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The Cauchy–Optimal Stability Results for Cauchy–Jensen Additive Mappings in the Fuzzy Banach Space and the Unital Fuzzy Banach Space

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Abstract: In this article, we apply a new class of fuzzy control functions to approximate a Cauchy additive mapping in fuzzy Banach space (FBS). Further, considering the unital FBS (UFBS), we will investigate the isomorphisms defined in this space. By introducing several specific functions and choosing the optimal control function from among these functions, we evaluate the Cauchy–Optimal stability (C–O-stability) for all defined mappings.

Keywords: Cauchy additive mapping; Jensen additive mapping; stability; isomorphism; fuzzy Banach space; unital space

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1. Introduction

The topic of stability first started with Ulam's famous question about additive mappings in 1940. After that, Hyers and Rassias, by expanding this topic, presented new definitions of stability for additive mappings and continuous maps, which were known as Hyers–Ulam and Hyers–Ulam–Rasias stability, respectively [1–3]. Since then, many researchers have conducted extensive research on the issue of stability for functional equations in different spaces. For example, in 1980, the stability of homomorphism equations was studied.

Researchers in [4] considered the function $\Delta : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$, which is defined as follows

$$\widetilde{\Delta}(\xi, \varrho) = \sum_{\ell=0}^{\infty} 2^{-\ell} \Delta(2^{\ell}\xi, 2^{\ell}\varrho) < \infty, \qquad \text{for all} \qquad \xi, \varrho \in \mathcal{Q},$$

and in [5], considered the function $\Delta : \mathcal{M} \setminus \{0\} \times \mathcal{M} \setminus \{0\} \to [0, \infty)$, which is defined as follows

$$\widetilde{\Delta}(\xi, \varrho) = \sum_{\ell=0}^{\infty} \frac{1}{3^{\ell}} \Delta(3^{\ell}x, 3^{\ell}y) < \infty, \qquad \text{for all} \qquad \xi, \varrho \in \mathcal{M} \setminus \{0\},$$

on the abelian group $\mathcal{Q}$ and the Banach space $\mathcal{M}$. They investigated the stability of two types of functions. In such a way that if for the function $\Psi : \mathcal{Q} \to \mathcal{M}$, we have $\|\Psi(\xi + \varrho) - \Psi(\xi) - \Psi(\varrho)\| \leq \Delta(\xi, \varrho)$, then there is a unique additive mapping such as

$\Phi : \mathcal{Q} \to \mathcal{M}$ such that $\|\Psi(\xi) - \Phi(\xi)\| \leq \frac{1}{2}\widetilde{\Delta}(\xi, \xi)$, and if for the function $\Psi : \mathcal{M} \to \mathcal{N}$, we have $\|2\Psi(\frac{\xi+\varrho}{2}) - \Psi(\xi) - \Psi(\varrho)\| \leq \Delta(\xi, \varrho)$, then there is a unique additive mapping such as $\Phi : \mathcal{M} \to \mathcal{N}$ such that $\|\Psi(\xi) - \Psi(0) - \Phi(\xi)\| \leq \frac{1}{3}\big(\widetilde{\Delta}(\xi, -\xi) + \widetilde{\Delta}(-\xi, 3\xi)\big)$. Additionally, in [6–12], authors have investigated different equations using fixed point theory in classical and fuzzy spaces. In 1965, the theory of fuzzy sets was introduced in [13]. After that, this new theory was applied to classical concepts. The concept of the fuzzy norm was introduced by Katsaras in 1984 [14]. Moreover, Kramosil and Michalek (1975) introduced the new concept of fuzzy metric space and provided many results. In 1994, George and Veeramani introduced a stronger form of fuzzy metric space [15,16]. Afterward, many mathematicians studied fixed-point theorems in related spaces. In [10], the authors considered a nonlinear single fractional differential equation and applied an alternative fixed-point theorem to prove the existence of a unique solution and the multiple stability for the NS–ABC–FDE in the symmetric matrix-valued FBS in [17], considering fuzzy measure theory and matrix-valued fuzzy norm spaces, they study a differential system of non-autonomous cellular neural networks with mixed delays. Specific functions are among the most widely used functions in mathematics and other sciences, which have attracted the attention of many researchers today. These functions are used in various fields, such as physical sciences, engineering, probability theory, decision theory, artificial intelligence, pattern recognition, image processing, etc. The Mittag–Leffler function, Gauss hypergeometric function, Wright function, H-Fox function, and aggregation functions are the most important types of these functions. Since our effort in the stability issue is to achieve the best approximation, we achieve this important goal by selecting the most optimal function among these specific functions and using it as the control function [8–10]. In the following, we will explain the different parts of this article:

In the first section, we state all the basic concepts, including definitions, lemmas, and basic theorems needed for the main steps. In the second section, we do the main proofs. In this way, considering the vector spaces $\mathcal{M}, \mathcal{N}$ and defining the function $\Psi : \mathcal{M} \to \mathcal{N}$, we consider the following equations

$$\Psi(\frac{\xi+\varrho}{2} + \tau) + \Psi(\frac{\xi-\varrho}{2} + \tau) = \Psi(\xi) + 2\Psi(\tau), \tag{1}$$

$$\Psi(\frac{\xi+\varrho}{2} + \tau) - \Psi(\frac{\xi-\varrho}{2} + \tau) = \Psi(\varrho), \tag{2}$$

$$2\Psi(\frac{\xi+\varrho}{2} + \tau) = \Psi(\xi) + \Psi(\varrho) + 2\Psi(\tau), \tag{3}$$

for all $\xi, \varrho, \tau \in \mathcal{M}$ and show that the function Ψ is an additive function. In the following, we prove the stability of functional Equations (1)–(3) by considering FBS. Additionally, all the proofs are done considering the UFBS for the isomorphisms defined in these spaces. In the third section, by choosing the aggregation function as the optimal control function, we investigate the O-stability of the functional equations. We end this article in the last section entitled Conclusion.

2. Basic Concepts

We first introduce the required spaces. These spaces are used in all parts of the article [6,8,10]. From here on, we consider $\mathcal{S}_1 = [0,1]$, $\mathcal{S}_2 = (0,\infty)$, $\mathcal{S}_3 = [0,\mathrm{a}]$ and $\mathcal{S}_4 = [0,\infty)$.

Definition 1. *On the interval $\mathcal{S}_1$, we define $\mathcal{G}_{\mathcal{S}_1}$ as follows*

$$\mathcal{G}_{\mathcal{S}_1} = \mathrm{diag}\ \mathrm{g}(\mathcal{S}_1) = \left\{ \mathrm{diag}[\mathrm{g}_1, \cdots, \mathrm{g}_\jmath] = \begin{bmatrix} \mathrm{g}_1 & & \\ & \ddots & \\ & & \mathrm{g}_\jmath \end{bmatrix}, \ \mathrm{g}_1, \ldots, \mathrm{g}_\jmath \in \mathcal{S}_1 \right\},$$

such that for any $\mathbf{g}, \mathbf{k} \in \mathcal{G}_{\mathcal{S}_1}$, *we have* $\mathbf{g} = \mathrm{diag}[\mathrm{g}_1, \cdots, \mathrm{g}_\jmath]$, $\mathbf{k} = \mathrm{diag}[\mathrm{k}_1, \cdots, \mathrm{k}_\jmath]$, $\mathrm{diag}[1, \ldots, 1] = \mathbf{1}$ *and* $\mathrm{diag}[0, \ldots, 0] = \mathbf{0}$. *Further,* $\mathbf{g} \preceq \mathbf{k}$ *means that* $\mathrm{g}_\iota \leq \mathrm{k}_\iota$ *for every* $\iota = 1, \ldots, \jmath$.

Definition 2. *We consider the mapping $\circledast$ from $\mathcal{G}_{\mathcal{S}_1} \times \mathcal{G}_{\mathcal{S}_1}$ to $\mathcal{G}_{\mathcal{S}_1}$. If for each $\mathbf{g}, \mathbf{k}, \mathbf{h}, \mathbf{l} \in \mathcal{G}_{\mathcal{S}_1}$ we have $\mathbf{g} \circledast \mathbf{1} = \mathbf{g}$, $\mathbf{g} \circledast \mathbf{k} = \mathbf{k} \circledast \mathbf{g}$, $\mathbf{g} \circledast (\mathbf{k} \circledast \mathbf{h}) = (\mathbf{g} \circledast \mathbf{k}) \circledast \mathbf{h}$, $\mathbf{g} \preceq \mathbf{k}$ and $\mathbf{h} \preceq \mathbf{l}$ implies that $\mathbf{g} \circledast \mathbf{h} \preceq \mathbf{k} \circledast \mathbf{l}$, we say that $\circledast$ is a generalized t-norm or briefly GTN. Additionally, we consider sequences $\{\mathbf{g}_\jmath\}$ and $\{\mathbf{k}_\jmath\}$ that converge to $\mathbf{g}$ and $\mathbf{k}$. If we have $\lim_{\mathrm{m}}(\mathbf{g}_\jmath \circledast \mathbf{k}_\jmath) = \mathbf{g} \circledast \mathbf{k}$, then $\circledast$ is a CGTN.*

There are different types of CGTN—minimum CGTN, product CGTN, and Lukasiewicz CGTN can be mentioned among the most important of them. In this work, we choose the minimum CGTN $\circledast_M : \mathcal{G}_{\mathcal{S}_1} \times \mathcal{G}_{\mathcal{S}_1} \to \mathcal{G}_{\mathcal{S}_1}$, which is defined as follows:

$$\mathbf{g} \circledast_M \mathbf{k} = \mathrm{diag}[\mathrm{g}_1, \cdots, \mathrm{g}_\jmath] \circledast_{\mathrm{M}} \mathrm{diag}[\mathrm{k}_1, \cdots, \mathrm{k}_\jmath] = \mathrm{diag}[\min\{\mathrm{g}_1, \mathrm{k}_1\}, \cdots, \min\{\mathrm{g}_\jmath, \mathrm{k}_\jmath\}].$$

We also provide the definition of product CGTN and Lukasiewicz CGTN as follows:

$$\begin{aligned} \mathbf{g} \circledast_P \mathbf{k} &= \mathrm{diag}[\mathrm{g}_1, \cdots, \mathrm{g}_\jmath] \circledast_{\mathrm{P}} \mathrm{diag}[\mathrm{k}_1, \cdots, \mathrm{k}_\jmath] = \mathrm{diag}[\mathrm{g}_1.\mathrm{k}_1, \cdots, \mathrm{g}_\jmath.\mathrm{k}_\jmath\}], \\ \mathbf{g} \circledast_L \mathbf{k} &= \mathrm{diag}[\mathrm{g}_1, \cdots, \mathrm{g}_\jmath] \circledast_{\mathrm{L}} \mathrm{diag}[\mathrm{k}_1, \cdots, \mathrm{k}_\jmath] \\ &= \mathrm{diag}[\max\{\mathrm{g}_1 + \mathrm{k}_1 - 1, 0\}, \cdots, \max\{\mathrm{g}_\jmath + \mathrm{k}_\jmath - 1, 0\}]. \end{aligned}$$

In the following, we provide examples of these CGTNs.

Example 1. *($\circledast_M$)* $\mathrm{diag}[0, \frac{1}{10}, \frac{1}{100}] \circledast_{\mathrm{M}} \mathrm{diag}[\frac{1}{5}, 1, \frac{1}{4}] = \mathrm{diag}[0, \frac{1}{10}, \frac{1}{100}]$;
($\circledast_P$) $\mathrm{diag}[0, \frac{1}{10}, \frac{1}{100}] \circledast_{\mathrm{P}} \mathrm{diag}[\frac{1}{5}, 1, \frac{1}{4}] = \mathrm{diag}[0, \frac{1}{10}, \frac{1}{400}]$;
($\circledast_L$) $\mathrm{diag}[0, \frac{1}{10}, \frac{1}{100}] \circledast_{\mathrm{L}} \mathrm{diag}[\frac{1}{5}, 1, \frac{1}{4}] = \mathrm{diag}[0, \frac{1}{10}, 0]$.

In the following, we will define the features of the matrix-type fuzzy functions and the matrix-type fuzzy norms spaces.

Definition 3. *The MVFF $\Delta : \mathcal{S}_3 \times \mathcal{S}_2 \to \mathcal{G}_{\mathcal{S}_1}$ is increasing and continuous, $\lim_{\varsigma \to +\infty} \Delta(\xi, \varsigma) = \mathbf{1}$ for every $\xi \in \mathcal{S}_3$ and $\varsigma \in \mathcal{S}_2$, $\mathcal{X} \precsim \Delta$ if and only if $\mathcal{X}(\xi, \varsigma) \preceq \Delta(\xi, \varsigma)$, for all $\varsigma \in \mathcal{S}_2$ and $\xi \in \mathcal{S}_3$ where $\mathcal{X}$ is the MVFF and $\preceq$ is the relation defined for this type of function.*

Definition 4. *Consider the linear space $\mathcal{Y}$, CGTN $\circledast$ and the MVFF $\mathrm{Y}_{\mathcal{Y}} : \mathcal{S} \times \mathcal{S}_2 \to \mathcal{G}_{\mathcal{S}_1}$, we define $(\mathcal{Y}, \mathrm{Y}_{\mathcal{Y}}, \circledast)$, which is called an MVFN-S and has the following properties,*

- $\mathrm{Y}_{\mathcal{Y}}(\xi, \varsigma) = \mathbf{1}$ *if and only if* $\xi = 0$ *for* $\varsigma \in \mathcal{S}_2$;
- $\mathrm{Y}_{\mathcal{Y}}(\epsilon\xi, \varsigma) = \mathrm{Y}_{\mathcal{Y}}(\xi, \frac{\varsigma}{|\epsilon|})$ *for all* $\xi \in \mathcal{Y}$ *and* $\gamma \neq 0 \in \mathbb{C}$;
- $\mathrm{Y}_{\mathcal{Y}}(\xi + \varrho, \varsigma + \beta) \succeq \mathrm{Y}_{\mathcal{Y}}(\xi, \varsigma) \circledast \mathrm{Y}_{\mathcal{Y}}(\xi, \beta)$ *for all* $\xi \in \mathcal{Y}$ *and any* $\varsigma, \beta \in \mathcal{S}_2$;
- $\lim_{\varsigma \to +\infty} \mathrm{Y}_{\mathcal{Y}}(\xi, \varsigma) = \mathbf{1}$ *for any* $\varsigma \in \mathcal{S}_2$.

When an MVFN-S is complete, we denote it by MVFB-S.

In the following, we investigate optimal stability by introducing a new optimal control function. For this purpose, we go to the definition of the aggregation function. Next, we provide a brief introduction of the special functions used in the optimal control function [6].

Definition 5. *If for any $(\xi_1, \cdots, \xi_{\mathrm{m}})$, $(\varrho_1, \cdots, \varrho_{\mathrm{m}}) \in \mathbb{R}^{\mathrm{m}}$ and $\iota \in \{1, \cdots, \mathrm{m}\}$, and an idempotent function $\mathrm{z}^{(\mathrm{m})} : \mathbb{R}^{\mathrm{m}} \longrightarrow \mathbb{R}$, we have $\xi_\iota \leq \varrho_\iota \Longrightarrow \mathrm{z}^{(\mathrm{m})}(\xi_1, \cdots, \xi_{\mathrm{m}}) \leq \mathrm{z}^{(\mathrm{m})}(\varrho_1, \cdots, \varrho_{\mathrm{m}})$,* *then the* m*-ary* $\mathrm{z}^{(\mathrm{m})}$ *is a generalized aggregation function where* $\mathrm{m} \in \mathbb{N}$*. For* $\mathrm{m} = 1$ *and each* $\xi \in \mathbb{R}$*, we have* $\mathrm{z}^{(1)}(\xi) = \xi$ *and for the convenience of writing we can remove* m *(*m *indicates the number of function variables).*

The famous functions, i.e., arithmetic mean function, projection function, order statistic function, median function, and minimum and maximum functions, are among the important functions of aggregation type. In [6], the authors showed a control function made by minimum aggregation function is the optimal controller. The minimum (MIN) is the smallest generalized aggregation function, and it is defined as follows

$$MIN(\xi) = OS_1(\xi) = \min\{\xi_1, \cdots, \xi_\jmath\} = \bigwedge_{\iota=1}^{\mathrm{m}} \xi_\iota. \tag{4}$$

Therefore, by studying [6,10], we consider the following function as the optimal controller:

$$\boldsymbol{MIN}\Big(\Omega(\xi,\varsigma)\Big) = \mathrm{diag}\Big[MIN(\Omega(\xi,\varsigma)), \cdots, MIN(\Omega(\xi,\varsigma))\Big]. \tag{5}$$

where

$$\Omega(\xi,\varsigma) = \Big(\mathbb{W}_{E_{\lambda,\mu}}, \mathbb{W}_{W_{\lambda,\mu}}, \mathbb{W}_{{}_2F_1}, \mathbb{W}_{H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}}, \mathbb{W}_{\exp}\Big). \tag{6}$$

In the proofs, we use the symbol $\boldsymbol{\Omega}_{\boldsymbol{OS_1}}(\xi,\varsigma)$ instead of the control function $\boldsymbol{MIN}\Big(\Omega(\xi,\varsigma)\Big)$. Additionally, all the variables in the above function are considered as $\frac{-\|\xi\|}{\varsigma}$. In the following, we introduce special functions used in $\boldsymbol{\Omega}_{\boldsymbol{OS_1}}(\xi,\varsigma)$ function [6,10].

Definition 6. *The real exponential function is defined by the following power series*

$$\mathbb{W}_{\exp}(\xi)) = \sum_{\jmath=0}^{\infty} \frac{\xi^\jmath}{\jmath!}, \quad \mathbb{W}_{\exp}(1) = \sum_{\jmath=0}^{\infty} \frac{1}{\jmath!} \tag{7}$$

since the radius of convergence of this power series is infinite, this definition is, in fact, applicable to all complex numbers $z \in \mathbb{C}$*.*

Definition 7. *For* $\lambda, \mu \in \mathbb{C},\ Re(\lambda), Re(\mu) > 0$*, the Mittag-Leffler functions are defined as follows*

$$\mathbb{W}_{E_\lambda}(\xi) = \sum_{\jmath=0}^{\infty} \frac{\xi^\jmath}{\Gamma(\jmath\lambda+1)}, \qquad \mathbb{W}_{E_{\lambda,\mu}}(\xi) = \sum_{\jmath=0}^{\infty} \frac{\xi^\jmath}{\Gamma(\jmath\lambda+\mu)},$$

where $\Gamma(.)$ *is the famous gamma function and* $\mathbb{W}_{\mathrm{U}_\lambda}$*,* $\mathbb{W}_{E_{\lambda,\mu}}$ *are the one- and two-parameter Mittag-Leffler functions, respectively.*

Definition 8. *For* $\lambda > -1, \mu > 0, \xi \in \mathbb{R}$*, the Wright function is defined as follows*

$$\mathbb{W}_{W_{\lambda,\mu}}(\xi) = \sum_{\jmath=0}^{\infty} \frac{\xi^\jmath}{\jmath!\Gamma(\lambda\jmath+\mu)},$$

such that it is of the $1/(1+\sigma)$ *order.*

Definition 9. *H-Fox function for* $0 \le \sigma_1 \le \sigma_2$*,* $1 \le \sigma_3 \le \sigma_4$*,* $\{\mathrm{b}_\iota, \mathrm{c}_\iota\} \in \mathbb{C}$ *and* $\{\mathrm{d}_\iota, \mathrm{e}_\iota\} \in \mathbb{R}^+$ *is defined as follows*

$$\mathbb{W}_{H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}}(\xi) = \mathbb{W}_{H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}}\left[\xi \left| \begin{matrix} (\mathrm{b}_\iota, \mathrm{d}_\iota)_{\iota=1,\cdots,\sigma_2} \\ (\mathrm{c}_\iota, \mathrm{e}_\iota)_{\iota=1,\cdots,\sigma_2} \end{matrix} \right.\right] = \frac{1}{2\pi i} \int_{\mathcal{A}} H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}(\varsigma)\xi^\varsigma d\mathrm{t}, \tag{8}$$

where $\mathcal{A} \in \mathbb{C}$ is a path that is deleted and $\mathcal{R}_1(\varsigma) = \prod_{\iota=1}^{\sigma_1} \Gamma(\mathrm{c}_\iota - \varrho_\iota \varsigma)$, $\mathcal{R}_2(\varsigma) = \prod_{\iota=1}^{\sigma_3} \Gamma(1 - \mathrm{b}_\iota + \xi_\iota \varsigma)$, $\mathcal{R}_3(\varsigma) = \prod_{\iota=\sigma_3+1}^{\sigma_3} \Gamma(1 - \mathrm{c}_\iota + \varrho_\iota \varsigma)$, $\mathcal{R}_4(\varsigma) = \prod_{\iota=\sigma_1+1}^{\sigma_2} \Gamma(\mathrm{b}_\iota - \xi_\iota \varsigma)$ and $\xi^\varsigma = \exp\{\varsigma(\log|\xi| + i \arg \xi)\}$. For these functions, there is a condition that $\sigma_1 = 0$ if and only if $\mathcal{R}_2(\varsigma) = 1$, $\sigma_3 = \sigma_4$ if and only if $\mathcal{R}_3(\varsigma) = 1$ and $\sigma_1 = \sigma_2$ if and only if $\sigma_4(\varsigma) = 1$. Further, $H_{\sigma_2,\sigma_4}^{\sigma_3,\sigma_1}(\varsigma) = \frac{\mathcal{R}_1(\varsigma)\mathcal{R}_2(\varsigma)}{\mathcal{R}_3(\varsigma)\mathcal{R}_4(\varsigma)}$.

Definition 10. *Considering $p, q, r > 0$, the Gauss hypergeometric function $\mathbb{W}_{2F_1} : \mathbb{R}^3 \times \mathcal{S}_3 \longrightarrow \mathcal{S}_2$ is defined as follows*

$$\mathbb{W}_{2F_1}(p, q, r; \xi) = \sum_{\jmath=0}^{\infty} \frac{(p)_\jmath (q)_\jmath}{(r)_\jmath} \frac{\xi^\jmath}{\jmath!} = \frac{\Gamma(r)}{\Gamma(p)\Gamma(q)} \sum_{\jmath=0}^{\infty} \frac{\Gamma(p+\jmath)\Gamma(q+\jmath)}{\Gamma(r+\jmath)} \frac{\xi^\jmath}{\jmath!}.$$

These functions are used in all the theorems presented in Section 4.

Considering the two vector spaces $\mathcal{M}$ and $\mathcal{N}$ along with FNS, for all $\xi, \varrho, \tau \in \mathcal{M}$, we define the following equations by considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$:

$$\mathcal{L}_{1\Psi}(\xi, \varrho, \tau) = \Psi(\frac{\xi+\varrho}{2} + \tau) + \Psi(\frac{\xi-\varrho}{2} + \tau) - \Psi(\xi) - 2\Psi(\tau), \tag{9}$$

$$\mathcal{L}_{2\Psi}(\xi, \varrho, \tau) = \Psi(\frac{\xi+\varrho}{2} + \tau) - \Psi(\frac{\xi-\varrho}{2} + \tau) - \Psi(\varrho), \tag{10}$$

$$\mathcal{L}_{3\Psi}(\xi, \varrho, \tau) = 2\Psi(\frac{\xi+\varrho}{2} + \tau) - \Psi(\xi) - \Psi(\varrho) - 2\Psi(\tau). \tag{11}$$

For the field $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$, we consider two UFBSs, $\mathcal{E}_1$ and $\mathcal{E}_2$ along with the unique terms E', E, and FNSs. For each $\delta \in \mathbb{F}$ and all $\xi, \varrho, \tau \in \mathcal{E}_1$ and for the mapping $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$, we consider the following equation

$$\mathrm{U}_\delta \Psi(\xi, \varrho, \tau) := \Psi(\frac{\delta\xi + \delta\varrho}{2} + \delta\tau) + \delta\Psi(\frac{\xi-\varrho}{2} + \tau) - \delta\Psi(\xi) - 2\delta\Psi(\tau). \tag{12}$$

We assume that $\mathbf{\Delta}_{OS_1} : \mathcal{M}^3 \times \mathcal{S}_2 \to \mathcal{S}_4$ is a function that for all $\xi, \varrho, \tau \in \mathcal{M}$, we define in the following 4 cases:

$$\widetilde{\mathbf{\Delta}_{OS_1}}\Big((\xi, \varrho, \tau), \varsigma\Big) = \mathbf{\Delta}_{OS_1}\Big((\frac{\xi}{2^\ell}, \frac{\varrho}{2^\ell}, \frac{\tau}{2^\ell}), \frac{\varsigma}{\sum_{\ell=1}^{\infty} 2^\ell}\Big) < \infty, \tag{13}$$

$$\widetilde{\mathbf{\Delta}_{OS_1}}\Big((\xi, \varrho, \tau), \varsigma\Big) = \mathbf{\Delta}_{OS_1}\Big((2^\ell\xi, 2^\ell\varrho, 2^\ell\tau), \frac{\varsigma}{\sum_{\ell=0}^{\infty} \frac{1}{2^\ell}}\Big) < \infty, \tag{14}$$

$$\widetilde{\mathbf{\Delta}_{OS_1}}\Big((\xi, \varrho, \tau), \varsigma\Big) = \mathbf{\Delta}_{OS_1}\Big((\frac{3^\ell}{2^\ell}\xi, \frac{3^\ell}{2^\ell}\varrho, \frac{3^\ell}{2^\ell}\tau), \frac{\varsigma}{\sum_{\ell=0}^{\infty} \frac{2^\ell}{3^\ell}}\Big) < \infty, \tag{15}$$

$$\widetilde{\mathbf{\Delta}_{OS_1}}\Big((\xi, \varrho, \tau), \varsigma\Big) = \mathbf{\Delta}_{OS_1}\Big((\frac{2^\ell}{3^\ell}\xi, \frac{2^\ell}{3^\ell}\varrho, \frac{2^\ell}{3^\ell}\tau), \frac{\varsigma}{\sum_{\ell=1}^{\infty} \frac{3^\ell}{2^\ell}}\Big) < \infty, \tag{16}$$

and we consider these functions to prove our results.

3. C–O-Stability for CJAM and Isomorphisms in FBS and UFBS

Theorem 1 ([7])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be vector spaces. If a mapping $\Psi : \mathcal{M} \to \mathcal{N}$ satisfies (1)–(3), then the mapping Ψ is Cauchy additive (CA).*

Proof. If we put $\xi = \varrho$ in (1), (2), and (3) respectively, we have

$$\Psi(\xi+\tau)+\Psi(\tau)=\Psi(\xi)+2\Psi(\tau), \tag{17}$$
$$\Psi(\xi+\tau)-\Psi(\tau)=\Psi(\xi), \tag{18}$$
$$2\Psi(\xi+\tau)=2\Psi(\xi)+2\Psi(\tau), \tag{19}$$

for all $\xi, \tau \in \mathcal{M}$. Therefore, in all three cases above, we conclude that $\Psi(\xi+\tau)=\Psi(\xi)+\Psi(\tau)$ and this means that $\Psi : \mathcal{M} \to \mathcal{N}$ is CA. □

Proposition 1 ([7])**.** *If $\Psi : \mathcal{M} \to \mathcal{N}$ are the functions used in theorem 1, then these functions are of Cauchy–Jensen additive mapping (CJAM) type. If we write $\xi = \varrho$ and $\tau = 0$ in (3), then we have the following Cauchy–Jensen additive mapping (CJAM), respectively*

$$\Psi(\xi+\tau)=\Psi(\xi)+\Psi(\tau),$$
$$2\Psi(\frac{\xi+\varrho}{2})=\Psi(\xi)+\Psi(\varrho).$$

Theorem 2. *Considering the mapping $\Psi : \mathcal{M} \to Y$ and function (13), for each $\xi, \varrho, \tau \in \mathcal{M}$, if we have*

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{1\Psi}(\xi,\varrho,\tau),\varsigma\Big) \succeq \mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big), \tag{20}$$

then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ such that for all $\xi \in \mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\Big) \succeq \widetilde{\mathbf{\Delta}_{OS_1}}\Big((\xi,\xi,\xi),2\varsigma\Big). \tag{21}$$

Proof. In the assumption of (20), we put $\xi = \varrho = \tau$. Therefore, for all $\xi \in \mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(2\xi)-2\Psi(\xi),\varsigma\Big) \succeq \mathbf{\Delta}_{OS_1}\Big((\xi,\xi,\xi),\varsigma\Big). \tag{22}$$

Then, for each $\xi \in \mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-2\Psi(\frac{\xi}{2}),\varsigma\Big) \succeq \mathbf{\Delta}_{OS_1}\Big((\frac{\xi}{2},\frac{\xi}{2},\frac{\xi}{2}),\varsigma\Big),$$

therefore, for all $\xi \in \mathcal{M}$ and for each $\kappa \in \mathbb{Z}^+$ and $\gamma \in \mathbb{Z}^+$ with $\kappa > \gamma$, we have

$$\Upsilon_{\mathcal{Y}}\Big(2^{\gamma}\Psi(\frac{\xi}{2^{\gamma}})-2^{\kappa}\Psi(\frac{\xi}{2^{\kappa}}),\varsigma\Big) \succeq \mathbf{\Delta}_{OS_1}\Big((\frac{\xi}{2^{\ell}},\frac{\xi}{2^{\ell}},\frac{\xi}{2^{\ell}})\frac{\varsigma}{\sum_{\ell=\gamma+1}^{\kappa}2^{\ell-1}}\Big). \tag{23}$$

Therefore, according to (13) and (23), for all $\xi \in \mathcal{M}$, the sequence $\{2^{\alpha}\Psi(\frac{\xi}{2^{\alpha}})\}$ is a Cauchy sequence, and since $\mathcal{N}$ is complete, the convergence of this sequence is the result. Therefore, for each $\xi \in \mathcal{M}$, we define the mapping $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ as $\mathcal{H}(\xi) = \lim_{\alpha\to\infty} 2^{\alpha}\Psi(\frac{\xi}{2^{\alpha}})$. Due to equations (13) and (20), for each $\xi, \varrho, \tau \in \mathcal{M}$, we get

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{\mathcal{H}}(\xi,\varrho,\tau),\varsigma\Big) = \lim_{\alpha\to\infty}\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{1\Psi}(\frac{\xi}{2^{\alpha}},\frac{\varrho}{2^{\alpha}},\frac{\tau}{2^{\alpha}}),\frac{\varsigma}{2^{\alpha}}\Big) \succeq \lim_{\alpha\to\infty}\mathbf{\Delta}_{OS_1}\Big((\frac{\xi}{2^{\alpha}},\frac{\varrho}{2^{\alpha}},\frac{\tau}{2^{\alpha}}),\frac{\varsigma}{2^{\alpha}}\Big) = \mathbf{1},$$

and then, $\mathcal{L}_{\mathcal{H}}(\xi,\varrho,\tau) = 0$. Now, using Theorem 1, we conclude that $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ is a CAM. If we consider (23) and assume that $\gamma = 0$ and we get limit when $m \to \infty$, we reach (21). Next, we prove the uniqueness of $\mathcal{H}$. For this purpose, we consider another CJAM $\Phi : \mathcal{M} \to \mathcal{N}$ and assume that it applies to (21). Therefore, we have the following inequality for every $\xi \in \mathcal{M}$, which tends to zero when $\alpha \to \infty$,

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{H}(\xi)-\Phi(\xi),2\varsigma\Big) = \Upsilon_{\mathcal{Y}}\Big(\mathcal{H}(\frac{\xi}{2^{\alpha}})-\Phi(\frac{\xi}{2^{\alpha}}),\frac{2\varsigma}{2^{\alpha}}\Big) \tag{24}$$

$$\succeq \Upsilon_{\mathcal{Y}}\Big(\mathcal{H}(\frac{\xi}{2^{\alpha}})-\Psi(\frac{\xi}{2^{\alpha}}),\frac{\varsigma}{2^{\alpha}}\Big)\circledast \Upsilon_{\mathcal{Y}}\Big(\Phi(\frac{\xi}{2^{\alpha}})-\Psi(\frac{\xi}{2^{\alpha}}),\frac{\varsigma}{2^{\alpha}}\Big) \tag{25}$$

$$\succeq \widetilde{\mathbf{\Delta}_{OS_1}}\Big((\frac{\xi}{2^{\alpha}},\frac{\xi}{2^{\alpha}},\frac{\xi}{2^{\alpha}}),\frac{\varsigma}{2^{\alpha}}\Big)\circledast \widetilde{\mathbf{\Delta}_{OS_1}}\Big((\frac{\xi}{2^{n}},\frac{\xi}{2^{\alpha}},\frac{\xi}{2^{\alpha}}),\frac{\varsigma}{2^{\alpha}}\Big) \tag{26}$$

$$\succeq \widetilde{\mathbf{\Delta}_{OS_1}}\Big((\frac{\xi}{2^{\alpha}},\frac{\xi}{2^{\alpha}},\frac{\xi}{2^{\alpha}}),\frac{\varsigma}{2^{\alpha}}\Big), \tag{27}$$

and this means $\mathcal{H}(\xi)=\Phi(\xi)$. □

Theorem 3. *Considering the mapping $\Psi:\mathcal{M}\to\mathcal{N}$ and function (14), for each $\xi,\varrho,\tau\in\mathcal{M}$, if (20) is established, then, there exists a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$ such that it applies to (21).*

Proof. Using (22) and for each $\xi\in\mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-\frac{1}{2}\Psi(2\xi),\varsigma\Big)\succeq \mathbf{\Delta}_{OS_1}\Big((\xi,\xi,\xi),2\varsigma\Big),$$

therefore, for all $\xi\in\mathcal{M}$ and for each $\kappa,\gamma\in\mathbb{Z}^+$ with $\kappa>\gamma$, we have

$$\Upsilon_{\mathcal{Y}}\Big(\frac{1}{2^{\gamma}}\Psi(2^{\gamma}\xi)-\frac{1}{2^{\kappa}}\Psi(2^{\kappa}\xi),\varsigma\Big)\succeq \mathbf{\Delta}_{OS_1}\Big((2^{\ell}\xi,2^{\ell}\xi,2^{\ell}\xi),\frac{\varsigma}{\sum_{\ell=\gamma}^{\kappa-1}\frac{1}{2^{\ell+1}}}\Big). \tag{28}$$

Therefore, according to (14) and (28), for all $\xi\in\mathcal{M}$, the sequence $\{\frac{1}{2^{\alpha}}\Psi(2^{\alpha}\xi)\}$ is a Cauchy sequence, and since $\mathcal{N}$ is complete, the convergence of this sequence is the result. Therefore, for each $\xi\in\mathcal{M}$, we define the mapping $\mathcal{H}:\mathcal{M}\to\mathcal{N}$ as $\mathcal{H}(\xi):=\lim_{\alpha\to\infty}\frac{1}{2^{\alpha}}\Psi(2^{\alpha}\xi)$. Due to the (14) and (20), for each $\xi,\varrho,\tau\in\mathcal{M}$, we get

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{\mathcal{H}}(\xi,\varrho,\tau),\varsigma\Big)=\lim_{\alpha\to\infty}\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{1\Psi}(2^{\alpha}\xi,2^{\alpha}\varrho,2^{\alpha}\tau),2^{\alpha}\varsigma\Big)\succeq\lim_{\alpha\to\infty}\mathbf{\Delta}_{OS_1}\Big((2^{\alpha}\xi,2^{\alpha}\varrho,2^{\alpha}\tau),2^{\alpha}\varsigma\Big)=\mathbf{1},$$

and then, $\mathcal{L}_{\mathcal{H}}(\xi,\varrho,\tau)=0$. Now, using Theorem 1, we conclude that $\mathcal{H}:\mathcal{M}\to\mathcal{N}$ is a CAM. If we consider (28) and assume that $\gamma=0$ and we get limit when $\kappa\to\infty$, we reach (21). To prove the uniqueness, we repeat all the steps taken in Theorem 2, and the proof is finished. □

Theorem 4. *Considering the mapping $\Psi:\mathcal{M}\to\mathcal{N}$ and function (15), for each $\xi,\varrho,\tau\in\mathcal{M}$, if (20) is established, then, there exists a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$ such that for all $\xi\in\mathcal{M}$*

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\Big)\succeq\widetilde{\mathbf{\Delta}_{OS_1}}\Big((\xi,0,\xi),3\varsigma\Big). \tag{29}$$

Proof. In the assumption of (20), we put $\xi=\tau,\ \varrho=0$. Therefore, for all $\xi\in\mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\Big(2\Psi(\frac{3}{2}\xi)-3\Psi(\xi),\varsigma\Big)\succeq\mathbf{\Delta}_{OS_1}\Big((\xi,0,\xi),\varsigma\Big). \tag{30}$$

Then, for each $\xi\in\mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-\frac{2}{3}\Psi(\frac{3}{2}\xi),\varsigma\Big)\succeq\mathbf{\Delta}_{OS_1}\Big((\xi,0,\xi),3\varsigma\Big),$$

and therefore, for all $\xi \in \mathcal{M}$ and for each $\kappa \in \mathbb{Z}^+$ and $\gamma \in \mathbb{Z}^+$ with $\kappa > \gamma$, we have

$$\Upsilon_{\mathcal{Y}}\left(\frac{2^\gamma}{3^\gamma}\Psi(\frac{3^\gamma}{2^\gamma}\xi) - \frac{2^\kappa}{3^\kappa}\Psi(\frac{3^\kappa}{2^\kappa}\xi), \varsigma\right) \succeq \mathbf{\Delta}_{\mathcal{OS}_1}\left((\frac{3^\ell}{2^\ell}\xi, 0, \frac{3^\ell}{2^\ell}\xi), \frac{\varsigma}{\sum_{\ell=\gamma}^{\kappa-1}\frac{2^\ell}{3^{\ell+1}}}\right). \tag{31}$$

Therefore, according to (15) and (31), for all $\xi \in \mathcal{M}$, the sequence $\{\frac{2^\alpha}{3^\alpha}\Psi(\frac{3^\alpha}{2^\alpha}\xi)\}$ is a Cauchy sequence, and since $\mathcal{N}$ is complete, the convergence of this sequence is the result. Therefore, for each $\xi \in \mathcal{M}$, we define the mapping $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ as $\mathcal{H}(\xi) := \lim_{\alpha\to\infty}\frac{2^\alpha}{3^\alpha}\Psi(\frac{3^\alpha}{2^\alpha}\xi)$. To continue the proof, we repeat all the steps taken in Theorem 2, and the proof is finished. □

Theorem 5. *Considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ and function (16), for each $\xi, \varrho, \tau \in \mathcal{M}$, if (20) is established, then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ such that it applies to (29).*

Proof. Using (30) and for each $\xi \in \mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\left(\Psi(\xi) - \frac{3}{2}\Psi(\frac{2}{3}\xi), \varsigma\right) \succeq \mathbf{\Delta}_{\mathcal{OS}_1}\left((\frac{2}{3}\xi, 0, \frac{2}{3}\xi), 2\varsigma\right),$$

therefore, for all $\xi \in \mathcal{M}$ and for each $\kappa, \gamma \in \mathbb{Z}^+$ with $\kappa > \gamma$, we have

$$\Upsilon_{\mathcal{Y}}\left(\frac{3^\gamma}{2^\gamma}\Psi(\frac{2^\gamma}{3^\gamma}\xi) - \frac{3^\kappa}{2^\kappa}\Psi(\frac{2^\kappa}{3^\kappa}\xi), \varsigma\right) \succeq \mathbf{\Delta}_{\mathcal{OS}_1}\left((\frac{2^\ell}{3^\ell}\xi, 0, \frac{2^\ell}{3^\ell}\xi), \frac{\varsigma}{\sum_{\ell=\gamma+1}^{\kappa}\frac{3^{\ell-1}}{2^\ell}}\right). \tag{32}$$

Therefore, according to (16) and (32), for all $\xi \in \mathcal{M}$, the sequence $\{\frac{3^\alpha}{2^\alpha}\Psi(\frac{2^\alpha}{3^\alpha}\xi)\}$ is a Cauchy sequence, and since $\mathcal{N}$ is complete, the convergence of this sequence is the result. Therefore, for each $\xi \in \mathcal{M}$, we define the mapping $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ as $\mathcal{H}(\xi) = \lim_{\alpha\to\infty}\frac{3^\alpha}{2^\alpha}\Psi(\frac{2^\alpha}{3^\alpha}\xi)$. To continue the proof, we repeat all the steps taken in Theorem 2, and the proof is finished. □

Theorem 6. *Considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ and function (13), for each $\xi, \varrho, \tau \in \mathcal{M}$, if for all $\xi, \varrho, \tau \in \mathcal{M}$, we have*

$$\Upsilon_{\mathcal{Y}}\left(\mathcal{L}_{2\Psi}(\xi, \varrho, \tau), \varsigma\right) \succeq \mathbf{\Delta}_{\mathcal{OS}_1}\left((\xi, \varrho, \tau), \varsigma\right), \tag{33}$$

then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to Y$ such that it applies to (21).

Proof. In the (33), we put $\xi = \varrho = \tau$. Therefore, for all $\xi \in \mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\left(\Psi(2\xi) - 2\Psi(\xi), \varsigma\right) \succeq \mathbf{\Delta}_{\mathcal{OS}_1}\left((\xi, \xi, \xi), \varsigma\right). \tag{34}$$

To continue the proof, all the steps we have to go through are similar to the steps of Theorem 2. □

Theorem 7. *Considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ and function (14), for each $\xi, \varrho, \tau \in \mathcal{M}$, if (33) is established, then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ such that it applies to (21).*

Proof. Using (34) and for each $\xi \in \mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\left(\Psi(\xi) - \frac{1}{2}\Psi(2\xi), \varsigma\right) \succeq \mathbf{\Delta}_{\mathcal{OS}_1}\left((\xi, \xi, \xi), 2\varsigma\right),$$

the continuation of the proof process is similar to the proof process of Theorems 2 and 3. □

Theorem 8. *Considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ and function (14), for each $\xi, \varrho, \tau \in \mathcal{M}$, if for all $\xi, \varrho, \tau \in \mathcal{M}$, we have*

$$\Upsilon_{\mathcal{Y}}\left(\mathcal{L}_{3\Psi}(\xi, \varrho, \tau), \varsigma\right) \succeq \mathbf{\Delta}_{\mathbf{OS}_1}\left((\xi, \varrho, \tau), \varsigma\right), \tag{35}$$

then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ such that for all $\xi \in \mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\left(\Psi(\xi) - \mathcal{H}(\xi), \varsigma\right) \succeq \widetilde{\mathbf{\Delta}_{\mathbf{OS}_1}}\left((\xi, \xi, \xi), 4\varsigma\right). \tag{36}$$

Proof. In the (35), we put $\xi = \varrho = \tau$. Therefore, for all $\xi \in \mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\left(\Psi(2\xi) - 2\Psi(\xi), \varsigma\right) \succeq \mathbf{\Delta}_{\mathbf{OS}_1}\left((\xi, \xi, \xi), 2\varsigma\right). \tag{37}$$

The continuation of the proof process is similar to the proof process of Theorem 2. □

Theorem 9. *Considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ and function (14), for each $\xi, \varrho, \tau \in \mathcal{M}$, if (35) is established, then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ such that it applies to (36).*

Proof. Using (37) and for each $\xi \in \mathcal{M}$, we have

$$\Upsilon_{\mathcal{Y}}\left(\Psi(\xi) - \frac{1}{2}\Psi(2\xi), \varsigma\right) \succeq \mathbf{\Delta}_{\mathbf{OS}_1}\left((\xi, \xi, \xi), 4\varsigma\right),$$

the continuation of the proof process is similar to the proof process of Theorems 2 and 3. □

Theorem 10. *Considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ and function (15), for each $\xi, \varrho, \tau \in \mathcal{M}$, if (35) is established, then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ such that it applies to (29).*

Proof. All the steps we have to go through to prove this theorem are similar to the steps of Theorems 2 and 4. □

Theorem 11. *Considering the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ and function (16), for each $\xi, \varrho, \tau \in \mathcal{M}$, if (35) is established, then, there exists a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$ such that it applies to (29).*

Proof. All the steps we have to go through to prove this theorem are similar to the steps of Theorems 2, 4, and 5. □

Theorem 12. *Considering the bijective multiplicative mapping $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ and the function $\mathbf{\Delta}_{\mathbf{OS}_1} : \mathcal{E}_1^3 \times \mathcal{S}_2 \to \mathcal{S}_4$ which satisfies (13), if for every $\delta \in \mathbb{F}$ and for all $\xi, \varrho, \tau \in \mathcal{M}$, we have*

$$\Upsilon_{\mathcal{Y}}\left(\mathrm{U}_{\delta}\Psi(\xi, \varrho, \tau), \varsigma\right) \succeq \mathbf{\Delta}_{\mathbf{OS}_1}\left((\xi, \varrho, \tau), \varsigma\right), \tag{38}$$

and

$$\lim_{\alpha \to \infty} 2^{\alpha}\Psi(\frac{\mathrm{E}}{2^{\alpha}}) = \mathrm{E}', \tag{39}$$

then, the BMM Ψ is an isomorphism.

Proof. To start the proof, we consider hypothesis (38) with $\delta = 1$. According to Theorem 2, for every $\xi \in \mathcal{E}_1$, there exists a unique AM $\mathcal{H} : \mathcal{E}_1 \to \mathcal{E}_2$ with the mapping

$$\mathcal{H}(\xi) = \lim_{\alpha \to \infty} 2^{\alpha}\Psi(\frac{\xi}{2^{\alpha}}), \tag{40}$$

such that it holds in (21). For all $\delta \in \mathbb{F}$ and each $\xi, \varrho, \tau \in \mathcal{E}_1$, using (38) and (39), we get

$$\mathrm{Y}_{\mathcal{Y}}\left(\mathrm{U}_\delta \mathcal{H}(\xi,\xi,\xi),\varsigma\right) = \lim_{n\to\infty}\left(\mathrm{U}_\delta \Psi(\frac{\xi}{2^\alpha},\frac{\xi}{2^\alpha},\frac{\xi}{2^\alpha}),\frac{\varsigma}{2^\alpha}\right) \succeq \lim_{\alpha\to\infty} \mathbf{\Delta}_{OS_1}\left((\frac{\xi}{2^\alpha},\frac{\xi}{2^\alpha},\frac{\xi}{2^\alpha}),\frac{\varsigma}{2^\alpha}\right) = \mathbf{1}.$$

Therefore, $\mathrm{U}_\delta \mathcal{H}(\xi,\xi,\xi) = 0$ and for all $\delta \in \mathbb{F}$ and all $\xi \in \mathcal{E}_1$, $\mathcal{H}(2\delta\xi) = 2\delta\mathcal{H}(\xi)$. Considering that $\mathcal{H}$ is additive, for every $\delta \in \mathbb{F}$ and all $\xi \in \mathcal{E}_1$, we have $\mathcal{H}(\delta\xi) = \delta\mathcal{H}(\xi)$ and this means that $\mathcal{H} : \mathcal{E}_1 \to \mathcal{E}_2$ is an $\mathbb{F}$-linear mapping. Now we use the multiplicative property of Ψ. As a result, for each $\xi, \varrho \in \mathcal{E}_1$, we have

$$\mathcal{H}(\xi\varrho) = \lim_{\alpha\to\infty} 2^\alpha \Psi(\frac{\xi\varrho}{2^\alpha}) = \lim_{\alpha\to\infty} 2^\alpha \Psi(\frac{\xi}{2^\alpha})\Psi(\varrho) = \mathcal{H}(\xi)\Psi(\varrho). \tag{41}$$

In the following, according to (39), we have

$$\mathcal{H}(\mathrm{E}) = \lim_{\alpha\to\infty} 2^\alpha \Psi(\frac{\mathrm{E}}{2^\alpha}) = \mathrm{E}', \tag{42}$$

and according to (41) and (42), for every $\xi \in \mathcal{E}_1$, we get $\mathcal{H}(\xi) = \mathcal{H}(\mathrm{E}\xi) = \mathcal{H}(\mathrm{E})\Psi(\xi) = \mathrm{E}'\Psi(\xi) = \Psi(\xi)$. Therefore, the BMM Ψ is an isomorphism. □

4. C–O–M-Stability for CJAM and Isomorphisms in FBS and UFBS

In this section, in Theorems 13–22, the function Ψ from $\mathcal{M}$ to $\mathcal{N}$, and also $\xi, \varrho, \tau \in \mathcal{M}$.

Theorem 13. *If we have the following condition for every $\eta, \xi \in \mathbb{R}^+$ and $\eta > 1$*

$$\mathrm{Y}_{\mathcal{Y}}\left(\mathcal{L}_{1\Psi}(\xi,\varrho,\tau),\varsigma\right) \succeq \mathbf{\Omega}_{OS_1}\left(\xi(||\xi||^\eta + ||\varrho||^\eta + ||\tau||^\eta),\varsigma\right),$$

then, we can say that there is a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$, such that for every $\xi \in \mathcal{M}$

$$\mathrm{Y}_{\mathcal{Y}}\left(\Psi(\xi) - \mathcal{H}(\xi),\varsigma\right) \succeq \mathbf{\Omega}_{OS_1}\left(\frac{3\varphi}{2^\eta - 2}||\xi||^\eta,\varsigma\right).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\left((\xi,\varrho,\tau),\varsigma\right)$ as $\mathbf{\Delta}_{OS_1}\left((\xi,\varrho,\tau),\varsigma\right) = \mathbf{\Omega}_{OS_1}\left(\varphi(||\xi||^\eta + ||\varrho||^\eta + ||\tau||^\eta),\varsigma\right)$ and put it in Theorem 2. □

Theorem 14. *If we have the following condition for every $\eta_1, \eta_2, \eta_3, \varphi \in \mathbb{R}^+$ and $\eta_1 + \eta_2 + \eta_3 > 1$*

$$\mathrm{Y}_{\mathcal{Y}}\left(\mathcal{L}_{1\Psi}(\xi,\varrho,\tau),\varsigma\right) \succeq \mathbf{\Omega}_{OS_1}\left(\varphi \cdot ||\xi||^{\eta_1} \cdot ||\varrho||^{\eta_2} \cdot ||\tau||^{\eta_3},\varsigma\right),$$

then, we can say that there is a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$, such that for every $\xi \in \mathcal{M}$

$$\mathrm{Y}_{\mathcal{Y}}\left(\Psi(\xi) - \mathcal{H}(\xi),\varsigma\right) \succeq \mathbf{\Omega}_{OS_1}\left(\frac{\varphi}{2^{\eta_1+\eta_2+\eta_3} - 2}||\xi||^{\eta_1+\eta_2+\eta_3},\varsigma\right).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\left((\xi,\varrho,\tau),\varsigma\right)$ as $\mathbf{\Delta}_{OS_1}\left((\xi,\varrho,\tau),\varsigma\right) = \mathbf{\Omega}_{OS_1}\left(\varphi \cdot ||\xi||^{\eta_1} \cdot ||\varrho||^{\eta_2} \cdot ||\tau||^{\eta_3},\varsigma\right)$ and put it in Theorem 2. □

Theorem 15. *If we have the following condition for every $\eta, \varphi \in \mathbb{R}^+$ and $\eta < 1$*

$$\Upsilon_{\mathcal{Y}}\left(\mathcal{L}_{1\Psi}(\xi,\varrho,\tau),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\right),$$

then, we can say that there is a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$*, such that for every* $\xi\in\mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\left(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\frac{3\varphi}{2-2^{\eta}}||\xi||^{\eta},\varsigma\right).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{\boldsymbol{OS}_1}\left((\xi,\varrho,\tau),\varsigma\right)$ as $\mathbf{\Delta}_{\boldsymbol{OS}_1}\left((\xi,\varrho,\tau),\varsigma\right)=\mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\right)$ and put it in Theorem 3. □

Theorem 16. *If we have the following condition for every* $\eta_1,\eta_2,\eta_3,\varphi\in\mathbb{R}^+$ *and* $\eta_1+\eta_2+\eta_3<1$

$$\Upsilon_{\mathcal{Y}}\left(\mathcal{L}_{1\Psi}(\xi,\varrho,\tau),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\varphi\cdot||\xi||^{\eta_1}\cdot||\varrho||^{\eta_2}\cdot||\tau||^{\eta_3},\varsigma\right),$$

then, we can say that there is a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$*, such that for every* $\xi\in\mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\left(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\frac{\varphi}{2-2^{\eta_1+\eta_2+\eta_3}}||\xi||^{\eta_1+\eta_2+\eta_3},\varsigma\right).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{\boldsymbol{OS}_1}\left((\xi,\varrho,\tau),\varsigma\right)$ as $\mathbf{\Delta}_{\boldsymbol{OS}_1}\left((\xi,\varrho,\tau),\varsigma\right)=\mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\varphi\cdot||\xi||^{\eta_1}\cdot||\varrho||^{\eta_2}\cdot||\tau||^{\eta_3},\varsigma\right)$ and put it in Theorem 3. □

Theorem 17. *If we have the following condition for every* $\eta,\varphi\in\mathbb{R}^+$ *and* $\eta<1$

$$\Upsilon_{\mathcal{Y}}\left(\mathcal{L}_{1\Psi}(\xi,\varrho,\tau),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\right),$$

then, we can say that there is a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$*, such that for every* $\xi\in\mathcal{M}$

$$\left(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\frac{2^{\eta+1}\varphi}{3\cdot 2^{\eta}-2\cdot 3^{\eta}}||\xi||^{\eta},\varsigma\right).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{\boldsymbol{OS}_1}\left((\xi,\varrho,\tau),\varsigma\right)$ as $\mathbf{\Delta}_{\boldsymbol{OS}_1}\left((\xi,\varrho,\tau),\varsigma\right)=\mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\right)$ and put it in Theorem 4. □

Theorem 18. *If we have the following condition for every* $\eta,\varphi\in\mathbb{R}^+$ *and* $\eta>1$

$$\Upsilon_{\mathcal{Y}}\left(\mathcal{H}_{1\Psi}(\xi,\varrho,\tau),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\right),$$

then, we can say that there is a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$*, such that for every* $\xi\in\mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\left(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\right) \succeq \mathbf{\Omega}_{\boldsymbol{OS}_1}\left(\frac{2^{\eta+1}\varphi}{2\cdot 3^{\eta}-3\cdot 2^{\eta}}||\xi||^{\eta},\varsigma\right).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)$ as $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)=\mathbf{\Omega}_{OS_1}\Big(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\Big)$ and put it in Theorem 5. □

Theorem 19. *If we have the following condition for every $\eta,\varphi\in\mathbb{R}^{+}$ and $\eta>1$*

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{3\Psi}(\xi,\varrho,\tau),\varsigma\Big)\succeq\mathbf{\Omega}_{OS_1}\Big(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\Big),$$

then, we can say that there is a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$, such that for every $\xi\in\mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\Big)\succeq\mathbf{\Omega}_{OS_1}\Big(\frac{3\varphi}{2^{\eta+1}-4}||\xi||^{\eta},\varsigma\Big).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)$ as $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)=\mathbf{\Omega}_{OS_1}\Big(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\Big)$ and put it in Theorem 8. □

Theorem 20. *If we have the following condition for every $\eta_1,\eta_2,\eta_3,\varphi\in\mathbb{R}^{+}$ and $\eta_1+\eta_2+\eta_3>1$*

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{3\Psi}(\xi,\varrho,\tau),\varsigma\Big)\succeq\mathbf{\Omega}_{OS_1}\Big(\varphi\cdot||\xi||^{\eta_1}\cdot||\varrho||^{\eta_2}\cdot||\tau||^{\eta_3},\varsigma\Big),$$

then, we can say that there is a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$, such that for every $\xi\in\mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\Big)\succeq\mathbf{\Omega}_{OS_1}\Big(\frac{\varphi}{2^{\eta_1+\eta_2+\eta_3+1}-4}||\xi||^{\eta_1+\eta_2+\eta_3},\varsigma\Big).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)$ as $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)=\mathbf{\Omega}_{OS_1}\Big(\varphi\cdot||\xi||^{\eta_1}\cdot||\varrho||^{\eta_2}\cdot||\tau||^{\eta_3},\varsigma\Big)$ and put it in Theorem 8. □

Theorem 21. *If we have the following condition for every $\eta,\varphi\in\mathbb{R}^{+}$ and $\eta<1$*

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{3\Psi}(\xi,\varrho,\tau),\varsigma\Big)\succeq\mathbf{\Omega}_{OS_1}\Big(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\Big),$$

then, we can say that there is a unique AM $\mathcal{H}:\mathcal{M}\to\mathcal{N}$, such that for every $\xi\in\mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi)-\mathcal{H}(\xi),\varsigma\Big)\succeq\mathbf{\Omega}_{OS_1}\Big(\frac{3\varphi}{4-2^{\eta+1}}||\xi||^{\eta},\varsigma\Big).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)$ as $\mathbf{\Delta}_{OS_1}\Big((\xi,\varrho,\tau),\varsigma\Big)=\mathbf{\Omega}_{OS_1}\Big(\varphi(||\xi||^{\eta}+||\varrho||^{\eta}+||\tau||^{\eta}),\varsigma\Big)$ and put it in Theorem 9. □

Theorem 22. *If we have the following condition for every $\eta_1,\eta_2,\eta_3,\varphi\in\mathbb{R}^{+}$ and $\eta_1+\eta_2+\eta_3<1$*

$$\Upsilon_{\mathcal{Y}}\Big(\mathcal{L}_{3\Psi}(\xi,\varrho,\tau),\Big)\succeq\mathbf{\Omega}_{OS_1}\Big(\varphi\cdot||\xi||^{\eta_1}\cdot||\varrho||^{\eta_2}\cdot||\tau||^{\eta_3},\varsigma\Big),$$

then, we can say that there is a unique AM $\mathcal{H} : \mathcal{M} \to \mathcal{N}$, such that for every $\xi \in \mathcal{M}$

$$\Upsilon_{\mathcal{Y}}\Big(\Psi(\xi) - \mathcal{H}(\xi), \varsigma\Big) \succeq \mathbf{\Omega}_{OS_1}\Big(\frac{\varphi}{4 - 2^{\eta_1+\eta_2+\eta_3+1}}||\xi||^{\eta_1+\eta_2+\eta_3}, \varsigma\Big).$$

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\Big((\xi, \varrho, \tau), \varsigma\Big)$ as $\mathbf{\Delta}_{OS_1}\Big((\xi, \varrho, \tau), \varsigma\Big) = \mathbf{\Omega}_{OS_1}\Big(\varphi \cdot ||\xi||^{\eta_1} \cdot ||\varrho||^{\eta_2} \cdot ||\tau||^{\eta_3}, \varsigma\Big)$ and put it in Theorem 9. □

Theorem 23. *We consider the BMM function $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ such that for every $\delta \in \mathbb{F}$ and all $\xi, \varrho, \tau \in \mathcal{E}_1$, it applies to the following inequalities*

$$\Upsilon_{\mathcal{Y}}\Big(\mathrm{U}_\delta\Psi(\xi, \varrho, \tau), \varsigma\Big) \succeq \mathbf{\Omega}_{OS_1}\Big(\varphi(||\xi||^{\eta} + ||\varrho||^{\eta} + ||\tau||^{\eta}), \varsigma\Big), \tag{43}$$

$$\lim_{\alpha\to\infty} 2^{\alpha}\Psi(\frac{\mathrm{E}}{2^{\alpha}}) = \mathrm{E}', \tag{44}$$

where $\eta, \varphi \in \mathbb{R}^+$ and $\eta > 1$. Then the BMM Ψ is an isomorphism.

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\Big((\xi, \varrho, \tau), \varsigma\Big)$ as $\mathbf{\Delta}_{OS_1}\Big((\xi, \varrho, \tau), \varsigma\Big) = \mathbf{\Omega}_{OS_1}\Big(\varphi(||\xi||^{\eta} + ||\varrho||^{\eta} + ||\tau||^{\eta}), \varsigma\Big)$ and put it in Theorem 12. □

Theorem 24. *We consider the BMM function $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ such that for every $\delta \in \mathbb{F}$ and all $\xi, \varrho, \tau \in \mathcal{E}_1$, it applies to the following inequalities*

$$\Upsilon_{\mathcal{Y}}\Big(\mathrm{U}_\delta\Psi(\xi, \varrho, \tau), \varsigma\Big) \succeq \mathbf{\Omega}_{OS_1}\Big(\varphi \cdot ||\xi||^{\eta_1} \cdot ||\varrho||^{\eta_2} \cdot ||\tau||^{\eta_3}, \varsigma\Big), \tag{45}$$

$$\lim_{\alpha\to\infty} 2^{\alpha}\Psi(\frac{\mathrm{E}}{2^{\alpha}}) = \mathrm{E}', \tag{46}$$

where $\eta_1, \eta_2, \eta_3, \varphi \in \mathbb{R}^+$ and $\eta_1 + \eta_2 + \eta_3 > 1$. Then the BMM Ψ is an isomorphism.

Proof. To prove it, it is enough to define the function $\mathbf{\Delta}_{OS_1}\Big((\xi, \varrho, \tau), \varsigma\Big)$ as $\mathbf{\Delta}_{OS_1}\Big((\xi, \varrho, \tau), \varsigma\Big) = \mathbf{\Omega}_{OS_1}\Big(\varphi \cdot ||\xi||^{\eta_1} \cdot ||\varrho||^{\eta_2} \cdot ||\tau||^{\eta_3}, \varsigma\Big)$ and put it in Theorem 12. □

5. Conclusions

The issue of the stability of equations has attracted the attention of many authors in the last few decades. In this article, we have tried to present a generalization of previous works in this field. For this purpose, we have first defined a new space called the matrix value fuzzy Banach spaces. In the stability of equations, the goal is to obtain the best approximation. Since the control function plays an important role in this field, we have selected the best and most optimal controller from among the specific functions. Choosing the aggregation control function brings us closer to the appropriate approximation. We have done all these proofs for functional Equations (1)–(3) and isomorphisms in FBS and UFBS.

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