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New Criteria for Convex-Exponent Product of Log-Harmonic Functions

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Abstract: In this study, we consider different types of convex-exponent products of elements of a certain class of log-harmonic mapping and then find sufficient conditions for them to be starlike log-harmonic functions. For instance, we show that, if f is a spirallike function, then choosing a suitable value of γ, the log-harmonic mapping $F(z) = f(z)|f(z)|^{2\gamma}$ is *α-spiralike* of order ρ. Our results generalize earlier work in the literature.

Keywords: product; log-harmonic function; convex-exponent combination; starlike and spirallike functions

MSC: 30C45; 30C80

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1. Introduction

Let E be the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(E)$ denote the linear space of all analytic functions defined on E. Additionally, let $\mathcal{A}$ be a subclass consisting of $f \in \mathcal{H}(E)$ such that $f(0) = f'(0) - 1 = 0$.

A C^2-function defined in E is said to be harmonic if $\Delta f = 0$, and a log-harmonic function f is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f}_{\overline{z}}}{\overline{f}} = a\frac{f_z}{f}, \tag{1}$$

where the second dilation function $a \in \mathcal{H}(E)$ is such that $|a(z)| < 1$ for all $z \in E$. In the above formula, $\overline{f}_{\overline{z}}$ means $\overline{(f_{\overline{z}})}$. Observe that f is log-harmonic if $\log f$ is harmonic. The authors in [1] have proven that, if f is a non-constant log-harmonic mapping that vanishes only at $z = 0$, then f should be in the form

$$f(z) = z^m|z|^{2m\beta}h(z)\overline{g}(z), \tag{2}$$

where m is a nonnegative integer, $\mathrm{Re}\beta > -\frac{1}{2}$, while h and g are analytic functions in $\mathcal{H}(E)$ satisfying $g(0) = 1$ and $h(0) \neq 0$. The exponent β in (2) depends only on $a(0)$ and is given by

$$\beta = \overline{a}(0)\frac{1 + a(0)}{1 - |a(0)|^2}. \tag{3}$$

We remark that $f(0) \neq 0$ if and only if $m = 0$ and that a univalent log-harmonic mapping in E vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta}h(z)\overline{g}(z),$$

where $\text{Re}\beta > -\frac{1}{2}$ and $0 \notin hg(E)$.

Recently, the class of log-harmonic functions has been extensively studied by many authors; for instance, see [1–10].

The Jacobian of log-harmonic function f is given by

$$J_f(z) = |f_z|^2(1 - |a(z)|^2) \tag{4}$$

and is positive. Therefore, all non-constant log-harmonic mappings are sense-preserving in the unit disk E. Let B denote the class of functions $a \in \mathcal{H}(E)$ with $|a(z)| < 1$ and B_0 denote $a \in B$ such that $a(0) = 0$.

It is easy to see that, if $f(z) = zh(z)\overline{g(z)}$, then the functions h and g, and the dilation a satisfy

$$\frac{zg'(z)}{g(z)} = a(z)\left(1 + \frac{zh'(z)}{h(z)}\right). \tag{5}$$

Definition 1. *(See [2].) Let $f = z|z|^{2\beta}h(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping of order α if*

$$\frac{\partial \arg f(re^{i\theta})}{\partial \theta} = \text{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha, \qquad 0 \leq \alpha < 1$$

for all $z \in E$. Denote by $ST_{LH}(\alpha)$ the class of all starlike log-harmonic mappings.

By taking $\beta = 0$ and $g(z) = 1$ in Definition 1, we obtain the class of starlike analytic functions in $\mathcal{A}$, which we denote by $S^*(\alpha)$.

The following lemma shows the relationship of the classes $ST_{LH}(\alpha)$ and $S^*(\alpha)$.

Lemma 1. *(See [2].) Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be a log-harmonic mapping on E, $0 \notin hg(E)$. Then, $f \in ST_{LH}(\alpha)$ if and only if $\varphi(z) = \frac{zh(z)}{g(z)} \in S^*(\alpha)$.*

In [2], the authors studied the class of $\alpha - spirallike$ functions and proved that, if $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ is a log-harmonic mapping on E, $0 \notin hg(E)$, then f is $\alpha - spirallike$ if

$$\text{Re}\left(e^{-i\alpha}\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > 0, \qquad 0 \leq \alpha < 1$$

for all $z \in E$. We remark that a simply connected domain Ω in $\mathbb{C}$ containing the origin is said to be $\alpha - spirallike$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ if $w\exp(-te^{i\alpha}) \in \Omega$ for all $t \geq 0$ whenever $w \in \Omega$ and that f is an $\alpha - spirallike$ function, if $f(E)$ is an α-*spiralike* domain. Motivated by this, we define the class of $\alpha - spirallike$ log-harmonic mappings of order ρ as follows:

Definition 2. *Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be a univalent log-harmonic mapping on E, with $0 \notin hg(E)$. Then, we say that f is an $\alpha - spirallike$ log-harmonic mapping of order ρ $(0 \leq \rho < 1)$ if*

$$\text{Re}\left(e^{-i\alpha}\frac{zf_z - \overline{z}f_{\overline{z}}}{f(z)}\right) > \rho\cos\alpha \qquad (z \in E)$$

for some real $\alpha(|\alpha| < \frac{\pi}{2})$. The class of these functions is denoted by $S^{\alpha}_{LH}(\rho)$. Furthermore, we define $S^{\alpha}_{LH}(1) = \bigcap_{0 \leq \rho < 1} S^{\alpha}_{LH}(\rho)$.

Additionally, we denote by $S^{\alpha}(\rho)$ the subclass of all $f \in \mathcal{A}$ such that f is α-*spiralike* of order ρ and $S^{\alpha}(1) = \bigcap_{0 \leq \rho < 1} S^{\alpha}(\rho)$.

Lemma 2. *([2]) If $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ is log-harmonic on E and $0 \notin hg(E)$, with $\mathrm{Re}\beta > -\frac{1}{2}$, then $f \in S^{\alpha}_{LH}(\rho)$ if and only if $\psi(z) = \frac{zh(z)}{g(z)^{e^{2i\alpha}}} \in S^{\alpha}(\rho)$.*

In the celebrated paper [11], the authors introduce a new way of studying harmonic functions in Geometric Function Theory. Additionally, many authors investigated the linear combinations of harmonic functions in a plane; see, for example, [12–14]. In Section 2 of this paper, taking the convex-exponent product combination of two elements, a specified class of new log-harmonic functions is constructed. Indeed, we show that, if $f(z) = zh(z)\bar{g}(z)$ is spirallike log-harmonic of order ρ, then by choosing suitable parameters of α and γ, the function $F(z) = f(z)|f(z|^{2\gamma}$ is log-harmonic spirallike of order α. Additionally, in Section 3, we provide some examples that are constructed from Section 2.

2. Main Results

Theorem 1. *Let $f(z) = zh(z)\overline{g(z)} \in ST_{LH}(\rho), (0 \leq \rho < 1)$ with respect to $a \in B_0$, $\phi \in S^*(\gamma), (0 \leq \gamma < 1)$ and α, β be real numbers with $\alpha + \beta = 1$. Then, $F(z) = f(z)^{\alpha}K(z)^{\beta}$ is starlike log-harmonic mapping of order $\alpha\rho + \beta\gamma$ with respect to a, where*

$$K(z) = \phi(z)\exp\left\{2\mathrm{Re}\int_0^z \frac{a(s)}{1-a(s)}\frac{\phi'(s)}{\phi(s)}ds\right\}.$$

Proof. By definition of F, we have

$$\frac{F_z}{F} = \alpha\frac{f_z}{f} + \beta\frac{K_z}{K} \quad and \quad \frac{F_{\bar{z}}}{F} = \alpha\frac{f_{\bar{z}}}{f} + \beta\frac{K_{\bar{z}}}{K}. \tag{6}$$

Additionally direct computations show that

$$\frac{K_z}{K} = \frac{1}{1-a(z)}\frac{\phi'(z)}{\phi(z)}, \quad and \quad \frac{\overline{K_{\bar{z}}}}{\overline{K}} = \frac{a(z)}{1-a(z)}\frac{\phi'(z)}{\phi(z)}. \tag{7}$$

Now, in view of Equations (6) and (7),

$$\hat{a}(z) = \frac{\overline{\frac{F_{\bar{z}}}{F}}}{\frac{F_z}{F}} = \frac{\alpha\overline{\frac{f_{\bar{z}}}{f}} + \beta\overline{\frac{K_{\bar{z}}}{K}}}{\alpha\frac{f_z}{f} + \beta\frac{K_z}{K}} = a(z)\frac{\alpha\frac{f_z}{f} + \beta\frac{K_z}{K}}{\alpha\frac{f_z}{f} + \beta\frac{K_z}{K}} = a(z).$$

On the other hand,

$$\begin{aligned}
\mathrm{Re}\frac{zF_z - \bar{z}F_{\bar{z}}}{F} &= \mathrm{Re}\left(\alpha\frac{zf_z}{f} + \beta\frac{zK_z}{K}\right) - \mathrm{Re}\left(\alpha\frac{z\overline{f_{\bar{z}}}}{\overline{f}} + \beta\frac{z\overline{K_{\bar{z}}}}{\overline{K}}\right) \\
&= \alpha\mathrm{Re}\left(\frac{zf_z}{f} - \frac{z\overline{f_{\bar{z}}}}{\overline{f}}\right) + \beta\mathrm{Re}\left(\frac{zK_z}{K} - \frac{z\overline{K_{\bar{z}}}}{\overline{K}}\right) \\
&> \alpha\rho + \beta\gamma.
\end{aligned}$$

The above relation shows that F is a log-harmonic starlike function of order $\alpha\rho + \beta\gamma$, and the proof is complete. □

Theorem 2. *Let $f(z) = zh(z)\overline{g(z)} \in S^{\beta}_{LH}(\rho)$ with respect to $a \in B_0$ and γ be a constant with $\mathrm{Re}\gamma > -\frac{1}{2}$. Then, $F(z) = f(z)|f(z)|^{2\gamma}$ is an $\alpha -$ spirallike log-harmonic mapping of order ρ with respect to*

$$\hat{a}(z) = \frac{(1+\bar{\gamma})a(z) + \bar{\gamma}}{1+\gamma+\gamma a(z)},$$

where $|\beta| < \frac{\pi}{2}$ and $\alpha = \tan^{-1}\left(\frac{\tan\beta + 2\mathrm{Im}\gamma}{1+2\mathrm{Re}\gamma}\right)$.

Proof. By definition of F, we have

$$F(z) = f(z)|f(z)|^{2\gamma} = z^{1+\gamma}\overline{z}^{\gamma}H(z)\overline{G(z)},$$

where

$$H(z) = h^{1+\gamma}(z)g^{\gamma}(z) \qquad \text{and} \qquad G(z) = h^{\bar{\gamma}}(z)g^{1+\bar{\gamma}}(z).$$

With a straightforward calculation and using Equation (5),

$$\frac{zF_z}{F} = (1+\gamma)\left(1+\frac{zh'(z)}{h(z)}\right) + \gamma\frac{zg'(z)}{g(z)} = \left(1+\frac{zh'(z)}{h(z)}\right)((1+\gamma)+\gamma a(z)),$$

and

$$\frac{\bar{z}F_{\bar{z}}}{F} = \gamma\left(1+\overline{\frac{zh'(z)}{h(z)}}\right) + (1+\gamma)\overline{\frac{zg'(z)}{g(z)}} = \left(1+\overline{\frac{zh'(z)}{h(z)}}\right)(\gamma+(1+\gamma)\overline{a(z)}).$$

If we consider

$$\hat{a}(z) = \frac{\overline{\left(\frac{\bar{z}F_{\bar{z}}(z)}{F(z)}\right)}}{\frac{zF_z(z)}{F(z)}},$$

then

$$\hat{a}(z) = \frac{\bar{\gamma}+(1+\bar{\gamma})a(z)}{(1+\gamma)+\gamma a(z)}.$$

Now, in view of $|a(z)| < 1$, it easy to see that $|\hat{a}(z)| < 1$ provided that $\left|\frac{\overline{\gamma}}{1+\overline{\gamma}}\right| < 1$, which evidently holds $|\gamma|^2 < |1+\overline{\gamma}|^2$ since $\mathrm{Re}\gamma > -\frac{1}{2}$, and this means that F is a log-harmonic function.

Additionally, by putting

$$\psi(z) = \frac{zH(z)}{G(z)^{e^{2i\alpha}}},$$

we have

$$\psi(z) = \frac{zH(z)}{G(z)^{e^{2i\alpha}}} = \frac{zh(z)^{1+\gamma}g(z)^{\gamma}}{(h^{\bar{\gamma}}(z)g^{1+\bar{\gamma}}(z))^{e^{2i\alpha}}}.$$

Then, we obtain

$$\begin{aligned}
e^{-i\alpha}\frac{z\psi'(z)}{\psi(z)} &= e^{-i\alpha} + [(1+\gamma)e^{-i\alpha} - \overline{\gamma}e^{i\alpha}]\frac{zh'(z)}{h(z)} - [(1+\overline{\gamma})e^{i\alpha} - \gamma e^{-i\alpha}]\frac{zg'(z)}{g(z)} \\
&= (-\gamma e^{-i\alpha} + \bar{\gamma}e^{i\alpha}) + [(1+\gamma)e^{-i\alpha} - \overline{\gamma}e^{i\alpha}]\left(1+\frac{zh'(z)}{h(z)}\right) \\
&- [(1+\overline{\gamma})e^{i\alpha} - \gamma e^{-i\alpha}]\frac{zg'(z)}{g(z)}.
\end{aligned}$$

The condition on α ensures that

$$(1+\gamma)e^{-i\alpha} - \overline{\gamma}e^{i\alpha} = \frac{\cos\alpha}{\cos\beta}e^{-i\beta} \quad \text{and} \quad (1+\overline{\gamma})e^{i\alpha} - \gamma e^{-i\alpha} = \frac{\cos\alpha}{\cos\beta}e^{i\beta},$$

because by letting $\gamma = \gamma_1 + i\gamma_2$, the first equality holds true if and only if

$$\cos\beta\cos\alpha - i(1+2\gamma_1)\sin\alpha\cos\beta + i2\gamma_2\cos\beta\cos\alpha = \cos\alpha\cos\beta - i\cos\alpha\sin\beta$$

or, equivalently, after simplification

$$2\gamma_2\cot\beta - (1+2\gamma_1)\tan\alpha\cot\beta = -1$$

or

$$\alpha = \tan^{-1}\left(\frac{\tan\beta + 2\mathrm{Im}\gamma}{1 + 2\mathrm{Re}\gamma}\right).$$

Thus, by hypothesis,

$$\mathrm{Re}\{e^{-i\alpha}\frac{z\psi'(z)}{\psi(z)}\} = \frac{\cos\alpha}{\cos\beta}\mathrm{Re}\left(e^{-i\beta}(1 + \frac{zh'(z)}{h(z)}) - e^{i\beta}\frac{zg'(z)}{g(z)}\right) > \rho\cos\alpha$$

and it follows that F is an α-spirallike log-harmonic mapping of order ρ in which the dilation is $\hat{a}(z)$. □

Theorem 3. *Let $f_k(z) = zh_k(z)\overline{g_k}(z) \in S^{\beta}_{LH}(\rho)$ with $k = 1,2$ and with respect to the same $a \in B_0$ and γ be a constant with $\mathrm{Re}\gamma > -\frac{1}{2}$. Moreover, let*

$$F_1(z) = f_1(z)|f_1(z)|^{2\gamma} \qquad \text{and} \qquad F_2(z) = f_2(z)|f_2(z)|^{2\gamma}.$$

Then, $F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z)$ is an α-spirallike log-harmonic mapping of order ρ with respect to

$$\hat{a}(z) = \frac{(1+\bar{\gamma})a(z) + \bar{\gamma}}{1 + \gamma + \gamma a(z)},$$

where $|\beta| < \frac{\pi}{2}$ and $\alpha = \tan^{-1}\left(\frac{\tan\beta + 2\mathrm{Im}\gamma}{1+2\mathrm{Re}\gamma}\right)$.

Proof. According to the definitions of F_1 and F_2, we have

$$\begin{aligned} F_1^{\lambda}(z) &= (f_1(z)|f_1(z)|^{2\gamma})^{\lambda} \\ &= (z|z|^{2\gamma}h_1^{1+\gamma}(z)g_1^{\gamma}(z)\overline{h_1^{\overline{\gamma}}(z)g_1^{1+\overline{\gamma}}(z))}^{\lambda} \end{aligned}$$

and

$$\begin{aligned} F_2^{1-\lambda}(z) &= (f_2(z)|f_2(z)|^{2\gamma})^{1-\lambda} \\ &= (z|z|^{2\gamma}h_2^{1+\gamma}(z)g_2^{\gamma}(z)\overline{h_2^{\overline{\gamma}}(z)g_2^{1+\overline{\gamma}}(z))}^{1-\lambda}. \end{aligned}$$

Putting the values of F_1^{λ} and $F_2^{1-\lambda}$ on F, we obtain

$$\begin{aligned} F(z) &= (z|z|^{2\gamma}h_1^{1+\gamma}(z)g_1^{\gamma}(z)\overline{h_1^{\overline{\gamma}}(z)g_1^{1+\overline{\gamma}}(z))}^{\lambda}(z|z|^{2\gamma}h_2^{1+\gamma}(z)g_2^{\gamma}(z)\overline{h_2^{\overline{\gamma}}(z)g_2^{1+\overline{\gamma}}(z))}^{1-\lambda} \\ &= z|z|^{2\gamma}H(z)\overline{G(z)}, \end{aligned}$$

where

$$H(z) = h_1(z)^{\lambda(1+\gamma)}g_1(z)^{\lambda\gamma}h_2(z)^{(1-\lambda)(1+\gamma)}g_2(z)^{(1-\lambda)\gamma} \tag{8}$$

and

$$G(z) = h_1(z)^{\lambda\overline{\gamma}}g_1(z)^{\lambda(1+\overline{\gamma})}h_2(z)^{(1-\lambda)\overline{\gamma}}g_2(z)^{(1-\lambda)(1+\overline{\gamma})}. \tag{9}$$

Now, we show that the second dilation of F i.e., $\mu(z)$ satisfies the condition $|\mu(z)| < 1$. For this, since

$$\mu(z) = \frac{\frac{\overline{F}_{\bar{z}}(z)}{\overline{F}(z)}}{\frac{F_z(z)}{F(z)}},$$

we have

$$
\begin{aligned}
\mu(z) &= \frac{\lambda \frac{\overline{F_{1\bar{z}}(z)}}{\overline{F_1}(z)} + (1-\lambda)\frac{\overline{F_{2\bar{z}}(z)}}{\overline{F_2}(z)}}{\lambda \frac{F_{1z}(z)}{F_1(z)} + (1-\lambda)\frac{F_{2z}(z)}{F_2(z)}} \\
&= \frac{\lambda[\overline{\gamma}(1+\frac{zh_1'}{h_1}) + (1+\overline{\gamma})\frac{zg_1'}{g_1}] + (1-\lambda)[\overline{\gamma}(1+\frac{zh_2'}{h_2}) + (1+\overline{\gamma})\frac{zg_2'}{g_2}]}{\lambda[(1+\gamma)(1+\frac{zh_1'}{h_1}) + \gamma\frac{zg_1'}{g_1}] + (1-\lambda)[(1+\gamma)(1+\frac{zh_2'}{h_2}) + \gamma\frac{zg_2'}{g_2}]} \\
&= \frac{\lambda(1+\frac{zh_1'}{h_1})[\overline{\gamma} + (1+\overline{\gamma})a(z)] + (1-\lambda)(1+\frac{zh_2'}{h_2})[\overline{\gamma} + (1+\overline{\gamma})a(z)]}{\lambda(1+\frac{zh_1'}{h_1})[(1+\gamma) + \gamma a(z)] + (1-\lambda)(1+\frac{zh_2'}{h_2})[(1+\gamma) + \gamma a(z)]} \\
&= \frac{[\lambda(1+\frac{zh_1'}{h_1}) + (1-\lambda)(1+\frac{zh_2'}{h_2})][\overline{\gamma} + (1+\overline{\gamma})a(z)]}{[\lambda(1+\frac{zh_1'}{h_1}) + (1-\lambda)(1+\frac{zh_2'}{h_2})][(1+\gamma) + \gamma a(z)]} \\
&= \frac{[\overline{\gamma} + (1+\overline{\gamma})a(z)]}{[(1+\gamma) + \gamma a(z)]} \\
&= \frac{(1+\overline{\gamma})}{(1+\gamma)} \frac{a(z) + \frac{\overline{\gamma}}{1+\overline{\gamma}}}{1 + \frac{a(z)\gamma}{1+\gamma}},
\end{aligned}
\tag{10}
$$

and the condition $\mathrm{Re}\gamma > -\frac{1}{2}$ ensures that $|\mu(z)| < 1$ in E, which implies that F is a locally univalent log-harmonic mapping. Now, to prove

$$F(z) = F_1^{\lambda}(z) F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(\rho),$$

we have to show that $\psi(z) = \frac{zH(z)}{G(z)^{e^{2i\alpha}}} \in S^{\alpha}(\rho)$. However, a direct calculation shows that

$$\psi(z) = \frac{zH(z)}{G(z)^{e^{2i\alpha}}} = \frac{[zh_1^{\lambda(1+\gamma)}(z)g_1^{\lambda\gamma}(z)h_2^{(1-\lambda)(1+\gamma)}(z)g_2^{(1-\lambda)\gamma}(z)]}{[h_1^{\lambda\overline{\gamma}}(z)g_1^{\lambda(1+\overline{\gamma})}(z)h_2^{(1-\lambda)\overline{\gamma}}(z)g_2^{(1-\lambda)(1+\overline{\gamma})}(z)]^{e^{2i\alpha}}}.$$

Now,

$$
\begin{aligned}
& e^{-i\alpha}\frac{z\psi'(z)}{\psi(z)} \\
&= e^{-i\alpha}\left[1 + \lambda(((1+\gamma) - e^{2i\alpha}\overline{\gamma})\frac{zh_1'(z)}{h_1(z)} - ((1+\overline{\gamma})e^{2i\alpha} - \gamma)\frac{zg_1'(z)}{g_1(z)})\right] \\
&+ e^{-i\alpha}\left[(1-\lambda)(((1+\gamma) - e^{2i\alpha}\overline{\gamma})\frac{zh_2'(z)}{h_2(z)} - ((1+\overline{\gamma})e^{2i\alpha} - \gamma)\frac{zg_2'(z)}{g_2(z)})\right] \\
&= -\gamma e^{-i\alpha} + e^{i\alpha}\bar{\gamma} \\
&+ \lambda\left[((1+\gamma)e^{-i\alpha} - e^{i\alpha}\overline{\gamma})(1 + \frac{zh_1'(z)}{h_1(z)}) - ((1+\overline{\gamma})e^{i\alpha} - \gamma e^{-i\alpha})\frac{zg_1'(z)}{g_1(z)}\right] \\
&+ (1-\lambda)\left[((1+\gamma)e^{-i\alpha} - e^{i\alpha}\overline{\gamma})(1 + \frac{zh_2'(z)}{h_2(z)}) - ((1+\overline{\gamma})e^{i\alpha} - \gamma e^{-i\alpha})\frac{zg_2'(z)}{g_2(z)}\right].
\end{aligned}
$$

By hypothesis, we know that

$$(1+\gamma)e^{-i\alpha} - \overline{\gamma}e^{i\alpha} = \frac{\cos\alpha}{\cos\beta}e^{-i\beta} \quad \text{and} \quad (1+\overline{\gamma})e^{i\alpha} - \gamma e^{-i\alpha} = \frac{\cos\alpha}{\cos\beta}e^{i\beta},$$

so

$$\begin{aligned}
&\mathrm{Re}\{e^{-i\alpha}\frac{z\psi'(z)}{\psi(z)}\} \\
&\qquad = \lambda\frac{\cos\alpha}{\cos\beta}\mathrm{Re}\left(e^{-i\beta}(1+\frac{zh_1'(z)}{h_1(z)}) - e^{i\beta}\frac{zg_1'(z)}{g_1(z)}\right) \\
&\qquad + (1-\lambda)\frac{\cos\alpha}{\cos\beta}\mathrm{Re}\left(e^{-i\beta}(1+\frac{zh_2'(z)}{h_2(z)}) - e^{i\beta}\frac{zg_1'(z)}{g_1(z)}\right) \\
&\qquad > \rho\cos\alpha
\end{aligned}$$

and the proof is completed. □

Theorem 4. *Let* $f_k(z) = zh_k(z)\overline{g}_k(z) \in S^{\beta}_{LH}(\rho)$ *with respect to* $a_k \in B_0 (k = 1, 2)$*. Moreover, suppose that* $\mathrm{Re}\gamma > -\frac{1}{2}$*,*

$$F_1(z) = f_1(z)|f_1(z)|^{2\gamma} \;\; and \;\; F_2(z) = f_2(z)|f_2(z)|^{2\gamma}.$$

If

$$\mathrm{Re}\left[(1 - a_1(z)\overline{a}_2(z))\left(1 + \frac{zh_1'(z)}{h_1(z)}\right)\overline{\left(1 + \frac{zh_2'(z)}{h_2(z)}\right)}\right] \geq 0 \qquad (for\ any\ z \in E),$$

then

$$F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z) \in S^{\alpha}_{LH}(\rho),$$

where $|\beta| < \frac{\pi}{2}, 0 \leq \lambda \leq 1$ *and* $\alpha = \tan^{-1}\left(\frac{\tan\beta+2Im\gamma}{1+2Re\gamma}\right)$.

Proof. Using the same argument as in Theorem 3, we have

$$F(z) = z|z|^{2\gamma}H(z)\overline{G(z)},$$

where $H(z)$ and $G(z)$ are defined by Equations (8) and (9). Now, we show that the second dilation of F, i.e., $\mu(z)$, satisfies the condition $|\mu(z)| < 1$. For this, since

$$\mu(z) = \frac{\frac{\overline{F}_{\overline{z}}(z)}{\overline{F}(z)}}{\frac{F_z(z)}{F(z)}},$$

using a similar argument to the relation Equation (10) of Theorem 3, we have

$$|\mu(z)| = \left|\frac{\lambda(1+\frac{zh_1'}{h_1})[\overline{\gamma} + (1+\overline{\gamma})a_1(z)] + (1-\lambda)(1+\frac{zh_2'}{h_2})[\overline{\gamma} + (1+\overline{\gamma})a_2(z)]}{\lambda(1+\frac{zh_1'}{h_1})[(1+\gamma) + \gamma a_1(z)] + (1-\lambda)(1+\frac{zh_2'}{h_2})[(1+\gamma) + \gamma a_2(z)]}\right|.$$

However, by hypothesis, we obtain

$$\begin{aligned}
&\left|\lambda(1+\frac{zh_1'}{h_1})[(1+\gamma) + \gamma a_1(z)] + (1-\lambda)(1+\frac{zh_2'}{h_2})[(1+\gamma) + \gamma a_2(z)]\right|^2 \\
&- \left|\lambda(1+\frac{zh_1'}{h_1})[\overline{\gamma} + (1+\overline{\gamma})a_1(z)] + (1-\lambda)(1+\frac{zh_2'}{h_2})[\overline{\gamma} + (1+\overline{\gamma})a_2(z)]\right|^2 \\
&= (2\mathrm{Re}\gamma+1)\left(\lambda^2\left|1+\frac{zh_1'}{h_1}\right|^2(1-|a_1|^2) + (1-\lambda)^2\left|1+\frac{zh_2'}{h_2}\right|^2(1-|a_2|^2)\right) \\
&+ (2\mathrm{Re}\gamma+1)\left(2\lambda(1-\lambda)\mathrm{Re}[(1-a_1\overline{a}_2)(1+\frac{zh_1'}{h_1})\overline{(1+\frac{zh_2'}{h_2})}]\right) > 0.
\end{aligned}$$

Therefore, $|\mu(z)| < 1$ in E, which implies that F is a locally univalent mapping. Moreover, by following a similar proof to that in Theorem 3, we observe that

$$F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(\rho),$$

and the proof is completed. □

Theorem 5. *Let $f_k(z) = zh_k(z)\overline{g}_k(z)$ be univalent log-harmonic functions with respect to $a_k \in B_0 (k = 1, 2)$ and $\mathrm{Re}\gamma > -\frac{1}{2}$. Moreover, suppose that $zh_k g_k = \phi_k(z)$, where*

$$\phi_k(z) = zexp\left\{2\int_0^z \frac{a_k(t)}{t(1-a_k(t))}dt\right\}$$

and

$$F_1(z) = f_1(z)|f_1(z)|^{2\gamma} \quad \text{and} \quad F_2(z) = f_2(z)|f_2(z)|^{2\gamma}.$$

Then,

$$F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(1)$$

where $0 \le \lambda \le 1$ and $\alpha = \tan^{-1}\left(\frac{2\mathrm{Im}\gamma}{1+2\mathrm{Re}\gamma}\right)$.

Proof. Since $zh_k g_k = \phi_k(z)$, by definition of $a_k(z)$ and $\phi_k(z)$, we obtain

$$1 + \frac{zh_k'(z)}{h_k(z)} = \frac{1}{1-a_k(z)} \quad (k = 1, 2).$$

Let

$$\mu(z) = \frac{\frac{\overline{F_{\overline{z}}}(z)}{\overline{F}(z)}}{\frac{F_z(z)}{F(z)}}.$$

Using a similar argument to the relation in Equation (10) of Theorem 3, we obtain

$$|\mu(z)| = \left|\frac{\lambda(1-a_2(z))[\overline{\gamma} + (1+\overline{\gamma})a_1(z)] + (1-\lambda)((1-a_1(z))[\overline{\gamma} + (1+\overline{\gamma})a_2(z)]}{\lambda(1-a_2(z))[(1+\gamma) + \gamma a_1(z)] + (1-\lambda)(1-a_1(z))[(1+\gamma) + \gamma a_2(z)]}\right|.$$

Now, $|\mu(z)| < 1$ is equivalent to

$$\begin{aligned}
&\psi(\lambda) := |\lambda(1-a_2(z))[(1+\gamma) + \gamma a_1(z)] + (1-\lambda)(1-a_1(z))[(1+\gamma) + \gamma a_2(z)]|^2 \\
&- |\lambda(1-a_2(z))[\overline{\gamma} + (1+\overline{\gamma})a_1(z)] + (1-\lambda)((1-a_1(z))[\overline{\gamma} + (1+\overline{\gamma})a_2(z)]|^2 \\
&= (2\mathrm{Re}\gamma + 1)[\lambda^2|1-a_2(z)|^2(1-|a_1(z)|^2) \\
&+ 2\lambda(1-\lambda)\mathrm{Re}[(1-a_2(z))(1-\overline{a_1(z)})(1-a_1(z)\overline{a_2(z)})] \\
&+ (1-\lambda)^2|1-a_1(z)|^2(1-|a_2(z)|^2)] > 0.
\end{aligned}$$

However, by taking the derivative of $\psi(\lambda)$, we have

$$\begin{aligned}
\psi'(\lambda) &= 2(2\mathrm{Re}\gamma + 1) \\
&\left[\mathrm{Re}[(1-a_2(z))(1-\overline{a_1(z)})(1-a_1(z)\overline{a_2(z)})] - |1-a_1(z)|^2(1-|a_2(z)|^2)\right],
\end{aligned}$$

which shows that ψ is a continuous monotonic function of λ in the interval $[0, 1]$. Since

$$\psi(0) = (2\mathrm{Re}\gamma + 1)|1-a_2(z)|^2(1-|a_1(z)|^2) > 0$$

and

$$\psi(1) = (2\text{Re}\gamma + 1)|1 - a_1(z)|^2(1 - |a_2(z)|^2) > 0,$$

we deduce that $\psi(\lambda) > 0$ for all $\lambda \in [0,1]$, which implies that F is a locally univalent mapping. Now, to prove

$$F = F_1^{\lambda} F_2^{1-\lambda} \in S_{LH}^{\alpha} \tag{11}$$

we have to show that $\psi(z) = \frac{zH(z)}{G(z)^{e^{2i\alpha}}} \in S^{\alpha}(1)$, where $H(z)$ and $G(z)$ are defined by Equations (8) and (9). A direct computation such as that in Theorem 3 shows that

$$\frac{(1+\gamma)e^{-i\alpha} - \overline{\gamma}e^{i\alpha}}{\cos\alpha} = \frac{(1+\overline{\gamma})e^{i\alpha} - \gamma e^{-i\alpha}}{\cos\alpha} = 1.$$

Additionally, we note that

$$1 + \frac{zh_1'}{h_1} - \frac{zg_1'}{g_1} = 1 + \frac{zh_2'}{h_2} - \frac{zg_2'}{g_2} = 1.$$

Using these relation and the same argument as that made in Theorem 3, we obtain $\psi(z) = \frac{zH(z)}{G(z)^{e^{2i\alpha}}} \in S^{\alpha}(1)$, and the proof is complete. □

Theorem 6. *Let $f_k(z) = zh_k(z)\overline{g}_k(z) (k = 1,2)$ be log-harmonic functions with respect to $a_k \in B_0$. Moreover, suppose that $zh_k g_k = z$ and*

$$F_1(z) = f_1(z)|f_1(z)|^{2\gamma} \qquad \text{and} \qquad F_2(z) = f_2(z)|f_2(z)|^{2\gamma}.$$

Then,

$$F(z) = F_1^{\lambda}(z) F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(1),$$

where $0 \le \lambda \le 1$ and $\alpha = \tan^{-1}\left(\frac{2\text{Im}\gamma}{(1+2\text{Re}\gamma)}\right)$.

Proof. Since $zh_k g_k = z$, by definition of $a_k(z)$, we obtain

$$1 + \frac{zh_k'(z)}{h_k(z)} = \frac{1}{1 + a_k(z)} \quad (k = 1,2).$$

Using the same argument as that in Theorem 5, we obtain our result, but we omit the details. □

3. Examples

We provide several examples in this section.

Example 1. *Let $\text{Re}\gamma > -\frac{1}{2}$ and*

$$f(z) = z\frac{(1+z)^{[\cos\beta(1-\rho)e^{i\beta}-1]}}{(1-z)^{(1-\rho)\cos\beta e^{i\beta}}}(1+\overline{z})^{[(1-\rho)\cos\beta e^{i\beta} - e^{2i\beta}]}(1-\overline{z})^{(1-\rho)\cos\beta e^{i\beta}}.$$

Then, it is easy to see that f is a β-spirallike log-harmonic mapping of order ρ with respect to $a(z) = -ze^{-2i\beta}$. Now, Theorem 2 implies that the function $F(z) = f(z)|f(z)|^{2\gamma}$ is a α-spirallike log-harmonic mapping of order ρ with respect to

$$\hat{a}(z) = \frac{-(1+\bar{\gamma})ze^{-2i\beta} + \bar{\gamma}}{(1+\gamma) - \gamma e^{-2i\beta}z},$$

where

$$\alpha = \tan^{-1}\left(\frac{\tan\beta + 2\mathrm{Im}\gamma}{1 + 2\mathrm{Re}\gamma}\right).$$

The image in Example 1 is shown in Figure 1.

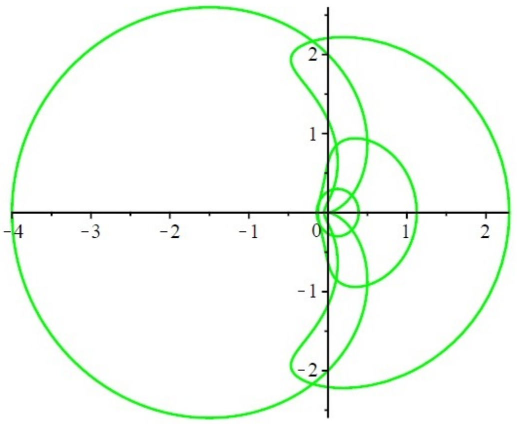

Figure 1. Image of $F(z)$ for $\beta = 0.5$, $\rho = 1$, and $\gamma = 0.25$ in Example 1.

Example 2. *Let* $\mathrm{Re}\gamma > -\frac{1}{2}, 0 < a < 1$, f_1 *be the function defined in Example 1 and*

$$f_2(z) = z\frac{(1+z)^{[\cos\beta\frac{(1+a-2\rho)}{1+a}e^{i\beta}-1]}}{(1-az)^{\frac{(1+a-2\rho)}{1+a}\cos\beta e^{i\beta}}}(1+\bar{z})^{[\frac{(1+a-2\rho)}{1+a}\cos\beta e^{i\beta}-e^{2i\beta}]}(1-a\bar{z})^{\frac{(1+a-2\rho)}{a(1+a)}\cos\beta e^{i\beta}}.$$

Then, it is easy to see that f_1 *and* f_2 *are* β*-spirallike log-harmonic mappings of order* ρ *with respect to* $a_2(z) = a_1(z) = -ze^{-2i\beta}$. *Additionally, suppose that*

$$F_1(z) = f_1(z)|f_1(z)|^{2\gamma} \text{ and } F_2(z) = f_2(z)|f_2(z)|^{2\gamma}.$$

Then, Theorem 3 shows that

$$F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(\rho),$$

where $0 \leq \lambda \leq 1$ *and* $\alpha = \tan^{-1}\left(\frac{\tan\beta+2\mathrm{Im}\gamma}{(1+2\mathrm{Re}\gamma)}\right)$.

Example 3. *Let* $\mathrm{Re}\gamma > -\frac{1}{2}$,

$$f_1(z) = \frac{z}{|1+z|}\sqrt{\frac{1-\bar{z}}{1-z}}$$

and

$$f_2(z) = \frac{z}{1-z}e^{\mathrm{Re}\frac{1}{1-z}}.$$

Firstly, we show that f_1 *and* f_2 *are log-harmonic starlike functions of order* $1/2$ *with respect to* $a_1(z) = -z$ *and* $a_2(z) = \frac{z}{2-z}$*, respectively. A direct computation shows that*

$$\frac{z(f_1)_z}{f_1} = \frac{1}{1-z^2}, \qquad \overline{\left(\frac{\bar{z}(f_1)_{\bar{z}}}{f_1}\right)} = \frac{-z}{1-z^2}$$

$$\frac{z(f_2)_z}{f_2} = \frac{2-z}{2(1-z^2)}, \qquad \overline{\left(\frac{\bar{z}(f_2)_{\bar{z}}}{f_2}\right)} = \frac{z}{2(1-z^2)}.$$

Therefore, we obtain

$$\overline{\left(\frac{\bar{z}(f_1)_{\bar{z}}}{f_1}\right)} = a_1(z)\frac{z(f_1)_z}{f_1} \qquad \text{and} \qquad \overline{\left(\frac{\bar{z}(f_2)_{\bar{z}}}{f_2}\right)} = a_2(z)\frac{z(f_2)_z}{f_2},$$

and this means that f_1 and f_2 are locally univalent log-harmonic functions. Additionally,

$$Re\frac{z(f_1)_z - \bar{z}(f_1)_{\bar{z}}}{f_1} = Re\left(\frac{1}{1-z^2} + \frac{z}{1-z^2}\right) = Re\frac{1}{1-z} > \frac{1}{2},$$

and

$$Re\frac{z(f_2)_z - \bar{z}(f_2)_{\bar{z}}}{f_2} = Re\left(\frac{2-z}{2(1-z^2)} - \frac{z}{2(1-z^2)}\right) = Re\frac{1}{1+z} > \frac{1}{2}.$$

Hence, f_1 and f_2 are starlike log-harmonic functions of order $1/2$. Additionally, let

$$F_1(z) = f_1(z)|f_1(z)|^{2\gamma} \text{ and } F_2(z) = f_2(z)|f_2(z)|^{2\gamma}.$$

Since for $z = re^{i\theta}$,

$$\begin{aligned}&\mathrm{Re}(1-a_1\bar{a}_2)(1+\frac{zh_1'}{h_1})\overline{(1+\frac{zh_2'}{h_2})}\\ &= (1-|z|^2)\mathrm{Re}\frac{1}{(1-\bar{z})^2}\frac{1}{1-z^2} = \frac{1-|z|^2}{|1-z|^2}\mathrm{Re}\frac{1}{(1-\bar{z})(1+z)}\\ &= \frac{1-r^2}{|1-re^i\theta|^2}(1-r^2) > 0.\end{aligned}$$

Theorem 4 implies that

$$F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(\frac{1}{2}),$$

where $0 \le \lambda \le 1$ and $\alpha = \tan^{-1}\left(\frac{2\mathrm{Im}\gamma}{1+2\mathrm{Re}\gamma}\right)$.

The images in Example 2–4 are shown in Figures 2–4.

Example 4. *Let $\mathrm{Re}\gamma > \frac{1}{2}$, $a_1(z) = z$, and $h_1(z) = g_1(z) = \frac{1}{1-z}$. Moreover, let $a_2(z) = -z$ and $h_2(z) = g_2(z) = \frac{1}{1+z}$. Then, it is easy to verify that all conditions of Theorem 5 are satisfied. Hence, according to Theorem 5, by taking*

$$F_1(z) = \frac{z|z|^{2\gamma}}{(1-z)^{1+2\gamma}(1-\bar{z})^{1+2\gamma}}$$

and

$$F_2(z) = \frac{z|z|^{2\gamma}}{(1+z)^{1+2\gamma}(1+\bar{z})^{1+2\gamma}},$$

we have

$$F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(1),$$

where $0 \le \lambda \le 1$ and $\alpha = \tan^{-1}\left(\frac{2\mathrm{Im}\gamma}{1-\rho+2\mathrm{Re}\gamma}\right)$.

Example 5. *Let $\mathrm{Re}\gamma > -\frac{1}{2}$, $a_1(z) = -z$ and $h_1(z) = \frac{1}{1-z}, g(z) = 1-z$. Moreover, let $a_2(z) = z$ and $h_2(z) = \frac{1}{1+z}$, $g_2(z) = 1+z$. Then, it is easy to verify that all conditions of Theorem 6 are satisfied. Hence, according to Theorem 6, by taking*

$$F_1(z) = \frac{z|z|^{2\gamma}(1-\bar{z})}{(1-z)} \qquad \text{and} \qquad F_2(z) = \frac{z|z|^{2\gamma}(1+\bar{z})}{(1+z)},$$

we have

$$F(z) = F_1^{\lambda}(z)F_2^{1-\lambda}(z) \in S_{LH}^{\alpha}(1),$$

where $0 \le \lambda \le 1$ *and* $\alpha = \tan^{-1}\left(\frac{2\text{Im}\gamma}{1-\rho+2\text{Re}\gamma}\right)$.

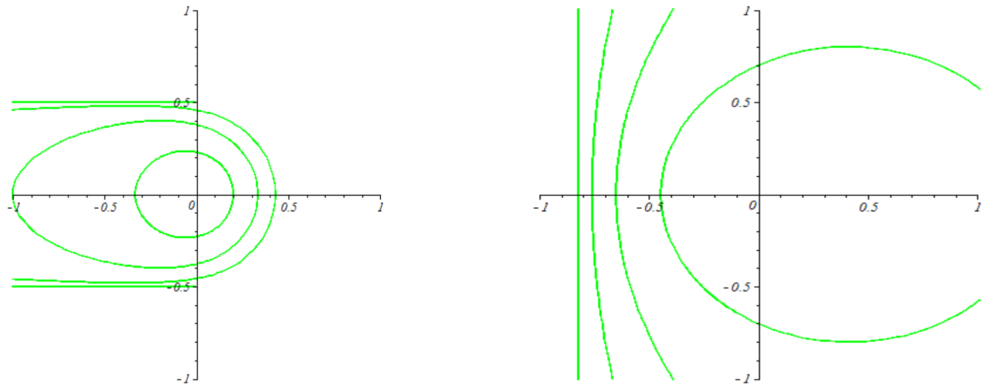

Figure 2. Images of $f_1(z)$ and $f_2(z)$ in Example 3.

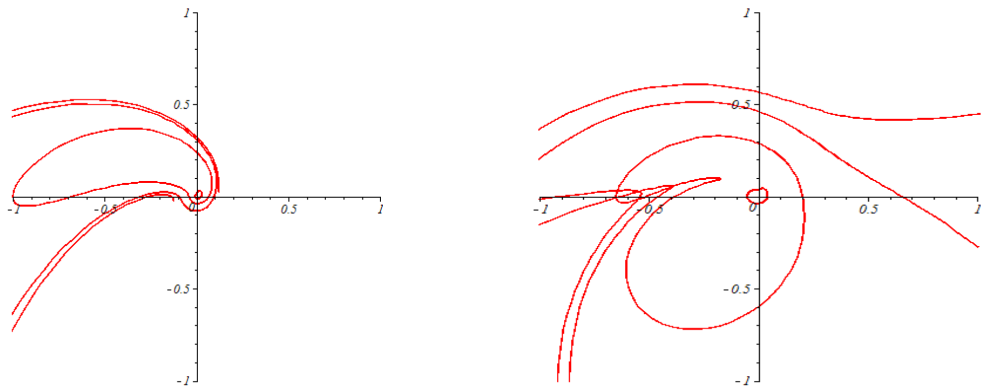

Figure 3. Images of $F_1(z)$ and $F_2(z)$ for $\gamma = 1 + i$ in Example 3.

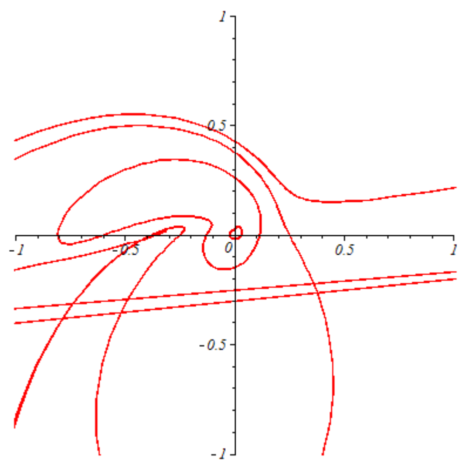

Figure 4. Image of $F(z)$ for $\gamma = 1 + i$ and $\lambda = 0.5$ in Example 3.

4. Conclusions

In this paper, we have shown that, if $f(z) = zh(z)\bar{g}(z)$ is spirallike log-harmonic of order ρ, then by choosing suitable parameters of α and γ, the function $F(z) = f(z)|f(z|^{2\gamma}$ is log-harmonic spirallike of order α. Moreover, we provide some examples for the obtained results.

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