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Spanning k-Ended Tree in 2-Connected Graph

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Abstract: Win proved a very famous conclusion that states the graph G with connectivity $\kappa(G)$, independence number $\alpha(G)$ and $\alpha(G) \leq \kappa(G) + k - 1$ $(k \geq 2)$ contains a spanning k-ended tree. This means that there exists a spanning tree with at most k leaves. In this paper, we strengthen the Win theorem to the following: Let G be a simple 2-connected graph such that $|V(G)| \geq 2\kappa(G) + k$, $\alpha(G) \leq \kappa(G) + k$ $(k \geq 2)$ and the number of maximum independent sets of cardinality $\kappa + k$ is at most $n - 2\kappa - k + 1$. Then, either G contains a spanning k-ended tree or a subgraph of $K_\kappa \vee ((k + \kappa - 1)K_1 \cup K_{n-2\kappa-k+1})$.

Keywords: connectivity; independence number; k-ended tree; maximum independent set

MSC: 05C10

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1. Introduction

Notation regarding graph theory is not covered in this paper. We refer the reader to [1]. Let $G = (V(G), E(G))$ be a graph satisfying vertex set $V(G)$ and edge set $E(G)$. We denote the set of vertices adjacent to v in G as $N(v)$. We write $N(X) = \bigcup_{x \in X} N(x)$ for $X \subseteq V(G)$. We also denote the subgraph of G induced by S as $G[S]$ for $S \subseteq V(G)$. Let H_1 and H_2 be two subgraphs of G which vertex disjoint, and P be a path of G. A path xPy in G with end vertices $x, y \in V(G)$ is called a path from H_1 to H_2 if $V(xPy) \cap V(H_1) = \{x\}$ and $V(xPy) \cap V(H_2) = \{y\}$. (x, U)-path is a path from $\{x\}$ to a vertex set U. We write an (x, U)-fan of width k for $F \subseteq G$ if F is a union of (x, U)-paths $P_1, P_2, \ldots, P_k$, where $V(P_i) \cap V(P_j) = \{x\}$ for $i \neq j$. Let G_1 and G_2 be two subgraphs of G. We denote by xG_1 (G_1x, respectively) the Hamilton path of $G[\{x\} \cup V(G_1)]$, which starts at x (terminates at x, respectively). We denote by xG_1y the Hamilton path of $G[V(G_1) \cup \{x, y\}]$, which starts at x and terminates at y. We denote by G_1xG_2 the Hamilton path of $G[V(G_1) \cup \{x\} \cup V(G_2)]$. A nontrivial graph, G, is considered k-connected if the maximum number of pairwise internally disjoint xy-paths for any two distinct vertices, x and y, is greater than or equal to k. A trivial graph is considered 0-connected or 1-connected, but it is not considered k-connected for any k greater than 1. The connectivity, $\kappa(G)$, of G is defined as the maximum value of k for which G is k-connected.

If a graph contains a Hamilton path, then the graph is said *traceable*, and if a graph contains a Hamilton cycle, then the graph is said *hamiltonian*. The sufficient conditions under which a graph can be traceable involving *connectivity* ($\kappa(G)$) and *independence number* ($\alpha(G)$) were given by Chvátal and Erdős in 1972.

Theorem 1. *(Chvátal and Erdős, [2]) If a graph G with $|V(G)| \geq 3$ satisfies the conditions $\alpha(G) \leq \kappa(G)$, $\alpha(G) \leq \kappa(G) + 1$, respectively, then G is Hamiltonian and traceable, respectively.*

Theorem 1 has been extended in various directions, as documented in previous studies [3–8]. For recent results, see [9–12]. Fouquet and Jolivet [13] conjecture whether a graph's circumference can have a best possible lower bound when its independence number exceeds its connectivity. This has been proved by Suil O et al.

Theorem 2. *(Suil O et al., [14]) If G is a simple graph such that* $|V(G)| = n$ *and* $\alpha(G) \geq \kappa(G)$*, then G contains a cycle with length of at least* $\frac{\kappa(G)(n-\alpha(G)-\kappa(G))}{\alpha(G)}$.

The number of maximum independent sets of H for a subgraph $H \subseteq G$ is denoted by $m(H)$. In their study [15], Chen et al. presented the following theorem that generalizes Theorem 1 by bound $m(G)$. Specifically, the authors demonstrated that expanding the independence number (i.e., $\alpha(G) \leq \kappa(G) + 2$) slightly and bounding $m(G)$ does not alter the traceability of G. It is worth noting that K_s represents a complete graph with s vertices, while $\overline{K_s}$ is the complement of K_s. Additionally, the *join* $G \vee H$ of disjoint graphs G and H is obtained by joining each vertex of G to each vertex of H in the graph $G + H$.

In the following, we construct three graphs which are excluded. Let $H_i(k_i)$ be a copy of $\overline{K}_{k_i}$ where $i = 1, 2$. The graph $F_0(k_1, k_2)$ is defined as $(H_1(k_1) \vee H_2(k_2)) \cup K_{n-k_1-k_2} \cup M_1(k_2)$, where $n - k_1 - k_2 \geq k_2$ and $M_1(k_2)$ is a matching of cardinality k_2 between $H_2(k_2)$ and $K_{n-k_1-k_2}$. If $n - k_1 - k_2 \geq k_2$, then $F_{11}(k_1, k_2)$ is obtained from $F_0(k_1, k_2)$ by joining exactly two (nonadjacent) vertices of $H_2(k_2)$) or by joining all vertices of $V(K_{n-k_1-k_2}) \setminus V(M_1(k_2))$ and some fixed vertex $w_0 \in H_2(k_2)$. Let $F_{00}(k_1, k_2)$ be the graph $(H_1(k_1) \vee H_2(k_2)) \cup K_{n-k_1-k_2} \cup M_2(k_2)$, where $n - k_1 - k_2 \leq k_2$ and $M_2(k_2)$ is a matching of cardinality $n - k_1 - k_2$ between $K_{n-k_1-k_2}$ and $H_2(k_2)$. Define the graph $F_2(k_1, k_2) = K_{k_2} \vee (\overline{K_{k_1}} \cup K_{n-k_1-k_2})$; see Figure 1.

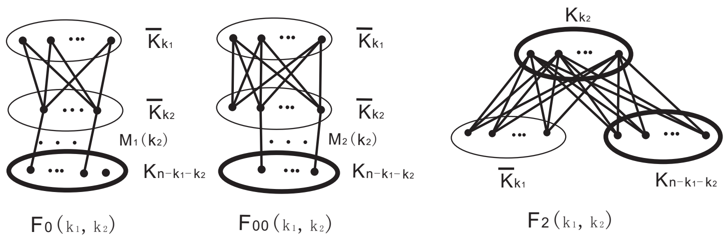

Figure 1. $F_0(k_1, k_2)$, $F_{00}(k_1, k_2)$ and $F_2(k_1, k_2)$.

Theorem 3. *(Chen et al., [15]) Let G be a 2-connected graph with* $|V(G)| \geq 2\kappa^2(G)$, $\kappa(G) = \kappa$, $\alpha(G) \leq \kappa + 1$ *and* $m(G) \leq n - 2\kappa$*. Then, either G is Hamiltonian or* $F_{11}(\kappa, \kappa) \subseteq G \subseteq F_2(\kappa, \kappa)$*, where* $F_{11}(\kappa, \kappa)$ *and* $F_2(\kappa, \kappa)$ *are two graphs defined above.*

Theorem 4. *(Chen et al., [15]) Let G be a connected graph with* $|V(G)| \geq 2\kappa^2(G)$, $\kappa(G) = \kappa$, $\alpha(G) \leq \kappa + 2$ *and* $m(G) \leq n - 2\kappa - 1$*. Then either G is traceable or* $F_{11}(\kappa+1, \kappa) \subseteq G \subseteq K_\kappa \vee ((\kappa+1)K_1 \cup K_{n-2\kappa-1})$*, where* $F_{11}(\kappa+1, \kappa)$ *is the graph defined above.*

A Hamilton path is viewed as a spanning tree with exactly two leaves. This perspective allows for the generalization of sufficient conditions for a graph to be traceable to those for a spanning tree with at most k leaves. A tree is called a *k-ended tree* if it has at most k leaves. Our focus now shifts to *spanning k-ended trees*. Clearly, if $s \leq t$, then a spanning s-ended tree is also a spanning t-ended tree. Theorem 1 demonstrates that each graph G such that $\alpha(G) \leq \kappa(G) + 1$ is traceable. In [16], Win proved the following theorem, which generalizes Theorem 1.

Theorem 5. *(Win, [16]) Let G be a connected graph and let* $k \geq 2$ *be an integer. If* $\alpha(G) \leq \kappa(G) + k - 1$*, then G contains a spanning k-ended tree.*

In [17], Lei et al. extend Theorem 5 in cases when $\kappa(G) = 1$ to the following direction.

Theorem 6. *(Lei et al., [17]) Let $k \geq 3$ and G be a connect graph with $|V(G)| \geq 2k+2$, $\alpha(G) \leq 1+k$ and $m(G) \leq n-2k-2$. Then G contains a spanning k-ended tree.*

In [15], Chen et al. generalize Theorem 1 by bound $m(G)$. The authors demonstrated that expanding the independence number (i.e., $\alpha(G) \leq \kappa(G)+2$) slightly and bounding $m(G)$ does not alter the traceability of G.

In this paper, our focus will be on the existence of spanning k-ended tree. We will work on extending Theorem 5 to a more general case. A natural question is whether expanding the independence number can alter the existence of the spanning k-ended tree. In the following section, we introduce the k-ended system, which is an important tool for studying the k-ended tree.

k-Ended System

If there exists a set of paths and cycles where the elements are pairwise vertex-disjoint, we refer to it as a system. This system is often viewed as a subgraph. Let $\mathcal{S}$ be a system in a graph. For $S \in \mathcal{S}$, we put $f(S) = 2$ if S is a path of order at least 3 and $f(S) = 1$ otherwise (i.e., S is a vertex, an edge or a cycle). We write $V(\mathcal{S}) = \bigcup_{S \in \mathcal{S}} V(S)$ and $f(\mathcal{S}) = \sum_{S \in \mathcal{S}} f(S)$. If $f(\mathcal{S}) \leq k$, $\mathcal{S}$ is called a *k-ended system*. Moreover, we call $\mathcal{S}$ a *spanning k-ended system* of G, if $V(\mathcal{S}) = V(G)$. Let

$$\mathcal{S}_P = \{S \in \mathcal{S} : f(S) = 2\},\ \mathcal{S}_C = \{S \in \mathcal{S} : f(S) = 1\}.$$

Then,

$$\mathcal{S} = \mathcal{S}_P \cup \mathcal{S}_C, \qquad V(\mathcal{S}) = \bigcup_{S \in \mathcal{S}} V(S).$$

Additionally, $V(\mathcal{S}_P)$ and $V(\mathcal{S}_C)$ can be defined in a similar manner. We use $|\mathcal{S}|$, $|\mathcal{S}_P|$ and $|\mathcal{S}_C|$ to represent the number of elements in $\mathcal{S}$, $\mathcal{S}_P$ and $\mathcal{S}_C$, respectively. For each $S \in \mathcal{S}$, we assign an orientation denoted by the symbol $<$, where $x < y$ if x precedes y in the orientation. Let $\overrightarrow{S}$ be the orientation of $S \in \mathcal{S}$ and let $\overleftarrow{S}$ be the reverse orientation of $\overrightarrow{S}$ for $S \in \mathcal{S}$. By assigning an orientation to each $S \in \mathcal{S}$, we identify $\mathcal{S}$ as a system with an orientation, where each element is ordered relative to the others.

Let $\mathcal{S}$ be a system with a defined orientation. For any $P \in \mathcal{S}_P$, we define $v_L(P)$ and $v_R(P)$ as the two end-vertices of P such that $v_L(P) < v_R(P)$. Additionally, for each $C \in \mathcal{S}_C$, we select an arbitrary vertex v_C within C. These definitions will be used in subsequent analyses. Then define

$$End(\mathcal{S}_P) = \bigcup_{P \in \mathcal{S}_P} \{v_L(P), v_R(P)\},\ End(\mathcal{S}_C) = \bigcup_{C \in \mathcal{S}_C} \{v_C\},\ \ End(\mathcal{S}) = End(\mathcal{S}_P) \cup End(\mathcal{S}_C).$$

For $S \in \mathcal{S}$ and $x \in V(S)$, we write the first, second and ith predecessor (successor, respectively) of x as x^-, x^{--} and x^{i-} (x^+, x^{++} and x^{i+}, respectively). For convenience, we write $x = x^+ = x^-$ for $K_1 = x$ and $y = x^+ = x^-$ for $K_2 = xy$.

For $P \in \mathcal{S}_P$, if $x = v_R(P)$ ($x = v_L(P)$, respectively), we have only the predecessor of $v_R(P)$ (successor of $v_L(P)$, respectively). For $\{x, y\} \subseteq V(P)$, we denote by the *section* $P(x, y)$ a path $x^+x^{2+}x^{3+}...x^{s+}(= y^-)$ of consecutive vertices of P and denote by the *section* $P[x, y]$ a path $xx^+x^{2+}...x^{s+}(= y)$ of consecutive vertices of P. Moreover, if $x = y$, then the section $P[x, y]$ is trivial.

The following lemma illustrates the importance of k-ended systems for spanning k-ended trees.

Lemma 1. (Win, [16]) Let $k \geq 2$ be an integer and let G be a connected simple graph. If G contains a spanning k-ended system, then G also contains a spanning k-ended tree.

A k-ended system $\mathcal{S}$ in G is considered a *maximal k-ended system* if there is no other k-ended system $\hat{\mathcal{S}}$ in G satisfying $V(\mathcal{S}) \subset V(\hat{\mathcal{S}})$. The following lemma presents some useful properties of k-ended systems. It is important to note that two distinct elements of $\mathcal{S}$ are connected by a path in $G - V(\mathcal{S})$ if there exists a path in G whose end-vertices are in elements of $\mathcal{S}$ and whose inner vertices are not all contained in $V(\mathcal{S})$. It is worth mentioning that a path may not have any inner vertex.

Lemma 2. (Akiyama and Kano, [18]) Let $k \geq 2$ be an integer and G be a connected simple graph. Assume that G does not contain a spanning k-ended system and let $\mathcal{S}$ be a maximal k-ended system of G satisfying the cardinality of the maximum value of $\mathcal{S}_P$ subject to the maximum value of $V(\mathcal{S})$. Then the following characterizations are true.

(i) There is no path connecting two distinct elements of $\mathcal{S}_C$ whose inner vertices are in $V(G) \setminus V(\mathcal{S})$.

(ii) There is no path connecting an element of $\mathcal{S}_C$ and one end-vertex of an element of $\mathcal{S}_P$ whose inner vertices are in $V(G) \setminus V(\mathcal{S})$.

(iii) There is no path connecting an end-vertex of an element of $\mathcal{S}_P$ and an end-vertex of another element of $\mathcal{S}_P$ whose inner vertices are in $V(G) \setminus V(\mathcal{S})$.

(iv) There are no two internally disjoint paths Q_1 and Q_2 connecting two distinct elements of $\mathcal{S}_C$ whose inner vertices are in $V(G) \setminus V(\mathcal{S})$ with $|V(Q_1) \cap V(Q_2)| = 1$.

2. Methods

In this paper, our focus will be on the existence of spanning k-ended tree. We will work on extending Theorem 5 to a more general case. We tried to prove that it does not change the existence of spanning k-ended tree if we expand the independent numbers a little bit and bound $m(G)$. The proof will follow an approach similar to Theorem 6, but with additional considerations for the increased connectivity of the graph. Our proof follows a method of contradiction. We primarily utilize the crucial tool of the maximal k-ended system, as mentioned above, to derive contradictions. The subsequent section is the crucial property of the maximal k-ended system which we obtained. This property plays a pivotal role in our proof.

Important Properties of Maximal k-Ended System

In this section, for convenience, we assume the following: Let $k \geq 2$ and G be a graph with $|V(G)| > 2\kappa(G) + k$, $\kappa(G) = \kappa \geq 2$, $\alpha(G) = \kappa + k$ and $m(G) \leq n - 2\kappa - k + 1$. Suppose that there is no spanning k-ended system in G and let $\mathcal{S}$ be a k-ended system of G satisfying the following:

(I) The cardinality of the set $V(\mathcal{S})$ is maximized.
(II) The cardinality of $\mathcal{S}_P$ is maximized subject to condition (I).

Then $\mathcal{S}$ is a set of subgraphs of G satisfying the hypothesis of Lemma 2. Let $H = G - V(\mathcal{S})$. Then $|V(H)| \geq 1$. Let $w \in (V(G) - V(\mathcal{S})$. The following lemma is easily obtained from the selection of $\mathcal{S}$ and we omit the proof.

Lemma 3. *The following characterizations are true.*

(1) For any $P \in \mathcal{S}_P$, $v_L(P)$ and $v_R(P)$ are not in $N(H)$.
(2) For any $C \in \mathcal{S}_C$ such that $C \cong K_1$, $N(H) \cap V(C) = \emptyset$.

By the Fan Lemma, there exists a $(w, V(\mathcal{S}))$-fan $\mathcal{L}$ with width κ. For $S \in \mathcal{S}$ with $V(S) \cap V(\mathcal{L}) \neq \emptyset$, let $V(S) \cap V(\mathcal{L}) = \{u_{S,1}, \cdots, u_{S,t_S}\}$ (where $u_{S,1}, \cdots, u_{S,t_S}$ are the vertices of S along the direction of S) and $L_{S,i}$ be the path of $\mathcal{L}$ between w and $u_{S,i}$. Then $U = \bigcup_{S \in \mathcal{S}} \{V(S) \cap V(\mathcal{L})\}$ and $L_S = \{L_{S,1}, \cdots, L_{S,t_S}\}$ is the set of paths between w and S. Denote $U^+ = \{u^+ : u \in U\}$ and $U^- = \{u^- : u \in U\}$. By Lemma 3(1),(2), U^+ and U^- are well defined and hence $|U^+| = |U^-| = |U|$.

The proof of the following lemmas can be easily obtained from the choice of $\mathcal{S}$ and we will omit it.

Lemma 4. *A graph G cannot have a k'-ended system $\mathcal{T}$ that includes all vertices in a k-ended system $\mathcal{S}$, where $k' < k$.*

Lemma 5. *The following characterizations are true:*

(1) *Both U^+ and $End(\mathcal{S})$ are independent sets of G.*
(2) $N(w) \cap U^+ = \emptyset$.
(3) $\mid \{C \in \mathcal{S}_C : V(C) \cap U \neq \emptyset\} \mid \leq 1$.
(4) *Let $x \in U^+ \cap V(P)$, where $P \in \mathcal{S}_P$. Then $N(x) \cap (End(\mathcal{S}_P) \setminus \{v_R(P)\}) = \emptyset$. Furthermore, if $x \neq u^+_{P,t_P}$, then $N(x) \cap End(\mathcal{S}_P) = \emptyset$.*

Lemma 6. *Let $v_1 \in End(\mathcal{S})$ and $S_1 \in \mathcal{S}$ with $v_1 \in V(S_1)$. Then the following statements are true:*

(1) $[(N(v_1) \cap V(S))^- \cup (N(v_1) \cap V(S))^+] \cap N(v) = \emptyset$ *for any* $v \in V(\mathcal{S}_C - \{S, S_1\}) \cup End(\mathcal{S} - \{S, S_1\})$ *and* $S \in \mathcal{S} - \{S_1\}$.
(2) *If $v_1 = v_L(P)$ ($v_1 = v_R(P)$, respectively), then $(N(v_1) \cap V(S_1))^- \cap N(v) = \emptyset$ ($N(v_1) \cap (N(v) \cap V(S_1))^- = \emptyset$, respectively) for any $v \in V(\mathcal{S}_C) \cup V(H) \cup (End(\mathcal{S}) \setminus \{v_1\})$.*

Let Y be an independent set of G with size $k + \kappa$. Then the following lemma holds.

Lemma 7. *Let S' belong to $V(G)$ which satisfies $S' \cap Y$ having precisely one vertex, denoted as z, i.e, $S' \cap Y = \{z\}$. If $N(x) \cap Y = \{z\}$ for each $x \in S' \setminus \{z\}$, then $G[S']$ forms a clique.*

Proof of Lemma 7. We begin by assuming the opposite and using a proof by contradiction. Suppose that $x_1x_2 \notin E(G)$ for some pair of vertices x_1 and x_2 in S', where $x_1 \neq x_2$. Then, $(Y \setminus \{z\}) \cup \{x_1, x_2\}$ forms an independent set of G with a size of $k + \kappa + 1$. This contradicts the fact that $\alpha(G) = k + \kappa$. Hence, $G[S']$ is a clique. □

For convenience, suppose that x is an element of $V(P)$. For each $P \in \mathcal{S}_P$, we define $\overrightarrow{Q}(x, P)$ as follows:

$$\overrightarrow{Q}(x,P) = \begin{cases} v_R(P), & \text{if} \quad x = v_R(P), \\ xv_R(P), & \text{if} \quad x^+ = v_R(P), \\ x\overrightarrow{P}v_R(P)x, & \text{if} \quad xv_R(P) \in E(G)\ (x \neq v_R(P), x^+ \neq v_R(P)). \end{cases}$$

Similarly, we define $\overleftarrow{Q}(x, P)$ as follows:

$$\overleftarrow{Q}(x,P) = \begin{cases} x = v_L(P), & \text{if} \quad x = v_L(P), \\ xv_L(P), & \text{if} \quad x^- = v_L(P), \\ x\overleftarrow{P}v_L(P)x, & \text{if} \quad xv_L(P) \in E(G)\ (x \neq v_L(P), x^- \neq v_L(P)). \end{cases}$$

Therefore, $f(\overrightarrow{Q}(x, P)) = 1$ and $f(\overleftarrow{Q}(x, P)) = 1$. We say that $C(G_0)$ is a spanning subgraph of G_0 satisfying $f(C(G_0)) = 1$ if $G_0 \subseteq G$.

Some properties of $\mathcal{S}$ are described in the following lemmas, as proved in Appendix B.

Lemma 8. *$U \subseteq V(\mathcal{S}_P)$ and $\mid \mathcal{S}_P \mid= 1$.*

By Lemma 8, $\mathcal{S}_P = \{P\}$ (say). Then $U = V(P) \cap V(\mathcal{L}) = \{u_{P,1}, \cdots, u_{P,t_P}\}$ and $t_P = \kappa$. For convenience, denote that $U = \{u_1, u_2, \cdots, u_\kappa\}$ and

$$\mathcal{X} = \begin{cases} End(\mathcal{S}) \cup U^+ \cup \{w\}, & \text{if} \quad End(\mathcal{S}) \cap U^+ \neq \emptyset, \\ (End(\mathcal{S}) \cup U^+ \cup \{w\}) \setminus \{u^+_\kappa\}, & if\ End(\mathcal{S}) \cap U^+ = \emptyset. \end{cases}$$

Lemma 9. *The following statements hold.*

(1) $N(u_\kappa^+) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overrightarrow{Q}(u_\kappa^+, P)) = 1$ and $N(u_1^-) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overleftarrow{Q}(u_1^-, P)) = 1$.

(2) *If* $f(\overrightarrow{Q}(u_\kappa^+, P)) \neq 1$ *and* $f(\overleftarrow{Q}(u_1^-, P)) \neq 1$, *then there exist at least two elements* $C, C' \in \mathcal{S}_C$ *such that* $u_\kappa^+ v_{C'} \in E(G)$, $u_1^- v_C \in E(G)$.

(3) $\mathcal{X}$ *forms an independent set of* G *with size* $k + \kappa$.

(4) *If* $N(u_\kappa^+) \cap End(\mathcal{S}_C) \neq \emptyset$ *and* $u_\kappa^{2+} \neq v_R(P)$, *then* $u_\kappa^{2+} v_R(P) \in E(G)$.

(5) *Let* $y \in V(P)$ *satisfy* $|V(P[y^+, v_R(P)])| \geq 1$, $y v_R(P) \in E(G)$ *and* $V(P[y, v_R(P)]) \cap U = \emptyset$. *Then* $G[V(P[y^+, v_R(P)])]$ *forms a clique. Additionally, if the intersection of* $N(y)$ *and* $\mathcal{X}$ *is* $\{v_R(P)\}$, *then the graph* $G[V(P[y, v_R(P)])]$ *forms a clique.*

(6) $G[V(C)]$ *forms a clique for any* $C \in \mathcal{S}_C$. *Furthermore,* $N(x) \cap \mathcal{X} = \{v_C\}$ *for any* $x \in V(C) \setminus \{v_C\}$.

3. Results and Discussion

In [17], the authors provide a novel extension by imposing a limit on the maximum number of independent sets, although the limit is not sharp. Note that G has no spanning $k_1 + 1 - k_2$-ended tree for each $G \in \{F_0(k_1, k_2), F_{00}(k_1, k_2), F_2(k_1, k_2)\}$. In this paper, we extend Theorem 5 to the case where $\kappa(G) \geq 2$ and the bound on the number of maximum independent sets is already sharp.

Theorem 7. *Let* $k \geq 2$ *and* G *be a graph with* $|V(G)| \geq 2\kappa(G) + k$, $\kappa(G) = \kappa \geq 2$, $\alpha(G) \leq \kappa + k$ *and* $m(G) \leq n - 2\kappa - k + 1$. *Then* G *contains a spanning* k*-ended tree, unless either* $F_0(k + \kappa - 1, \kappa) \subseteq G \subseteq F_2(k + \kappa - 1, \kappa)$ *for* $n > 3\kappa + k - 1$, *or* $F_{00}(k + \kappa - 1, \kappa) \subseteq G \subseteq F_2(k + \kappa - 1, \kappa)$ *for* $2\kappa + k \leq n \leq 3\kappa + k - 1$.

Note that a spanning tree having exactly two leaves is called a Hamilton path. Then, we can immediately obtain the following result.

Corollary 1. *Let* G *be a graph with* $|V(G)| \geq 2\kappa(G) + 2$, $\kappa(G) = \kappa \geq 2$, $\alpha(G) \leq \kappa + 2$ *and* $m(G) \leq n - 2\kappa - 1$. *Then* G *is traceable, unless either* $F_0(\kappa + 1, \kappa) \subseteq G \subseteq F_2(\kappa + 1, \kappa)$ *for* $n > 3\kappa + 1$, *or* $F_{00}(\kappa + 1, \kappa) \subseteq G \subseteq F_2(\kappa + 1, \kappa)$ *for* $2\kappa + 2 \leq n \leq 3\kappa + 1$.

In the case of 2-connected, the bounds of $|V(G)|$ can do better. Clearly, Corollary 1 improves the result of Theorem 4. It demonstrated that expanding the independence number slightly and bounding $m(G)$ also does not alter the traceability in highly connected graphs.

If $F_0(k + \kappa - 1, \kappa) \subseteq G \subseteq F_2(k + \kappa - 1, \kappa)$ for $n > 3\kappa + k - 1$, or $F_{00}(k + \kappa - 1, \kappa) \subseteq G \subseteq F_2(k + \kappa - 1, \kappa)$ for $2\kappa + k \leq n \leq 3\kappa + k - 1$, then $m(G) = n - 2\kappa - k + 1$. Hence, we can obtain the following result immediately.

Corollary 2. *Let* $k \geq 2$ *and* G *be a graph of order* $n \geq 2\kappa(G) + k$ *such that* $\kappa(G) = \kappa \geq 2$, $\alpha(G) \leq \kappa + k$ *and* $m(G) \leq n - 2\kappa - k$. *Then* G *contains a spanning* k*-ended tree.*

4. Proof of Theorem 7

In this section, we employ the same terminology and notation in Section 2.

Proof of Theorem 7. Let $k \geq 2$ and G be a graph with $|V(G)| > 2\kappa(G) + k$, $\kappa(G) = \kappa \geq 2$, $\alpha(G) \leq \kappa + k$ and $m(G) \leq n - 2\kappa - k + 1$. We begin by assuming the opposite and using a proof by contradiction. Suppose that G does not have a spanning k-ended tree. This assumption, along with Theorem 5, implies the following equation:

$$\alpha(G) = \kappa(G) + k. \tag{1}$$

Thus, by Lemma 1, G cannot have a spanning k-ended system. We select a maximal k-ended system $\mathcal{S}$ of G that satisfies conditions (I) and (II) outlined in Section 2. Define $H = G - V(\mathcal{S})$. Clearly $|V(H)| \geq 1$. Let $w \in (V(G) - V(\mathcal{S}))$.

We will show that $F_0(k+\kappa-1,\kappa) \subseteq G \subseteq F_2(k+\kappa-1,\kappa)$ for $n > 3\kappa+k-1$, or $F_{00}(k+\kappa-1,\kappa) \subseteq G \subseteq F_2(k+\kappa-1,\kappa)$ for $2\kappa+k \leq n \leq 3\kappa+k-1$.

Fact 1. $m(G) \geq n-2\kappa-k+1$.

Proof of Fact 1. We consider a $(w, V(\mathcal{S}))$-fan $\mathcal{L}$ in Section 2. By Lemma 8, we choose $U = \{u_1, u_2, \cdots, u_\kappa\}$ and $\mathcal{S}_P = \{P\}$ in Section 2.

We consider a $(w, V(\mathcal{S}))$-fan $\mathcal{L}$ in Section 2. By Lemma 8, we choose $U = \{u_1, u_2, \cdots, u_\kappa\}$ and $\mathcal{S}_P = \{P\}$ in Section 2.

Claim 1. $N(u_\kappa^+) \cap V(\mathcal{S}_C) = \emptyset$ *and* $N(u_1^-) \cap V(\mathcal{S}_C) = \emptyset$.

Proof of Claim 1. Using symmetry, we can focus on proving that $N(u_\kappa^+) \cap V(\mathcal{S}_C) = \emptyset$. By contradiction, suppose that $N(u_\kappa^+) \cap V(\mathcal{S}_C) \neq \emptyset$; say $v \in N(u_\kappa^+) \cap V(\mathcal{S}_C)$ and $v \in V(C')$. By Lemma 2(ii), $u_\kappa^+ \neq v_R(P)$.

Denote

$$A = \begin{cases} V(P[u_\kappa^{3+}, v_R(P)]), & \text{if } u_\kappa^+ v_R(P) \notin E(G), \\ V(P[u_\kappa^{2+}, v_R(P)]), & \text{if } u_\kappa^+ v_R(P) \in E(G). \end{cases}$$

If $u_\kappa^+ v_R(P) \notin E(G)$, then, by Lemma 9(4), $u_\kappa^{2+} v_R(P) \in E(G)$. Therefore, by Lemma 9(5), $G[V(P[u_\kappa^{3+}, v_R(P)])]$ forms a clique. If $u_\kappa^+ v_R(P) \in E(G)$, then, according to Lemma 9(5), $G[V(P[u_\kappa^{2+}, v_R(P)])]$ forms a clique. Hence,

$$G[A] \text{ forms a clique.} \tag{2}$$

As G is a connected graph, $N(A) \cap V(G-A) \neq \emptyset$. For $y \in N(A) \cap V(G-A)$, there exists a vertex $x \in A$ with $xy \in E(G)$. We will show that

$$y \in \{u_\kappa^+, u_\kappa^{2+}\}. \tag{3}$$

By contradiction, suppose that $y \notin \{u_\kappa^+, u_\kappa^{2+}\}$. By Lemma 6(2) and (2), $y \notin V(\mathcal{S}_C) \cup V(H)$. We will examine the following two scenarios to reach a contradiction:

- Suppose that $y \in V(P[v_L(P), u_1^-])$. By Lemma 9(1), $f(\overleftarrow{Q}(u_1^-, P)) = 1$ or $N(u_1^-) \cap V(\mathcal{S}_C) \neq \emptyset$. We will show that $u_1^{2-} v_L(P) \in E(G)$. If $f(\overleftarrow{Q}(u_1^-, P)) = 1$, then, by symmetry and Lemma 9(5), $u_1^{2-} v_L(P) \in E(G)$. If $f(\overleftarrow{Q}(u_1^-, P)) \neq 1$, then, by Lemma 9(1), $N(u_1^-) \cap V(\mathcal{S}_C) \neq \emptyset$. By symmetry and Lemma 9(4), $u_1^{2-} v_L(P) \in E(G)$. Then, by symmetry and Lemma 9(5) and (2), either the set of vertices of the subgraph $C' u_\kappa^+ \overrightarrow{P} x^- v_R(P) \overleftarrow{P} xy \overrightarrow{P} u_\kappa L_\kappa w$ and $\overleftarrow{Q}(y^-, P)$ is equal to $V(P \cup C') \cup \{w\}$, which contradicts (I); or $C' u_\kappa^+ \overrightarrow{P} x^- v_R(P) \overleftarrow{P} x v_L(P) \overrightarrow{P} u_\kappa$ in G is equal to $V(P \cup C')$, which contradicts Lemma 4.
- Assume that $y \in V(P[u_1, u_\kappa])$. Then, by Lemma 9(1)(2) and (2), the set of vertices of the subgraph

$$Q_1 = \begin{cases} C' u_\kappa^+ \overrightarrow{P} x^- v_R(P) \overleftarrow{P} xy \overrightarrow{P} u_\kappa L_\kappa w L_1 u_1 \overrightarrow{P} y^-, & \text{if } y \in V(P(u_1, u_\kappa)), \\ C' u_\kappa^+ \overrightarrow{P} x^- v_R(P) \overleftarrow{P} x u_1 \overrightarrow{P} u_\kappa L_\kappa w, & \text{if } y = u_1, \\ C' u_\kappa^+ \overrightarrow{P} x^- v_R(P) \overleftarrow{P} x u_\kappa \overleftarrow{P} u_1 L_1 w, & \text{if } y = u_\kappa, \end{cases}$$

and

$$Q_2 = \begin{cases} C u_{Q,1}^- \overleftarrow{Q} v_L(Q), & \text{if } f(\overleftarrow{Q}(u_{Q,1}^-, Q)) \neq 1, \\ \overleftarrow{Q}(u_{Q,1}^-, Q), & \text{if } f(\overleftarrow{Q}(u_{Q,1}^-, Q)) = 1. \end{cases}$$

is equal to $V(P \cup C \cup C') \cup \{w\}$ or $V(P \cup C') \cup \{w\}$, which contradicts (I).

This contradiction shows that (3) holds.

If $y = u_\kappa^+$ and $u_\kappa^+ v_R(P) \notin E(G)$, then, $u_\kappa^{2+} v_R(P) \in E(G)$. By (2), $G[V(P[u_\kappa^+, v_R(P)])]$ has a cycle $C_\kappa = v_R(P) \overleftarrow{P} x u_\kappa^+ \overrightarrow{P} x^- v_R(P)$. By (2), we structure a new path P' such that $P' = v_L(P) \overrightarrow{P} u_\kappa^+ x \overrightarrow{P} v_R(P) x^- \overleftarrow{P} u_\kappa^{2+}$ by rearranging the order of the vertices in P. Then $v_R(P') = u_\kappa^{2+}$. It is easy to verify that $G[V(P[u_\kappa^+, v_R(P)])] \cong G[V(P'[u_\kappa^+, v_R(P')])]$. Note that $u_\kappa^+ v_R(P') \in E(G)$. By Lemma 9(5), the subgraph $G[V(P'[x, v_R(P')])]$ forms a clique. Let $A' = V(P'[u_\kappa^+, v_R(P')]) \setminus \{u_\kappa^+\} = V(P'[x, v_R(P')])$. Since G is connected, $N(A') \cap V(G - A') \neq \emptyset$. For $y' \in N(A') \cap V(G - A')$, there exists a vertex $x' \in A$ with $x'y' \in E(G)$. By the proof of (3), $y' = u_\kappa^+$. That means $|N(A') \cap V(G - A')| = 1$, contradicting $|N(A') \cap V(G - A')| \geq \kappa \geq 2$. Therefore, by (3), we have either $y = u_\kappa^{2+}$ and $u_\kappa^+ v_R(P) \notin E(G)$ or $y = u_\kappa^+$ and $u_\kappa^+ v_R(P) \in E(G)$. Then, $|N(A) \cap V(G - A)| = 1$, contradicting $|N(A) \cap V(G - A)| \geq \kappa \geq 2$. This contradiction indicates that Claim 1 is true. □

According to Claim 1 and Lemma 9(1), $f(\overrightarrow{\mathcal{Q}}(u_\kappa^+, P)) = 1$ and $f(\overleftarrow{\mathcal{Q}}(u_1^-, P)) = 1$. Denote

$$\mathcal{X} = \begin{cases} End(\mathcal{S}) \cup U^+ \cup \{w\}, & \text{if} \quad End(\mathcal{S}) \cap U^+ \neq \emptyset, \\ (End(\mathcal{S}) \cup U^+ \cup \{w\}) \setminus \{u_\kappa^+\}, & otherwise (i.e., if\ End(\mathcal{S}) \cap U^+ = \emptyset). \end{cases}$$

By Lemma 9(3), $\mathcal{X}$ is an independent set of G with size $\kappa + k$. Thus,

$$N(v) \cap \mathcal{X} \neq \emptyset \ \ for\ \ any\ \ v \in V(G) \setminus \mathcal{X}. \tag{4}$$

Claim 2. *$G[V(P[u_\kappa^+, v_R(P)])]$ and $G[V(P[v_L(P), u_1^-])]$ form cliques.*

Proof of Claim 2. By virtue of symmetry, we may restrict our consideration to prove that $G[V(P[u_\kappa^+, v_R(P)])]$ forms a clique. As $G[V(P[u_\kappa^+, v_R(P)])]$ is connected, we can assume that $|V(P[u_\kappa^+, v_R(P)])| \geq 3$. According to Lemma 5(1) (4) and Claim 1, $N(u_\kappa^+) \cap (\mathcal{X} \setminus \{v_R(P)\}) = \emptyset$. By (4), $N(u_\kappa^+) \cap \mathcal{X} = \{v_R(P)\}$. By Lemma 9(5), $G[V(P[u_\kappa^+, v_R(P)])]$ forms a clique. □

Claim 3. $N(V(H)) \cap V(\mathcal{S}_C) = \emptyset$.

Proof of Claim 3. By contradiction, suppose that $N(V(H)) \cap V(\mathcal{S}_C) \neq \emptyset$; say $x \in N(V(H)) \cap V(C)$ for some $C \in \mathcal{S}_C$. This implies that there is a vertex $v \in V(H)$ with $e = vx$. By Lemma 8, $v \neq w$.

We will show that

$$v \notin V(\mathcal{L}). \tag{5}$$

Suppose, by way of contradiction, that $v \in V(L_{i_0})$ for some $i_0 \in \{1, \cdots, \kappa\}$. Suppose that $v \in V(L_1) \cup V(L_\kappa)$. By symmetry, we may only think of $v \in V(L_1)$. Then, by Claim 2, $v_R(P) \overleftarrow{P} u_1 L_1 v C$ and $\overleftarrow{\mathcal{Q}}(u_1^-, P)$ cover $V(P) \cup V(C) \cup \{v\}$, contradicting (I). Therefore, $v \in V(L_{i_0})$ for some $i_0 \in \{2, \cdots, \kappa - 1\}$. Then, by Claim 2, the set of vertices of the subgraph $v_L(P) \overrightarrow{P} u_\kappa L_\kappa w L_{i_0} v C$ and $\overrightarrow{\mathcal{Q}}(u_{i_0}^+, P)$ in G is equal to $V(P) \cup V(C) \cup \{w, v\}$, which again contradicts (I). Thus, we have shown that (5) holds.

Next, we will prove that

$$vw \in E(G). \tag{6}$$

By contradiction, suppose that $vw \notin E(G)$. Note that $x \in V(C)$. We consider the neighbourhood of the vertex x^+. According to Lemma 2$(i)(ii)$, $N(x^+) \cap (End(\mathcal{S}) \setminus \{v_C\}) = \emptyset$. If x^+ and $u_i^+ \in V(P)$ for some $i \in \{1, \cdots, \kappa - 1\}$ are adjacent in G, then, by Claim 2 and (5), $v_L(P) \overrightarrow{P} u_i L_i w L_\kappa u_\kappa \overleftarrow{P} u_i^+ C v$ and $\overrightarrow{\mathcal{Q}}(u_\kappa^+, P)$ in G covers $V(P) \cup V(C) \cup \{w, v\}$, contradicting (I). Hence, $N(x^+) \cap (U^+ \setminus \{u_\kappa^+\}) = \emptyset$. If v and $u_i^+ \in V(P)$ for some $i \in \{1, \cdots, \kappa - 1\}$ are adjacent in G, then, by Claim 2 and (5), $v_L(P) \overrightarrow{P} u_i L_i w L_\kappa u_\kappa \overleftarrow{P} u_i^+ v C$ and $\overrightarrow{\mathcal{Q}}(u_\kappa^+, P)$ in G covers $V(P) \cup V(C) \cup \{w, v\}$, contradicting (I). Hence, $N(v) \cap (U^+ \setminus \{u_\kappa^+\}) = \emptyset$. Note that $vw \notin E(G)$. Therefore, by Lemma 2$(i)(ii)$, $\{v_L(P), v_R(P), x^+, w, v\} \cup (U^+ \setminus \{u_\kappa^+\}) \cup (End(\mathcal{S}_C) \setminus \{v_C\})$ forms an independent set of size $\kappa + k + 1$. This contradicts the fact that $\alpha(G) = \kappa + k$ and thus establishes that (6) holds.

Then, by Claim 2, (5) and (6), the set of vertices of the subgraph $v_L(P)\overrightarrow{P}u_\kappa L_\kappa wvC$ and $\overrightarrow{\mathcal{Q}}(u_\kappa^+, P)$ is equal to $V(P \cup C) \cup \{w, v\}$. This contradicts (I) and establishes that $N(V(H)) \cap V(C) = \emptyset$. □

Claim 4. *$N(V(C)) \cap (V(G) \setminus V(C)) = U$ for each element $C \in \mathcal{S}_C$.*

Proof of Claim 4. Since G is connected, $N(V(C)) \cap (V(G) \setminus V(C)) \neq \emptyset$ for any $C \in \mathcal{S}_C$. For $z \in N(V(C)) \cap (V(G) \setminus V(C))$, there exists a vertex $x \in V(C)$ with $xz \in E(G)$. According to Lemma 2 $(i), (ii)$, $z \notin V(\mathcal{S}_C \setminus \{C\})$. By Claim 3, $z \notin V(H)$. This implies that

$$z \in V(P). \tag{7}$$

Next, we will show that $z \in U$. By contradiction, suppose that $z \notin U$. By Lemma 6(2) and Claim 2, z does not belong to $V(P(u_\kappa^+, v_R(P)]) \cup V(P[v_L(P), u_1^-))$. To arrive at a contradiction, we will examine the following three scenarios using (7):

- Suppose that $z \in \{u_\kappa^+, u_1^-\}$. Then $N(u_\kappa^+) \cap V(\mathcal{S}_C) \neq \emptyset$ or $N(u_1^-) \cap V(\mathcal{S}_C) \neq \emptyset$, contradicting Claim 1.
- Suppose that $z \in U^+ \setminus \{u_\kappa^+\}$ or $U^- \setminus \{u_1^-\}$. By symmetry, we consider that $z \in U^+ \setminus \{u_\kappa^+\}$ say $z = u_i^+$ for some $i \in \{1, \cdots, \kappa - 1\}$. Then, by Claim 2, $Cu_i^+\overrightarrow{P}u_\kappa L_\kappa wL_i u_i\overleftarrow{P}v_L(P)$ and $\overrightarrow{\mathcal{Q}}(u_\kappa^+, P)$ cover $V(P \cup C) \cup \{w\}$, contradicting (I).
- Suppose that $z \in V(P[u_i^{2+}, u_{i+1}^{2-}])$ for some $i \in \{1, \cdots, \kappa - 1\}$. We consider the neighbourhood of the vertex z^+. We claim that $N(z^+) \cap \mathcal{X} = \{v_C\}$. Suppose otherwise that there exists a vertex $y \in N(z^+) \cap \mathcal{X}$ such that $y \neq v_C$. By Lemma 6(1), $y \notin End(\mathcal{S}_C) \setminus \{v_C\}$. According to the definition of $\mathcal{X}$, we will examine the following two scenarios to reach a contradiction.
  - Assume that $y \in U^+ \cap \mathcal{X}$; say $y = u_j^+$ for some $j \in \{1, \cdots, \kappa - 1\}$. If $j > i$, then, by Claim 2, the set of vertices of the subgraph $v_R(P)\overleftarrow{P}u_j^+z^+\overrightarrow{P}u_jL_jwL_1u_1\overrightarrow{P}zC$ and $\overleftarrow{\mathcal{Q}}(u_1^-, P)$ is equal to $V(P \cup C) \cup \{w\}$, which contradicts (I). If $j \leq i$, then, by Claim 2, the set of vertices of the subgraph $v_L(P)\overrightarrow{P}u_jL_jwu_\kappa\overleftarrow{P}z^+u_j^+\overrightarrow{P}zC \cup \overrightarrow{\mathcal{Q}}(u_\kappa^+, P)$ is equal to $V(P \cup C) \cup \{w\}$, which again contradicts (I).
  - Assume that $y \in End(\mathcal{S}_P)$. By Lemma 6(2), $y = v_R(P)$. Then, by Claim 2, the set of vertices of the subgraph $u_\kappa^-\overleftarrow{P}z^+v_R(P)\overleftarrow{P}u_\kappa L_\kappa wL_1u_1\overrightarrow{P}zC \cup \overleftarrow{\mathcal{Q}}(u_1^-, P)$ is equal to $V(P \cup C) \cup \{w\}$, which contradicts (I).

  This contradiction establishes that $N(z^+) \cap \mathcal{X} \subseteq \{v_C\}$. By (4), $N(z^+) \cap \mathcal{X} = \{v_C\}$. If $x \neq v_C$ and $|V(C)| > 1$ or $|V(C)| = 1$, then, by Lemma 9(6), the set of vertices of the subgraph $v_L(P)\overrightarrow{P}zCz^+\overrightarrow{P}v_R(P)$ is equal to $V(P \cup C)$, which contradicts Lemma 4. If $x = v_C$ and $|V(C)| > 1$, then, according to Lemma 9(6), $N(v_C^+) \cap \mathcal{X} = \{v_C\}$. Note that $N(z^+) \cap \mathcal{X} = \{v_C\}$. If $z^+v_C^+ \notin E(G)$, then, by Lemma 9(6), $(\mathcal{X} \setminus \{v_C\}) \cup \{z^+, v_C^+\}$ would be an independent set of cardinality $\kappa + k + 1$, contradicting (1). Therefore, $z^+v_C^+ \in E(G)$. Then the set of vertices of the subgraph $v_L(P)\overrightarrow{P}zCz^+\overrightarrow{P}v_R(P)$ is equal to $V(P \cup C)$, which contradicts Lemma 4.

This contradiction establishes that $z \in U$. Since $|N(V(C)) \cap (V(G) \setminus V(C))| \geq \kappa$ and $|U| = \kappa$, $N(V(C)) \cap (V(G) \setminus V(C)) = U$ for any element $C \in \mathcal{S}_C$. □

Claim 5. *Let $C \in S_C$ with $|V(C)| > 1$. For any two disjoint vertices u_i, $u_j \in U$, there exist two disjoint vertices v, $v' \in V(C)$ such that u_iv, $u_jv' \in E(G)$.*

Proof of Claim 5. We establish Claim 5 by contradiction. Suppose that either $N(u_{i_0}) \cap V(C) = N(u_{j_0}) \cap V(C) = \emptyset$ or $N(u_{i_0}) \cap V(C) = N(u_{j_0}) \cap V(C) = \{v\}$ and $v \in V(C)$ for some $u_{i_0}, u_{j_0} \in U$.

If $N(u_{i_0}) \cap V(C) = N(u_{j_0}) \cap V(C) = \emptyset$, then $N(V(C)) \cap (V(G) \setminus V(C)) \neq U$, contradicting Claim 4. Now suppose that $N(u_{i_0}) \cap V(C) = N(u_{j_0}) \cap V(C) = \{v\}$. Let $\hat{U} = (U \setminus \{u_{i_0}, u_{j_0}\}) \cup \{v\}$ and $\hat{C} = C - v$. Since $|V(C)| > 1$, $\hat{C} \neq \emptyset$. Then, by hypothesis and Claim 4, $N(V(\hat{C})) \cap (V(G) \setminus V(\hat{C})) \subseteq \hat{U}$. However, $|\hat{U}| = \kappa - 1$, contradicting the hypothesis that G is κ-connected. These contradictions establish that Claim 5 is true. □

Claim 6. *$G[V(H)]$ forms a clique.*

Proof of Claim 6. We will only focus on the case where $|V(H)| \geq 2$. For every vertex $v \in V(H) \setminus \{w\}$, $N(v) \cap \mathcal{X} \neq \emptyset$. We assume that there is at least one vertex $x \in N(v) \cap \mathcal{X}$ with $x \neq w$. By Claim 3, x is not an element of $End(\mathcal{S}_C)$. By Lemma 3(1), $x \notin End(\mathcal{S}_P)$. Then, $x \in \mathcal{X} \cap U^+$; say $x = u_i^+$ for some $i \in \{1, \cdots, \kappa - 1\}$. Then, by Claims 2, 4 and 5, there exist a path Q and $\overrightarrow{Q}(u_\kappa^+, P)$ cover $V(P) \cup V(C) \cup \{v\}$, see Figure 2, contradicting (I). This contradiction shows that $N(v) \cap \mathcal{X} \subseteq \{w\}$. By (4), $N(v) \cap \mathcal{X} = \{w\}$ for every vertex $v \in V(H) \setminus \{w\}$. Let $S' = V(H)$. Then, according to Lemma 7, $G[V(H)]$ forms a clique. □

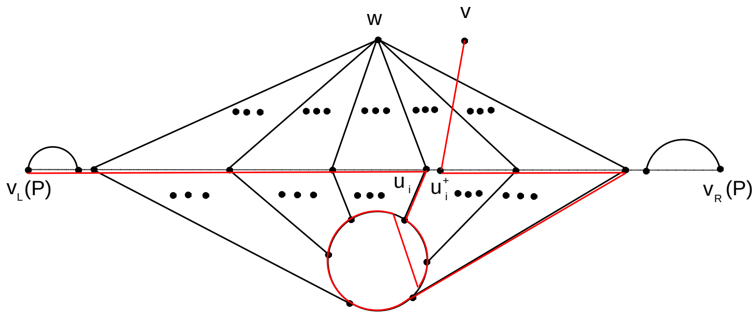

Figure 2. $vu_i^+ \in E(G)$ $(i \neq \kappa)$.

Denote $A_1 = V(P[v_L(P), u_1^-])$ and $A_2 = V(P[u_\kappa^+, v_R(P)])$.

Claim 7. *The following two statements are true.*

(1) $N(A_i) \cap (G - A_i) = U$ *for* $i \in \{1, 2\}$;
(2) $N(V(H)) \cap V(\mathcal{S}) = U$.

Proof of Claim 7. We will prove the first statement. By symmetry, we have only proved that $N(A_2) \cap (G - A_2) = U$. Let $C^* = G[A_2]$. By Claim 2(1), $f(C^*) = 1$. We pick an element $C \in \mathcal{S}_C$, by Claim 4, a new path $Q = v_L(P) \overrightarrow{P} u_\kappa C$ would be obtained. We structure a new system $\mathcal{S}^*$ such that $\mathcal{S}^* = \mathcal{S}_C^* \cup \mathcal{S}_P^*$, $\mathcal{S}_P^* = \{Q\}$ and $\mathcal{S}_C^* = (\mathcal{S}_C \setminus \{C\}) \cup \{C^*\}$. It is easy to verify that $V(\mathcal{S}^*) = V(\mathcal{S})$, $|\mathcal{S}_C^*| = |\mathcal{S}_C|$ and $|\mathcal{S}_P^*| = |\mathcal{S}_P|$. Hence, $\mathcal{S}^*$ is also a k-ended system satisfying (I), (II). Then, by Claim 4, $N(A_2) \cap (G - A_2) = U$.

Next, we need to prove the second statement. The proof here is similar to Claim 4. (For details, see Appendix A.) □

Claim 8. *Suppose that $|V(H)| > 1$. For any two disjoint vertices u_i, $u_j \in U$, there exist two disjoint vertices $v, v' \in V(H)$ such that $u_i v, u_j v' \in E(G)$.*

Proof of Claim 8. The proof here is similar to Claim 5. (For details, see Appendix A.) □

Claim 9. $u_{i+1}^- u_i^+ \in E(G)$ *for each* $i \in \{1, \cdots, \kappa - 1\}$.

Proof of Claim 9. Since $G[V(P[u_i^+, u_{i+1}^-])]$ is connected, we only need to focus on the case where $|V(P[u_i^+, u_{i+1}^-])| \geq 3$. By contradiction, suppose that $u_{i_0+1}^- u_{i_0}^+ \notin E(G)$ for some

$i_0 \in \{1, \cdots, \kappa-1\}$. By (4), there exists at least one vertex $y \in N(u^-_{i_0+1}) \cap \mathcal{X}$ satisfying $y \neq u^+_{i_0}$. By Claim 4, $y \notin End(\mathcal{S}_C)$. We will examine the following two scenarios to reach a contradiction, based on the definition of $\mathcal{X}$:

- Assume that $y \in \mathcal{X} \cap (U^+ \setminus \{u^+_{i_0}\})$; say $y = u^+_j$ for some $j \in \{1, \cdots, \kappa-1\} \setminus \{i_0\}$. If $j > i_0$, then, by Claim 4, the set of vertices of the subgraph $v_L(P) \overrightarrow{P} u^-_{i_0+1} u^+_j \overrightarrow{P} u_\kappa L_\kappa w$ $L_{i_0+1} u_{i_0+1} \overrightarrow{P} u_j C \cup \overrightarrow{Q}(u^+_\kappa, P)$ is equal to $V(P \cup C) \cup \{w\}$, which contradicts (I). If $j < i_0$, then, by Claims 4 and 5, the set of vertices of the subgraph $v_L(P) \overrightarrow{P} u_j L_j w L_\kappa u_\kappa \overleftarrow{P} u_{i_0+1} C$ $u_{i_0} \overrightarrow{P} u^-_{i_0+1} u^+_j \overrightarrow{P} u^-_{i_0} \cup \overrightarrow{Q}(u^+_\kappa, P)$ is equal to $V(P \cup C) \cup \{w\}$, which again contradicts (I).
- Assume that $y \in End(\mathcal{S}_P)$. If $y = v_L(P)$, then $v_R(P) \overleftarrow{P} u_{i_0+1} L_{i_0+1} w L_{i_0} u_{i_0} \overleftarrow{P} v_L(P) u^-_{i_0+1}$ $\overleftarrow{P} u^+_{i_0}$ covers $V(P) \cup \{w\}$, which contradicts (I). Therefore, $y = v_R(P)$. Then $v_L(P)$ $\overrightarrow{P} u^-_{i_0+1} v_R(P) \overleftarrow{P} u_{i_0+1}$ $L_{i_0+1} w$ covers $V(P) \cup \{w\}$, which again contradicts (I).

This contradiction demonstrates that $N(u^-_{i+1}) \cap \mathcal{X} \subseteq \{u^+_i\}$. By (4), $N(u^-_{i+1}) \cap \mathcal{X} = \{u^+_i\}$. $\square$

By Claim 9, it holds that $u^-_{i+1} u^+_i \in E(G)$ for every $i \in \{1, \cdots, \kappa-1\}$. Let $C_i = G[E(P[u^+_i, u^-_{i+1}] \cup \{u^+_i u^-_{i+1}\})]$ for every $i \in \{1, \cdots, \kappa-1\}$.

Claim 10. *For each section $P[u^+_i, u^-_{i+1}]$, the following two statements are true.*

- *$G[V(P[u^+_i, u^-_{i+1}])]$ forms a clique;*
- *$N(V(P[u^+_i, u^-_{i+1}])) \cap (V(G) \setminus V(P[u^+_i, u^-_{i+1}])) = U$.*

Proof of Claim 10. We pick an element $C \in \mathcal{S}_C$; by Claims 4 and 5, a new path $Q_i = v_L(P) \overrightarrow{P} u_i C u_{i+1} \overrightarrow{P} v_R(P)$ would be obtained. We structure a new system $\mathcal{S}_i$ such that $\mathcal{S}_i = \mathcal{S}_{iC} \cup \mathcal{S}_{iP}$, $\mathcal{S}_{iP} = \{Q_i\}$ and $\mathcal{S}_{iC} = (\mathcal{S}_C \setminus \{C\}) \cup \{C_i\}$. It is easy to verify that $V(\mathcal{S}_i) = V(\mathcal{S})$, $|\mathcal{S}_{iC}| = |\mathcal{S}_C|$ and $|\mathcal{S}_{iP}| = |\mathcal{S}_P|$. Hence, $\mathcal{S}_i$ is also a k-ended system satisfying (I), (II). According to Lemma 9(6), $G[V(C_i)]$ forms a clique; by Claim 4, $N(V(C_i)) \cap (V(G) \setminus V(C_i)) = U$. Claim 10 is proved. $\square$

By Claims 2–10 and Lemma 9(6), $\omega(G-U) = k+\kappa$ and every component of $G-U$ forms a clique. Then,

$$m(G) = |V(H)| \cdot |V(P[v_L(P), u^-_1])| \cdot |V(P[u^+_\kappa, v_R(P)])| \cdot \prod_{i=1}^{\kappa-1} |V(P[u^+_i, u^-_{i+1}])| \cdot \prod_{C \in \mathcal{S}_C} |V(C)| \geq \underbrace{1 \cdot 1 \cdot \cdots \cdot 1 \cdot 1}_{\kappa+k} \cdot [n-2\kappa-k+1] = n-2\kappa-k+1.$$

This completes the proof of Fact 1. $\square$

Fact 2. $|V(H)| = 1$ and $m(G) = n-2\kappa-k+1$.

Proof of Fact 2. By Fact 1 and the condition of Theorem 5, $m(G) = n-2\kappa-k+1$. Then, we will show that $|V(H)| = 1$.

By contradiction, suppose that $|V(H)| \geq 2$. Then $|V(P[u^+_i, u^-_{i+1}])| = 1$ for any $i \in \{1, \cdots, \kappa-1\}$. Otherwise, $m(G) > n-2\kappa-k+1$, contradicting $m(G) = n-2\kappa-k+1$. Let $x = V(P[u^+_{i_0}, u^-_{i_0+1}])$ for some $i_0 \in \{1, \cdots, \kappa-1\}$. By Claims 7(2) and 8, $v_L(P) \overrightarrow{P} u_{i_0} H u_{i_0+1} \overrightarrow{P} v_R(P)$ in G cover $(V(P) \setminus \{x\}) \cup V(H)$, which contradicts (I). This contradiction shows that $|V(H)| = 1$. $\square$

Denote

$$J(r) = \begin{cases} G[V(P[u^+_i, u^-_{i+1}])], & \text{if} \quad r \in U^+ \setminus \{u^+_\kappa\}, \text{ say } r = u^+_i, \\ G[V(P[v_L(P), u^-_1])], & \text{if} \quad r = v_L(P), \\ G[V(P[u^+_\kappa, v_R(P)])], & \text{if} \quad r = v_R(P), \\ G[V(C)]. & \text{if} \quad r = v_C. \end{cases}$$

Finally, we need to prove that G is isomorphic to one of those graphs F with $F_0(\kappa+k-1,\kappa) \subseteq F \subseteq F_2(\kappa+k-1,\kappa)$ or $F_{00}(\kappa+k-1,\kappa) \subseteq F \subseteq F_2(\kappa+k-1,\kappa)$. Denote $R = End(\mathcal{S}) \cup (U^+ \setminus \{u_\kappa^+\})$. By Fact 2, $|V(\mathcal{S})| = n-1$ and there exists at most one vertex $r_0 \in R$ such that $|J(r_0)| \geq 2$. Then $|J(r_0)| = n - 2\kappa - k + 1 = m(G)$. Let $W_1(G) = \{w\} \cup (R \setminus \{r_0\})$. It follows that $W_1(G)$ is an independent set of G with a cardinality of $k+\kappa-1$. Additionally, $W_1(G) \cup \{x\}$ is a maximum independent set of G for any vertex $x \in J(r_0)$. By Claims 4, 7 and 10, $y \in R \setminus \{r_0\}$ is not adjacent to any vertex in $J(r_0) \cup \{w\}$; it should be adjacent to u_i for all $i \in \{1, \cdots, \kappa\}$. Now let $H_1 = G[W_1(G)]$ and $H_2 = G[U]$. This implies that $F_0(\kappa+k-1,\kappa) \subseteq G \subseteq F_2(\kappa+k-1,\kappa)$ and $n > 3\kappa+k-1$ or $F_{00}(\kappa+k-1,\kappa) \subseteq G \subseteq F_2(\kappa+k-1,\kappa)$ and $2\kappa+k \leq n \leq 3\kappa+k-1$ (note that $J(r_0) \cong K_{n-2\kappa-k+1}$), which completes the proof of Theorem 7. □

5. Conclusions

We demonstrats that it does not change the existence of spanning k-ended tree if we expand the independent numbers a little bit and bound $m(G)$. Therefore, we generalize Theorem 5 and the bound on the number of maximum independent sets is already sharp. Note that a Hamilton path is viewed as a spanning tree with exactly two leaves; in other words, a Hamilton path is a spanning 2-ended tree. Hence, our results extend Theorem 4, which has significant implications for traceability and the existence of spanning trees. Moreover, we extend Theorem 5 to the case where $\kappa(G) \geq 2$. This extension has important implications for the study of independent sets in highly connected graphs.

The proof of the results is currently too complex and difficult. We hope to find a more clever and concise proof technique for Theorem 7 in the future.

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Appendix A. Some Proofs of Claims of Theorem 7

Proof of Claim 7(2). Since G is connected, $N(V(H)) \cap V(\mathcal{S}) \neq \emptyset$. For $z \in N(V(H)) \cap (V(G) \setminus V(H))$, there exists a vertex $v \in V(H)$ with $vz \in E(G)$. By Claim 3, $z \notin V(\mathcal{S}_C)$. This implies that $z \in V(P)$. We will prove that $z \in U$. Suppose, by way of contradiction, that $z \notin U$. We will examine the following three scenarios to reach a contradiction.

- Assume that $z \in V(P(u_\kappa^+, v_R(P)])$ or $V(P[v_L(P), u_1^-))$. By symmetry, it would therefore suffice to consider that $z \in V(P(u_\kappa^+, v_R(P)])$. (By) Claim 2(1), the set of vertices of the subgraph $v_L(P)\overrightarrow{P}z^- v_R(P)\overleftarrow{P}zv$ is equal to $V(P) \cup \{v\}$, which contradicts (I).
- Assume that $z \in U^+$ or U^-. By symmetry, it would therefore suffice to think about $z \in U^+$; say $z = u_i^+$ for some $i \in \{1, \cdots, \kappa\}$. If $v = w$, then $v_L(P)\overrightarrow{P}u_i L_i w u_i^+ \overrightarrow{P} v_R(P)$ covers $V(P) \cup \{w\}$, contradicting (I). If $v \neq w$, then, by Claim 6, the set of vertices of the subgraph $v_L(P)\overrightarrow{P}u_i H u_i^+ \overrightarrow{P} v_R(P)$ is equal to $V(P \cup H)$, which contradicts (I).

- Assume z belongs to $V(P[u_i^{2+}, u_{i+1}^{2-}])$ for some $i \in \{1, \cdots, \kappa - 1\}$. We consider the neighbourhood of the vertex z^+. By (4), $N(z^+) \cap \mathcal{X} \neq \emptyset$; say $y \in N(z^+) \cap \mathcal{X}$. By Claim 4, $y \notin End(\mathcal{S}_C)$. We will consider the following two cases to obtain a contradiction.
  - Assume that $y \in \mathcal{X} \cap U^+$; say $y = u_j^+$ for some $j \in \{1, \cdots, \kappa - 1\}$. Suppose, first, that $j > i$. If $v \neq w$, then, by Claim 6, the set of vertices of the subgraph $v_L(P) \overrightarrow{P} zHu_j \overleftarrow{P} z^+ u_j^+ \overrightarrow{P} v_R(P)$ is equal to $V(P \cup H)$, which contradicts (I). If $v = w$, then the set of vertices of the subgraph $v_L(P) \overrightarrow{P} zwL_j u_j \overleftarrow{P} z^+ u_j^+ \overrightarrow{P} v_R(P)$ is equal to $V(P) \cup \{w\}$, which also contradicts (I). Suppose, now, that $j \leq i$. If $v \neq w$, then, by Claim 6, $v_L(P) \overrightarrow{P} u_j Hz \overleftarrow{P} u_j^+ z^+ \overrightarrow{P} v_R(P)$ covers $V(P) \cup V(H)$, which contradicts (I). If $v = w$, then $v_L(P) \overrightarrow{P} u_j wz \overleftarrow{P} u_j^+ z^+ \overrightarrow{P} v_R(P)$ covers $V(P) \cup \{w\}$, which also contradicts (I).
  - Assume that $y \in End(\mathcal{S}_P)$. Let us take $y = v_R(P)$ without loss of generality. If $v \neq w$, then, by Claim 6, the set of vertices of the subgraph $v_L(P) \overrightarrow{P} zHu_\kappa \overleftarrow{P} z^+ v_R(P) \overleftarrow{P} u_\kappa^+$ is equal to $V(P \cup H)$, which contradicts (I). If $v = w$, then $v_L(P) \overrightarrow{P} zwL_\kappa u_\kappa \overleftarrow{P} z^+ v_R(P) \overleftarrow{P} u_\kappa^+$ covers $V(P) \cup \{w\}$, which also contradicts (I).

  This contradiction shows that $N(z^+) \cap \mathcal{X} = \emptyset$, contradicting (4).

This contradiction shows $z \in U$. Since $|N(V(H)) \cap (V(G) \setminus V(H))| \geq \kappa$ and $|U| = \kappa$, $N(V(H)) \cap (V(G) \setminus V(H)) = U$. □

Proof of Claim 8. By contradiction, suppose that either $N(u_{i_0}) \cap V(H) = N(u_{j_0}) \cap V(H) = \emptyset$ or $N(u_{i_0}) \cap V(H) = N(u_{j_0}) \cap V(H) = \{v\}$ and $v \in V(H)$ for some $u_{i_0}, u_{j_0} \in U$.

Suppose first that $N(u_{i_0}) \cap V(H) = N(u_{j_0}) \cap V(H) = \emptyset$. Then $N(V(H)) \cap (V(G) \setminus V(H)) \neq U$, contradicting Claim 7(2). Suppose now that $N(u_{i_0}) \cap V(H) = N(u_{j_0}) \cap V(H) = \{v\}$. Let $\hat{U} = (U \setminus \{u_{i_0}, u_{j_0}\}) \cup \{v\}$ and $\hat{H} = H - v$. Since $|V(H)| > 1$, $\hat{H} \neq \emptyset$. Then, by hypothesis and Claim 7(2), $N(V(\hat{H})) \cap (V(G) \setminus V(\hat{H})) \subseteq \hat{U}$. However, $|\hat{U}| = \kappa - 1$, contradicting the hypothesis that G is κ-connected. These contradictions show that Claim 8 holds. □

Appendix B. Proof of Lemmas 8 and 9

In this section, we employ the same terminology and notation in Section 2.

In order to prove Lemmas 8 and 9, we first do some preparatory work.

Denote $X^+ = End(\mathcal{S}) \cup U^+ \cup \{w\}$ and $X^- = End(\mathcal{S}) \cup U^- \cup \{w\}$. If $U \cap V(\mathcal{S}_C) \neq \emptyset$, then, by Lemma 5(3), $|\{C : C \in \mathcal{S}_C \text{ and } U \cap V(C) \neq \emptyset\}| = 1$; say $C_U \in \mathcal{S}_C$.

Lemma A1. *(Akiyama and Kano, [18]) The following statements are true.*

(1) *If $U \cap V(\mathcal{S}_C) \neq \emptyset$, then $X^+ \setminus \{v_{C_U}\}$ forms an independent set of G with a size of $k + \kappa$.*

(2) *If $U \subseteq V(\mathcal{S}_P)$ and $End(\mathcal{S}) \cap U^+ \neq \emptyset$, then $End(\mathcal{S}) \cap U^+ = \{v_R(P)\}$ for some $P \in \mathcal{S}_P$ and X^+ forms an independent set of G with a size of $\kappa + k$.*

(3) *If $U \subseteq V(\mathcal{S}_P)$ and $End(\mathcal{S}) \cap U^+ = \emptyset$, then:*
(i) The set X^+ does not include four distinct vertices x_1, x_2, x_3, x_4 with $\{x_1x_2, x_3x_4\} \subseteq E(G)$;
(ii) $G[X^+]$ is triangle-free;
(iii) U^+ is an independent set of G.

Lemma A2. *Suppose that $U \subseteq V(\mathcal{S}_P)$. The following statements are true.*

(1) *If $End(\mathcal{S}) \cap U^+ = \emptyset$, then $G[X^+]$ has exactly one nontrivial component denoted by $S(X^+)$ such that $S(X^+)$ is a star with $S(X^+) = (V(S(X^+)) \cap U^+) \vee (V(S(X^+)) \cap End(\mathcal{S}))$;*

(2) *If $End(\mathcal{S}) \cap U^- = \emptyset$, then $G[X^-]$ has exactly one nontrivial component denoted by $S(X^-)$ such that $S(X^-)$ is a star with $S(X^-) = (V(S(X^-)) \cap U^-) \vee (V(S(X^-)) \cap End(\mathcal{S}))$.*

Proof of Lemma A2. By symmetry, it would therefore suffice to show that (1) is true. By Lemmas 3 and 5(1), $End(\mathcal{S}) \cup \{w\}$ is an independent set of G. By Lemma 5(1)(2), $U^+ \cup \{w\}$ is an independent set of G. Since $|X^+| = k + \kappa + 1$, there must exist some edges between U^+ and $End(\mathcal{S})$. By Lemma A1(3)$(i)(ii)$, $G[X^+]$ has exactly one nontrivial component $S(X^+)$ and $S(X^+)$ is a star. □

Remark A1. *$S(X^+)$ and $S(X^-)$ always denote the stars in Lemma A2 in the following. From Lemma A3 to Lemma A6, for convenience, we assume $U \subseteq V(\mathcal{S}_P)$ and $End(\mathcal{S}) \cap U^+ = \emptyset$.*

Lemma A3. *Let $C \in \mathcal{S}_C$ and $P \in \mathcal{S}_P$. Then for each vertex $v \in V(C)$ with $vu^+_{P,i} \in E(G)$ for some $i \in \{1, \cdots, t_P\}$, it holds that $N(v^+) \cap (U^+ \setminus \{u^+_{P,i}\}) = \emptyset$.*

Proof of Lemma A3. By contradiction, suppose that $N(v^+) \cap (U^+ \setminus \{u^+_{P,i}\}) \neq \emptyset$. Then there exists a vertex $x \in N(v^+) \cap (U^+ \setminus \{u^+_{P,i}\})$; say $x = u^+_{P',j}$ for some $j \in \{1, \cdots, t_{P'}\}$. Suppose first that $P' = P$. If $i < j$, then the set of vertices of the subgraph $v_L(P) \overrightarrow{P} u_{P,i} L_{P,i} w L_{P,j} u_{P,j} \overleftarrow{P} u^+_{P,i} C u^+_{P,j} \overrightarrow{P} v_R(P)$ is equal to $V(P \cup C) \cup \{w\}$, which contradicts Lemma 4. If $i > j$, then $v_L(P) \overrightarrow{P} u_{P,j} L_{P,j} w L_{P,i} u_{P,i} \overleftarrow{P} u^+_{P,j} C u^+_{P,i} \overrightarrow{P} v_R(P)$ covers $V(P \cup C) \cup \{w\}$, contradicting Lemma 4. Now suppose that $P' \neq P$. Then $v_L(P) \overrightarrow{P} u_{P,i} L_{P,i} w L_{P',j} u_{P',j} \overleftarrow{P'} v_L(P')$ and $v_R(P) \overleftarrow{P} u^+_{P,i} C u^+_{P',j} \overrightarrow{P'} v_R(P')$ cover $V(P \cup P' \cup C) \cup \{w\}$, contradicting Lemma 4. These contradictions show that Lemma A3 holds. □

Lemma A4. *The cardinality of the set $V(S(X^+)) \cap U^+$ is equal to one and $|V(S(X^+)) \cap End(\mathcal{S})| \geq 1$.*

Proof of Lemma A4. By Lemma A2(1), $|V(S(X^+)) \cap U^+| \geq 1$ and $|V(S(X^+)) \cap End(\mathcal{S})| \geq 1$. In other words, we need to prove that $|V(S(X^+)) \cap U^+| = 1$.

By contradiction, suppose that $|V(S(X^+)) \cap U^+| \neq 1$. Then, by Lemma A2(1), there exists a vertex $x \in End(\mathcal{S})$ such that $xu^+_{P,i}, xu^+_{P',j} \in E(G)$ and $u^+_{P,i} \neq u^+_{P',j}$ for some $i \in \{1, \cdots, t_P\}$ and some $j \in \{1, \cdots, t_{P'}\}$. By Lemma 5(4), $x \notin End(\mathcal{S}_P)$. Then, $x \in End(\mathcal{S}_C)$; say $x = v_C$. By Lemma A2(1), $X^+ \setminus \{v_C\}$ is an independent set of G with size $k + \kappa$. By Lemma A3, $|V(C)| \geq 2$. We consider the neighbourhood of the vertex v^+_C. By Lemma A3, $N(v^+_C) \cap (U^+ \setminus \{u^+_{P,i}\}) = \emptyset$ and $N(v^+_C) \cap (U^+ \setminus \{u^+_{P',j}\}) = \emptyset$. Then, $N(v^+_C) \cap U^+ = \emptyset$. According to Lemma 2 $(i)(ii)$, $N(v^+_C) \cap (End(\mathcal{S}) \setminus \{v_C\}) = \emptyset$. Hence, $(X^+ \setminus \{v_C\}) \cup \{v^+_C\}$ forms an independent set with a cardinality of $\kappa + k + 1$, which contradicts $\alpha(G) = \kappa + k$. This contradiction show that Lemma A4 holds. □

Remark A2. *If $U \subseteq V(\mathcal{S}_P)$ and $End(\mathcal{S}) \cap U^+ = \emptyset$, then, by Lemma A4, $|V(S(X^+)) \cap U^+| = 1$; say $V(S(X^+)) \cap U^+ = \{u^+_{P,i}\}$ for some $P \in \mathcal{S}_P$ and some $i \in \{1, \cdots, t_P\}$. Denote $X^+_i = X^+ \setminus \{u^+_{P,i}\}$. Then, by Lemmas A1(3), A2(1) and A4, X^+_i is an independent set of G with size $k + \kappa$. If $End(\mathcal{S}_P) \cap V(S(X^+)) \neq \emptyset$, then, by Lemmas 5(4) and A4,*

$$u^+_{P,i} = u^+_{P,t_P} \text{ and } u^+_{P,t_P} v_R(P) \in E(G). \tag{A1}$$

Lemma A5. *Let $C \in \mathcal{S}_C$. Then, for some $P \in \mathcal{S}_P$, the following statements are true.*

(1) If $v_C u^+_{P,i} \in E(G)$ for some $i \in \{1, \cdots, t_P\}$, then $N(x) \cap (U^+ \setminus \{u^+_{P,i}\}) = \emptyset$ for each $x \in V(C) \setminus \{v_C\}$;

(2) If $S(X^+) = u^+_{P,i} v_C$ for some $i \in \{1, \cdots, t_P\}$, then $u^+_{P,i}$ is adjacent to all vertices in C.

Proof of Lemma A5. First, we will show that (1) holds. Let $|V(C)| \geq 2$ and $C = v_C v^+_C \cdots v^{t+}_C \cdots v^{(|V(C)|-1)+}_C v_C$. We prove Lemma A5(1) by induction on t. Note that $U \subseteq V(\mathcal{S}_P)$, $End(\mathcal{S}) \cap U^+ = \emptyset$ and $v_C u^+_{P,i} \in E(G)$. According to Lemma A3, $N(v^+_C) \cap (U^+ \setminus \{u^+_{P,i}\}) = \emptyset$.

This implies that Lemma A5(1) holds for $t = 1$. Next, we assume that Lemma A5(1) holds for all positive integers $t \leq t_0$. Then $N(v_C^{t_0+}) \cap (U^+ \setminus \{u_{P,i}^+\}) = \emptyset$. We need to prove that it holds for $t = t_0 + 1$. By Lemma 2$(i)(ii)$, $N(v_C^{t_0+}) \cap (End(\mathcal{S}) \setminus \{v_C\}) = \emptyset$. Note that $v_C u_{P,i}^+ \in E(G)$. By Lemma A4, $V(S(X^+)) \cap U^+ = \{u_{P,i}^+\}$. Then X_i^+ is an independent set of G with size $k + \kappa$. Hence, $v_C^{t_0+} v_C \in E(G)$. Otherwise, $X_i^+ \cup \{v_C^{t_0+}\}$ is an independent set of G with size $\kappa + k + 1$, contradicting $\alpha(G) = \kappa + k$.

We claim that $G[\{v_C, v_C^+, \ldots, v_C^{t_0+}\}]$ forms a clique. Since $G[\{v_C, v_C^+, \ldots, v_C^{t_0+}\}]$ is connected, we only need to focus on the case when $|V(G[\{v_C, v_C^+, \ldots, v_C^{t_0+}\}])| \geq 3$. By contradiction, suppose that $v_C^{t_1+} v_C^{t_2+} \notin E(G)$ for some pair of vertices $v_C^{t_1+}, v_C^{t_2+} \in \{v_C, v_C^+, \ldots, v_C^{t_0+}\}$ with $v_C^{t_1+} \neq v_C^{t_2+}$, then $(X_i \setminus \{v_C\}) \cup \{v_C^{t_1+}, v_C^{t_2+}\}$ is an independent set of G with size $k + \kappa + 1$, contradicting $\alpha(G) = k + \kappa$. Hence, $G[\{v_C, v_C^+, \ldots, v_C^{t_0+}\}]$ is a clique.

By our claim, $G[\{v_C^+, \ldots, v_C^{t_0+}\}]$ contains a subgraph $C(G[\{v_C^+, \ldots, v_C^{t_0+}\}])$ such that $f(C(G[\{v_C^+, \ldots, v_C^{t_0+}\}])) = 1$ and $V(G[\{v_C^+, \ldots, v_C^{t_0+}\}]) = V(C(G[\{v_C^+, \ldots, v_C^{t_0+}\}]))$.

Next, we will show that Lemma A5(1) holds for $t = t_0 + 1$. By contradiction, suppose that $N(v_C^{(t_0+1)+}) \cap (U^+ \setminus \{u_{P,i}^+\}) \neq \emptyset$. Then there exists a vertex $x \in N(v_C^{(t_0+1)+}) \cap (U^+ \setminus \{u_{P,i}^+\})$; say $x = u_{P',j}^+$ for some $j \in \{1, \cdots, t_{P'}\}$. Suppose first that $P' = P$. Then, by our claim, the set of vertices of the subgraph

$$Q_2 = \begin{cases} v_L(P) \overrightarrow{P} u_{P,j} L_{P,j} w L_{P,i} u_{P,i} \overleftarrow{P} u_{P,j}^+ v_C^{(t_0+1)+} \overrightarrow{C} v_C u_{P,i}^+ \overrightarrow{P} v_R(P), & \text{if } j < i, \\ v_L(P) \overrightarrow{P} u_{P,i} L_{P,i} w L_{P,j} u_{P,j} \overleftarrow{P} u_{P,i}^+ v_C \overleftarrow{C} v_C^{(t_0+1)+} u_{P,j}^+ \overrightarrow{P} v_R(P), & \text{if } j > i, \end{cases}$$

and $C(G[\{v_C^+, \ldots, v_C^{t_0+}\}])$ is equal to $V(P \cup C) \cup \{w\}$, which contradicts (I). Now suppose that $P' \neq P$. Then, by our claim, the set of vertices of the subgraph $v_L(P) \overrightarrow{P} u_{P,i} L_{P,i} w L_{P',j} u_{P',j} \overleftarrow{P'} v_L(P')$, $v_R(P) \overleftarrow{P} u_{P,i}^+ v_C \overleftarrow{C} v_C^{(t_0+1)+} u_{P',j}^+ \overrightarrow{P'} v_R(P')$ and $C(G[\{v_C^+, \ldots, v_C^{t_0+}\}])$ is equal to $V(P \cup P' \cup C) \cup \{w\}$, which contradicts (I). These contradictions show that Lemma A5(1) holds for $t = t_0 + 1$. Thus, Lemma A5(1) is proved.

Now, we start to prove Lemma A5(2). If $S(X^+) = u_{P,i}^+ v_C$, then $X^+ \setminus \{u_{P,i}^+\}$ and $X^+ \setminus \{v_C\}$ are two independent sets of G. For any $x \in V(C) \setminus \{v_C\}$, $N(x) \cap (U^+ \setminus \{u_{P,i}^+\}) = \emptyset$ by Lemma A5(1). According to Lemma 2$(i)(ii)$, $N(x) \cap (End(\mathcal{S}) \setminus \{v_C\}) = \emptyset$. Hence, $x u_{P,i}^+ \in E(G)$. Otherwise, $(X^+ \setminus \{v_C\}) \cup \{x\}$ is an independent set of G with size $\kappa + k + 1$, contradicting $\alpha(G) = \kappa + k$. Lemma A5(2) is proved. □

Remark A3. *Suppose $U \subseteq V(\mathcal{S}_P)$ and $End(\mathcal{S}) \cap U^+ = \emptyset$. For some $P \in \mathcal{S}_P$ and some $i \in \{1, \cdots, t_P\}$, if $S(X^+) = u_{P,i}^+ v_C$ for $C \in S_C$, then, by Lemma A5(2), $G[V(C) \cup \{u_{P,i}^+\}]$ contains a spanning subgraph $C(G[V(C) \cup \{u_{P,i}^+\}])$ with $f(C(G[V(C) \cup \{u_{P,i}^+\}])) = 1$.*

For some $P \in \mathcal{S}_P$ and $i \in \{1, 2, \ldots, t_P\}$, we consider the following configuration.

(i) $S(X^+) = u_{P,i}^+ v_C$, for some $C \in \mathcal{S}_C$.

(ii) $G[\{u_{P,i}^+, v_C, v_{C'}\}] \subseteq S(X^+)$, for some $\{C, C'\} \subseteq \mathcal{S}_C$.

(iii) For given $P' \in \mathcal{S}_P \setminus \{P\}$ and $C \in \mathcal{S}_C$, $S(X^+) = u_{P,i}^+ v_C$ and $\{u_{P,i}^+ v_C, x v_C\} \subseteq E(G)$ for some $x \in V(P')$.

(iv) For some $P' \in \mathcal{S}_P \setminus \{P\}$ and $\{C, C'\} \subseteq \mathcal{S}_C$, $\{u_{P,i}^+ v_C, x v_{C'}\} \subseteq E(G)$ for some $x \in V(P')$.

Then we denote

$$Q_* = \begin{cases} C(G[V(C) \cup \{u_{P,i}^+\}]), & \text{if } (i) \text{ occurs}, \\ C u_{P,i}^+ C', & \text{if } (ii) \text{ occurs}, \end{cases}$$

$$Q_\star = \begin{cases} v_R(P)\overleftarrow{P}u^+_{P,i}Cx\overleftarrow{P'}v_L(P'), & \text{if } (iii) \text{ occurs,} \\ Cu^+_{P,i}\overrightarrow{P}v_R(P) \text{ and } C'x\overleftarrow{P'}v_L(P'), & \text{if } (iv) \text{ occurs,} \end{cases}$$

Lemma A6. *Suppose that* $U^+ \cap V(S(X^+)) = \{u^+_{P,i}\}$ *and* $End(\mathcal{S}_C) \cap V(S(X^+)) \neq \emptyset$ *for some* $P \in \mathcal{S}_P$ *and* $i \in \{1, \cdots, t_P\}$. *If* $u^{2+}_{P,i} \notin \{v_R(P), u_{P,i+1}\}$, *then* $u^{2+}_{P,i}v_R(P) \in E(G)$.

Proof of Lemma A6. By contradiction, suppose that $u^{2+}_{P,i}v_R(P) \notin E(G)$. Note that X^+_i is an independent set of G with size $k+\kappa$. Then there exists at least one vertex $x \in N(u^{2+}_{P,i}) \cap X^+_i$ with $x \neq v_R(P)$. Recall that $End(\mathcal{S}_C) \cap V(S(X^+)) \neq \emptyset$, we assume that $v_C \in End(\mathcal{S}_C) \cap V(S(X^+))$. In other words, $v_Cu^+_{P,i} \in E(G)$. By Lemma 6(1)(2), $x \notin End(\mathcal{S}_P) \setminus \{v_R(P)\}$. According to the definition of X^+_i, we will consider the following three cases in order to arrive at a contradiction.

- Assume that $x \in End(\mathcal{S}_C)$. According to Lemma 6(1), $N(u^+_{P,i}) \cap End(\mathcal{S}_C) = N(u^{2+}_{P,i}) \cap End(\mathcal{S}_C) = \{v_C\}$, then $S(X^+) = v_Cu^+_{P,i}$. By (A1) and Lemma 6(2), $End(\mathcal{S}_P) \cap V(S(X^+)) = \emptyset$. Note that $End(\mathcal{S}_C) \cap V(S(X^+)) \neq \emptyset$. By Lemma A5(2), the set of vertices of the subgraph $v_L(P)\overrightarrow{P}u^+_{P,i}Cu^{2+}_{P,i}\overrightarrow{P}v_R(P)$ in G is equal to $V(P \cup C)$, which contradicts Lemma 4.
- Assume that $x \in (X^+_i \cap U^+) \cap V(P)$; say $x = u^+_{P,j}$ for some $j \in \{1, \cdots, t_P\} \setminus \{i\}$. If $End(\mathcal{S}_P) \cap V(S(X^+)) \neq \emptyset$, then, by (A1), the set of vertices of the subgraph $u^+_{P,t_P}v_R(P)\overleftarrow{P}u^{2+}_{P,t_P}u^+_{P,j}\overrightarrow{P}u_{P,t_P}L_{P,t_P}wL_{P,j}u_{P,j}\overleftarrow{P}v_L(P)$ in G is equal to $V(P) \cup \{w\}$, which contradicts Lemma 4. Therefore, $End(\mathcal{S}_P) \cap V(S(X^+)) = \emptyset$. Note that $End(\mathcal{S}_C) \cap V(S(X^+)) \neq \emptyset$. Then (i) or (ii) occurs. By Lemma A5(2),

$$Q_3 = \begin{cases} v_R(P)\overleftarrow{P}u^{2+}_{P,i}u^+_{P,j}\overrightarrow{P}u_{P,i}L_{P,i}wL_{P,j}u_{P,j}\overleftarrow{P}v_L(P), & \text{if } j < i, \\ v_R(P)\overleftarrow{P}u^+_{P,j}u^{2+}_{P,i}\overrightarrow{P}u_{P,j}L_{P,j}wL_{P,i}u_{P,i}\overleftarrow{P}v_L(P), & \text{if } j > i, \end{cases}$$

  Q_* cover $V(P \cup C) \cup \{w\}$ or $V(P \cup C \cup C') \cup \{w\}$, which contradicts (I).
- Suppose that $x \in U^+ \cap V(P')$ for $P' \in \mathcal{S}_P \setminus \{P\}$; say $x = u^+_{P',j}$ for some $j \in \{1, \cdots, t_{P'}\}$. If $End(\mathcal{S}_P) \cap V(S(X^+)) \neq \emptyset$, then, by (A1), the set of vertices of the subgraph $Cu^+_{P,t_P}v_R(P)\overleftarrow{P}u^{2+}_{P,t_P}u^+_{P',j}\overrightarrow{P'}v_R(P') \cup v_L(P)\overrightarrow{P}u_{P,t_P}L_{P,t_P}wL_{P',j}u_{P',j}\overleftarrow{P'}v_L(P')$ is equal to $V(P \cup P' \cup C) \cup \{w\}$, which contradicts Lemma 4. Therefore, $End(\mathcal{S}_P) \cap V(S(X^+)) = \emptyset$. Note that $End(\mathcal{S}_C) \cap V(S(X^+)) \neq \emptyset$. Then (i) or (ii) occurs. By Lemma A5(2), the set of vertices of the subgraph $v_R(P)\overleftarrow{P}u^{2+}_{P,i}u^+_{P',j}\overrightarrow{P'}v_R(P') \cup v_L(P)\overrightarrow{P}u_{P,i}L_{P,i}wL_{P',j}u_{P',j}\overleftarrow{P'}v_L(P') \cup Q_*$ is equal to $V(P \cup P' \cup C) \cup \{w\}$ or $V(P \cup P' \cup C \cup C') \cup \{w\}$, which contradicts (I).

The contradiction indicates that $N(u^{2+}_{P,i}) \cap X^+_i \subseteq \{v_R(P)\}$. Note that $N(u^{2+}_{P,i}) \cap X^+_i \neq \emptyset$. Therefore, $N(u^{2+}_{P,i}) \cap X^+_i = \{v_R(P)\}$. Lemma A6 holds. □

Lemma A7. *If* $U \subseteq V(\mathcal{S}_P)$, *then there exists exactly one path* $Q \in \mathcal{S}_P$ *such that* $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$ *or* $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) = 1$.

Proof of Lemma A7. First, we claim that there is a path $Q \in \mathcal{S}_P$ with $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) = 1$. To prove this claim, we will consider the following two cases.

- Suppose that $U \subseteq V(\mathcal{S}_P)$ and $End(\mathcal{S}) \cap U^+ \neq \emptyset$. Then, by Lemma A1(2), there is a path $Q \in \mathcal{S}_P$ with $u^+_{Q,t_Q} = v_R(Q)$, i.e., $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) = 1$.
- Assume that $U \subseteq V(\mathcal{S}_P)$ and $End(\mathcal{S}) \cap U^+ = \emptyset$. According to Lemma A4, $End(\mathcal{S}) \cap V(S(X^+)) \neq \emptyset$. Suppose first that $End(\mathcal{S}_P) \cap V(S(X^+)) \neq \emptyset$. By Lemmas 5(4) and A4, there is a path $Q \in \mathcal{S}_P$ satisfying $End(\mathcal{S}_P) \cap V(S(X^+)) = \{v_R(Q)\}$ and

$U^+ \cap V(S(X^+)) = \{u^+_{Q,t_Q}\}$, i.e., $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) = 1$. Now, suppose that $End(\mathcal{S}_P) \cap V(S(X^+)) = \emptyset$, i.e., $End(\mathcal{S}_C) \cap V(S(X^+)) \neq \emptyset$; say $v_C \in End(\mathcal{S}_C) \cap V(S(X^+))$. By Lemma A4, $|V(S(X^+)) \cap U^+| = 1$; say $V(S(X^+)) \cap U^+ = \{u^+_{Q,i}\}$ for some $Q \in \mathcal{S}_P$ and $i \in \{1, \cdots, t_Q\}$. It should be noted that $X^+ \setminus \{u^+_{Q,i}\}$ forms an independent set of G with a cardinality of $k + \kappa$. By Lemmas A2 and A4, $N(v_C) \cap U^+ = \{u^+_{Q,i}\}$. We will show that $u^+_{Q,i} = u^+_{Q,t_Q}$. Suppose otherwise that $u^+_{Q,i} \neq u^+_{Q,t_Q}$. By (A1), $End(\mathcal{S}_P) \cap V(S(X^+)) = \emptyset$. Note that $u^+_{Q,i} v_C \in E(G)$. Then (i) or (ii) occurs. By Lemmas A5(2) and A6, the set of vertices of the subgraph

$$Q_4 = \begin{cases} v_L(Q)\overrightarrow{Q}u_{Q,i}L_{Q,i}wL_{Q,i+1}u_{Q,i+1}\overrightarrow{Q}v_R(Q), & \text{if } |V(Q[u^+_{Q,i}, u^-_{Q,i+1}])| = 1, \\ u^+_{Q,t_Q}\overrightarrow{Q}v_R(Q)u^{2+}_{Q,i}\overrightarrow{Q}u_{Q,t_Q}L_{Q,t_Q}wL_{Q,i}u_{Q,i}\overleftarrow{Q}v_L(Q), & \text{if } |V(Q[u^+_{Q,i}, u^-_{Q,i+1}])| > 1 \end{cases}$$

$\cup Q_*$ in G is equal to $V(Q \cup C) \cup \{w\}$ or $V(Q \cup C' \cup C) \cup \{w\}$, which contradicts (I). The contradiction indicates that $u^+_{Q,i} = u^+_{Q,t_Q}$. Then $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$.

Hence, our claim is proved.

Now, we will prove Lemma A7. We begin by assuming the opposite and using a proof by contradiction. Suppose that there exists another path $P \in \mathcal{S}_P$ with $N(u^+_{P,t_P}) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overrightarrow{Q}(u^+_{P,t_P}, P)) = 1$. To arrive at a contradiction, we consider the following two cases:

- Assume that $End(\mathcal{S}) \cap U^+ \neq \emptyset$. If $u^+_{P,t_P} = v_R(P)$ and $u^+_{Q,t_Q} = v_R(Q)$, then X^+ does not form an independent set of G with a cardinality of $k + \kappa$, contradicting Lemma A1(2). Therefore, $u^+_{P,t_P} \neq v_R(P)$ or $u^+_{Q,t_Q} \neq v_R(Q)$. Without loss of generality, suppose that $u^+_{P,t_P} \neq v_R(P)$, then $u^+_{P,t_P} v_R(P) \in E(G)$ or $N(u^+_{P,t_P}) \cap End(\mathcal{S}_C) \neq \emptyset$. Hence, there exist two adjacent vertices in X^+, contradicting Lemma A1(2).
- Assume that $End(\mathcal{S}) \cap U^+ = \emptyset$. Then $u^+_{Q,t_Q} \neq v_R(Q)$ and $u^+_{P,t_P} \neq v_R(P)$. Then, $G[X^+]$ has at least two stars, contradicting Lemma A2(1).

This statement indicates that Lemma A7 is true. □

Let

$$X = \begin{cases} X^+ \setminus \{v_{C_U}\}, & \text{if } U \cap V(\mathcal{S}_C) \neq \emptyset, \\ X^+ \setminus \{u^+_{Q,t_Q}\}, & \text{if } U \subseteq V(\mathcal{S}_P) \text{ and } U^+ \cap End(\mathcal{S}) = \emptyset, \\ X^+, & \text{if } U \subseteq V(\mathcal{S}_P) \text{ and } U^+ \cap End(\mathcal{S}) \neq \emptyset. \end{cases}$$

Then, by Lemmas A1, A4 and A7, X forms an independent set of G satisfying size $k + \kappa$ and

$$N(v) \cap X \neq \emptyset \;\; for \;\; any \;\; v \in V(G) \setminus X. \tag{A2}$$

Otherwise, there is a vertex $v_0 \in V(G) \setminus X$ satisfying $X \cup \{v_0\}$ being an independent set of G with size $\kappa + k + 1$, which contradicts $\alpha(G) = \kappa + k$.

Lemma A8. *Suppose that $U \subseteq V(\mathcal{S}_P)$. The following two statements are true.*

(1) $\mathcal{S}_P$ contains exactly one path Q such that $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) = 1$ and $N(u^-_{Q,1}) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overleftarrow{Q}(u^-_{Q,1}, Q)) = 1$;

(2) If $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) \neq 1$ and $f(\overleftarrow{Q}(u^-_{Q,1}, Q)) \neq 1$, then there exist at least two elements C, $C' \in \mathcal{S}_C$ with $u^+_{Q,t_Q} v_{C'} \in E(G)$, $u^-_{Q,1} v_C \in E(G)$.

Proof of Lemma A8. By symmetry and Lemma A7, $\mathcal{S}_P$ has exactly one path P (say) such that $N(u^-_{P,1}) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overleftarrow{Q}(u^-_{P,1}, P)) = 1$. First, we will show $Q = P$. By contradiction, suppose that $Q \neq P$. Denote $Q_5 = v_L(Q)\overrightarrow{Q}u_{Q,t_Q}L_{Q,t_Q}wL_{P,1}u_{P,1}\overrightarrow{P}v_R(P)$. To arrive at a contradiction, we consider the following three cases using Lemma A7:

- Assume that $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) = 1$ and $f(\overleftarrow{\mathcal{Q}}(u^-_{P,1}, P)) = 1$. Then the set of vertices of the subgraph $Q_5 \cup \overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q) \cup \overleftarrow{\mathcal{Q}}(u^-_{P,1}, P)$ in G is equal to $V(Q \cup P) \cup \{w\}$, which contradicts (I).
- Assume that either $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) \neq 1$ and $f(\overleftarrow{\mathcal{Q}}(u^-_{P,1}, P)) = 1$ or $f(\overleftarrow{\mathcal{Q}}(u^-_{P,1}, P)) \neq 1$ and $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) = 1$. By symmetry, suppose that $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) \neq 1$ and $f(\overleftarrow{\mathcal{Q}}(u^-_{P,1}, P)) = 1$. According to Lemma A7, $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$; say $v_{C'} \in N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C)$. Then the set of vertices of the subgraph $Q_5 \cup v_R(Q)\overleftarrow{Q}u^+_{Q,t_Q}C' \cup \overleftarrow{\mathcal{Q}}(u^-_{P,1}, P)$ in G is equal to $V(Q \cup P \cup C') \cup \{w\}$, which contradicts (I).
- Suppose that $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) \neq 1$ and $f(\overleftarrow{\mathcal{Q}}(u^-_{P,1}, P)) \neq 1$. Applying symmetry and using Lemma A7, $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$ and $N(u^-_{P,1}) \cap End(\mathcal{S}_C) \neq \emptyset$. Then (iii) or (iv) occurs. By Lemma A5(2), Q_5 and $Q_\star$ in G cover $V(Q \cup P \cup C) \cup \{w\}$ or $V(Q \cup P \cup C' \cup C) \cup \{w\}$, contradicting Lemma 4 or (I).

This contradiction shows that Lemma A8(1) holds.

Next, we will demonstrate Lemma A8(2). By Lemma A8(1), $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$ and $N(u^-_{Q,1}) \cap End(\mathcal{S}_C) \neq \emptyset$. We begin by assuming the opposite and using a proof by contradiction. Suppose that there is precisely one element $C \in \mathcal{S}_C$ with $u^+_{Q,t_Q}v_C \in E(G)$ and $u^-_{Q,1}v_C \in E(G)$. Note that $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) \neq 1$ and $f(\overleftarrow{\mathcal{Q}}(u^-_{Q,1}, Q)) \neq 1$. By Lemma A4, $S(X^+) = u^+_{Q,t_Q}v_C$ and $S(X^-) = u^-_{Q,1}v_C$. Then, by Lemmas A5(2) and A6, the set of vertices of the subgraph $v_L(Q)\overrightarrow{Q}u^-_{Q,1}Cu^+_{Q,t_Q}\overleftarrow{Q}u_{Q,1}L_{Q,1}w \cup \overrightarrow{\mathcal{Q}}(u^{2+}_{Q,t_Q}, Q)$ in G is equal to $V(Q \cup C) \cup \{w\}$, which contradicts (I). This contradiction demonstrates that Lemma A8(2) is true. □

Remark A4. *If $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) \neq 1$, then, by Lemma A7, $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$; say $v_{C'} \in N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C)$. If $f(\overleftarrow{\mathcal{Q}}(u^-_{Q,1}, Q)) \neq 1$, then, by Lemma A8(1), $N(u^-_{Q,1}) \cap End(\mathcal{S}_C) \neq \emptyset$; say $v_C \in N(u^-_{Q,1}) \cap End(\mathcal{S}_C)$. If $f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) \neq 1$ and $f(\overleftarrow{\mathcal{Q}}(u^-_{Q,1}, Q)) \neq 1$, then, by Lemma A8(2), $V(C) \cap V(C') = \emptyset$. For convenience, we denote*

$$Q_6 = \begin{cases} C'u^+_{Q,t_Q}\overrightarrow{Q}v_R(Q), & \text{if } f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) \neq 1, \\ \overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q) & \text{if } f(\overrightarrow{\mathcal{Q}}(u^+_{Q,t_Q}, Q)) = 1. \end{cases}$$

$$Q_7 = \begin{cases} Cu^-_{Q,1}\overleftarrow{Q}v_L(Q), & \text{if } f(\overleftarrow{\mathcal{Q}}(u^-_{Q,1}, Q)) \neq 1, \\ \overleftarrow{\mathcal{Q}}(u^-_{Q,1}, Q), & \text{if } f(\overleftarrow{\mathcal{Q}}(u^-_{Q,1}, Q)) = 1. \end{cases}$$

Let

$$\mathcal{S}' = \begin{cases} \mathcal{S} \setminus \{C_U\}, & \text{if } U \cap V(\mathcal{S}_C) \neq \emptyset, \\ \mathcal{S} \setminus \{Q\}, & \text{if } U \cap V(\mathcal{S}_C) = \emptyset, \end{cases}$$

and $\mathcal{S}'_C = \mathcal{S}' \cap \mathcal{S}_C$, $\mathcal{S}'_P = \mathcal{S}' \cap \mathcal{S}_P$.

Lemma A9. *Let $C \in \mathcal{S}'_C$ and $x \in V(C) \setminus \{v_C\}$. It follows that $N(x) \cap X = \{v_C\}$.*

Proof of Lemma A9. First, we assert that for each vertex $x \in V(C) \setminus \{v_C\}$, $N(x) \cap (X \cap U^+) = \emptyset$. To prove this, we will use a proof by contradiction. Assume that $N(x) \cap (X \cap U^+) \neq \emptyset$. According to Lemma 2$(i)$, $N(x) \cap (V(\mathcal{S}_C) \cap U^+) = \emptyset$. Then, there is at least one vertex $u^+_{P,i} \in N(x) \cap (X \cap V(\mathcal{S}_P) \cap U^+)$ for some $P \in \mathcal{S}_P$ and some $i \in \{1, \cdots, t_P\}$. Assuming $U \cap V(\mathcal{S}_C) \neq \emptyset$. Then, the set of vertices of the subgraph $v_L(P)\overrightarrow{P}u_{P,i}L_{P,i}wG[V(L_{C_U,1}) \cup V(C_U)] \cup Cu^+_{P,i}\overrightarrow{P}v_R(P)$ in G is equal to $V(P \cup C_U \cup C) \cup \{w\}$, which contradicts (I). Therefore, $U \subseteq V(\mathcal{S}_P)$. Suppose first that $P = Q$. If $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$, (say $v_{C'} \in N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C)$) and $C = C'$, then the set of vertices of the subgraph $v_L(Q)\overrightarrow{Q}u_{Q,i}L_{Q,i}wL_{Q,t_Q}u_{Q,t_Q}\overleftarrow{Q}u^+_{Q,i}Cu^+_{Q,t_Q}\overrightarrow{Q}v_R(Q)$ in G is equal to $V(Q \cup C) \cup \{w\}$ or $V(Q \cup C) \cup \{w\}$; see Lemma 4. Otherwise, by Lemma A7, $v_L(Q)\overrightarrow{Q}u_{Q,i}L_{Q,i}wL_{Q,t_Q}u_{Q,t_Q}\overleftarrow{Q}u^+_{Q,i}C$ and Q_6 in G cover $V(Q \cup C \cup C') \cup \{w\}$ or $V(Q \cup C) \cup \{w\}$, contradicting (I). Suppose now that $P \neq Q$, i.e., $P \in \mathcal{S}'_P$. Let $Q' = v_L(P)\overrightarrow{P}u_{P,i}L_{P,i}wL_{Q,t_Q}u_{Q,t_Q}\overleftarrow{Q}v_L(Q)$. If $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$, (say $v_{C'} \in N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C)$) and $C = C'$, then the set of vertices of the subgraph $Q' \cup v_R(Q)\overleftarrow{Q}u^+_{Q,t_Q}Cu^+_{P,i}\overrightarrow{P}v_R(P)$ in G is equal to $V(P \cup Q \cup C) \cup \{w\}$, which contradicts Lemma 4. Otherwise, by Lemma A7, Q', Q_6 and $Cu^+_{P,i}\overrightarrow{P}v_R(P)$ in G cover $V(P \cup Q \cup C \cup C') \cup \{w\}$ or $V(P \cup Q \cup C) \cup \{w\}$, contradicting (I). These contradictions show that our claim holds.

According to Lemma 2$(i)(ii)$, $N(x) \cap (End(\mathcal{S}) \setminus \{v_C\}) = \emptyset$. Combining this with our claim, we arrive at $N(x) \cap (X \setminus \{v_C\}) = \emptyset$. By (A2), $N(x) \cap X = \{v_C\}$. □

Lemma A10. *For any $C \in \mathcal{S}'_C$, $G[V(C)]$ forms a clique.*

Proof of Lemma A10. As $G[V(C)]$ is connected, we only need to focus on the case when $|V(C)| \geq 3$. It is worth noting that $V(C) \cap X = \{v_C\}$. According to Lemma A9, $N(x) \cap X = \{v_C\}$ for every vertex $x \in V(C) \setminus \{v_C\}$. Let $S' = V(C)$. Then, according to Lemma 7, $G[V(C)]$ forms a clique. □

Lemma A11. *Suppose that $V(\mathcal{S}'_P) \cap U = \emptyset$. The following two statements are true.*

(1) *Let $P \in \mathcal{S}_P$ and $y \in V(P)$ such that $yv_R(P) \in E(G)$, $|V(P[y^+, v_R(P)])| \geq 1$ and $V(P[y, v_R(P)]) \cap U = \emptyset$. Then $G[V(P[y^+, v_R(P)])]$ forms a clique. Moreover, if $N(y) \cap X = \{v_R(P)\}$, then $G[V(P[y, v_R(P)])]$ also forms a clique;*

(2) *Let $P \in \mathcal{S}_P$ and $x \in V(P)$ such that $xv_L(P) \in E(G)$, $|V(P[v_L(P), x^-])| \geq 1$ and $V(P[v_L(P), x]) \cap U = \emptyset$. Then $G[V(P[v_L(P), x^-])]$ forms a clique. Moreover, if $N(x) \cap X = \{v_L(P)\}$, then $G[V(P[v_L(P), x])]$ also forms a clique.*

Proof of Lemma A11. By virtue of symmetry, we may restrict our consideration to demonstrate the truth of (1). As $G[V(P[y^+, v_R(P)])]$ is connected, it is sufficient to focus on the case where $|V(P[y^+, v_R(P)])| \geq 3$. Suppose that there is at least one vertex $v \in N(y^+) \cap X$ with $v \neq v_R(P)$. According to Lemma 6(1)(2), $v \in U^+ \cap X \cap V(\mathcal{S}_P)$. Note that $V(\mathcal{S}'_P) \cap U = \emptyset$. Then $U \subseteq V(\mathcal{S}_P)$. We assume that $v = u^+_{Q,j}$ for some $j \in \{1, \cdots, t_Q - 1\}$. If $P \neq Q$, then the set of vertices of the subgraph $v_L(P)\overrightarrow{P}yv_R(P)\overleftarrow{P}y^+u^+_{Q,j}\overrightarrow{Q}v_R(Q) \cup v_L(Q)\overrightarrow{Q}u_{Q,j}L_{Q,j}w$ in G is equal to $V(Q \cup P) \cup \{w\}$, which contradicts (I). If $P = Q$, then $v_L(Q)\overrightarrow{Q}u_{Q,j}L_{Q,j}wL_{Q,t_Q}u_{Q,t_Q}\overleftarrow{Q}u^+_{Q,j}y^+\overrightarrow{Q}v_R(Q)\,y\overleftarrow{Q}u^+_{Q,t_Q}$ in G covers $V(Q) \cup \{w\}$, contradicting (I). Therefore, we have $N(y^+) \cap X = \{v_R(P)\}$, which is a contradiction. According to (A2),

$$N(y^+) \cap X = \{v_R(P)\}. \tag{A3}$$

Note that $V(P[y^+, v_R(P)]) \cap X = \{v_R(P)\}$. Let $S' = V(P[y^+, v_R(P)])$. According to Lemma 7, it would therefore suffice to show that the following characterization holds,

$$N(y') \cap X = \{v_R(P)\} \text{ for every vertex } y' \in V(P[y^+, v_R(P))). \tag{A4}$$

We apply (A3) repeatedly to obtain (A4).

Next, we will demonstrate that if $N(y) \cap X = \{v_R(P)\}$ and $V(P[y, v_R(P)]) \cap U = \emptyset$, then $G[V(P[y, v_R(P)])]$ forms a clique. Since $G[V(P[y, v_R(P)])]$ is connected, we can assume that $|V(P[y, v_R(P)])| \geq 3$. It is important to note that $N(y) \cap X = \{v_R(P)\}$, which combined with (A4) implies that $N(x) \cap X = \{v_R(P)\}$ for every vertex $x \in V(P[y, v_R(P)))$. Let $S' = V(P[y, v_R(P)])$. According to Lemma 7, we can conclude that $G[V(P[y, v_R(P)])]$ forms a clique. □

Denote

$T_1(P) := \{x \in V(P) : P \in \mathcal{S}'_P,\ f(P[v_L(P), x]) = 1,\ V(P[v_L(P), x^+]) \cap U = \emptyset,\ x^+ \neq v_R(P)\}$;

$T_2(P) := \{x \in V(P) : P \in \mathcal{S}'_P,\ N(x) \cap End(\mathcal{S}'_C) \neq \emptyset,\ V(P[v_L(P), x^+]) \cap U = \emptyset,\ x^+ \neq v_R(P)\}$.

Remark A5. *If $x \in T_2(P)$, then, according to the definition of $T_2(P)$, there is at least one vertex $v_C \in N(x) \cap End(\mathcal{S}'_C)$. Let*

$$Q_8 = \begin{cases} \overleftarrow{Q}(x, P), & \text{if } x \in T_1(P), \\ Cx\overleftarrow{P}v_L(P), & \text{if } x \in T_2(P) \ (\text{say } v_C \in N(x) \cap End(\mathcal{S}'_C)). \end{cases}$$

Lemma A12. *Let $P \in \mathcal{S}'_P$, $x \in V(P)$ and $P' \in \mathcal{S}'_P \setminus \{P\}$. Then the following three characterizations are true:*

(1) If $x \in T_1(P) \cup T_2(P)$, then $N(x^+) \cap (U^+ \cap V(P')) = \emptyset$;
(2) If $x \in T_1(P)$, then $N(x^+) \cap (X \setminus (End(\mathcal{S}'_C) \cup \{v_L(P), v_R(P)\})) = \emptyset$;
(3) If $x \in T_2(P)$, then $N(x^+) \cap (X \setminus (End(\mathcal{S}'_C) \cup \{v_R(P)\})) = \emptyset$.

Proof of Lemma A12. First, we will prove Lemma A12(1). We begin by assuming the opposite, i.e., $N(x^+) \cap (U^+ \cap V(P')) \neq \emptyset$; say $u^+_{P',i} \in N(x^+) \cap (U^+ \cap V(P'))$ for some $i \in \{1, \cdots, t_{P'}\}$. Denote $Q_9 = v_R(P')\overleftarrow{P'}u^+_{P',i}x^+\overrightarrow{P}v_R(P)$. To arrive at a contradiction, we will consider the following two situations.

- Suppose that $U \cap V(\mathcal{S}_C) \neq \emptyset$. Then Q_9, $v_L(P')\overrightarrow{P'}u_{P',i}L_{P',i}wG[V(L_{C_U,1}) \cup V(C_U)]$ and Q_8 in G cover $V(P' \cup P) \cup C_U) \cup \{w\}$ or $V(P' \cup P \cup C \cup C_U) \cup \{w\}$, which contradicts (I).
- Assume that $U \subseteq V(\mathcal{S}_P)$. Then, by Lemma A7, there exists exactly one path $Q \in \mathcal{S}_P$ such that $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$ (say $v_{C'} \in N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C)$) or $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) = 1$. Denote $Q_{10} = v_L(P')\overrightarrow{P'}u_{P',i}L_{P',i}wL_{Q,t_Q}u_{Q,t_Q}\overleftarrow{Q}v_L(Q)$. To arrive at a contradiction, we differentiate between the following two cases:
  - Assume that $x \in T_1(P)$. Then, by Lemma A7, the set of vertices of the subgraph $Q_9 \cup Q_{10} \cup \overleftarrow{Q}(x, P) \cup Q_6$ is equal to $V(Q \cup P' \cup P \cup C') \cup \{w\}$ or $V(Q \cup P' \cup P) \cup \{w\}$, which contradicts (I).
  - Assume that $x \in T_2(P)$. Then $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) \neq 1$. Otherwise the set of vertices of the subgraph $Q_9 \cup Q_{10} \cup Cx\overleftarrow{P}v_L(P) \cup \overrightarrow{Q}(u^+_{Q,t_Q}, Q)$ is equal to $V(Q \cup P' \cup P \cup C) \cup \{w\}$, which contradicts (I). Then, by Lemma A7, $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$. Then (iii) or (iv) occurs. By Lemma A5(2), the set of vertices of the subgraph $Q_9 \cup Q_{10} \cup Q_\star$ in G is equal to $V(Q \cup P' \cup P \cup C') \cup \{w\}$ or $V(Q \cup P' \cup P \cup C' \cup C) \cup \{w\}$, which contradicts Lemma 4 or (I).

This statement indicates that Lemma A12(1) is true.

Next, we assert that if $x \in T_1(P) \cup T_2(P)$, then $N(x^+) \cap (X \setminus (End(\mathcal{S}'_C) \cup \{v_L(P), v_R(P)\})) = \emptyset$.

Assuming a contradiction, let us suppose that $N(x^+) \cap (X \setminus (End(\mathcal{S}'_C) \cup \{v_L(P), v_R(P)\})) \neq \emptyset$; say $z \in N(x^+) \cap (X \setminus (End(\mathcal{S}'_C) \cup \{v_L(P), v_R(P)\}))$. By Lemma 6(1), $z \notin$

$End(\mathcal{S}_P) \setminus \{v_L(P), v_R(P)\}$. To derive a contradiction, we differentiate between the following two cases based on the definition of X:

- Assuming $z \in U^+ \cap V(\mathcal{S}_C)$; say $z = u^+_{C_U,i}$ for some $i \in \{1, \cdots, t_{C_U}\}$. Then the set of vertices of the subgraph $Q_8 \cup v_R(P) \overleftarrow{P} x^+ u^+_{C_U,i} \overrightarrow{C_U} u_{C_U,i} L_{C_U,i} w$ in G is equal to $V(P) \cup V(C) \cup V(C_U) \cup \{w\}$ or $V(P) \cup V(C_U) \cup \{w\}$, which contradicts (I).
- Assuming $z \in (X \cap U^+) \cap V(\mathcal{S}_P)$. To arrive at a contradiction by Lemma A12(1), we differentiate between the following two cases:
  - Assume that $U \cap V(\mathcal{S}_C) \neq \emptyset$; say $z = u^+_{P,j}$ for some $j \in \{1, \cdots, t_P\}$. Then the set of vertices of the subgraph $v_R(P) \overleftarrow{P} u^+_{P,j} x^+ \overrightarrow{P} u_{P,j} L_{P,j} w G[V(L_{C_U,1}) \cup V(C_U)]$ and Q_8 is equal to $V(P \cup C \cup C_U) \cup \{w\}$ or $V(P \cup C_U) \cup \{w\}$, which contradicts (I).
  - Assume that $U \subseteq V(\mathcal{S}_P)$. Let

$$Q_{11} = \begin{cases} v_R(P) \overleftarrow{P} u^+_{P,j} x^+ \overrightarrow{P} u_{P,j} L_{P,j} w L_{Q,t_Q} u_{Q,t_Q} \overleftarrow{Q} v_L(Q), & \text{if } z = u^+_{P,j} \text{ for some } j \in \{1, \cdots, t_P\}, \\ v_R(P) \overleftarrow{P} x^+ u^+_{Q,j} \overrightarrow{Q} u_{Q,t_Q} L_{Q,t_Q} w L_{Q,j} u_{Q,j} \overleftarrow{Q} v_L(Q), & \text{if } z = u^+_{Q,j} \text{ for some } j \in \{1, \cdots, t_Q - 1\}. \end{cases}$$

  Assume that $x \in T_1(P)$. Then, by Lemma A7, the set of vertices of the subgraph $Q_{11} \cup Q_6 \cup \overleftarrow{Q}(x, P)$ is equal to $V(Q \cup P \cup C') \cup \{w\}$ or $V(Q \cup P) \cup \{w\}$, which contradicts (I). Now suppose that $x \in T_2(P)$. Then $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) \neq 1$. Otherwise, the set of vertices of the subgraph $Q_{11} \cup \overrightarrow{Q}(u^+_{Q,t_Q}, Q) \cup Cx \overleftarrow{P} v_L(P)$ is equal to $V(Q \cup P \cup C) \cup \{w\}$, which contradicts (I). Then, by Lemma A7, $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$. Then (iii) or (iv) occurs. By Lemma A5(2), Q_{11} and $Q_\star$ in G cover $V(Q \cup P \cup C) \cup \{w\}$ or $V(Q \cup P \cup C \cup C') \cup \{w\}$, contradicting Lemma 4 or (I).

This contradiction demonstrates the validity of our claim. Therefore, Lemma A12(2) is true.

Final, we will prove Lemma A12(3). By our claim, if $x \in T_2(P)$, then $N(x^+) \cap (X \setminus (End(\mathcal{S}'_C) \cup \{v_L(P), v_R(P)\})) = \emptyset$. Hence, we only prove that $x^+ v_L(P) \notin E(G)$. By contradiction, suppose that $x^+ v_L(P) \in E(G)$. Then $Cx \overleftarrow{P} v_L(P) x^+ \overrightarrow{P} v_R(P)$ in G covers $V(P \cup C)$, which contradicts Lemma 4. This contradiction demonstrates that Lemma A12(3) is true. □

Lemma A13. *Let $P \in \mathcal{S}'_P$ and $x \in V(P)$ with $V(P(x^+, v_R(P)]) \cap U \neq \emptyset$. If $x \in T_1(P) \cup T_2(P)$, then $x^+ v_R(P) \notin E(G)$.*

Proof of Lemma A13. By contradiction, suppose that $x^+ v_R(P) \in E(G)$. To arrive at a contradiction, we differentiate between the following two cases:

- Assume that $U \cap V(\mathcal{S}_C) \neq \emptyset$. Then the set of vertices of the subgraph $G[V(L_{C_U,1}) \cup V(C_U)] w L_{P,1} u_{P,1} \overleftarrow{P} x^+ v_R(P) \overleftarrow{P} u^+_{P,1}$ and Q_8 in G cover $V(P \cup C \cup C_U) \cup \{w\}$ or $V(P \cup C_U) \cup \{w\}$, which contradicts (I).
- Suppose that $U \subseteq V(\mathcal{S}_P)$. Let $Q_{12} = u^-_{P,1} \overleftarrow{P} x^+ v_R(P) \overleftarrow{P} u_{P,1} L_{P,1} w L_{Q,t_Q} u_{Q,t_Q} \overleftarrow{Q} v_L(Q)$. Suppose first that $x \in T_1(P)$. Then, by Lemma A7, the set of vertices of the subgraph $Q_{12} \cup Q_6 \cup \overleftarrow{Q}(x, P)$ is equal to $V(Q \cup P \cup C') \cup \{w\}$ or $V(Q \cup P) \cup \{w\}$, which contradicts (I). Now suppose that $x \in T_2(P)$. Then $f(\overrightarrow{Q}(u^+_{Q,t_Q}, Q)) \neq 1$. Otherwise, the set of vertices of the subgraph $Q_{12} \cup \overrightarrow{Q}(u^+_{Q,t_Q}, Q) \cup Cx \overleftarrow{P} v_L(P)$ is equal to $V(Q \cup P \cup C) \cup \{w\}$, which contradicts (I). Then, by Lemma A7, $N(u^+_{Q,t_Q}) \cap End(\mathcal{S}_C) \neq \emptyset$. Then (iii) or (iv) occurs. By Lemma A5(2), the set of vertices of the subgraph $Q_{12} \cup Q_\star$ is equal to $V(Q \cup P \cup C) \cup \{w\}$ or $V(Q \cup P \cup C \cup C') \cup \{w\}$, which contradicts Lemma 4 or (I).

This contradiction shows that $x^+v_R(P) \notin E(G)$. □

Lemma A14. *Let $P \in \mathcal{S}'_P$, $x \in T_2(P)$. If $x^+v_R(P) \notin E(G)$, then $N(x^+) \cap End(\mathcal{S}'_C) = \emptyset$.*

Proof of Lemma A14. Assuming a contradiction, let us suppose that $N(x^+) \cap End(\mathcal{S}'_C) \neq \emptyset$. By Lemma 2$(ii)$, $x^+ \notin \{v_L(P), v_R(P)\}$. Note that $N(x) \cap End(\mathcal{S}'_C) \neq \emptyset$. We assume that $v_C \in N(x) \cap End(\mathcal{S}'_C)$. By Lemma 6(1), $N(x^+) \cap End(\mathcal{S}'_C) = \{v_C\}$. If $x^+v_C^+ \in E(G)$, then the set of vertices of the subgraph $v_L(P)\overrightarrow{P}xCx^+\overrightarrow{P}v_R(P)$ in G is equal to $V(P \cup C)$, which contradicts Lemma 4. Therefore, $|V(C)| \geq 2$ and $x^+v_C^+ \notin E(G)$. By Lemma A9, $N(v_C^+) \cap X = \{v_C\}$. By Lemma A12(3), $N(x^+) \cap X = \{v_C\}$. Then $(X \setminus \{v_C\}) \cup \{x^+, v_C^+\}$ forms an independent set of size $\kappa + k + 1$; this would contradict the fact that $\alpha(G) = \kappa + k$. This contradiction demonstrates that Lemma A14 is true. □

Lemma A15. *Let $P \in \mathcal{S}'_P$ with $V(P) \cap U^+ \neq \emptyset$. Then $u^-_{P,1} \neq v_L(P)$ and $N(u^-_{P,1}) \cap (End(\mathcal{S}'_C) \cup \{v_L(P)\}) = \emptyset$.*

Proof of Lemma A15. Denote

$$X' = \begin{cases} X^- \setminus \{v_{C_U}\}, & \text{if } U \cap V(\mathcal{S}_C) \neq \emptyset, \\ X^- \setminus \{u^-_{Q,1}\}, & \text{if } U \subseteq V(\mathcal{S}_P) \text{ and } U^- \cap End(\mathcal{S}) = \emptyset, \\ X^-, & \text{if } U \subseteq V(\mathcal{S}_P) \text{ and } U^- \cap End(\mathcal{S}) \neq \emptyset. \end{cases}$$

By symmetry and Lemmas A1, A4 and A8, X' is an independent set of G with size $k + \kappa$. Then, $u^-_{P,1} \neq v_L(P)$; otherwise, X' is an independent set of G with size $k + \kappa - 1$, contradicting $\alpha(G) = \kappa + k$. Moreover, $N(u^-_{P,1}) \cap (End(\mathcal{S}'_C) \cup \{v_L(P)\}) = \emptyset$. Otherwise, X' is not an independent set of G. □

Lemma A16. $\mathcal{S}'_P = \emptyset$.

Proof of Lemma A16. By contradiction, suppose that $\mathcal{S}'_P \neq \emptyset$.

Claim A1. *$V(P) \cap U = \emptyset$ for any $P \in \mathcal{S}'_P$.*

Proof of Claim A1. By contradiction, suppose that $V(P) \cap U \neq \emptyset$ for some $P \in \mathcal{S}'_P$. Now, we consider the section $P[v_L(P), u^-_{P,1}]$. By Lemma A15, $|V(P[v_L(P), u^-_{P,1}])| \geq 3$.

Suppose that $v_L(P)u^{2-}_{P,1} \in E(G)$. Then, by Lemmas A12(2), A13 and A15, we have $N(u^-_{P,1}) \cap X = \emptyset$, contradicting (A2). This contradiction shows that

$$v_L(P)u^{2-}_{P,1} \notin E(G). \tag{A5}$$

Hence, $|V(P[v_L(P), u^-_{P,1}])| \geq 4$. Then there exists a vertex $v_L(P)^{i+} \in V(P[v_L(P), u^{2-}_{P,1}))$ such that $v_L(P)v_L(P)^{i+} \in E(G)$ for $i \geq 1$. By Lemmas A12(2) and A13, $N(v_L(P)^{(i+1)+}) \cap (X \setminus (End(\mathcal{S}'_C) \cup \{v_L(P)\})) = \emptyset$. By (A2), $N(v_L(P)^{(i+1)+}) \cap X \neq \emptyset$. Then there exists at least one vertex $v \in N(v_L(P)^{(i+1)+}) \cap (End(\mathcal{S}'_C) \cup \{v_L(P)\})$. Suppose that $v \in End(\mathcal{S}'_C)$. We know $N(v_L(P)^{(i+2)+}) \cap X = \emptyset$ by Lemmas A12(3), A13 and A14, contradicting (A2). This contradiction shows that $v \notin End(\mathcal{S}'_C)$. Combining this with (A2) and Lemmas A12(2), A13, we obtain that $N(v_L(P)^{(i+1)+}) \cap X = \{v_L(P)\}$. Thus, $v_L(P)u^{2-}_{P,1} \in E(G)$, contradicting (A5). Claim A1 is proved. □

According to Lemma 4, for any path $P \in \mathcal{S}_P$, $v_L(P)v_R(P) \notin E(G)$. We can select the vertex x_P from $V(P)$ such that $V(P[v_L(P), x^-_P]) \subseteq N(v_L(P))$ and $x_P \notin N(v_L(P))$. Denote

$$\widehat{x}_P = \begin{cases} x^+_P, & \text{if } x_P \neq v_R(P) \text{ and } x_P v_R(P) \notin E(G); \\ x_P, & \text{if } x_P = v_R(P) \text{ or } x_P v_R(P) \in E(G). \end{cases}$$

If $x_P \neq v_R(P)$ and $x_P v_R(P) \notin E(G)$, then, by the definition of x_P and Lemma A12(2), (A2), $N(x_P) \cap End(\mathcal{S}'_C) \neq \emptyset$.

Claim A2. *For any path $P \in \mathcal{S}'_P$, the following two characterizations are true.*

(1) $f(G[V(P[v_L(P), x_P^-])]) = 1$ *and* $f(G[V(P[\widehat{x}_P, v_R(P)])]) = 1$;
(2) *Either x_P^- or x_P is a cut vertex of G.*

Proof of Claim A2. First, we will prove Claim A2(1). If $x_P = v_R(P)$ or $x_P v_R(P) \in E(G)$, then Claim A2(1) holds. Therefore, $x_P \neq v_R(P)$ and $x_P v_R(P) \notin E(G)$. Note that $N(x_P) \cap End(\mathcal{S}'_C) \neq \emptyset$. Suppose first that $x_P^+ = v_R(P)$. Claim A2(1) holds. Suppose now that $x_P^+ \neq v_R(P)$. Then we consider the neighbourhood of the vertex x_P^+. If $x_P^+ v_R(P) \notin E(G)$, then, by Lemmas A12(3) and A14, $N(x_P^+) \cap X = \emptyset$, contradicting (A2). Therefore, $x_P^+ v_R(P) \in E(G)$. Claim A2(1) holds.

Next, we will prove Claim A2(2). Since G is connected, $N(V(P[v_L(P), x_P^-])) \cap (V(G) \setminus V(P[v_L(P), x_P^-])) \neq \emptyset$. For $z \in N(V(P[v_L(P), x_P^-])) \cap (V(G) \setminus V(P[v_L(P), x_P^-]))$, there exists a vertex $x' \in V(P[v_L(P), x_P^-])$ with $x'z \in E(G)$. By the definition of x_P and Claim A2(1), $v_L(P) x_P^- \in E(G)$. According to Claim A1 and Lemma A11(2),

$$G[V(P[v_L(P), x_P^-])] \text{ is a clique.} \tag{A6}$$

We will demonstrate that z belongs to $V(P)$. To begin, we assume the opposite, z is not an element of $V(P)$. By Lemma 6(2) and (A6), $z \notin V(\mathcal{S}_C) \cup V(H)$. To arrive at a contradiction, we differentiate between the following two cases:

- Assume that $z \in V(P')$ with $P' \in \mathcal{S}'_P \setminus \{P\}$. By Lemma 6(2), $z \notin \{v_L(P'), v_R(P')\}$. Therefore, $z \in V(P') \setminus \{v_L(P'), v_R(P')\}$. By the definition of $x_{P'}$ and Claim A2(1), $v_L(P') x_{P'}^- \in E(G)$ and $f(G[V(P'[\widehat{x}_{P'}, v_R(P')])]) = 1$. Then, by Claim A1 and Lemmas A11(1)(2), $G[V(P'[v_L\ (P'), x_{P'}^-))]$ and $G[V(P'(\widehat{x}_{P'}, v_R(P')])]$ are cliques. Hence, there exists a $Q' \in \{v_L(P') \overrightarrow{P'} z^- v_L\ (P'), z^+ \overrightarrow{P'}\ v_R(P') z^+\}$ with $f(Q') = 1$. By Lemma A11(2), the set of vertices of the subgraph $G[E(P' \setminus Q')] x' G[V(P[v_L(P), x_P^-) \setminus \{x'\})] x_P^-$
$\overrightarrow{P} v_R(P)$ and Q' in G is equal to $V(P \cup P')$, which contradicts Lemma 4.
- Suppose that $z \in V(Q)$. To arrive at a contradiction, we differentiate between the following two cases:
  - Assume that $z \in V(Q[u_{Q,1}, u_{Q,t_Q}])$. Then, by Lemmas A8(1)(2), A11(1)(2) and (A6), the set of vertices of the subgraph

$$Q_{13} = \begin{cases} v_R(P) \overleftarrow{P} x_P^- G[V(P[v_L(P), x_P^-))] z \overrightarrow{Q} u_{Q,t_Q} L_{Q,t_Q} w L_{Q,1} u_{Q,1} \overrightarrow{Q} z^-, & \text{if } z \in \\ V(Q(u_{Q,1}, u_{Q,t_Q})), & \\ v_R(P) \overleftarrow{P} x_P^- G[V(P[v_L(P), x_P^-))] z \overrightarrow{Q} u_{Q,t_Q} L_{Q,t_Q} w, & \text{if } z = \\ u_{Q,1}, & \\ v_R(P) \overleftarrow{P} x_P^- G[V(P[v_L(P), x_P^-))] z \overleftarrow{Q} u_{Q,1} L_{Q,1} w, & \text{if } z = \\ u_{Q,t_Q}, & \end{cases}$$

  Q_6 and Q_7 are equal to $V(P \cup Q \cup C \cup C') \cup \{w\}$ or $V(P \cup Q \cup C) \cup \{w\}$ or $V(P \cup Q \cup C') \cup \{w\}$ or $V(P \cup Q) \cup \{w\}$, which contradicts (I).
  - Suppose that $z \in V(P[v_L(Q), u_{Q,1}^-])$ or $V(P[u_{Q,t_Q}^+, v_R(Q)])$. By virtue of symmetry, we may restrict our consideration to $z \in V(P[v_L(Q), u_{Q,1}^-])$. By Lemma A8(1), $N(u_{Q,1}^-) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overleftarrow{Q}(u_{Q,1}^-, Q)) = 1$. Combining this with Lemma A6, we obtain that $u_{Q,1}^{2-} v_L(Q) \in E(G)$. By Claim A1 and Lemma A11(2), $G[V(P[v_L(Q), u_{Q,1}^{2-}))]$ is a clique. Then, by (A6), either $v_R(P) \overleftarrow{P} x_P^- G[\ V(P[v_L(P), x_P^-))] z \overrightarrow{Q} v_R(Q)$

and $\overleftarrow{Q}(z^-, Q)$ in G cover $V(P) \cup V(Q)$, or $v_R(P)\overleftarrow{P}x_P^- G[V(P[v_L(P), x_P^-))]v_L(Q)\overrightarrow{Q}v_R(Q)$ in G covers $V(P \cup Q)$, which contradicts Lemma 4.

This contradiction demonstrates that

$$z \in V(P). \tag{A7}$$

To prove Claim A2(2), we differentiate between the following two cases:

- Suppose that $\hat{x}_P = x_P$. We will show that there is no pair of edges $x_P x_1$ and $x_P^- x_2$ with $x_1 \in V(P[v_L(P), x_P^-))$ and $x_2 \in V(P(x_P, v_R(P)])$. Suppose otherwise that $x_P x_1 \in E(G)$ and $x_P^- x_2 \in E(G)$. Note that $G[V(P(x_P, v_R(P)])]$ and $G[V(P[v_L(P), x_P^-))]$ form cliques. Then the set of vertices of the subgraph $x_P^- G[V(P(x_P, v_R(P)])]x_P G[V(P[v_L(P), x_P^-))]x_P^-$ in G is equal to $V(P)$, which contradicts Lemma 4. If either $x_P^- x_2 \in E(G)$ and $x_P x_1 \notin E(G)$ or $x_P x_1 \notin E(G)$ and $x_P^- x_2 \notin E(G)$, then, by Lemma 6(2) and (A7), $N(x') \subseteq V(P[v_L(P), x_P^-])$. Therefore, x_P^- is a cut vertex of G. If $x_P x_1 \in E(G)$ and $x_P^- x_2 \notin E(G)$, then, by Lemma 6(2) and (A7), $N(x') \subseteq V(P[v_L(P), x_P])$. Therefore, x_P is a cut vertex of G.
- Suppose that $\hat{x}_P = x_P^+$. Then $x_P v_R(P) \notin E(G)$ and $N(x_P) \cap End(\mathcal{S}'_C) \neq \emptyset$; say $v_C \in N(x_P) \cap End(\mathcal{S}'_C)$. Suppose, first, that $N(x_P) \cap V(P[v_L(P), x_P^-)) = \emptyset$. Then $x_P^+ x' \notin E(G)$; otherwise, the set of vertices of the subgraph $Cx_P G[V(P[v_L(P), x_P^-])]x_P^+ \overrightarrow{P} v_R(P)$ in G is equal to $V(P \cup C)$, which contradicts Lemma 4. Combining this with Lemma 6(2) and (A7), we obtain that $N(x') \subseteq V(P[v_L(P), x_P^-])$. Then, x_P^- is a cut vertex of G. Suppose, now, that $N(x_P) \cap V(P[v_L(P), x_P^-)) \neq \emptyset$; say $x_0 \in N(x_P) \cap V(P[v_L(P), x_P^-))$. By (A6), $G[P[v_L(P), x_P]]$ has a cycle $C_{x_0} = x_0 \overleftarrow{P} v_L(P) x_0^+ \overrightarrow{P} x_P x_0$ with $V(P[v_L(P), x_P]) = V(C_{x_0})$. By (A6), we structure a new path P' such that $P' = x_0 \overleftarrow{P} v_L(P) x_0^+ \overrightarrow{P} x_P \overrightarrow{P} v_R(P)$ by rearranging the order of the vertices in P. Then $v_L(P') = x_0$. It is easy to verify that $G[V(P)] \cong G[V(P')]$. We will prove that there is no pair of edges $x_P^+ x'_1$, $x_P x'_2$ such that $x'_1 \in V(P'[v_L(P'), x_P))$ and $x'_2 \in V(P'(x_P^+, v_R(P')])$. Suppose otherwise that $x_P^+ x'_1 \in E(G)$ and $x_P x'_2 \in E(G)$. Then $x_P^+ G[V(P'(x_P^+, v_R(P')])]x_P G[V(P'[v_L(P'), x_P^-])]x_P^+$ in G cover $V(P')$, contradicting Lemma 4. Let $x'' \in V(P[v_L(P), x_P^-])$. By (A7), $N(x'') \subseteq V(P)$. If $x_P x'_2 \in E(G)$, then $x_P^+ x'_1 \notin E(G)$. By Lemma 6(2) and (A7), $N(x'') \subseteq V(P[v_L(P), x_P])$. Therefore, x_P is a cut vertex of G. If $x_P^+ x'_1 \in E(G)$, then, $x_P x'_2 \notin E(G)$. By Lemma 6(2) and (A7), $N(x'') \subseteq V(P[v_L(P), x_P^+])$. Therefore, x_P^+ is a cut vertex of G. If $x_P^+ x'_1 \notin E(G)$ and $x_P x'_2 \notin E(G)$, then, according to Lemma 6(2) and (A7), $N(x'') \subseteq V(P[v_L(P), x_P])$. Therefore, x_P is a cut vertex of G.

Claim A2(2) is proved. □

Claim A2(2) contradicts $\kappa \geq 2$. Hence, Lemma A16 is proved. □

Now, let us prove Lemmas 8 and 9 which are mentioned in Section 2.

Proof of Lemma 8. By contradiction, suppose that $U \cap V(\mathcal{S}_C) \neq \emptyset$. According to Lemma A16, $|\mathcal{S}_P| = 0$. As G is connected and $k \geq 2$, there are at least two elements of $\mathcal{S}_C$ connected by a path whose inner vertices are in $V(G) \setminus V(\mathcal{S})$, contradicting Lemma 2$(i)$. Therefore, $U \subseteq V(\mathcal{S}_P)$. By Lemma A16, $|\mathcal{S}_P| = 1$. □

Proof of Lemma 9. By Lemma A8(1)(2), Lemma 9(1)(2) holds. Suppose first that $End(\mathcal{S}) \cap U^+$ is not empty. Then, by Lemma A1(2), $\mathcal{X}$ forms an independent set of G with size $\kappa + k$. Suppose now that $End(\mathcal{S}) \cap U^+ = \emptyset$. By Lemma 9(1), $N(u_\kappa^+) \cap End(\mathcal{S}_C) \neq \emptyset$ or $f(\overrightarrow{\mathcal{P}}(u_\kappa^+, P)) = 1$. By Lemmas A1(3), A2(1) and A4, $\mathcal{X}$ forms an independent set of G with size $k + \kappa$. Therefore, Lemma 9(3) holds. Furthermore, by Lemma A6, Lemma 9(4) holds. By Lemma A11(1), Lemma 9(5) holds. By Lemmas A9 and A10, Lemma 9(6) holds. □

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