l,m\in\Omega$, where $\psi\in\Psi$ and $\phi,\varphi\in\Phi$. Then, there exists a unique $u\in\Omega$ such that $u=\Upsilon u$ and $\varphi(u)=0$.

By taking $\phi(t)=0$, for all $t\in\mathbb{R}^+$, we have the next result.

Corollary 2. *Let (Ω,σ) be a σ-complete MlS. Assume that the self-mapping Υ satisfies the following condition:*

$$\psi(\sigma(\Upsilon l,\Upsilon l)+\sigma(\Upsilon l,\Upsilon m)+\varphi(\Upsilon l)+\varphi(\Upsilon m))\leq\psi(\mathcal{C}(l,m,\varphi)),\tag{25}$$

for all $l,m\in\Omega$, where $\psi\in\Psi$. Then, there exists a unique $u\in\Omega$ such that $u=\Upsilon u$ and $\varphi(u)=0$.

Corollary 3. *Let (Ω,σ) be a σ-complete MlS. Suppose that the self-mapping Υ satisfies the following condition:*

$$\sigma(\Upsilon l,\Upsilon l)+\sigma(\Upsilon l,\Upsilon m)+\varphi(\Upsilon l)+\varphi(\Upsilon m))\leq\mathcal{C}(l,m,\varphi),\tag{26}$$

for all $l,m\in\Omega$. Then there exists a unique $u\in\Omega$ such that $u=\Upsilon u$ and $\varphi(u)=0$.

Proof. Take $\psi(t)=t$ for all $t\in\mathbb{R}^+$ in Corollary 2. □

Corollary 4. *Let (Ω,σ) be a σ-complete MlS. Suppose that the self-mapping Υ satisfies the following condition:*

$$\begin{aligned}&\psi(\sigma(\Upsilon l,\Upsilon l)+\sigma(\Upsilon l,\Upsilon m)+\varphi(\Upsilon l)+\varphi(\Upsilon m))\\&\leq\ \psi(\sigma(l,l)+\sigma(l,m)+\varphi(l)+\varphi(m))-\phi(\sigma(l,l)+\sigma(l,m)+\varphi(l)+\varphi(m)),\end{aligned}$$

for all $l,m\in\Omega$, where $\psi\in\Psi$ and $\phi,\varphi\in\Phi$. Then, there exists a unique $u\in\Omega$ such that $u=\Upsilon u$ and $\varphi(u)=0$.

Corollary 5. *Let (Ω,σ) be a σ-complete MlS. Suppose that the self-mapping Υ satisfies the following condition:*

$$\begin{aligned}&\psi(\sigma(\Upsilon^k x,\Upsilon^k l)+\sigma(\Upsilon^k x,\Upsilon^k m)+\varphi(\Upsilon^k l)+\varphi(\Upsilon^k m))\\&\leq\ \psi(\mathcal{C}(l,m,\varphi))-\phi(\mathcal{L}(l,m,\varphi)),\end{aligned}$$

for all $l,m\in\Omega$, where $\psi\in\Psi$, $\phi,\varphi\in\Phi$ and k is a positive integer. Then, there exists a unique $u\in\Omega$ such that $u=\Upsilon u$, and $\varphi(u)=0$.

Proof. Let $S=\Upsilon^k$. Then, by Theorem 2, S has a unique fixed point, say u. Then $\Upsilon^k u=Su=u$ and

$$\varphi(u)=\varphi(Su)=\varphi(\Upsilon^k u)=0.$$

Since $\Upsilon^{k+1}u=\Upsilon u$,

$$ST u=\Upsilon^k(\Upsilon u)=\Upsilon^{k+1}u=\Upsilon u,$$

and so Υu is a fixed point of S. By the uniqueness of a fixed point of S, $\Upsilon u=u$. □

We construct the following example to support the hypothesis of Corollary 1.

Example 3. *Let $\Omega=\{0,1,2,3\}$ and $\sigma(l,m)=l+m$, for all $l,m\in\Omega$. Then, (Ω,σ) is a σ-complete MlS. Note that σ is not a metric, since for $l=1=m$, $\sigma(1,1)=2>0$. Similarly, σ is not a partial metric, since for $l=2$ and $m=1$, $\sigma(2,2)>\sigma(2,1)$. Now, define the self-mapping $\Upsilon:\Omega\longrightarrow\Omega$ by $\Upsilon l=\frac{l^2}{3(1+l)}$ for all $l\in\Omega$. In addition, let $\psi(t)=\frac{4}{5}t$, $\phi(t)=\frac{t}{2}$ and $\varphi(t)=\frac{9}{10}t$.*

Obviously, $\psi \in \Psi$, $\phi, \varphi \in \Phi$. To show that the contractive inequality (24) *holds, we consider the following cases:*

Case 1: $l, m \in \Omega, l = m$;
Case 2: $l, m \in \Omega, l \neq m$.

We demonstrate using the following Table 2 that inequality (24) *is satisfied for each of the above cases.*

Table 2. Illustration of the contractive inequality (24).

Cases	l	m	LHS of (24)	RHS of (24)
Case 1	0	0	0	0
	1	1	0.7733	1.74
	2	2	2.0622	3.48
	3	3	3.48	5.22
Case 2	0	1	0.2533	1.265
	0	2	0.6755	2.5933
	0	3	1.14	3.9375
	1	0	0.52	1.17
	1	2	1.1955	2.5933
	1	3	1.66	3.9375
	2	0	1.3866	2.34
	2	1	1.64	2.91
	2	3	2.5266	4.05
	3	0	2.34	3.51
	3	1	2.5933	4.08
	3	2	3.0155	4.65

In the following Figure 2, we illustrate that under the above cases, inequality (24) *using Example 3 is satisfied.*

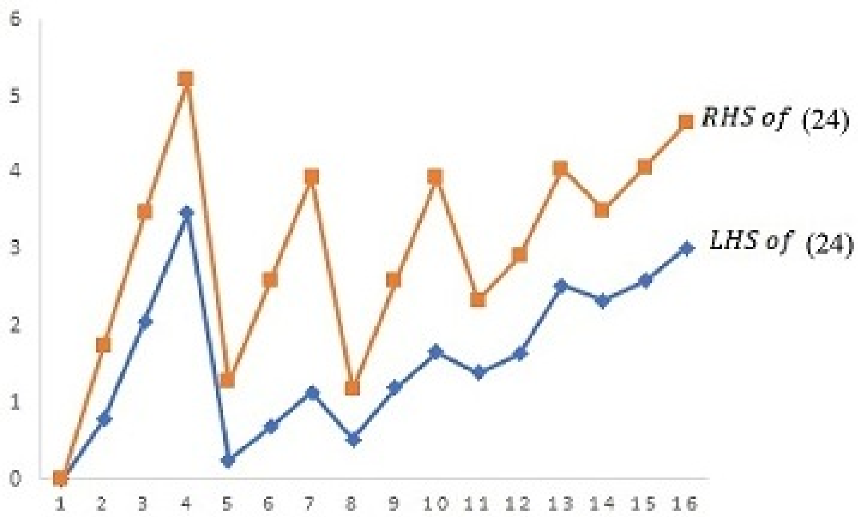

Figure 2. Illustration of contractive inequality (24) using Example 3.

Hence, all the assumptions of Theorem 2 are satisfied. We therefore see that $u = 0$ is a fixed point of Υ.

4. Applications to Fixed Point Results in Partial MS

In this section, we give some applications to fixed point theorems in partial metric spaces. To deduce partial metric version of our results, we consider an auxiliary function $\rho_q : \Omega \times \Omega \longrightarrow \mathbb{R}^+$ given as

$$\rho_q(l,l) + \rho_q(l,m) = 2\rho(l,m) - 2\rho(l,l) + \rho(m,m). \tag{27}$$

It is clear that the mapping ρ_q is an Ml on Ω.

Consistent with [6], we have the following observation.

Remark 4. *Let $\{l_y\}_{y\in\mathbb{N}}$ be a sequence in Ω. If the sequence $\{l_y\}_{y\in\mathbb{N}}$ is convergent to l in (Ω, ρ_q), then it is convergent to l in (Ω, ρ), and the converse is not always true.*

Theorem 3. *Let Ω be a complete partial MS. Suppose that $\Upsilon : \Omega \longrightarrow \Omega$ is a mapping such that*

$$\psi(\rho(\Upsilon l, \Upsilon l) + \rho(\Upsilon l, \Upsilon m) + \varphi(\Upsilon l) + \varphi(\Upsilon m)) \leq \psi(\mathcal{C}_q(l,m,\varphi)) - \phi(\mathcal{L}(l,m,\varphi)), \tag{28}$$

for all $l,m \in \Omega$, where $\psi \in \Psi$, $\varphi, \phi \in \Phi$, $\mathcal{C}_q = \max\left\{\rho(l,m), \rho(m,\Upsilon m), \frac{\rho(m,\Upsilon l)}{2}\right\}$ and $\mathcal{L} = \max\{\rho(l,\Upsilon m), \rho(m,\Upsilon l)\}$.

Then, there exists a unique $u \in \Omega$ such that $u = \Upsilon u$ and $\rho(u,u) = 0$.

Proof. From (27), we have

$$\rho(l,m) = \frac{\rho_q(l,l) + \rho_q(l,m) + 2\rho(l,l) - \rho(m,m)}{2},$$

for all $l,m \in \Omega$. Let $\sigma(l,l) = \frac{\rho_q(l,l)}{2}$, $\sigma(l,m) = \frac{\rho_q(l,m)-\rho(m,m)}{2}$ and $\varphi(l) = \rho(l,l)$ for all $l,m \in \Omega$. Then Ω is a σ-complete MlS with metric σ, and $\varphi : X \longrightarrow \mathbb{R}^+$ is a lower semi-continuous function. By these transformations, (28) reduces to (7). By Theorem 2, there exists a unique $u \in \Omega$ such that $u = \Upsilon u$ and $\rho(u,u) = 0$. □

In line with the method of deducing Theorem 3, we can also obtain the following results which are improvements of some ideas in [9,13].

Corollary 6. *Let Ω be a σ-complete with partial metric ρ. Suppose that $\Upsilon : \Omega \longrightarrow \Omega$ is a mapping such that*

$$\begin{aligned}\psi(\rho(\Upsilon l,\Upsilon l) + \rho(\Upsilon l,\Upsilon m)) \leq{}& \psi\left(\max\left\{\rho(l,m), \rho(l,\Upsilon l), \rho(m,\Upsilon m), \frac{1}{2}[\rho(l,\Upsilon m) + \rho(m,\Upsilon l)]\right\}\right) \\ &-\phi\left(\max\left\{\rho(l,m), \rho(l,\Upsilon l), \rho(m,\Upsilon m), \frac{1}{2}[\rho(l,\Upsilon m) + \rho(m,\Upsilon l)]\right\}\right),\end{aligned}$$

for all $l,m \in \Omega$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Then, there exists a unique $u \in \Omega$ such that $u = \Upsilon u$ and $\rho(u,u) = 0$.

Remark 5. *If ϕ is continuous in Corollary 6, then we obtain Theorem 2.5 of [11].*

Corollary 7. *Let Ω be a complete partial MS. Suppose that $\Upsilon : \Omega \longrightarrow \Omega$ is a mapping such that*

$$\psi(\rho(\Upsilon l,\Upsilon l) + \rho(\Upsilon l,\Upsilon m)) \leq \psi(\rho(l,m)) - \phi(\rho(l,m))$$

for all $l,m \in \Omega$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Then, there exists a unique $u \in \Omega$ such that $u = \Upsilon u$ and $\rho(u,u) = 0$.

5. Applications to Boundary Value Problem

In recent years, there has been a growing interest in studying integral equations to prove the existence and uniqueness of a fixed point. Mohammed et al. [14] in 2021, investigated sufficient criteria for the existence and uniqueness of solutions to nonlinear Fredholm integral equations of the second kind on time scales. Specifically, they proposed a new Lipschitz condition on the kernel that guarantees the existence and uniqueness of solutions. This result is important because it provides a new tool for solving nonlinear integral equations on time scales. In later development, Jiddah et al. [15] and Jiddah et al. [16] in 2022 obtained unprecedented existence conditions for the solution of a family of integral equations. They used a fixed point theorem in generalized metric space to prove the existence of solutions of the examined equation.

In this section, Corollary 3 is applied to examine existence criteria for a solution to a boundary value problem. To this effect, consider the following boundary value problem of a second order differential equation

$$\begin{cases} \frac{-d^2 l}{dt^2} = \tau(t, l(t)) \;\; t \in [0,1],\; x \in \mathbb{R}_+ \\ l(0) = l(1) = 0, \end{cases} \tag{29}$$

where $\tau : [0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function. This problem is equivalent to the integral equation:

$$l(t) = \int_0^1 \xi(t,s)\tau(s, l(s))ds\;,\; t \in [0,1] \tag{30}$$

where $\xi(t,s)$ is called the Green function, defined by

$$\xi(t,s) = \begin{cases} t(1-s), & \text{if } 0 \le s < t \le 1 \\ s(1-t), & \text{if } 0 \le t < s \le 1. \end{cases}$$

Let $\Omega = C([0,1], \mathbb{R})$ be the set of all continuous real-valued functions defined on $[0,1]$. We equip Ω with the mapping

$$\sigma(l, m) = \sup_{t \in [0,1]} (|l(t) + |y(t)|) \tag{31}$$

Then (Ω, σ) is a complete MlS. Consider the self-mapping $\Upsilon : \Omega \longrightarrow \Omega$ defined by

$$\Upsilon l(t) = \int_0^1 \xi(t,s)\tau(s, l(s))ds\;,\; t \in [0,1] \tag{32}$$

Then, obviously l is a fixed point of Υ if and only if l is a solution to (29) We now study existence conditions of the boundary value problem (29) under the following hypotheses.

Theorem 4. *Let $\varphi \in \Phi$ and $\Upsilon : \Omega \longrightarrow \Omega$ be a self-mapping on Ω. Assume further that the following conditions are satisfied:*

(1) $|\tau(s,a)| + |\tau(s,b)| \le |a| + |b|$ *for all* $a, b \in \Omega$;
(2) $|\tau(s,a)| \le |a|$, *for all* $a \in \Omega$.

Then, the boundary value problem (29) *has a solution in* Ω.

Proof. Taking (31) and (32) into account, let $l, m \in \Omega$. Then,

$$
\begin{aligned}
|\Upsilon l(t)| + |\Upsilon y(t)| &= \left| \int_0^1 \xi(t,s)\tau(s,l(s))\,ds \;+\; \int_0^1 \xi(t,s)\tau(s,y(s))\,ds \right| \\
&\leq \int_0^1 \xi(t,s)ds|\tau(s,l(s))| \;+\; \int_0^1 \xi(t,s)ds|\tau(s,y(s))| \\
&\leq \int_0^1 \xi(t,s)\,ds\,(|\tau(s,l(s))\;+\;\tau(s,y(s))|) \\
&\leq \int_0^1 \xi(t,s)\,ds\,(|l(s)+|y(s)|).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sigma(\Upsilon l,\Upsilon m) = \sup_{t\in[0,1]}(|\Upsilon l(t)| + |\Upsilon y(t)|) \;\leq\; &\int_0^1 \xi(t,s)ds \sup_{t\in[0,1]}(|l(t)| + |y(t)|) \\
\leq\; &\int_0^1 \xi(t,s)ds\,\sigma(l,m).
\end{aligned}
$$

Notice that,

$$
\begin{aligned}
\varphi(\Upsilon l) \leq \sup_{t\in[0,1]}(\varphi(\Upsilon l(t))) = &\;\varphi(\sup_{t\in[0,1]} \Upsilon l(t)) \\
\leq &\;\varphi\left(\sup_{t\in[0,1]} \int_0^1 \xi(t,s)\tau(s,l(s))ds\right) \\
\leq &\;\varphi\left(\sup_{t\in[0,1]} \int_0^1 \xi(t,s)l(s)ds\right) \\
\leq &\;\varphi(l).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sigma(\Upsilon l,\Upsilon l) + \sigma(\Upsilon l,\Upsilon m) + \varphi(\Upsilon l) + \varphi(\Upsilon m) \\
&\quad\leq \int_0^1 \xi(t,s)ds\,\sigma(l,l) + \int_0^1 \xi(t,s)ds\,\sigma(l,m) + \varphi(\Upsilon l) + \varphi(\Upsilon m) \\
&\quad\leq \int_0^1 \xi(t,s)ds(\sigma(l,l) + \sigma(l,m)) \\
&\quad\quad + \varphi\left(\int_0^1 \xi(t,s)\tau(s,l(s))ds\right) + \varphi\left(\int_0^1 \xi(t,s)\tau(s,y(s))ds\right) \\
&\quad\leq \int_0^1 \xi(t,s)ds\,\sigma(l,l) + \int_0^1 \xi(t,s)ds\,\sigma(l,m) \\
&\quad\quad + \varphi\left(\int_0^1 \xi(t,s)l(s)ds\right) + \varphi\left(\int_0^1 y(s)ds\right) \\
&\quad\leq \sigma(l,l) + \sigma(l,m) + \varphi(l) + \varphi(m) \\
&\quad\leq \mathcal{C}(l,m,\varphi).
\end{aligned}
$$

□

This corresponds to the inequality (26) of Corollary 3. It follows that there is a fixed point of Υ, l in Ω which is equivalent to a solution of (29).

Conversely, if l is a solution of (29), then l is also a solution of (32), so that $\Upsilon l = l$, that is, l is a fixed point of Υ.

6. Conclusions

As a generalization of Banach's fixed point theorem, Amini-Harandi introduced the concept of MlS and derived some related fixed-point results in such space. In this manuscript, the notion of generalized quasi-weakly contractive operators in MlS is introduced and conditions for the existence of fixed points for such mappings are investigated. Non-trivial comparative examples have been presented to illustrate the proposed ideas and to show that they are indeed generalizations of a few concepts in the literature. As an application, one of our results is utilized to examine novel criteria for the existence of solutions to a class of boundary value problems. The concepts examined in this work improve some known corresponding results in metric and dislocated metric spaces. While the presented ideas are theoretical, we hope that they will encourage further research in the proposed directions and also find applications in the areas where non-zero self distance is needed.

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