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Properties of Convex Fuzzy-Number-Valued Functions on Harmonic Convex Set in the Second Sense and Related Inequalities via Up and Down Fuzzy Relation

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Abstract: In this paper, we provide different variants of the Hermite–Hadamard ($H \cdot H$) inequality using the concept of a new class of convex mappings, which is referred to as up and down harmonically s-convex fuzzy-number-valued functions ($\mathcal{UDH}$ s-convex $\mathcal{FNVM}$) in the second sense based on the up and down fuzzy inclusion relation. The findings are confirmed with certain numerical calculations that take a few appropriate examples into account. The results deal with various integrals of the $\frac{2\rho\sigma}{\rho+\sigma}$ type and are innovative in the setting of up and down harmonically s-convex fuzzy-number-valued functions. Moreover, we acquire classical and new exceptional cases that can be seen as applications of our main outcomes. In our opinion, this will make a significant contribution to encouraging more research.

Keywords: up and down harmonically s-convex fuzzy-number-valued function in the second sense; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

MSC: 26A33; 26A51; 26D10

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1. Introduction

The theory of convexity is a fascinating and active field of research. Many researchers use innovative ideas and effective approaches to broaden and generalize its diverse forms in different ways. This theory allows us to design and organize a large range of extremely efficient numerical algorithms to address and solve issues that arise in both pure and applied sciences. Convexity has recently experienced significant development, generalization, and extension. Numerous studies have shown that theories of inequalities and convex functions are intimately connected. Due to numerous generalizations and extensions, the study of convex analysis and inequalities has become an attractive, exciting, and beneficial topic for scholars. An analogous type of a convex function, the Hermite–Hadamard inequality, must fulfill generalized convexity in order to be established. Readers who are interested are referred to [1–10], which explore convex functions and the related inequalities that have recently been studied.

However, when measuring uncertainty problems, interval analysis can be a useful technique. Its rich history extends back to Archimedes' measurement of p, but it did not receive the attention it merited until Moore's [11], which was the first application of interval analysis for automated error analysis. Numerous conventional integral inequalities have been extended by Costa et al. [12], Flores-Franuli et al. [13], Chalco-cano et al. [14], and others to interval-valued functions and fuzzy-valued functions. Specifically, Zhao et al. [15]

developed an interval h-convex function and illustrated the associated integral inequality using the interval inclusion relation. Khan et al. [16] defined an h-convex interval-valued function in 2021 using the Kulisch–Miranker order and developed several inequalities for these kinds of convex functions. Any two intervals might not be comparable because of the partial order in which these two relations exist. Finding a useful order to examine inequalities related to interval-valued functions is therefore a challenging but interesting task. Bhunia et al. [17] calculated the cr-order, a novel rank relationship, in 2014 using the interval's center–radius. This connection allows for the comparison of two intervals because it is a full order. Refer to [18–22] for related papers on interval-valued inequalities.

Fractional calculus is the study of arbitrary order integrals and derivatives. Fractional calculus was developed not long after conventional calculus, but many scientists and academics are now interested in learning more about its roots and fundamentals, especially in light of the shortcomings of conventional calculus. See [23–28] and a recent survey explanatory review paper [29] for examples. It is important to note that fractional integral inequalities can be utilized to check the uniqueness of fractional ordinary and partial differential equations. Integral inequalities have connections to mathematical analysis, differential equations, discrete fractional calculus, difference equations, mathematical physics, and convexity theory, according to [30].

In recent years, it has become clear that mathematicians strongly prefer to present well-known inequalities using a variety of cutting-edge theories of fractional integral operators. The books [31–33] mentioned in this context may be consulted. Işcan [34] has initiated the exploration of the concept of a harmonic set and finds its application in the field of inequalities. He introduced the classical Hermite–Hadamard inequalities to harmonically convex functions. Mihai et al. [35] proposed the definition of *h*-harmonically convex functions and related inequalities. Similarly, Khan et al. [36] introduced harmonic convex functions in a fuzzy environment and explore these concepts by proposing a novel version of Hermite–Hadamard inequalities for harmonically convex fuzzy-number-valued mappings. For more information, we refer the readers to the following articles, [37–48], and the references therein.

The main goal of this article is to use up and down inclusion relations, more specifically, up and down fuzzy inclusion relations, to establish a connection between the ideas of fuzzy-number-valued analysis and fuzzy Aumann integral inequalities. We also present a new midpoint-type H-H inequality for fuzzy-number-valued functions with up and down convex properties. Then, using differing integrals of the $\frac{2\rho\sigma}{\rho+\sigma}$ type, we provide midpoint inequalities for the up and down harmonically convex fuzzy-valued functions. For more studies related to convexity and nonconvexity, see [49,50].

This work is set up as follows: After examining the prerequisite material and important details on inequalities and fuzzy-number-valued analysis in Section 2, we discuss $\mathcal{UDH}$ s-convex $\mathcal{FNVM}$s with numerical estimates in Section 3. Moreover, in Section 3, we derive fuzzy-number-valued $H \cdot H$-type inequalities for $\mathcal{UDH}$ s-convex $\mathcal{FNVM}$s. To decide whether the predefined results are advantageous, numerical estimations of the supplied results are also taken into consideration. Section 4 explores a quick conclusion and potential study directions connected to the findings in this work before we finish.

2. Preliminaries

Let $\mathcal{X}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\mathfrak{p} \in \mathcal{X}_C$ be defined as

$$\mathfrak{p} = \left[\mathfrak{p}_*, \mathfrak{p}^*\right] = \left\{\varkappa \in \mathbb{R} \middle| \mathfrak{p}_* \leq \varkappa \leq \mathfrak{p}^*\right\}, \left(\mathfrak{p}_*, \mathfrak{p}^* \in \mathbb{R}\right). \tag{1}$$

If $Ƿ_* = Ƿ^*$, then $Ƿ$ is referred to as degenerate. In this article, all intervals are non-degenerate intervals. If $Ƿ_* \geq 0$, then $\left[Ƿ_*, Ƿ^*\right]$ is referred to as a positive interval. The set of all positive intervals is denoted as $\mathcal{X}_C^+$ and defined as

$$\mathcal{X}_C^+ = \left\{\left[Ƿ_*, Ƿ^*\right] : \left[Ƿ_*, Ƿ^*\right] \in \mathcal{X}_C \text{ and } Ƿ_* \geq 0\right\}. \tag{2}$$

Let $\epsilon \in \mathbb{R}$ and $\epsilon \cdot Ƿ$ be defined as

$$\epsilon \cdot Ƿ = \begin{cases} \left[\epsilon Ƿ_*, \epsilon Ƿ^*\right] \text{ if } \epsilon > 0, \\ \{0\} \text{ if } \epsilon = 0, \\ \left[\epsilon Ƿ^*, \epsilon Ƿ_*\right] \text{ if } \epsilon < 0. \end{cases} \tag{3}$$

Then, the Minkowski difference $Ƕ - Ƿ$ and the addition $Ƿ + Ƕ$ and multiplication $Ƿ \times Ƕ$ for $Ƿ, Ƕ \in \mathcal{X}_C$ are defined as

$$[Ƕ_*, Ƕ^*] + \left[Ƿ_*, Ƿ^*\right] = \left[Ƕ_* + Ƿ_*, Ƕ^* + Ƿ^*\right], \tag{4}$$

$$[Ƕ_*, Ƕ^*] \times \left[Ƿ_*, Ƿ^*\right] = \left[\min\left\{Ƕ_*Ƿ_*, Ƕ^*Ƿ_*, Ƕ_*Ƿ^*, Ƕ^*Ƿ^*\right\}, \max\left\{Ƕ_*Ƿ_*, Ƕ^*Ƿ_*, Ƕ_*Ƿ^*, Ƕ^*Ƿ^*\right\}\right] \tag{5}$$

$$[Ƕ_*, Ƕ^*] - \left[Ƿ_*, Ƿ^*\right] = \left[Ƕ_* - Ƿ^*, Ƕ^* - Ƿ_*\right], \tag{6}$$

Remark 1 ([49]). *For a given* $[Ƕ_*, Ƕ^*], \left[Ƿ_*, Ƿ^*\right] \in \mathbb{R}_I$, *we say that* $[Ƕ_*, Ƕ^*] \leq_I \left[Ƿ_*, Ƿ^*\right]$ *if and only if* $Ƕ_* \leq Ƿ_*, Ƕ^* \leq Ƿ^*$, *and it is a partial interval order relation.*

For $[Ƕ_*, Ƕ^*], \left[Ƿ_*, Ƿ^*\right] \in \mathcal{X}_C$, *the Hausdorff–Pompeiu distance between intervals* $[Ƕ_*, Ƕ^*]$ *and* $\left[Ƿ_*, Ƿ^*\right]$ *is defined as*

$$d_H\left([Ƕ_*, Ƕ^*], \left[Ƿ_*, Ƿ^*\right]\right) = max\left\{\left|Ƕ_* - Ƿ_*\right|, \left|Ƕ^* - Ƿ^*\right|\right\}. \tag{7}$$

It is a familiar fact that $(\mathcal{X}_C, d_H)$ *is a complete metric space [42,45,46].*

Definition 1 ([42,43]). *A fuzzy subset L of* $\mathbb{R}$ *is distinguished via the mapping* $\widetilde{Ƿ} : \mathbb{R} \to [0,1]$ *called the membership mapping of L. That is, a fuzzy subset L of* $\mathbb{R}$ *is the mapping* $\widetilde{Ƿ} : \mathbb{R} \to [0,1]$. *Therefore, for further study, we have chosen this notation. We appoint* $\mathbb{E}$ *to denote the set of all fuzzy subsets of* $\mathbb{R}$.

Let $\widetilde{Ƿ} \in \mathbb{E}$. *Then,* $\widetilde{Ƿ}$ *is referred to as a fuzzy number or fuzzy interval if the following properties are satisfied by* $\widetilde{Ƿ}$:

(1) $\widetilde{Ƿ}$ *should be normal if there exist* $\varkappa \in \mathbb{R}$ *and* $\widetilde{Ƿ}(\varkappa) = 1$;

(2) $\widetilde{Ƿ}$ *should be upper semi-continuous on* $\mathbb{R}$ *if for a given* $\varkappa \in \mathbb{R}$, *there exists* $\varepsilon > 0$ *and there exists* $\delta > 0$, *such that* $\widetilde{Ƿ}(\varkappa) - \widetilde{Ƿ}(\mathrm{y}) < \varepsilon$ *for all* $\mathrm{y} \in \mathbb{R}$ *with* $|\varkappa - \mathrm{y}| < \delta$;

(3) $\widetilde{Ƿ}$ *should be fuzzy convex, that is,* $\widetilde{Ƿ}((1-\partial)\varkappa + \partial\mathrm{y}) \geq min\left(\widetilde{Ƿ}(\varkappa), \widetilde{Ƿ}(\mathrm{y})\right)$ *for all* $\varkappa, \mathrm{y} \in \mathbb{R}$, *and* $\partial \in [0,1]$;

(4) $\widetilde{Ƿ}$ *should be compactly supported, that is,* $cl\left\{\varkappa \in \mathbb{R} \middle| \widetilde{Ƿ}(\varkappa) > 0\right\}$ *is compact.*

We appoint $\mathbb{E}_C$ to denote the set of all fuzzy numbers of $\mathbb{R}$.

Definition 2 ([38,43]). *Given $\widetilde{\mathfrak{p}} \in \mathbb{E}_C$, the level sets or cut sets are given as $\left[\widetilde{\mathfrak{p}}\right]^{\epsilon} = \left\{ \varkappa \in \mathbb{R} \middle| \widetilde{\mathfrak{p}}(\varkappa) > \epsilon \right\}$ for all $\epsilon \in [0,1]$ and as $\left[\widetilde{\mathfrak{p}}\right]^{0} = \left\{ \varkappa \in \mathbb{R} \middle| \widetilde{\mathfrak{p}}(\varkappa) > 0 \right\}$. These sets are known as ϵ-level sets or ϵ-cut sets of $\widetilde{\mathfrak{p}}$.*

Proposition 1 ([47]). *Let $\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{H}} \in \mathbb{E}_C$. Then, the relation " $\leq_{\mathbb{F}}$ " is given on $\mathbb{E}_C$ as*

$\widetilde{\mathfrak{p}} \leq_{\mathbb{F}} \widetilde{\mathfrak{H}}$ when and only when $\left[\widetilde{\mathfrak{p}}\right]^{\epsilon} \leq_I \left[\widetilde{\mathfrak{H}}\right]^{\epsilon}$ for every $\epsilon \in [0,1]$, and it is a partial-order relation.

Proposition 2 ([42]). *Let $\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{H}} \in \mathbb{E}_C$. Then, inclusion relation " $\supseteq_{\mathbb{F}}$ " is given on $\mathbb{E}_C$ as $\widetilde{\mathfrak{p}} \supseteq_{\mathbb{F}} \widetilde{\mathfrak{H}}$ when and only when $\left[\widetilde{\mathfrak{p}}\right]^{\epsilon} \supseteq_I \left[\widetilde{\mathfrak{H}}\right]^{\epsilon}$ for every $\epsilon \in [0,1]$, and it is an up and down fuzzy inclusion relation.*

Remember the approaching notions, which are offered in the literature. If $\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{H}} \in \mathbb{E}_C$ and $\epsilon \in \mathbb{R}$, then, for every $\epsilon \in [0,1]$, the arithmetic operations are defined as

$$\left[\widetilde{\mathfrak{p}} \oplus \widetilde{\mathfrak{H}}\right]^{\epsilon} = \left[\widetilde{\mathfrak{p}}\right]^{\epsilon} + \left[\widetilde{\mathfrak{H}}\right]^{\epsilon}, \tag{8}$$

$$\left[\widetilde{\mathfrak{p}} \otimes \widetilde{\mathfrak{H}}\right]^{\epsilon} = \left[\widetilde{\mathfrak{p}}\right]^{\epsilon} \times \left[\widetilde{\mathfrak{H}}\right]^{\epsilon}, \tag{9}$$

$$\left[\partial \odot \widetilde{\mathfrak{p}}\right]^{\epsilon} = \partial.\left[\widetilde{\mathfrak{p}}\right]^{\epsilon} \tag{10}$$

These operations follow directly from the Equations (2)–(4), respectively.

Theorem 1 ([42]). *The space $\mathbb{E}_C$ dealing with a supremum metric, i.e., such that for $\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{H}} \in \mathbb{E}_C$,*

$$d_{\infty}\left(\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{H}}\right) = \sup_{0 \leq \epsilon \leq 1} d_H\left(\left[\widetilde{\mathfrak{p}}\right]^{\epsilon}, \left[\widetilde{\mathfrak{H}}\right]^{\epsilon}\right), \tag{11}$$

is a complete metric space, where H denotes the well-known Hausdorff metric in a space of intervals.

Now we define and discuss some properties of Riemann integral operators of interval- and fuzzy-number-valued mappings.

Theorem 2 ([42,44]). *If $Y : [\rho, \sigma] \subset \mathbb{R} \rightarrow \mathcal{X}_C$ is an interval-valued mapping (I·V·M) satisfying $Y(\varkappa) = [Y_*(\varkappa), Y^*(\varkappa)]$, then Y is Aumann-integrable (IA-integrable) on $[\rho, \sigma]$ when and only when $Y_*(\varkappa)$ and $Y^*(\varkappa)$ are both integrable on $[\rho, \sigma]$, such that*

$$(IA)\int_{\rho}^{\sigma} Y(\varkappa)d\varkappa = \left[\int_{\rho}^{\sigma} Y_*(\varkappa)d\varkappa, \int_{\rho}^{\sigma} Y^*(\varkappa)d\varkappa\right]. \tag{12}$$

Definition 3 ([48]). *Let $\widetilde{Y} : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ be referred to as F·N-V·M. Then, for every $\epsilon \in [0,1]$ as well as ϵ-levels, define the family of I·V·Ms $Y_\epsilon : \mathbb{I} \subset \mathbb{R} \to \mathcal{X}_C$ satisfying that $Y_\epsilon(\varkappa) = [Y_*(\varkappa,\epsilon), Y^*(\varkappa,\epsilon)]$ for every $\varkappa \in \mathbb{I}$. Herein, for every $\epsilon \in [0,1]$, the end point real-valued mappings $Y_*(\bullet,\epsilon), Y^*(\bullet,\epsilon) : \mathbb{I} \to \mathbb{R}$ are called the lower and upper mappings of Y.*

Definition 4 ([48]). *Let $\widetilde{Y} : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ be an F·N-V·M. Then, $\widetilde{Y}(\varkappa)$ is referred to as continuous at $\varkappa \in \mathbb{I}$ if for every $\epsilon \in [0,1]$, $Y_\epsilon(\varkappa)$ is continuous when and only when both the end point mappings $Y_*(\varkappa,\epsilon)$ and $Y^*(\varkappa,\epsilon)$ are continuous at $\varkappa \in \mathbb{I}$.*

Definition 5 ([44]). *Let $\widetilde{Y} : [\rho,\sigma] \subset \mathbb{R} \to \mathbb{E}_C$ be F·N-V·M. The fuzzy Aumann integral ((FA) integral) of $\widetilde{Y}$ on $[\rho,\sigma]$, denoted as $(FA)\int_\rho^\sigma \widetilde{Y}(\varkappa)d\varkappa$, is defined level-wise as*

$$\left[(FA)\int_\rho^\sigma \widetilde{Y}(\varkappa)d\varkappa\right]^\epsilon = (IA)\int_\rho^\sigma Y_\epsilon(\varkappa)d\varkappa = \left\{\int_\rho^\sigma Y(\varkappa,\epsilon)d\varkappa : Y(\varkappa,\epsilon) \in S(Y_\epsilon)\right\}, \quad (13)$$

where $S(Y_\epsilon) = \{Y(.,\epsilon) \to \mathbb{R} : Y(.,\epsilon)$ is integrable and $Y(\varkappa,\epsilon) \in Y_\epsilon(\varkappa)\}$ for every $\epsilon \in [0,1]$. $\widetilde{Y}$ is (FA)-integrable on $[\rho,\sigma]$ if $(FA)\int_\rho^\sigma \widetilde{Y}(\varkappa)d\varkappa \in \mathbb{E}_C$.

Theorem 3 ([47]). *Let $Y : [\rho,\sigma] \subset \mathbb{R} \to \mathbb{E}_C$ be an F·N-V·M and ϵ-levels define the family of I·V·Ms $Y_\epsilon : [\rho,\sigma] \subset \mathbb{R} \to \mathcal{X}_C$ satisfying that $Y_\epsilon(\varkappa) = [Y_*(\varkappa,\epsilon), Y^*(\varkappa,\epsilon)]$ for every $\varkappa \in [\rho,\sigma]$ and for every $\epsilon \in [0,1]$. Then, Y is (FA)-integrable on $[\rho,\sigma]$ when and only when $Y_*(\varkappa,\epsilon)$ and $Y^*(\varkappa,\epsilon)$ are both integrable on $[\rho,\sigma]$. Moreover, if Y is (FA)-integrable on $[\rho,\sigma]$, then*

$$\left[(FA)\int_\rho^\sigma Y(\varkappa)d\varkappa\right]^\epsilon = \left[\int_\rho^\sigma Y_*(\varkappa,\epsilon)d\varkappa, \int_\rho^\sigma Y^*(\varkappa,\epsilon)d\varkappa\right] = (IA)\int_\rho^\sigma Y_\epsilon(\varkappa)d\varkappa, \quad (14)$$

for every $\epsilon \in [0,1]$.

Definition 6 ([34]). *A set $K = [\rho,\sigma] \subset \mathbb{R}^+ = (0,\infty)$ is referred to as a convex set if for all $\nu, \mathfrak{y} \in K, \omega \in [0,1]$ we obtain*

$$\frac{\nu\mathfrak{y}}{\omega\nu + (1-\omega)\mathfrak{y}} \in K. \quad (15)$$

Definition 7 ([34]). *$Y : [\rho,\sigma] \to \mathbb{R}^+$ is referred to as a harmonically convex function on $[\rho,\sigma]$ if*

$$Y\left(\frac{\nu\mathfrak{y}}{\omega\nu + (1-\omega)\mathfrak{y}}\right) \le (1-\omega)Y(\nu) + \omega Y(\mathfrak{y}), \quad (16)$$

for all $\nu, \mathfrak{y} \in [\rho,\sigma], \omega \in [0,1]$, where $Y(\nu) \ge 0$ for all $\nu \in [\rho,\sigma]$. If (11) is reversed, then Y is referred to as a harmonically concave function on $[\rho,\sigma]$.

Definition 8. ([35]). *The positive real-valued function $Y : [\rho,\sigma] \to \mathbb{R}^+$ is referred to as a harmonically s-convex function in the second sense on $[\rho,\sigma]$ if*

$$Y\left(\frac{\nu\mathfrak{y}}{\omega\nu + (1-\omega)\mathfrak{y}}\right) \le (1-\omega)^s Y(\nu) + \omega^s Y(\mathfrak{y}), \quad (17)$$

for all $\nu, \mathfrak{y} \in [\rho,\sigma], \omega \in [0,1]$, where $Y(\nu) \ge 0$ for all $\nu \in [\rho,\sigma]$ and $s \in [0,1]$. If (12) is reversed, then Y is referred to as a harmonically s-concave function in the second sense on $[\rho,\sigma]$. The set of all harmonically s-convex (harmonically s-concave) functions is denoted as

$$HSX\big([\rho,\sigma],\mathbb{R}^+,s\big)\big(HSV\big([\rho,\sigma],\mathbb{R}^+,s\big)\big).$$

Definition 9. **([37]).** *The* $\mathcal{FNVM}$ $\widetilde{Y}:[\rho,\sigma]\to\mathbb{E}_C$ *is referred to as a convex* $\mathcal{FNVM}$ *in the second sense on* $[\rho,\sigma]$ *if*

$$\widetilde{Y}((1-\omega)r+\omega\mathfrak{y})\leq_{\mathbb{F}}(1-\omega)^s\odot\widetilde{Y}(r)\oplus\omega^s\odot\widetilde{Y}(\mathfrak{y}),\tag{18}$$

for all $r,\mathfrak{y}\in[\rho,\sigma]$, $\omega\in[0,1]$, *where* $Y(r)\geq 0$ *for all* $r\in[\rho,\sigma]$ *and* $s\in[0,1]$. *If (18) is reversed, then* $\widetilde{Y}$ *is referred to as a concave* $\mathcal{FNVM}$ *on* $[\rho,\sigma]$. *The set of all convex (concave)* $\mathcal{FNVM}$*s is denoted as*

$$FSX([\rho,\sigma],\mathbb{E}_C,s),\ (FSV([\rho,\sigma],\mathbb{E}_C,s)).$$

Definition 10. *The* $\mathcal{FNVM}$ $\widetilde{Y}:[\rho,\sigma]\to\mathbb{E}_C$ *is referred to as an up and down harmonically s-convex* $\mathcal{FNVM}$ *in the second sense on* $[\rho,\sigma]$ *if*

$$\widetilde{Y}\left(\frac{r\mathfrak{y}}{\omega r+(1-\omega)\mathfrak{y}}\right)\supseteq_{\mathbb{F}}(1-\omega)^s\odot\widetilde{Y}(r)\oplus\omega^s\odot\widetilde{Y}(\mathfrak{y}),\tag{19}$$

for all $r,\mathfrak{y}\in[\rho,\sigma]$, $\omega\in[0,1]$,*where* $\widetilde{Y}(r)\geq_{\mathbb{F}}\widetilde{0}$ *for all* $r\in[\rho,\sigma]$ *and* $s\in[0,1]$. *If (19) is reversed, then* $\widetilde{Y}$ *is referred to as an up and down harmonically s-concave* $\mathcal{FNVM}$ *in the second sense on* $[\rho,\sigma]$. *The set of all up and down harmonically s-convex (up and down harmonically s-concave)* $\mathcal{FNVM}$*s is denoted as*

$$\mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)(\mathcal{UDHFSV}([\rho,\sigma],\mathbb{E}_C,s)).$$

Theorem 4. *Let* $[\rho,\sigma]$ *be a harmonically convex set, and let* $\widetilde{Y}:[\rho,\sigma]\to\mathbb{E}_C$ *be an* $\mathcal{FNVM}$ *whose parametrized form is given as* $Y_\epsilon:[\rho,\sigma]\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ *and defined as*

$$Y_\epsilon(r)=[Y_*(r,\epsilon),Y^*(r,\epsilon)],\forall r\in[\rho,\sigma].\tag{20}$$

for all $r\in[\rho,\sigma]$, $\epsilon\in[0,1]$. *Then,* $\widetilde{Y}\in\mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$ *if and only if for all* $\in[0,1]$, $Y_*(r,\epsilon)\in HSX([\rho,\sigma],\mathbb{R}^+,s)$ *and* $Y^*(r,\epsilon)\in(HSV([\rho,\sigma],\mathbb{R}^+,s))$.

Proof. The proof is similar to the proof of Theorem 2.12 (see [23]). □

Example 1. *We consider the* $\mathcal{FNVM}$*s* $\widetilde{Y}:\left[\frac{1}{2},1\right]\to\mathbb{E}_C$ *defined as*

$$\widetilde{Y}(r)(\partial)=\begin{cases}\dfrac{\partial^{-r^2}}{1-r^2}\partial\in[r^2,1],\\ \frac{5-e^r-\partial}{4-e^r}\partial\in(1,5-e^r],\\ 0\text{ otherwise.}\end{cases}$$

Then, for each $\epsilon\in[0,1]$, *we obtain* $Y_\epsilon(r)=\left[(1-\epsilon)r^2+\epsilon,(1-\epsilon)(5-e^r)+\epsilon\right]$. $Y_*(r,\epsilon)\in HSX([\rho,\sigma],\mathbb{R}^+,s)$ *and* $Y^*(r,\epsilon)\in(HSV([\rho,\sigma],\mathbb{R}^+,s))$ *with* $s=1$ *for each* $\epsilon\in[0,1]$. *Hence,* $\widetilde{Y}\in\mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$.

Remark 2. *If* $s=1$, *then Definition 10 reduces to the definition of a* $\mathcal{UDH}$ *convex* $\mathcal{FNVM}$.

If $Y_*(r,\epsilon)=Y^*(r,\epsilon)$ *with* $\epsilon=1$, *then a* $\mathcal{UDH}$ *s-convex* $\mathcal{FNVM}$ *in the second sense reduces to the classical harmonically s-convex function in the second sense (see [35]).*

If $Y_*(r,\epsilon)=Y^*(r,\epsilon)$ *with* $\epsilon=1$ *and* $s=1$, *then a* $\mathcal{UDH}$ *s-convex* $\mathcal{FNVM}$ *in the second sense reduces to the classical harmonically convex function (see [34]).*

If $\Upsilon_*(\varkappa, \epsilon) = \Upsilon^*(\varkappa, \epsilon)$ with $\epsilon = 1$ and $s = 0$, then a $\mathcal{UDH}$ s-convex $\mathcal{FNVM}$ in the second sense reduces to the classical harmonically P-function (see [35]).

Herein, we define some new outcomes by applying some mild restriction on the endpoint functions.

Definition 11. *Let $\widetilde{\Upsilon} : [\rho, \sigma] \to \mathbb{E}_C$ be an $\mathcal{FNVM}$ whose parametrized form is given as $\Upsilon_\epsilon : [\rho, \sigma] \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ and defined as*

$$\Upsilon_\epsilon(\varkappa) = \left[\Upsilon_*(\varkappa, \epsilon), \Upsilon^*(\varkappa, \epsilon)\right], \forall \varkappa \in [\rho, \sigma].$$

for all $\varkappa \in [\rho, \sigma]$ and for all $\epsilon \in [0, 1]$. Then, $\widetilde{\Upsilon}$ is said to be a lower $\mathcal{UDH}$ s-convex (s-concave) $\mathcal{FNVM}$ on $[\rho, \sigma]$ if

$$\Upsilon_*\left(\frac{\varkappa\mathfrak{y}}{\omega\varkappa + (1-\omega)\mathfrak{y}}\right) \leq (\geq)(1-\omega)^s \Upsilon_*(\varkappa) + \omega^s \Upsilon_*(\mathfrak{y}),$$

and

$$\Upsilon^*\left(\frac{\varkappa\mathfrak{y}}{\omega\varkappa + (1-\omega)\mathfrak{y}}\right) = (1-\omega)^s \Upsilon^*(\varkappa) + \omega^s \Upsilon^*(\mathfrak{y})$$

for all $\epsilon \in [0, 1]$.

Definition 12. *Let $\widetilde{\Upsilon} : [\rho, \sigma] \to \mathbb{E}_C$ be an $\mathcal{FNVM}$ whose parametrized form is given as $\Upsilon_\epsilon : [\rho, \sigma] \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ and defined as*

$$\Upsilon_\epsilon(\varkappa) = \left[\Upsilon_*(\varkappa, \epsilon), \Upsilon^*(\varkappa, \epsilon)\right], \forall \varkappa \in [\rho, \sigma].$$

for all $\varkappa \in [\rho, \sigma]$ and for all $\epsilon \in [0, 1]$. Then, $\widetilde{\Upsilon}$ is said to be a lower upper $\mathcal{UDH}$ s-convex (s-concave) $\mathcal{FNVM}$ on $[\rho, \sigma]$ if

$$\Upsilon_*\left(\frac{\varkappa\mathfrak{y}}{\omega\varkappa + (1-\omega)\mathfrak{y}}\right) = (1-\omega)^s \Upsilon_*(\varkappa) + \omega^s \Upsilon_*(\mathfrak{y}),$$

and

$$\Upsilon^*\left(\frac{\varkappa\mathfrak{y}}{\omega\varkappa + (1-\omega)\mathfrak{y}}\right) \leq (\geq)(1-\omega)^s \Upsilon^*(\varkappa) + \omega^s \Upsilon^*(\mathfrak{y}).$$

for all $\epsilon \in [0, 1]$.

Remark 3. *If $\widetilde{\Upsilon}$ is alower $\mathcal{UDH}$ s-convex (s-concave) $\mathcal{FNVM}$ with $s = 1$, then Definition 12 reduces to Definition 9.*

If $\widetilde{\Upsilon}$ is alower $\mathcal{UDH}$ s-convex (s-concave) $\mathcal{FNVM}$ with $s = 1$, then Definition 12 reduces to the definition of an $\mathcal{H}$- s-convex (s-concave) $\mathcal{FNVM}$.

3. Fuzzy-Number Hermite–Hadamard Inequalities

In this section, inequalities of the Hermite–Hadamard type are established including the $\mathcal{UDH}$ s-convex fuzzy-number-valued mapping for the products of two $\mathcal{UDH}$ s-convex fuzzy-number-valued mappings.

Theorem 5. *Let* $\widetilde{Y} \in \mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$*, whose parametrized form is given as* $Y_\epsilon : [\rho,\sigma] \subset \mathbb{R} \to \mathcal{K}_C^+$ *and defined as* $Y_\epsilon(r) = [Y_*(r,\epsilon), Y^*(r,\epsilon)]$ *for all* $r \in [\rho,\sigma]$ *,* $\epsilon \in [0,1]$*. If* $\widetilde{Y} \in \mathcal{F}r_{([\rho,\sigma],\epsilon)}$*, then*

$$2^{s-1} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot \int_\rho^\sigma \frac{\widetilde{Y}(r)}{r^2} dr \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)}{1+s}. \tag{21}$$

If $\widetilde{Y} \in \mathcal{UDHFSV}([\rho,\sigma],\mathbb{E}_C,s)$*, then*

$$2^{s-1} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \subseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot \int_\rho^\sigma \frac{\widetilde{Y}(r)}{r^2} dr \subseteq_{\mathbb{F}} \frac{\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)}{1+s}. \tag{22}$$

Proof. Let $\widetilde{Y} \in \mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$. Then, by hypothesis, we obtain

$$2^s \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \widetilde{Y}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right) \oplus \widetilde{Y}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right).$$

Therefore, for each $\epsilon \in [0,1]$, we obtain

$$\begin{aligned} 2^s Y_*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\le Y_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right) + Y_*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right),\\ 2^s Y^*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\ge Y^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right) + Y^*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right). \end{aligned}$$

Then,

$$\begin{aligned} 2^s \int_0^1 Y_*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) d\omega &\le \int_0^1 Y_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right) d\omega + \int_0^1 Y_*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right) d\omega,\\ 2^s \int_0^1 Y^*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) d\omega &\ge \int_0^1 Y^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right) d\omega + \int_0^1 Y^*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right) d\omega. \end{aligned}$$

It follows that

$$2^{s-1} Y_*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) \le \frac{\rho\sigma}{\sigma-\rho} \int_\rho^\sigma \tfrac{Y_*(r,\epsilon)}{r^2} dr,$$

$$2^{s-1} Y^*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) \ge \frac{\rho\sigma}{\sigma-\rho} \int_\rho^\sigma \tfrac{Y^*(r,\epsilon)}{r^2} dr.$$

That is,

$$2^{s-1}\left[Y_*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right), Y^*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\right] \supseteq_I \frac{\rho\sigma}{\sigma-\rho}\left[\int_\rho^\sigma \frac{Y_*(r,\epsilon)}{r^2} dr, \int_\rho^\sigma \frac{Y^*(r,\epsilon)}{r^2} dr\right].$$

Via Theorem 4, we obtain

$$2^{s-1} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_\rho^\sigma \frac{\widetilde{Y}(r)}{r^2} dr. \tag{23}$$

In a similar way as above, we obtain

$$\frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_\rho^\sigma \frac{\widetilde{Y}(r)}{r^2} dr \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)}{s+1}. \tag{24}$$

Combining (23) and (24), we obtain

$$2^{s-1} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot \int_{\rho}^{\sigma} \frac{\widetilde{Y}(\varkappa)}{\varkappa^2} d\varkappa \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)}{s+1}$$

Hence, we obtain the required result. □

Remark 4. *If* $s = 1$, *therefore, from (21), we obtain the new version of inequality that follows:*

$$\widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_{\rho}^{\sigma} \frac{\widetilde{Y}(\varkappa)}{\varkappa^2} d\varkappa \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)}{2}.$$

If $s \equiv 0$, *therefore, from (21), we obtain the new version of inequality that follows:*

$$\frac{1}{2} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_{\rho}^{\sigma} \frac{\widetilde{Y}(\varkappa)}{\varkappa^2} d\varkappa \supseteq_{\mathbb{F}} \widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma).$$

If $\widetilde{Y}$ *is a lower* $\mathcal{UDH}$ *s-convex* $\mathcal{FNVM}$ *with* $s = 1$, *therefore, from (21), we obtain the inequality that follows (see [36]):*

$$\widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \leq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_{\rho}^{\sigma} \frac{\widetilde{Y}(\varkappa)}{\varkappa^2} d\varkappa \leq_{\mathbb{F}} \frac{\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)}{2}.$$

If $\widetilde{Y}$ *is a lower* $\mathcal{UDH}$ *s-convex* $\mathcal{FNVM}$ *with* $s \equiv 0$, *therefore, from (21), we obtain the inequality that follows (see [36]):*

$$\frac{1}{2} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \leq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_{\rho}^{\sigma} \frac{\widetilde{Y}(\varkappa)}{\varkappa^2} d\varkappa \leq_{\mathbb{F}} \widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma).$$

Let $Y_*(\varkappa,\epsilon) = Y^*(\varkappa,\epsilon)$ *with* $\epsilon = 1$. *Then, from (21) we obtain the inequality that follows (see [35]):*

$$2^{s-1}Y\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \leq \frac{\rho\sigma}{\sigma-\rho}\int_{\rho}^{\sigma} \frac{Y(\varkappa)}{\varkappa^2} d\varkappa \leq \frac{1}{s+1}[Y(\rho) + Y(\sigma)].$$

If $Y_*(\varkappa,\epsilon) = Y^*(\varkappa,\epsilon)$ *with* $\epsilon = 1$ *and* $s = 1$, *therefore, from (21), we obtain the inequality that follows (see [34]):*

$$Y\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \leq \frac{\rho\sigma}{\sigma-\rho}\int_{\rho}^{\sigma} \frac{Y(\varkappa)}{\varkappa^2} d\varkappa \leq \frac{Y(\rho) + Y(\sigma)}{2}.$$

Let $Y_*(\varkappa,\epsilon) = Y^*(\varkappa,\epsilon)$ *with* $\epsilon = 1$ *and* $s = 0$. *Then, from (21), we obtain the inequality that follows (see [35]):*

$$\frac{1}{2}Y\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \leq \frac{\rho\sigma}{\sigma-\rho}\int_{\rho}^{\sigma} \frac{Y(\varkappa)}{\varkappa^2} d\varkappa \leq Y(\rho) + Y(\sigma).$$

Theorem 6. *Let* $\widetilde{Y} \in \mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$ *whose parametrized form is given as* $Y_\epsilon : [\rho,\sigma] \subset \mathbb{R} \to \mathcal{K}_C^+$ *and defined as* $Y_\epsilon(r) = [Y_*(r,\epsilon), Y^*(r,\epsilon)]$ *for all* $r \in [\rho,\sigma]$, $\epsilon \in [0,1]$. *If* $\widetilde{Y} \in \mathcal{F}r_{([\rho,\sigma],\epsilon)}$, *then*

$$4^{s-1} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \Omega_2 \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_\rho^\sigma \frac{\widetilde{Y}(r)}{r^2}dr \supseteq_{\mathbb{F}} \Omega_1 \supseteq_{\mathbb{F}} \frac{1}{s+1} \odot \left[\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)\right] \odot \left[\frac{1}{2} + \frac{1}{2^s}\right], \tag{25}$$

where

$$\Omega_1 = \frac{1}{s+1} \odot \left[\frac{\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)}{2} \oplus \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\right],$$

$$\Omega_2 = 2^{s-2} \odot \left[\widetilde{Y}\left(\frac{4\rho\sigma}{\rho+3\sigma}\right) \oplus \widetilde{Y}\left(\frac{4\rho\sigma}{3\rho+\sigma}\right)\right],$$

and $\Omega_1 = [\Omega_{1*}, \Omega_1{}^*]$, $\Omega_2 = [\Omega_{2*}, \Omega_2{}^*]$.

If $\widetilde{Y} \in \mathcal{UDHFSV}([\rho,\sigma],\mathbb{E}_C,s)$, *then inequality (21) is reversed.*

Proof. Taking $\left[\rho, \frac{2\rho\sigma}{\rho+\sigma}\right]$, we obtain

$$2^s \odot \widetilde{Y}\left(\frac{\rho\frac{4\rho\sigma}{\rho+\sigma}}{\omega\rho + (1-\omega)\frac{2\rho\sigma}{\rho+\sigma}} + \frac{\rho\frac{4\rho\sigma}{\rho+\sigma}}{(1-\omega)\rho + \omega\frac{2\rho\sigma}{\rho+\sigma}}\right) \supseteq_{\mathbb{F}} \widetilde{Y}\left(\frac{\rho\frac{2\rho\sigma}{\rho+\sigma}}{\omega\rho + (1-\omega)\frac{2\rho\sigma}{\rho+\sigma}}\right) \oplus \widetilde{Y}\left(\frac{\rho\frac{2\rho\sigma}{\rho+\sigma}}{(1-\omega)\rho + \omega\frac{2\rho\sigma}{\rho+\sigma}}\right).$$

Therefore, for every $\epsilon \in [0,1]$, we obtain

$$2^s Y_*\left(\frac{\rho\frac{4\rho\sigma}{\rho+\sigma}}{\omega\rho+(1-\omega)\frac{2\rho\sigma}{\rho+\sigma}} + \frac{\rho\frac{4\rho\sigma}{\rho+\sigma}}{(1-\omega)\rho+\omega\frac{2\rho\sigma}{\rho+\sigma}}, \epsilon\right) \le Y_*\left(\frac{\rho\frac{2\rho\sigma}{\rho+\sigma}}{\omega\rho+(1-\omega)\frac{2\rho\sigma}{\rho+\sigma}}, \epsilon\right) + Y_*\left(\frac{\rho\frac{2\rho\sigma}{\rho+\sigma}}{(1-\omega)\rho+\omega\frac{2\rho\sigma}{\rho+\sigma}}, \epsilon\right),$$
$$2^s Y^*\left(\frac{\rho\frac{4\rho\sigma}{\rho+\sigma}}{\omega\rho+(1-\omega)\frac{2\rho\sigma}{\rho+\sigma}} + \frac{\rho\frac{4\rho\sigma}{\rho+\sigma}}{(1-\omega)\rho+\omega\frac{2\rho\sigma}{\rho+\sigma}}, \epsilon\right) \ge Y^*\left(\frac{\rho\frac{2\rho\sigma}{\rho+\sigma}}{\omega\rho+(1-\omega)\frac{2\rho\sigma}{\rho+\sigma}}, \epsilon\right) + Y^*\left(\frac{\rho\frac{2\rho\sigma}{\rho+\sigma}}{(1-\omega)\rho+\omega\frac{2\rho\sigma}{\rho+\sigma}}, \epsilon\right).$$

In consequence, we obtain

$$2^{s-2} Y_*\left(\frac{4\rho\sigma}{\rho+3\sigma}, \epsilon\right) \le \frac{\rho\sigma}{\sigma-\rho}\int_\rho^{\frac{2\rho\sigma}{\rho+\sigma}} \frac{Y_*(r,\epsilon)}{r^2}dr,$$

$$2^{s-2} Y^*\left(\frac{4\rho\sigma}{\rho+3\sigma}, \epsilon\right) \ge \frac{\rho\sigma}{\sigma-\rho}\int_\rho^{\frac{2\rho\sigma}{\rho+\sigma}} \frac{Y^*(r,\epsilon)}{r^2}dr.$$

That is,

$$2^{s-2}\left[Y_*\left(\frac{4\rho\sigma}{\rho+3\sigma}, \epsilon\right), Y^*\left(\frac{4\rho\sigma}{\rho+3\sigma}, \epsilon\right)\right] \supseteq_I \frac{\rho\sigma}{\sigma-\rho}\left[\int_\rho^{\frac{2\rho\sigma}{\rho+\sigma}} \frac{Y_*(r,\epsilon)}{r^2}dr, \int_\rho^{\frac{2\rho\sigma}{\rho+\sigma}} \frac{Y^*(r,\epsilon)}{r^2}dr\right].$$

It follows that

$$2^{s-2} \odot \widetilde{Y}\left(\frac{4\rho\sigma}{\rho+3\sigma}\right) \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot \int_\rho^{\frac{2\rho\sigma}{\rho+\sigma}} \frac{\widetilde{Y}(r)}{r^2}dr. \tag{26}$$

In a similar way as above, we obtain

$$2^{s-2} \odot \widetilde{Y}\left(\frac{4\rho\sigma}{3\rho+\sigma}\right) \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot \int_{\frac{2\rho\sigma}{\rho+\sigma}}^\sigma \frac{\widetilde{Y}(r)}{r^2}dr. \tag{27}$$

Combining (26) and (27), we obtain

$$2^{s-2} \odot \left[\widetilde{Y}\left(\frac{4\rho\sigma}{\rho+3\sigma}\right) \oplus \widetilde{Y}\left(\frac{4\rho\sigma}{3\rho+\sigma}\right)\right] \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot \int_{\rho}^{\sigma} \frac{\widetilde{Y}(r)}{r^2} dr. \quad (28)$$

Therefore, for every $\epsilon \in [0,1]$, by using Theorem 5, we obtain

$$\begin{aligned}
4^{s-1} Y_*\left(\tfrac{2\rho\sigma}{\rho+\sigma}, \epsilon\right) &\leq 4^{s-1}\left[\tfrac{1}{2^s} Y_*\left(\tfrac{4\rho\sigma}{\rho+3\sigma}, \epsilon\right) + \tfrac{1}{2^s} Y_*\left(\tfrac{4\rho\sigma}{3\rho+\sigma}, \epsilon\right)\right], \\
4^{s-1} Y^*\left(\tfrac{2\rho\sigma}{\rho+\sigma}, \epsilon\right) &\geq 4^{s-1}\left[\tfrac{1}{2^s} Y^*\left(\tfrac{4\rho\sigma}{\rho+3\sigma}, \epsilon\right) + \tfrac{1}{2^s} Y^*\left(\tfrac{4\rho\sigma}{3\rho+\sigma}, \epsilon\right)\right], \\
&= \Omega_{2*}, \\
&= \Omega_2{}^*, \\
&\leq \tfrac{\rho\sigma}{\sigma-\rho} \int_{\rho}^{\sigma} \tfrac{Y_*(r,\epsilon)}{r^2} dr, \\
&\geq \tfrac{\rho\sigma}{\sigma-\rho} \int_{\rho}^{\sigma} \tfrac{Y^*(r,\epsilon)}{r^2} dr, \\
&\leq \tfrac{1}{s+1}\left[\tfrac{Y_*(\rho,\epsilon)+Y_*(\sigma,\epsilon)}{2} + Y_*\left(\tfrac{2\rho\sigma}{\rho+\sigma}, \epsilon\right)\right], \\
&\geq \tfrac{1}{s+1}\left[\tfrac{Y^*(\rho,\epsilon)+Y^*(\sigma,\epsilon)}{2} + Y^*\left(\tfrac{2\rho\sigma}{\rho+\sigma}, \epsilon\right)\right], \\
&= \Omega_{1*}, \\
&= \Omega_1{}^*, \\
&\leq \tfrac{1}{s+1}\left[\tfrac{Y_*(\rho,\epsilon)+Y_*(\sigma,\epsilon)}{2} + \tfrac{1}{2^s}(Y_*(\rho,\epsilon) + Y_*(\sigma,\epsilon))\right], \\
&\geq \tfrac{1}{s+1}\left[\tfrac{Y^*(\rho,\epsilon)+Y^*(\sigma,\epsilon)}{2} + \tfrac{1}{2^s}(Y^*(\rho,\epsilon) + Y^*(\sigma,\epsilon))\right], \\
&= \tfrac{1}{s+1}[Y_*(\rho,\epsilon) + Y_*(\sigma,\epsilon)]\left[\tfrac{1}{2} + \tfrac{1}{2^s}\right], \\
&= \tfrac{1}{s+1}[Y^*(\rho,\epsilon) + Y^*(\sigma,\epsilon)]\left[\tfrac{1}{2} + \tfrac{1}{2^s}\right],
\end{aligned}$$

that is,

$$4^{s-1} \odot \widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \supseteq_{\mathbb{F}} \Omega_2 \supseteq_{\mathbb{F}} \frac{\rho\sigma}{\sigma-\rho} \odot (FA)\int_{\rho}^{\sigma} \frac{\widetilde{Y}(r)}{r^2} dr \supseteq_{\mathbb{F}} \Omega_1 \supseteq_{\mathbb{F}} \frac{1}{s+1} \odot \left[\widetilde{Y}(\rho) \oplus \widetilde{Y}(\sigma)\right] \odot \left[\frac{1}{2} + \frac{1}{2^s}\right].$$

Hence, Theorem 6 has been proved. □

Theorem 7. *Let* $\widetilde{Y} \in \mathcal{UDHFSX}([\rho,\sigma], \mathbb{E}_C, s)$ *and* $\widetilde{\mathcal{U}} \in \mathcal{UDHFSX}([\rho,\sigma], \mathbb{E}_C, s)$, *whose parametrized forms are given as* $Y_\epsilon, \mathcal{U}_\epsilon : [\rho,\sigma] \subset \mathbb{R} \to \mathcal{K}_C^+$ *and defined as* $Y_\epsilon(r) = [Y_*(r,\epsilon), Y^*(r,\epsilon)]$ *and* $\mathcal{U}_\epsilon(r) = [\mathcal{U}_*(r,\epsilon), \mathcal{U}^*(r,\epsilon)]$ *for all* $r \in [\rho,\sigma]$, $\epsilon \in [0,1]$, *respectively. If* $\widetilde{Y} \otimes \widetilde{\mathcal{U}} \in \mathcal{F}r_{([\rho,\sigma],\epsilon)}$, *then*

$$\frac{\rho\sigma}{\sigma-\rho}(FA)\int_{\rho}^{\sigma} \frac{\widetilde{Y}(r) \otimes \widetilde{\mathcal{U}}(r)}{r^2} dr \supseteq_{\mathbb{F}} \widetilde{\Lambda}(\rho,\sigma) \int_0^1 \omega^s \cdot \omega^s d\omega \oplus \widetilde{\partial}(\rho,\sigma) \int_0^1 \omega^s (1-\omega)^s d\omega, \quad (29)$$

where $\widetilde{\Lambda}(\rho,\sigma) = \widetilde{Y}(\rho) \otimes \widetilde{\mathcal{U}}(\rho) \oplus \widetilde{Y}(\sigma) \otimes \widetilde{\mathcal{U}}(\sigma)$, $\widetilde{\partial}(\rho,\sigma) = \widetilde{Y}(\rho) \otimes \widetilde{\mathcal{U}}(\sigma) \oplus \widetilde{Y}(\sigma) \otimes \widetilde{\mathcal{U}}(\rho)$, *and* $\Lambda_\epsilon(\rho,\sigma) = [\Lambda_*((\rho,\sigma),\epsilon), \Lambda^*((\rho,\sigma),\epsilon)]$ *and* $\partial_\epsilon(\rho,\sigma) = [\partial_*((\rho,\sigma),\epsilon), \partial^*((\rho,\sigma),\epsilon)]$.

Proof. Since $\widetilde{Y}, \widetilde{\mathcal{U}}$ are $\mathcal{UDH}$ s-convex $\mathcal{FNVM}$ s-convex $\mathcal{FNVM}s$, then, for each $\epsilon \in [0,1]$, we obtain

$$Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \leq \omega^s Y_*(\rho,\epsilon) + (1-\omega)^s Y_*(\sigma,\epsilon),$$
$$Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \geq \omega^s Y^*(\rho,\epsilon) + (1-\omega)^s Y^*(\sigma,\epsilon).$$

Moreover,

$$\mathcal{U}_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \leq \omega^s \mathcal{U}_*(\rho,\epsilon) + (1-\omega)^s \mathcal{U}_*(\sigma,\epsilon),$$
$$\mathcal{U}^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \geq \omega^s \mathcal{U}^*(\rho,\epsilon) + (1-\omega)^s \mathcal{U}^*(\sigma,\epsilon).$$

From the definition of the $\mathcal{UDH}$ s-convex $\mathcal{FNVM}$ s-convexity of $\mathcal{FNVM}s$, it follows that $\widetilde{Y}(r) \geq_{\mathbb{F}} \widetilde{0}$ and $\widetilde{\mathcal{U}}(r) \geq_{\mathbb{F}} \widetilde{0}$, so

$$\begin{gathered}
Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \times \mathcal{U}_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \\
\leq \left(\omega^s Y_*(\rho,\epsilon) + (1-\omega)^s Y_*(\sigma,\epsilon)\right)\left(\omega^s \mathcal{U}_*(\rho,\epsilon) + (1-\omega)^s \mathcal{U}_*(\sigma,\epsilon)\right) \\
= Y_*(\rho,\epsilon) \times \mathcal{U}_*(\rho,\epsilon)[\omega^s \cdot \omega^s] + Y_*(\sigma,\epsilon) \times \mathcal{U}_*(\sigma,\epsilon)\left[(1-\omega)^s(1-\omega)^s\right] \\
+ Y_*(\rho,\epsilon)\mathcal{U}_*(\sigma,\epsilon)\omega^s(1-\omega)^s + Y_*(\sigma,\epsilon) \times \mathcal{U}_*(\rho,\epsilon)(1-\omega)^s \cdot \omega^s, \\
Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \times \mathcal{U}^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \\
\geq \left(\omega^s Y^*(\rho,\epsilon) + (1-\omega)^s Y^*(\sigma,\epsilon)\right)\left(\omega^s \mathcal{U}^*(\rho,\epsilon) + (1-\omega)^s \mathcal{U}^*(\sigma,\epsilon)\right) \\
= Y^*(\rho,\epsilon) \times \mathcal{U}^*(\rho,\epsilon)[\omega^s \cdot \omega^s] + Y^*(\sigma,\epsilon) \times \mathcal{U}^*(\sigma,\epsilon)\left[(1-\omega)^s(1-\omega)^s\right] \\
+ Y^*(\rho,\epsilon) \times \mathcal{U}^*(\sigma,\epsilon)\omega^s(1-\omega)^s + Y^*(\sigma,\epsilon) \times \mathcal{U}^*(\rho,\epsilon)(1-\omega)^s \omega^s.
\end{gathered}$$

Integrating both sides of the above inequality on [0, 1] we obtain

$$\begin{gathered}
\int_0^1 Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \times \mathcal{U}_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) = \frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma \frac{Y_*(r,\epsilon) \times \mathcal{U}_*(r,\epsilon)}{r^2} dr \\
\leq \left(Y_*(\rho,\epsilon) \times \mathcal{U}_*(\rho,\epsilon) + Y_*(\sigma,\epsilon) \times \mathcal{U}_*(\sigma,\epsilon)\right)\int_0^1 \omega^s \cdot \omega^s d\omega \\
+ \left(Y_*(\rho,\epsilon) \times \mathcal{U}_*(\sigma,\epsilon) + Y_*(\sigma,\epsilon) \times \mathcal{U}_*(\rho,\epsilon)\right)\int_0^1 \omega^s(1-\omega)^s d\omega,
\end{gathered}$$

$$\begin{gathered}
\int_0^1 Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) \times \mathcal{U}^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}, \epsilon\right) = \frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma \frac{Y^*(r,\epsilon) \times \mathcal{U}^*(r,\epsilon)}{r^2} dr \\
\geq \left(Y^*(\rho,\epsilon) \times \mathcal{U}^*(\rho,\epsilon) + Y^*(\sigma,\epsilon) \times \mathcal{U}^*(\sigma,\epsilon)\right)\int_0^1 \omega^s \cdot \omega^s d\omega \\
+ \left(Y^*(\rho,\epsilon) \times \mathcal{U}^*(\sigma,\epsilon) + Y^*(\sigma,\epsilon) \times \mathcal{U}^*(\rho,\epsilon)\right)\int_0^1 \omega^s(1-\omega)^s d\omega.
\end{gathered}$$

It follows that

$$\begin{gathered}
\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y_*(r,\epsilon) \times \mathcal{U}_*(r,\epsilon) dr \leq \Lambda_*((\rho,\sigma),\epsilon)\int_0^1 \omega^s \cdot \omega^s d\omega \\
+ \partial_*((\rho,\sigma),\epsilon)\int_0^1 \omega^s(1-\omega)^s d\omega,
\end{gathered}$$

$$\begin{gathered}
\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y^*(r,\epsilon) \times \mathcal{U}^*(r,\epsilon) dr \geq \Lambda^*((\rho,\sigma),\epsilon)\int_0^1 \omega^s \cdot \omega^s d\omega \\
+ \partial^*((\rho,\sigma),\epsilon)\int_0^1 \omega^s(1-\omega)^s d\omega,
\end{gathered}$$

that is,

$$\begin{gathered}
\frac{\rho\sigma}{\sigma-\rho}\left[\int_\rho^\sigma Y_*(r,\epsilon) \times \mathcal{U}_*(r,\epsilon) dr, \int_\rho^\sigma Y^*(r,\epsilon) \times \mathcal{U}^*(r,\epsilon) dr\right] \\
\supseteq_I \left[\Lambda_*((\rho,\sigma),\epsilon), \Lambda^*((\rho,\sigma),\epsilon)\right]\int_0^1 \omega^s \cdot \omega^s d\omega \\
+ \left[\partial_*((\rho,\sigma),\epsilon), \partial^*((\rho,\sigma),\epsilon)\right]\int_0^1 \omega^s(1-\omega)^s d\omega.
\end{gathered}$$

Thus,

$$\frac{\rho\sigma}{\sigma-\rho}(FA)\int_{\rho}^{\sigma}\frac{\widetilde{Y}(\varkappa)\otimes\widetilde{\mathcal{U}}(\varkappa)}{\varkappa^{2}}d\varkappa \supseteq_{\mathbb{F}} \widetilde{\Lambda}(\rho,\sigma)\int_{0}^{1}\omega^{s}\cdot\omega^{s}d\omega \oplus \widetilde{\partial}(\rho,\sigma)\int_{0}^{1}\omega^{s}(1-\omega)^{s}d\omega.$$

□

Theorem 8. *Let* $\widetilde{Y} \in \mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$, $\widetilde{\mathcal{U}} \in \mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$, *whose parametrized forms are given as* $Y_\epsilon,\mathcal{U}_\epsilon : [\rho,\sigma] \subset \mathbb{R} \to \mathcal{K}_C^+$ *and defined as* $Y_\epsilon(\varkappa) = [Y_*(\varkappa,\epsilon), Y^*(\varkappa,\epsilon)]$ *and* $\mathcal{U}_\epsilon(\varkappa) = [\mathcal{U}_*(\varkappa,\epsilon),\mathcal{U}^*(\varkappa,\epsilon)]$ *for all* $\varkappa \in [\rho,\sigma]$, $\epsilon \in [0,1]$, *respectively. If* $\widetilde{Y}\otimes\widetilde{\mathcal{U}} \in \mathcal{F}\varkappa_{([\rho,\sigma],\epsilon)}$, *then*

$$\begin{aligned}&2^{2s-1}\widetilde{Y}\left(\tfrac{2\rho\sigma}{\rho+\sigma}\right)\otimes\widetilde{\mathcal{U}}\left(\tfrac{2\rho\sigma}{\rho+\sigma}\right)\\ &\supseteq_{\mathbb{F}} \tfrac{\rho\sigma}{\sigma-\rho}(FA)\int_{\rho}^{\sigma}\tfrac{\widetilde{Y}(\varkappa)\widetilde{\times}\widetilde{\mathcal{U}}(\varkappa)}{\varkappa^{2}}d\varkappa \oplus \widetilde{\Lambda}(\rho,\sigma)\int_{0}^{1}\omega^{s}(1-\omega)^{s}d\omega \oplus \widetilde{\partial}(\rho,\sigma)\int_{0}^{1}\omega^{s}\cdot\omega^{s}d\omega,\end{aligned} \tag{30}$$

where $\widetilde{\Lambda}(\rho,\sigma) = \widetilde{Y}(\rho)\otimes\widetilde{\mathcal{U}}(\rho)\oplus\widetilde{Y}(\sigma)\otimes\widetilde{\mathcal{U}}(\sigma)$,

$\widetilde{\partial}(\rho,\sigma) = \widetilde{Y}(\rho)\otimes\widetilde{\mathcal{U}}(\sigma)\oplus\widetilde{Y}(\sigma)\otimes\widetilde{\mathcal{U}}(\rho)$, *and* $\Lambda_\epsilon(\rho,\sigma) = [\Lambda_*((\rho,\sigma),\epsilon),\Lambda^*((\rho,\sigma),\epsilon)]$ *and* $\partial_\epsilon(\rho,\sigma) = [\partial_*((\rho,\sigma),\epsilon),\partial^*((\rho,\sigma),\epsilon)]$.

Proof. Via hypothesis, for each $\epsilon \in [0,1]$, we obtain

$$\begin{aligned}
&Y_*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\\
&Y^*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\times\mathcal{U}^*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right)
\end{aligned}$$

$$\begin{aligned}
&\leq 2^{2s}\begin{bmatrix} Y_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\\ +Y_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\end{bmatrix}\\
&+2^{2s}\begin{bmatrix} Y_*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\\ +Y_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\end{bmatrix},\\
&\geq 2^{2s}\begin{bmatrix} Y^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\\ +Y^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\end{bmatrix}\\
&+2^{2s}\begin{bmatrix} Y^*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\times\mathcal{U}^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\\ +Y^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}^*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&\leq 2^{2s}\begin{bmatrix} Y_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\\ +Y_*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\end{bmatrix}\\
&+2^{2s}\begin{bmatrix} \left(\omega^{s}Y_*(\rho,\epsilon)+(1-\omega)^{s}Y_*(\sigma,\epsilon)\right)\\ \times\left((1-\omega)^{s}\mathcal{U}_*(\rho,\epsilon)+\omega^{s}\mathcal{U}_*(\sigma,\epsilon)\right)\\ +\left((1-\omega)^{s}Y_*(\rho,\epsilon)+\omega^{s}Y_*(\sigma,\epsilon)\right)\\ \times\left(\omega^{s}\mathcal{U}_*(\rho,\epsilon)+(1-\omega)^{s}\mathcal{U}_*(\sigma,\epsilon)\right)\end{bmatrix},\\
&\geq 2^{2s}\begin{bmatrix} Y^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\\ +Y^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\epsilon\right)\times\mathcal{U}^*\left(\tfrac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\end{bmatrix}\\
&+2^{2s}\begin{bmatrix} \left(\omega^{s}Y^*(\rho,\epsilon)+(1-\omega)^{s}Y^*(\sigma,\epsilon)\right)\\ \times\left((1-\omega)^{s}\mathcal{U}^*(\rho,\epsilon)+\omega^{s}\mathcal{U}^*(\sigma,\epsilon)\right)\\ +\left((1-\omega)^{s}Y^*(\rho,\epsilon)+\omega^{s}Y^*(\sigma,\epsilon)\right)\\ \times\left(\omega^{s}\mathcal{U}^*(\rho,\epsilon)+(1-\omega)^{s}\mathcal{U}^*(\sigma,\epsilon)\right)\end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
&= 2^{2s}\begin{bmatrix} Y_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right) \\ +Y_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\end{bmatrix} \\
&+2^{2s}\begin{bmatrix}\left\{\omega^s\cdot\omega^s+(1-\omega)^s(1-\omega)^s\right\}\partial_*((\rho,\sigma),\epsilon) \\ +\left\{\omega^s(1-\omega)^s+(1-\omega)^s\omega^s\right\}\Lambda_*((\rho,\sigma),\epsilon)\end{bmatrix}, \\
&= 2^{2s}\begin{bmatrix} Y^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right) \\ +Y^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\times\mathcal{U}^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\end{bmatrix} \\
&+2^{2s}\begin{bmatrix}\left\{\omega^s\cdot\omega^s+(1-\omega)^s(1-\omega)^s\right\}\partial^*((\rho,\sigma),\epsilon) \\ +\left\{\omega^s(1-\omega)^s+(1-\omega)^s\omega^s\right\}\Lambda^*((\rho,\sigma),\epsilon)\end{bmatrix},
\end{aligned}
$$

Integrating this on $[0,1]$, we obtain

$$
\begin{aligned}
2^{2s-1}Y_*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\times\mathcal{U}_*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\le \tfrac{1}{\sigma-\rho}\int_\rho^\sigma Y_*(r,\epsilon)\times\mathcal{U}_*(r,\epsilon)dr \\
&+\Lambda_*((\rho,\sigma),\epsilon)\int_0^1\omega^s(1-\omega)^s d\omega \\
&+\partial_*((\rho,\sigma),\epsilon)\int_0^1\omega^s\cdot\omega^s d\omega, \\
2^{2s-1}Y^*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\times\mathcal{U}^*\left(\tfrac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\ge \tfrac{1}{\sigma-\rho}\int_\rho^\sigma Y^*(r,\epsilon)\times\mathcal{U}^*(r,\epsilon)dr \\
&+\Lambda^*((\rho,\sigma),\epsilon)\int_0^1\omega^s(1-\omega)^s d\omega \\
&+\partial^*((\rho,\sigma),\epsilon)\int_0^1\omega^s\cdot\omega^s d\omega,
\end{aligned}
$$

that is,

$$2^{2s-1}\widetilde{Y}\left(\tfrac{2\rho\sigma}{\rho+\sigma}\right)\otimes\widetilde{\mathcal{U}}\left(\tfrac{2\rho\sigma}{\rho+\sigma}\right)$$

$$\supseteq_{\mathbb{F}}\ \frac{\rho\sigma}{\sigma-\rho}(FA)\int_\rho^\sigma\frac{\widetilde{Y}(r)\otimes\widetilde{\mathcal{U}}(r)}{r^2}dr\oplus\widetilde{\Lambda}(\rho,\sigma)\int_0^1\omega^s(1-\omega)^s d\omega\oplus\widetilde{\partial}(\rho,\sigma)\int_0^1\omega^s\cdot\omega^s d\omega$$

Theorem 8 has been proved.

First, we will derive the following inequality, which is referred to as the right (or first) fuzzy $H\cdot H$ Fejér inequality, which is related to the right portion of the classical $H\cdot H$ Fejér inequality for $\mathcal{UDH}$ s-convex $\mathcal{FNVM}s$ via up and down fuzzy order relations. □

Theorem 9. *Let $\widetilde{Y}\in\mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$, whose parametrized form is given as $Y_\epsilon:[\rho,\sigma]\subset\mathbb{R}\to\mathcal{K}_C^+$ and defined as $Y_\epsilon(r)=[Y_*(r,\epsilon),Y^*(r,\epsilon)]$ for all $r\in[\rho,\sigma]$, $\epsilon\in[0,1]$. If $\widetilde{Y}\in\mathcal{F}r_{([\rho,\sigma],\epsilon)}$ and $\mathfrak{B}:[\rho,\sigma]\to\mathbb{R}$, $\mathfrak{B}\left(\frac{1}{\frac{1}{\rho}+\frac{1}{\sigma}-\frac{1}{r}}\right)=\mathfrak{B}(r)\ge 0$, then*

$$\frac{\rho\sigma}{\sigma-\rho}(FA)\int_\rho^\sigma\frac{\widetilde{Y}(r)}{r^2}\mathfrak{B}(r)dr\supseteq_{\mathbb{F}}\left[\widetilde{Y}(\rho)\oplus\widetilde{Y}(\sigma)\right]\int_0^1\omega^s\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega, \tag{31}$$

and if $\widetilde{Y}\in\mathcal{UDHFSV}([\rho,\sigma],\mathbb{E}_C,s)$, then inequality (31) is reversed.

Proof. Let Y be an s-convex $\mathcal{FNVM}$. Then, for each $\epsilon\in[0,1]$, we obtain

$$
\begin{aligned}
&Y_*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)B\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right) \\
&\le\left(\omega^sY_*(\rho,\epsilon)+(1-\omega)^sY_*(\sigma,\epsilon)\right)B\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right), \\
&Y^*\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)B\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right) \\
&\ge\left(\omega^sY^*(\rho,\epsilon)+(1-\omega)^sY^*(\sigma,\epsilon)\right)B\left(\tfrac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right).
\end{aligned} \tag{32}
$$

In addition,

$$\begin{gathered}
Y_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right) \\
\leq \left((1-\omega)^s Y_*(\rho,\epsilon)+\omega^s Y_*(\sigma,\epsilon)\right)B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right), \\
Y^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right) \\
\geq \left((1-\omega)^s Y^*(\rho,\epsilon)+\omega^s Y^*(\sigma,\epsilon)\right)B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right).
\end{gathered} \tag{33}$$

After adding (32) and (33) and integrating on $[0,1]$, we obtain

$$\begin{gathered}
\int_0^1 Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega \\
+\int_0^1 Y_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\
\leq \int_0^1\left[\begin{array}{c} Y_*(\rho,\epsilon)\left\{\omega^s B\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)+(1-\omega)^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)\right\} \\ +Y_*(\sigma,\epsilon)\left\{(1-\omega)^s B\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)+\omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)\right\}\end{array}\right]d\omega, \\
\int_0^1 Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega \\
+\int_0^1 Y^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\
\geq \int_0^1\left[\begin{array}{c} Y^*(\rho,\epsilon)\left\{\omega^s B\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)+(1-\omega)^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)\right\} \\ +Y^*(\sigma,\epsilon)\left\{(1-\omega)^s B\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)+\omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)\right\}\end{array}\right]d\omega.
\end{gathered}$$

$$\begin{gathered}
=2Y_*(\rho,\epsilon)\int_0^1 \omega^s\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega \\
+2Y_*(\sigma,\epsilon)\int_0^1 \omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega, \\
=2Y^*(\rho,\epsilon)\int_0^1 \omega^s\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega \\
+2Y^*(\sigma,\epsilon)\int_0^1 \omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega.
\end{gathered}$$

Since $\mathfrak{B}$ is symmetric, then

$$\begin{gathered}
=2[Y_*(\rho,\epsilon)+Y_*(\sigma,\epsilon)]\int_0^1 \omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega, \\
=2[Y^*(\rho,\epsilon)+Y^*(\sigma,\epsilon)]\int_0^1 \omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega.
\end{gathered} \tag{34}$$

As such,

$$\begin{gathered}
\int_0^1 Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega \\
=\int_0^1 Y_*\left(\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right),\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\
=\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y_*(\varkappa,\epsilon)\mathfrak{B}(\varkappa)d\varkappa \\
\int_0^1 Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega \\
=\int_0^1 Y^*\left(\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right),\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\
=\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y^*(\varkappa,\epsilon)\mathfrak{B}(\varkappa)d\varkappa.
\end{gathered} \tag{35}$$

From (34) and (35), we obtain

$$\begin{gathered}
\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y_*(\varkappa,\epsilon)\mathfrak{B}(\varkappa)d\varkappa \\
\leq [Y_*(\rho,\epsilon)+Y_*(\sigma,\epsilon)]\int_0^1 \omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega, \\
\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y^*(\varkappa,\epsilon)\mathfrak{B}(\varkappa)d\varkappa \\
\geq [Y^*(\rho,\epsilon)+Y^*(\sigma,\epsilon)]\int_0^1 \omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega,
\end{gathered}$$

that is,

$$\begin{gathered}\left[\frac{\rho\sigma}{\sigma-\rho}\int_{\rho}^{\sigma} Y_*(r,\epsilon)\mathfrak{B}(r)dr, \frac{\rho\sigma}{\sigma-\rho}\int_{\rho}^{\sigma} Y^*(r,\epsilon)\mathfrak{B}(r)dr\right] \\ \supseteq_I [Y_*(\rho,\epsilon)+Y_*(\sigma,\epsilon), Y^*(\rho,\epsilon)+Y^*(\sigma,\epsilon)]\int_0^1 \omega^s B\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega\end{gathered}$$

and, hence,

$$\frac{\rho\sigma}{\sigma-\rho}(FA)\int_{\rho}^{\sigma}\frac{\widetilde{Y}(r)}{r^2}\mathfrak{B}(r)dr \supseteq_{\mathbb{F}} \left[\widetilde{Y}(\rho)\oplus\widetilde{Y}(\sigma)\right]\int_0^1 \omega^s\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega,$$

and this concludes the proof.

Next, we construct the first $H{\cdot}H$ Fejer inequality for a $\mathcal{UDH}$ s-convex $\mathcal{FNVM}$ s-convex $\mathcal{FNVM}$, which first generalizes the $H{\cdot}H$ Fejer inequality for the classical harmonically convex function. □

Theorem 10. *Let* $\widetilde{Y} \in \mathcal{UDHFSX}([\rho,\sigma],\mathbb{E}_C,s)$*, whose parametrized form is given as* $Y_\epsilon : [\rho,\sigma] \subset \mathbb{R} \to \mathcal{K}_C^+$ *and defined as* $Y_\epsilon(r) = [Y_*(r,\epsilon), Y^*(r,\epsilon)]$ *for all* $r \in [\rho,\sigma]$, $\epsilon \in [0,1]$*. If* $\widetilde{Y} \in \mathcal{F}r_{([\rho,\sigma],\epsilon)}$ *and* $\mathfrak{B} : [\rho,\sigma] \to \mathbb{R}, \mathfrak{B}\left(\frac{1}{\frac{1}{\rho}+\frac{1}{\sigma}-\frac{1}{r}}\right) = \mathfrak{B}(r) \geq 0,$ *then*

$$2^{s-1}\widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\int_{\rho}^{\sigma}\frac{\widetilde{Y}(r)}{r^2}dr \supseteq_{\mathbb{F}} (FA)\int_{\rho}^{\sigma}\frac{\widetilde{Y}(r)}{r^2}\mathfrak{B}(r)dr. \tag{36}$$

If $\widetilde{Y} \in \mathcal{UDHFSV}([\rho,\sigma],\mathbb{E}_C,s)$*, then inequality (36) is reversed.*

Proof. Since Y is s-convex, then for $\epsilon \in [0,1]$, we obtain

$$\begin{aligned} Y_*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\leq \frac{1}{2^s}\left(Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)+Y_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\right) \\ Y^*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\geq \frac{1}{2^s}\left(Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)+Y^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\right), \end{aligned} \tag{37}$$

and by multiplying (37) by $\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right) = \mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)$ and integrating it with ω on $[0,1]$, we obtain

$$\begin{aligned} &Y_*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\int_0^1\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\ &\leq \frac{1}{2^s}\left(\begin{array}{c}\int_0^1 Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\ +\int_0^1 Y_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega\end{array}\right) \\ &Y^*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\int_0^1\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\ &\geq \frac{1}{2^s}\left(\begin{array}{c}\int_0^1 Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega \\ +\int_0^1 Y^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega\end{array}\right) \end{aligned} \tag{38}$$

Since

$$\begin{aligned}
&\int_0^1 Y_*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega\\
&=\int_0^1 Y_*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega,\\
&=\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y_*(\imath,\epsilon)\mathfrak{B}(\imath)d\imath,\\
&\int_0^1 Y^*\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{\omega\rho+(1-\omega)\sigma}\right)d\omega\\
&=\int_0^1 Y^*\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma},\epsilon\right)\mathfrak{B}\left(\frac{\rho\sigma}{(1-\omega)\rho+\omega\sigma}\right)d\omega,\\
&=\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma Y^*(\imath,\epsilon)\mathfrak{B}(\imath)d\imath,
\end{aligned} \tag{39}$$

from (38) and (39), we obtain

$$\begin{aligned}
Y_*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\le \frac{2^{1-s}}{\int_\rho^\sigma \mathfrak{B}(\imath)d\imath}\int_\rho^\sigma Y_*(\imath,\epsilon)\mathfrak{B}(\imath)d\imath,\\
Y^*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right) &\ge \frac{2^{1-s}}{\int_\rho^\sigma \mathfrak{B}(\imath)d\imath}\int_\rho^\sigma Y^*(\imath,\epsilon)\mathfrak{B}(\imath)d\imath.
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
&\left[Y_*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right),Y^*\left(\frac{2\rho\sigma}{\rho+\sigma},\epsilon\right)\right]\\
&\supseteq_I \frac{2^{1-s}}{\int_\rho^\sigma \mathfrak{B}(\imath)d\imath}\left[\int_\rho^\sigma Y_*(\imath,\epsilon)\mathfrak{B}(\imath)d\imath,\int_\rho^\sigma Y^*(\imath,\epsilon)\mathfrak{B}(\imath)d\imath\right],
\end{aligned}$$

that is,

$$2^{s-1}\widetilde{Y}\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\int_\rho^\sigma \frac{\widetilde{Y}(\imath)}{\imath^2}d\imath \supseteq_{\mathbb{F}} (FA)\int_\rho^\sigma \frac{\widetilde{Y}(\imath)}{\imath^2}\mathfrak{B}(\imath)d\imath$$

Then, we complete the proof. □

Remark 5. *If $\mathfrak{B}(\imath)=1$, therefore, from (31) and (36), we obtain inequality (21).*

If $s=1$, then from inequalities (31) and (36), we acquire the inequality for harmonically convex $\mathcal{FNVM}$s (see [36]).

If $Y_*(\imath,\epsilon)=Y^*(\imath,\epsilon)$ with $\epsilon=1$ and $s=1$, then from inequalities (31) and (36), we acquire the inequality forthe classical harmonically convex function.

4. Conclusions

Incorporating an up and down fuzzy relation and the integral inequalities that come with it is a novel strategy that was examined in this paper. The fuzzy Aumann integral operator with fuzzy number values was used to generalize Hermite–Hadamard inequalities. Future research on the Hadamard–Mercer-type and other related integral inequalities will be very fascinating to see how the concepts of cr-convex fuzzy-number-valued functions and interval-valued functions are applied.

The techniques and ideas discussed in this work can be used to examine distinct convex inequalities, with possible applications in optimization and differential equations with convex shapes.

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