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First Entire Zagreb Index of Fuzzy Graph and Its Application

Umapada Jana and Ganesh Ghorai *

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721 102, India; umapada867032@gmail.com
* Correspondence: math.ganesh@mail.vidyasagar.ac.in

Abstract: The first entire Zagreb index (FEZI) is a graph parameter that has proven to be essential in various real-life scenarios, such as networking businesses and traffic management on roads. In this research paper, the FEZI was explored for a variety of fuzzy graphs, including star, firefly graph, cycle, path, fuzzy subgraph, vertex elimination, and edge elimination. This study presented several results, including determining the relationship between two isomorphic fuzzy graphs and between a path and cycle (connecting both end vertices of the path). This research also deals with the analysis of α-cut fuzzy graphs and establishes bounds for some fuzzy graphs. To apply these findings to modern life problems, the research team utilized the results to identify areas that require more development in internet systems. These results have practical implications for enhancing the efficiency and effectiveness of internet systems. The conclusion drawn from this research can be used to inform future research and aid in the development of more efficient and effective systems in various fields.

Keywords: first entire Zagreb index; second entire Zagreb index; first entire Zagreb index of a vertex; isomorphic graph; internet system

MSC: 05C09; 05C72

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1. Introduction

1.1. Research Background

In the modern day, fuzzy graph (FG) theory is one of the most applicable to regular life. So there are many researchers who are implementing FG theory, especially topological indices of fuzzy graphs. Rosenfeld [1], inspired by Zadeh's [2] classical set (fuzzy set) in 1975, introduced the fuzziness for a graph, then, it is called a fuzzy graph. Additionally, this time he introduced several connective parameters of an FG and some applications of these parameters by Yeh et al. [3]. In [4,5], Sunitha et al. studied fuzzy block, fuzzy bridge, FSG, CFG, PFSG, fuzzy tree, fuzzy forest, fuzzy cut vertex, etc. The degree of a vertex $\left(d(v)/deg(v)\right)$ in an FG is also discussed In [6]. The degree of an edge $(d_G(e))$ is also discussed in [7]. Further information on the FG hypothesis is provided in [8–11]. Bipolar and m-polar fuzzy graphs are discussed in [12,13]. The first Zagreb index is discussed in [14] and was inspired by the paper [15]. In a molecular graph of a chemical compound, we can calculate molecular descriptors by finding topological indices of this graph. A graph's topology is described by these topological indices, which are numerical numbers. The Zagreb index, established in 1972 by Gutman and Trinajstic [16], is a degree-based topological index. The *pi*-electron energy of a conjugate system is determined using topological indices. In a crisp graph, many indices are defined but several issues in real life cannot be handled by these indices. So we can generalize these indices in fuzzy graphs. In this piece, we have introduced the FEZI in fuzzy graphs which is a major generalization of FEZI for crisp graphs. These indices are investigated from both a theoretical and an application point of view.

1.2. Research Question

These questions are covered in this paper:

1. What is the FEZI upper bound for fuzzy graphs?
2. What are the precise values or boundaries of FEZI for the firefly graph, star, path, cycle, etc.?
3. What is the relation between the value of FEZI of a graph and its sub graph?
4. What is the relation between the value of FEZI for two isomorphic graph?
5. What is the relation for several graphs between the first Zagreb index and FEZI ?
6. What are the applications of this index?

1.3. Objective of the Work

Various types of topological indices of a graph can be used for a variety of purposes and yield a wide range of outcomes for crisp graphs. However, in numerous applications, a crisp graph is not enough to solve it. We need to define a fuzzy graph to answer this question. In this paper, FEZI is defined and some results relating to sub graphs, paths, stars, firefly graphs, cycles, isomorphic graphs, etc. are given. At the end of this paper, we applied the first entire Zagreb index in internet network systems.

1.4. Structure of the Study

The structure of the article is as follows: in Section 2, some definitions are provided which are necessary for this study. In Section 3, we studied the FEZI of a fuzzy graph and provided some results on sub graphs, paths, stars, firefly graphs, cycles, etc. Also some relation between fuzzy graphs are provided. In Section 4, an application of the first entire Zagreb index in development in internet networking system is discussed.

2. Preliminaries

Here, we provide some fundamental definitions and theorems which are crucial to developing the later sections.

Let U be a universal set. An FS A on U is a mapping $\sigma : U \to [0,1]$. Here, σ is the membership function of the FS A. A FS generally indicated by $A = (u, \sigma)$.

Assume that $F(\neq 0)$ is a known finite set. Then the fuzzy graph (FG) is a triplet, $G = (U, \omega, \varrho)$, where $U(\neq 0) \subseteq F$ with $\omega : U \to [0,1]$ and $\varrho : U \times U \to [0,1]$ satisfying $\varrho(x_1, x_2) \leq \omega(x_1) \wedge \omega(x_2)$. The set U is the set of vertices and $\varepsilon := ((x_1, x_2) : \varrho(x_1, x_2) > 0)$ is the set of edges of the FG. $\omega(x_1)$ represents the membership value (MV) of the vertex x_1 and ϱ represents the MV of the edge (x_1, x_2) (or x_1x_2).

Let $G = (U, \omega, \varrho)$ be an FG. Then, $H = (U', \omega', \varrho')$ is called the PFSG of the FG G if $U' \subset U, \omega'(x_1) \leq \omega(x_1), \varrho'(x_1, x_2) \leq \varrho(x_1, x_2)$ for all $x_1, x_2 \in U'$. If $\omega'(x_1) = \omega(x_1)$ and $\varrho'(x_1, x_2) \leq \varrho(x_1, x_2)$ for all $x_1, x_2 \in U'$ then H is called FSG of the graph of the FG G.

For $x_1 \in V$, we denote G_{x_1} as an FSG of the FG $G = (V, \omega, \varrho)$ with $\omega(x_1) = 0$ and for $x_1x_2 \in \varepsilon$, $G_{x_1x_2}$ represents the FSG of the FG G with $\varrho(x_1x_2) = 0$.

Let $x_0, x_1, \ldots, x_n$ all be vertices of an FG G. Then, the collection of vertices $P(x_0, x_1, \ldots, x_n)$ is a path in G if $\varrho(x_i, x_{i+1}) \neq 0$ (for all i). The path's length is n in this case. If $\varrho(x_0, x_n) > 0$ for the path $P(x_0, x_1, \ldots, x_n)$ then it is called a cycle.

Let $x_0, x_1, \ldots, x_n$ be distinct vertices of a fuzzy graph G. Then, $G = (x_0, x_1, \ldots, x_n)$ is called a star if $\varrho(x_0, x_i) \neq 0$ for $i = 1, 2, \ldots, n$ and for all vertex except x_0 there is no edge between every two vertices, where x_0 is the center of the star.

In a graph G if $\varrho(x_i, x_j) \geq 0$ for all vertices x_i and x_j, then it is called a complete graph. In a fuzzy graph $G = (V, \omega, \varrho)$ if for every two vertex x and y satisfy the condition $\varrho(x, y) = \omega(x) \wedge \omega(y)$ then the graph is called a CFG.

Two FG $G_1 = (V_1, \omega_1, \varrho_1)$ and $G_2 = (V_2, \omega_2, \varrho_2)$ are called isomorphic to each other if there exists a bijective mapping $g : V_1 \to V_2$ for any $x, y \in V_1$, $\omega_1(x) = \omega_2(g(x))$ and $\varrho_2(g(x), g(y)) = \varrho_1(x, y)$.

Let $G = (V, E)$ be a crisp graph. Then, the first and second ZI are defined by $M_1^\varepsilon = \sum_{x \in V(G) U E(G)} (deg(x))^2$ and $M_2^\varepsilon = \sum_{\substack{x \text{ is either adjacent} \\ \text{or incident to } y}} deg(x)deg(y)$ [17].

The total degree of an FG with respect to vertices is indicated by $T_v(G)$ and is defined by $T_v(G) = \sum_{v \in V(G)} d_G(v)$ [7]. Additionally, the total degree of an FG with respect to edge is indicated by $T_e(G)$ and is defined by $T_e(G) = \sum_{e \in E(G)} d_G(e)$.

Example 1. *Let G be an FG with vertex set $V = \{p, q, r, s\}$ such that $\omega(p) = 0.8, \omega(q) = 0.7, \omega(r) = 0.6, \omega(s) = 0.5, \varrho(p, q) = 0.4, \varrho(p, r) = 0.5, \varrho(q, r) = 0.5, \varrho(q, s) = 0.3, \varrho(r, s) = 0.4$, as shown in Figure 1. Then, $d(p) = 0.9, d(q) = 1.2, d(r) = 1.4, d(s) = 0.7, d(p, q) = 1.3, d(p, r) = 1.3, d(q, r) = 1.6, d(q, s) = 1.3, d(r, s) = 1.3$ and $T_v(G) = 0.9 + 1.2 + 1.4 + 0.7 = 4.2$ and $T_e(G) = 1.3 + 1.3 + 1.3 + 1.6 + 1.3 + 1.3 = 6.8$.*

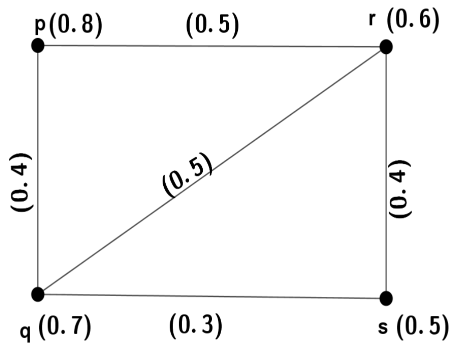

Figure 1. A fuzzy graph.

3. First Entire Zagreb Index of Fuzzy Graphs

First entire Zagreb index (FEZI) have an important role for finding strength of vertices. The strength of vertices is important in fuzzy graph theory. Thus, in this section the FEZI of a fuzzy graph is initiated. Various properties and an application of FEZI for fuzzy graphs is given.

Definition 1. *Let G be an FG. Then, the FEZI of G is indicated by M_1^z and is defined by : $M_1^z := \sum_{x \in V(G)} (\omega(x)d(x))^2 + \sum_{e \in E(G)} (\varrho(e)d(e))^2$.*

Example 2. *Let G be an FG with vertex set $V = \{p, q, r, s\}$ such that $\omega(p) = 0.8, \omega(q) = 0.7, \omega(r) = 0.6, \omega(s) = 0.5, \varrho(p, q) = 0.4, \varrho(p, r) = 0.5, \varrho(q, r) = 0.5, \varrho(q, s) = 0.3$ and $\varrho(r, s) = 0.4$, shown in Figure 1. Then $d(p) = 0.9, d(q) = 1.2, d(r) = 1.4, d(s) = 0.7, d(p, q) = 1.3, d(p, r) = 1.3, d(q, r) = 1.6, d(q, s) = 1.3, d(r, s) = 1.3$.*

Now, $M_1^z = \sum_{x \epsilon V(G)} (\omega(x)d(x))^2 + \sum_{e \epsilon E(G)} (\varrho(e)d(e))^2$

$= (0.8 \times 0.9)^2 + (0.7 \times 1.2)^2 + (0.6 \times 1.4)^2 + (0.5 \times 0.7)^2 + (0.4 \times 1.3)^2 + (0.5 \times 1.3)^2 + (0.5 \times 1.6)^2 + (0.3 \times 1.3)^2 + (0.4 \times 1.3)^2$

= *3.7135.*

Definition 2. *Let (G, ω, ϱ) be an FG. Then, the second entire Zagreb index of G is indicated by M_2^ε and is defined by $M_2^z := \sum_{\substack{u,v \text{are adjacent to} \\ \text{each other}}} \omega(u)d(u)\omega(v)d(v) + \sum_{\substack{v \text{ is incident} \\ \text{to } e}} \omega(v)d(v)\varrho(e)d(e)$, where ω is the MV of a vertex and ϱ is the MV of an edge.*

Example 3. *Let G be an FG with vertex set $V = \{p, q, r, s\}$ such that, $\omega(p) = 0.6, \omega(q) = 0.5, \omega(r) = 0.4, \omega(s) = 0.8, \varrho(p, r) = 0.4, \varrho(q, r) = 0.4, \varrho(r, s) = 0.3$ shown in Figure 2. Then*

$d(p) = 0.4, d(q) = 0.4, d(r) = 1.1, d(s) = 0.3, d(p,r) = 0.7, d(q,r) = 0.7, d(r,s) = 0.8$

Now, $M_2^z = \sum\limits_{\substack{u,v \text{are adjacent} \\ \text{to each other}}} \omega(u)d(u)\omega(v)d(v) + \sum\limits_{\substack{v \text{ is incident} \\ \text{to } e}} \omega(v)d(v)\omega(e)d(e)$

$= (0.4 \times 0.5 \times 0.4 \times 1.1) + (0.4 \times 0.6 \times 0.4 \times 1.1) + (0.3 \times 0.8 \times 0.4 \times 1.1) + (0.4 \times 0.7 \times 0.4 \times 0.7) + (0.4 \times 0.3 \times 0.7 \times 0.8) + (0.4 \times 0.3 \times 0.7 \times 0.8) + (0.4 \times 0.6 \times 0.4 \times 0.7) + (0.4 \times 0.5 \times 0.4 \times 0.7) + (0.8 \times 0.3 \times 0.3 \times 0.8) + (0.4 \times 0.7 \times 0.4 \times 1.1) + (0.4 \times 0.7 \times 0.4 \times 1.1) + (0.4 \times 0.3 \times 0.8 \times 1.1)$

$= 0.088 + 0.1056 + 0.1056 + 0.0784 + 0.0672 + 0.0672 + 0.0672 + 0.056 + 0.0576 + 0.1232 + 0.1232 + 0.1056$

$= 1.0448.$

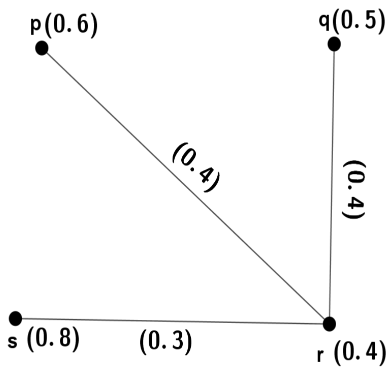

Figure 2. A fuzzy graph G with $M_2^z = 1.0448$.

Theorem 1. *Let the FG G have n vertices and m edges, then* $M_1^z \leq n^2 T_v^2(G) + m^2 T_e^2(G)$, *where* $T_v^2(G)$ *and* $T_e^2(G)$ *represent the total degree with respect to vertex and total degree with respect to edge.*

Proof. Now, the FEZI of G is given by

$$M_1^z = \sum_{x \epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum_{e \epsilon E(G)} \big(\varrho(e)d(e)\big)^2.$$

Since $\sum (pq)^2 \leq \sum p^2 \sum q^2$, we have

$$\sum_{x \epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum_{e \epsilon E(G)} \big(\varrho(e)d(e)\big)^2 \leq \sum_{x \epsilon V(G)} (\omega(x))^2 \sum_{x \epsilon V(G)} (d(x))^2 + \sum_{e \epsilon E(G)} (\varrho(e))^2 \sum_{e \epsilon E(G)} (d(e))^2$$

$$\leq \Big(\sum_{x \epsilon V(G)} \omega(x)\Big)^2 \Big(\sum_{x \epsilon V(G)} d(x)\Big)^2 + \Big(\sum_{e \epsilon E(G)} \varrho(e)\Big)^2 \Big(\sum_{e \epsilon E(G)} d(e)\Big)^2$$

$$\leq \Big(\sum_{x \epsilon V(G)} \omega(x)\Big)^2 T_v^2(G) + \Big(\sum_{e \epsilon E(G)} \varrho(e)\Big)^2 T_e^2(G).$$

Since $0 \leq \omega(x) \leq 1$ and $0 \leq \varrho(e) \leq 1$, therefore $M_1^z \leq n^2 T_v^2(G) + m^2 T_e^2(G)$. □

Definition 3. *Let* $G = (V, \omega, \varrho)$ *be an FG. Then FEZI at a vertex of G is indicated by* $M_1^z(v)$ *and is defined by* $M_1^z(v) := M_1^z(G) - M_1^z(G_v)$ *where* $G_v = \big(V(G) - v, \omega, \varrho\big)$.

Example 4. *Let G be an FG with vertex set* $V = \{p, q, r, s\}$ *such that* $\omega(p) = 0.8, \omega(q) = 0.7, \omega(r) = 0.5, \omega(s) = 0.6, \varrho(p,q) = 0.4, \varrho(p,r) = 0.5, \varrho(q,r) = 0.5, \varrho(q,s) = 0.3$ *and* $\varrho(r,s) = 0.4$ *shown in Figure 3. Then,* $d(p) = 0.9, d(q) = 1.2, d(r) = 1.4, d(s) = 0.7, d(p,q) = 1.3, d(p,r) = 1.3, d(q,r) = 1.6, d(q,s) = 1.3, d(r,s) = 1.3$.

Now, $M_1^z = \sum\limits_{x \epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e \epsilon E(G)} \big(\varrho(e)d(e)\big)^2$

$= (0.8 \times 0.9)^2 + (0.7 \times 1.2)^2 + (0.5 \times 1.4)^2 + (0.6 \times 0.7)^2 + (0.4 \times 1.3)^2 + (0.5 \times 1.3)^2 + (0.5 \times 1.6)^2 + (0.3 \times 1.3)^2 + (0.4 \times 1.3)^2$

So $M_1^z(G) = 3.6458$.

Now the vertex p is removed from the graph G, then the graph of G_p is shown in Figure 4. From the figure, $d(q) = 0.6, d(r) = 0.7, d(s) = 0.7, d(q,r) = 0.7, d(q,s) = 0.7, d(r,s) = 0.6$.

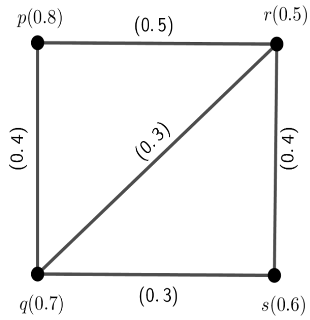

Figure 3. A fuzzy graph G with $M_1^z(G)$ = 3.6458.

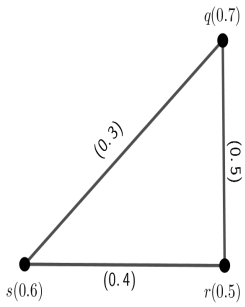

Figure 4. The FG G_p obtained by deleting the vertex p from the above graph.

Now, $M_1^z(G_p) = \sum\limits_{x\epsilon V(G_p)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e\epsilon E(G_p)} \big(\varrho(e)d(e)\big)^2$
$= (0.7 \times 0.6)^2 + (0.5 \times 0.7)^2 + (0.6 \times 0.7)^2 + (0.3 \times 0.7)^2 + (0.6 \times 0.4)^2 + (0.3 \times 0.7)^2$. *Therefore,* $M_1^z(G_p) = 0.6211$.

Then, the FEZI at p is $M_1^z(p) = 3.6458 - 0.6211 = 3.0247$.

Theorem 2. *Suppose that H is an FG that is created by removing an edge from G. Then,* $M_1^z(H) \leq M_1^z(G)$.

Proof. Since $G = (V, \omega, \varrho)$ is an FG and $H = (V', \omega', \varrho')$ is a graph that is created by removing an edge from G so the MV of a vertex is the same in both graphs and the MV of edges are the same if it contains both E and E'.

Then, the relation between the membership values of G and H is $\omega(x) \geq \omega'(x)$ for all vertices x and $\varrho(e) \geq \varrho'(e)$ for all edges.

This shows that $d(x) \geq d'(x)$ for all vertices x. Additionally, $d(e) \geq d'(e)$ for all edge e, where d and d' represent the degree of G and H.

Now, $M_1^z(G) = \sum\limits_{x\epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e\epsilon E(G)} \big(\varrho(e)d(e)\big)^2$
$\geq \sum\limits_{x\epsilon V(G)} \big(\omega'(x)d'(x)\big)^2 + \sum\limits_{e\epsilon E(G)} \big(\varrho'(e)d'(e)\big)^2$
$= \sum\limits_{x\epsilon V(H)} \big(\omega'(x)d'(x)\big)^2 + \sum\limits_{e\epsilon E(H)} \big(\varrho'(e)d'(e)\big)^2$
$= M_1^z(H)$.
Hence, $M_1^z(G) \geq M_1^z(H)$. □

Theorem 3. *If H is an FG that is obtained from G by deleting a vertex from G. Then,* $M_1^z(H) \leq M_1^z(G)$.

Proof. Since $G = (V, \omega, \varrho)$ is an FG and $H = (V', \omega', \varrho')$ is a graph that is created by removing a vertex from G.

So, $\omega(x) = \omega'(x)$ if $x \in V \cap V'$ otherwise $\omega(x) > \omega'(x)$. Additionally, $\varrho(e) = \varrho'(e)$ if $e \in E \cap E'$ otherwise $\varrho(e) > \varrho'(e)$. Then the relation between the membership values of $G = (V, \omega, \varrho)$ and $H = (V', \omega', \varrho')$ is $\omega(x) \geq \omega'(x)$ for all vertices x and $\varrho(e) \geq \varrho'(e)$ for all edges e. This shows that $d(x) \geq d'(x)$ for all vertices x. Additionally, $d(e) \geq d'(e)$ for all edges e. Here, d and d' represent the degrees of G and H.

Now, $M_1^z(G) = \sum\limits_{x \epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e \epsilon E(G)} \big(\varrho(e)d(e)\big)^2$

$\geq \sum\limits_{x \epsilon V(G)} \big(\omega'(x)d'(x)\big)^2 + \sum\limits_{e \epsilon E(G)} \big(\varrho'(e)d'(e)\big)^2$

$= \sum\limits_{x \epsilon V(H)} \big(\omega'(x)d'(x)\big)^2 + \sum\limits_{e \epsilon E(H)} \big(\varrho'(e)d'(e)\big)^2$

$= M_1^z(H)$. So, $M_1^z(G) \geq M_1^z(H)$. $\square$

Example 5. *Let G be an FG with vertex set* $V = \{p, q, r, s\}$ *such that* $\omega(p) = 0.8, \omega(q) = 0.7, \omega(r) = 0.5, \omega(s) = 0.6, \varrho(p, q) = 0.4, \varrho(p, r) = 0.5, \varrho(q, r) = 0.5, \varrho(q, s) = 0.3, \varrho(r, s) = 0.4$ *shown in Figure 3. Then* $d(p) = 0.9, d(q) = 1.2, d(r) = 1.4, d(s) = 0.7, d(p, q) = 1.3, d(p, r) = 1.3, d(q, r) = 1.6, d(q, s) = 1.3$ *and* $d(r, s) = 1.3$.

Now, $M_1^z(G) = \sum\limits_{x \epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e \epsilon E(G)} \big(\varrho(e)d(e)\big)^2$.

$= (0.8 \times 0.9)^2 + (0.7 \times 1.2)^2 + (0.5 \times 1.4)^2 + (0.6 \times 0.7)^2 + (0.4 \times 1.3)^2 + (0.5 \times 1.3)^2 + (0.5 \times 1.6)^2 + (0.3 \times 1.3)^2 + (0.4 \times 1.3)^2 = 3.6458$.

The graph of H where the vertex p is deleted from G shown in Figure 4. From the figure, $d(q) = 0.6, d(r) = 0.7, d(s) = 0.7, d(q, r) = 0.7, d(q, s) = 0.7, d(r, s) = 0.6$.
Now, $M_1^z(H) = \sum\limits_{x \epsilon V(H)} (\omega(x)d(x))^2 + \sum\limits_{e \epsilon E(H)} (\varrho(e)d(e))^2$

$= (0.7 \times 0.6)^2 + (0.5 \times 0.7)^2 + (0.6 \times 0.7)^2 + (0.3 \times 0.7)^2 + (0.6 \times 0.4)^2 + (0.3 \times 0.7)^2 = 0.6211$. *The graph of K where the edge pr is deleted from G is shown in Figure 5.*

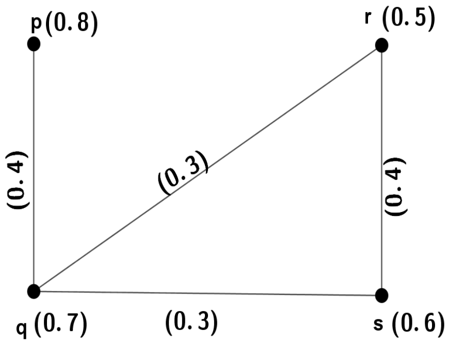

Figure 5. The FG obtained by deleting an edge from the graph in Figure 3.

From this figure, $d(p) = 0.4, d(q) = 1, d(r) = 0.7, d(s) = 0.7, d(p, q) = 0.6, d(q, r) = 1.1, d(q, s) = 1.1, d(r, s) = 1$
Now, $M_1^z(K) = \sum\limits_{x \epsilon V(K)} (\omega(x)d(x))^2 + \sum\limits_{e \epsilon E(K)} (\varrho(e)d(e))^2$

$= (0.8 \times 0.4)^2 + (0.7 \times 1)^2 + (0.5 \times 0.7)^2 + (0.6 \times 0.7)^2 + (0.6 \times 0.4)^2 + (0.3 \times 1.1)^2 + (0.3 \times 1.1)^2 + (0.4 \times 1)^2 = 1.3267$.
This shows that

1. $M_1^z(G) \geq M_1^z(H)$
2. $M_1^z(G) \geq M_1^z(K)$.

Theorem 4. *Let G be an FG and H be an FSG of G, then* $M_1^z(G) \geq M_1^z(H)$.

Proof. Since $H = (V', \omega', \varrho')$ is an FSG of $G = (V, \omega, \varrho)$, therefore

$\omega(x) \geq \omega'(x)$ for all vertices x and $\varrho(e) \geq \varrho'(e)$ for all edges e. This sows that $d(x) \geq d'(x)$ for all vertices x.

Additionally, $d(e) \geq d'(e)$ for all edges e, where d and d' represent the degrees of G and H. Now, $M_1^z(G) = \sum\limits_{x\epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e\epsilon E(G)} \big(\varrho(e)d(e)\big)^2$

$\geq \sum\limits_{x\epsilon V(G)} \big(\omega'(x)d'(x)\big)^2 + \sum\limits_{e\epsilon E(G)} \big(\varrho'(e)d'(e)\big)^2$

$= \sum\limits_{x\epsilon V(H)} \big(\omega'(x)d'(x)\big)^2 + \sum\limits_{e\epsilon E(H)} \big(\varrho'(e)d'(e)\big)^2$

$= M_1^z(H)$

Hence, $M_1^z(G) \geq M_1^z(H)$. □

Theorem 5. *Let $G = (V, \omega, \varrho)$ be an FG and $F = (V, \omega, \varrho')$ be the corresponding MST of G then $M_1^z(G) \geq M_1^z(F)$.*

Proof. Since $F = (V, \omega, \varrho')$ is an MST of $G = (V, \omega, \varrho)$ therefore F is an FSG of G. Then we can say from the above Theorem, $M_1^z(G) \geq M_1^z(F)$. □

Theorem 6. *Let G be an FG and let $G^\alpha = (V', \omega', \varrho')$ be an α cut FG of $G = (V, \omega, \varrho)$. Then, $M_1^z(G) \geq M_1^z(G^\alpha)$ where the FG G^α is defined as $V' = v\epsilon V : \omega(v) \geq \alpha$ and $\omega'(v) = \omega(v), \varrho\prime(u, v) = \varrho(u, v)$ for all $u, v\epsilon V'$.*

Proof. Since G^α is an FSG of the FG G, then by the above Theorem, $M_1^z(G) \geq M_1^z(G^\alpha)$. □

Theorem 7. *Let G be an FG and let $0 \leq p_1 \leq p_2 \leq 1$ Then $M_1^z(G^{p_1}) \geq M_1^z(G^{p_2})$.*

Proof. Since $0 \leq p_1 \leq p_2 \leq 1$, therefore G^{p_2} is a PFSG of G^{p_1}. Then, by the above Theorem, $M_1^z(G^{p_1}) \geq M_1^z(G^{p_2})$. □

Corollary 1. *Let G be an FG and let $0 \leq p_1 \leq p_2 \leq p_3 \leq \ldots \leq p_n \leq 1$.*
Then, $M_1^z(G^{p_n}) \leq M_1^z(G^{p_{n-1}}) \leq \ldots \leq M_1^z(G^{p_2}) \leq M_1^z(G^{p_1})$.

Theorem 8. *Let $G = P(v_0, v_1, v_2, \ldots, v_n)$ be a path. Then,*

1. $M_1^z(G) = ZF_1(G) + \big(\varrho(e_1)\varrho(e_2)\big)^2 + \big(\varrho(e_n)\varrho(e_{n-1})\big)^2 + \sum\limits_{i=2}^{i=n-1} \varrho^2(e_i)\big(\varrho(e_{i-1}) + \varrho(e_{i+1})\big)^2$.
2. $M_1^z(G) \leq 8(n-1)$.

Proof. Given that $G = P(v_0, v_1, v_2, \ldots, v_n)$ is a path, there are $(n+1)$ vertices and n edges.

1. Now,

$$M_1^z(G) = \sum_{x\epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum_{e\epsilon E(G)} \big(\varrho(e)d(e)\big)^2 \tag{1}$$

Here, the degree of each vertex v_i, except v_0 and v_n, is $\big(\varrho(e_i) + \varrho(e_{i+1})\big)$ and the degree of v_0 is $\varrho(e_1)$ and degree of v_n is $\varrho(e_n)$.
Additionally, the degree of each edge e_i, except e_1 and e_n, is $\big(\varrho(e_{i-1}) + \varrho(e_{i+1})\big)$ and the degree of e_1 is e_2, the degree of e_n is e_{n-1}.
Using this result, we have from (1),

$M_1^z(G) = (\omega_0\varrho_1)^2 + (\omega_n\varrho_n)^2 + \sum\limits_{i=1}^{i=n-1} \omega^2(v_i)\big(\varrho(e_i) + \varrho(e_{i+1})\big)^2 + \big(\varrho(e_1)\varrho(e_2)\big)^2 +$
$\big(\varrho(e_n)\varrho(e_{n-1})\big)^2 + \sum\limits_{i=2}^{i=n-1} \varrho^2(e_i)\big(\varrho(e_{i-1}) + \varrho(e_{i+1})\big)^2$

$$= ZF_1(G) + \big(\varrho(e_1)\varrho(e_2)\big)^2 + \big(\varrho(e_n)\varrho(e_{n-1})\big)^2 + \sum_{i=2}^{i=n-1} \varrho^2(e_i)\big(\varrho(e_{i-1}) + \varrho(e_{i+1})\big)^2 \tag{2}$$

2. From Equation (2), we get

$M_1^z(G) = ZF_1(G) + \big(\varrho(e_1)\varrho(e_2)\big)^2 + \big(\varrho(e_n)\varrho(e_{n-1})\big)^2 + \sum_{i=2}^{i=n-1} \varrho^2(e_i)\big(\varrho(e_{i-1}) + \varrho(e_{i+1})\big)^2.$

As $0 \le \varrho(e_i) \le 1$ and $0 \le \omega(v_i) \le 1$,

$M_1^z(G) \le 2(2n-1) + 1 + 1 + 4(n-2)$ (where $ZF_1(G) \le 2(2n-1)$)

$= 4n - 2 + 2 + 4n - 8$

$= 8n - 8.$

So, $M_1^z(G) \le 8(n-1)$.

□

Example 6. *Let $A = P(p,q,r,s,e)$ be an FG with vertex set $V = p,q,r,s,e$ such that $\omega(p) = 0.8, \omega(q) = 0.7, \omega(r) = 0.9, \omega(s) = 0.5, \omega(e) = 0.6, \varrho(p,q) = 0.6, \varrho(q,r) = 0.6, \varrho(r,s) = 0.4, \varrho(s,e) = 0.5$ shown in Figure 6. Then, $d(p) = 0.6, d(q) = 1.2, d(r) = 1.0, d(s) = 0.9, d(e) = 0.5, d(p,q) = 0.6, d(q,r) = 1.0, d(r,s) = 1.1$ and $d(s,e) = 0.4$.*

Now, $M_1^z(A) = \sum_{x \epsilon V(A)} \big(\omega(x)d(x)\big)^2 + \sum_{e \epsilon E(A)} \big(\varrho(e)d(e)\big)^2$

$= (0.8 \times 0.6)^2 + (0.7 \times 1.2)^2 + (0.9 \times 1.0)^2 + (0.5 \times 0.9)^2 + (0.6 \times 0.5)^2 + (0.6 \times 0.6)^2 + (0.6 \times 1.0)^2 + (0.4 \times 1.1)^2 + (0.5 \times 0.4)^2$

$= 0.2304 + 0.7056 + 0.81 + 0.2025 + 0.09 + 0.1296 + 0.36 + 0.1936 + 0.04 = 2.7617.$

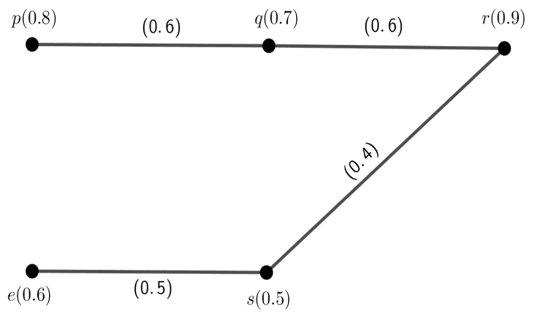

Figure 6. A fuzzy path G with $M_1^z(A) = 2.7617$.

Theorem 9. *Let $G = P(v_0, v_1, v_2, \ldots, v_n)$ be a path. If we take $v_0 = v_n$, it becomes a cycle $H = C(v_0, v_1, v_2, \ldots, v_n)$. Then $M_1^z(G) \le M_1^z(H)$.*

Proof. Let d and d' denote the degree of a vertex/edge in G and H respectively. Here, $d(v) = d'(v)$ for all vertices except v_0 and v_n. Now, $d(v_0) = e_1$ and $d(v_n) = e_n$ and $d'(v_0) = e_1 + e_n$. Additionally, $d(e) = d'(e)$ for all edges except e_1 and e_n. Now, $d(e_1) = e_2$ and $d(e_n) = e_{n-1}$ and $d'(e_1) = e_2 + e_n, d'(e_n) = e_1 + e_{n-1}$.

Now, $M_1^z(G) = \sum_{v \epsilon V(G)} \big(\omega(v)d(v)\big)^2 + \sum_{e \epsilon E(G)} \big(\varrho(e)d(e)\big)^2$

$= \big(\omega(v_0)d(v_0)\big)^2 + \big(\omega(v_n)d(v_n)\big)^2 + \sum_{v \epsilon V(G) - v_0, v_n} \big(\omega(v)d(v)\big)^2 + \big(\varrho(e_1)d(e_1)\big)^2 + \big(\varrho(e_n)d(e_n)\big)^2 + \sum_{e \epsilon E(G) - e_1, e_n} \big(\varrho(e)d(e)\big)^2$

$= \big(\omega'(v_0)e_1\big)^2 + \big(\omega'(v_n)e_n\big)^2 + \sum_{v \epsilon V(H) - v_0} \big(\omega'(v)d'(v)\big)^2 + \big(\varrho'(e_1)e_2\big)^2 + \big(\varrho'(e_n)e_{n-1}\big)^2 + \sum_{e \epsilon E(H) - e_1, e_n} \big(\varrho'(e)d'(e)\big)^2$

$= \big(\omega'(v_0)e_1\big)^2 + \big(\omega'(v_0)e_n\big)^2 + \sum_{v \epsilon V(H) - v_0} \big(\omega'(v)d'(v)\big)^2 + \big(\varrho'(e_1)e_2\big)^2 + \big(\varrho'(e_n)e_{n-1}\big)^2 + \sum_{e \epsilon E(H) - e_1, e_n} \big(\varrho'(e)d'(e)\big)^2$

$= \omega'(v_0)\big((e_1)^2 + (e_n)^2\big) + \sum_{v \epsilon V(H) - v_0} \big(\omega'(v)d'(v)\big)^2 + \big(\varrho'(e_1)e_2\big)^2 + \big(\varrho'(e_n)e_{n-1}\big)^2 + \sum_{e \epsilon E(H) - e_1, e_n} \big(\varrho'(e)d'(e)\big)^2$

$\leq \omega'(v_0)\big((e_1)+(e_n)\big)^2 + \sum\limits_{v\epsilon V(H)-v_0} \big(\omega'(v)d'(v)\big)^2 + \big(\varrho'(e_1)(e_2+e_n)\big)^2 + \big(\varrho'(e_n)(e_{n-1}+e_1)\big)^2 +$
$\sum\limits_{e\epsilon E(H)-e_1,e_n} \big(\varrho'(e)d'(e)\big)^2$
$= M_1^z(H).$
So, $M_1^z(G) \leq M_1^z(H)$. $\square$

Example 7. *Let G be a path with vertex set* $V = p,q,r,s,e$ *such that* $\omega(p) = 0.8, \omega(q) = 0.7, \omega(r) = 0.9, \omega(s) = 0.5, \omega(e) = 0.8, \varrho(p,q) = 0.6, \varrho(q,r) = 0.6, \varrho(r,s) = 0.4$ *and* $\varrho(s,e) = 0.5$ *shown in Figure 7.*

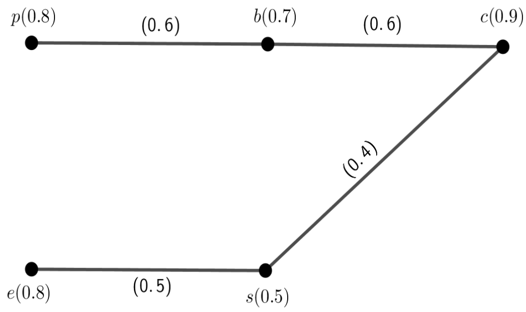

Figure 7. A fuzzy path G with $M_1^z(G)$ = 2.8317.

Then, $d(p) = 0.6, d(q) = 1.2, d(r) = 1.0, d(s) = 0.9, d(e) = 0.5, d(p,q) = 0.6, d(q,r) = 1.0, d(r,s) = 1.1, d(s,e) = 0.4$
Now, $M_1^z(G) = \sum\limits_{x\epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e\epsilon E(G)} \big(\varrho(e)d(e)\big)^2$
$= (0.8\times 0.6)^2 + (0.7\times 1.2)^2 + (0.9\times 1.0)^2 + (0.5\times 0.9)^2 + (0.8\times 0.5)^2 + (0.6\times 0.6)^2 + (0.6\times 1.0)^2 + (0.4\times 1.1)^2 + (0.5\times 0.4)^2$
$= 0.2304 + 0.7056 + 0.81 + 0.2025 + 0.16 + 0.1296 + 0.36 + 0.1936 + 0.04 = 2.8317.$

Marge two vertex a and e in G, as shown in Figure 8.

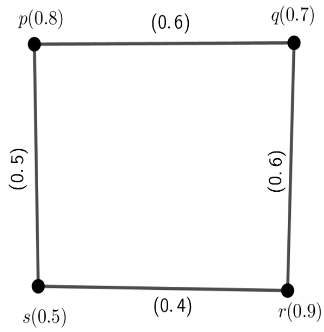

Figure 8. A fuzzy graph H with $M_1^z(H) = 3.7317$.

Then, $d(p) = 1.1, d(q) = 1.2, d(r) = 1.0, d(s) = 0.9, d(p,q) = 1.1, d(q,r) = 1.0, d(r,s) = 1.1, d(s,p) = 1.0.$
Now, $M_1^z(H) = \sum\limits_{x\epsilon V(H)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e\epsilon E(H)} \big(\varrho(e)d(e)\big)^2$
$= (0.8\times 1.1)^2 + (0.7\times 1.2)^2 + (0.9\times 1.0)^2 + (0.5\times 0.9)^2 + (0.6\times 1.1)^2 + (0.6\times 1.0)^2 + (0.4\times 1.1)^2 + (0.5\times 1.0)^2$
$= 0.7744 + 0.7056 + 0.81 + 0.2025 + 0.4356 + 0.36 + 0.1936 + 0.25 = 3.7317.$

Theorem 10. *Let* $G = C(v_0, v_1, \ldots, v_n)$ *be a cycle. Then,*

1. $M_1^z(G) = ZF_1(G) + \big(\varrho(e_1)\big)^2\big(\varrho(e_2)+\varrho(e_n)\big)^2 + \sum\limits_{i=2}^{i=n} \big(\varrho(e_i)\big)^2\big(\varrho(e_{i-1})+\varrho(e_{i+1})\big)^2$

2. $M_1^z(G) \leq 8(n+1) - 4$.

Proof. Similar proof to Theorem 8. □

Theorem 11. *Let G_1 and G_2 be two fuzzy graphs and they are isomorphic to each other. Then, $M_1^z(G_1) = M_1^z(G_2)$.*

Proof. As G_1 and G_2 are isomorphic to each other, then there exists a bijective mapping $\phi : V_1 \rightarrow V_2$ and for all $u, v \epsilon V_1$ then $\omega_1(v) = \omega_2(\phi(v))$ and $\varrho_1(uv) = \varrho_2(\phi(u)\phi(v))$
Then, $d_{G_1}(v) = \sum_{u \in V_1} \varrho_1(uv) = \sum_{\phi(u) \in V_2} \varrho_2(\phi(u)\phi(v)) = d_{G_2}(\phi(v))$.
Now, $M_1^z(G) = \sum_{v \in V(G_1)} (\omega(v) d_{G_1}(v))^2 + \sum_{uv \in E(G_1)} (\varrho(uv) d_{G_1}(uv))^2$
$= \sum_{v \in V(G_2)} \Big(\omega(\phi(v)) d_{G_2}(\phi(v))\Big)^2 + \sum_{\phi(u)\phi(v) \in E(G_2)} \Big(\varrho(\phi(u)\phi(v)) d_{G_2}(\phi(u)\phi(v))\Big)^2$
$= M_1^z(G_2)$.
So, $M_1^z(G_1) = M_1^z(G_2)$. □

Theorem 12. *If $G = K_{1,p-1}$ (see Figure 9) is a star and satisfies the condition $\omega(o) \leq \omega(v)$, where o is the center of the star, then the value of the FEZI is*
$M_1^z(G) = (\omega(o))^2 (\sum_{v \epsilon V(G) - o} \omega(v)^2 + \omega(o)^2 (p-1)(p^2 - 3p + 3))$.

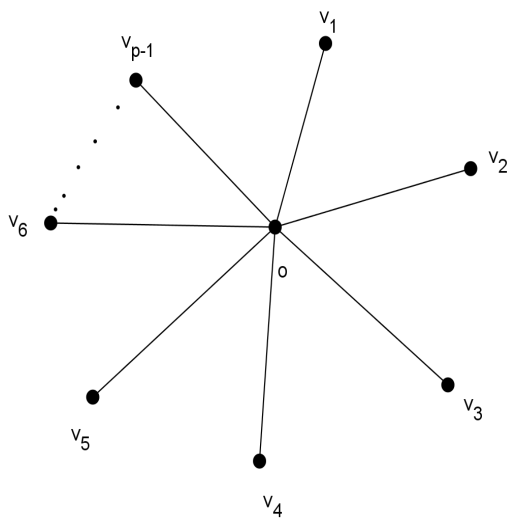

Figure 9. Graph of $K_{1,p-1}$.

Proof. Given that $\omega(o) \leq \omega(v)$, where o is the center of the star, so $\varrho(e) = \omega(o)$ for all edges. Now, $deg(v)$ for all $v \epsilon V - o$ is given by $\varrho(ov) = \omega(o)$ and $deg(o) = \sum_{v \epsilon V - o} \varrho(ov) = (p-1)\omega(o)$ also $deg(e) = (p-2)\omega(o)$.

Now, $M_1^z(G) = \sum_{v \epsilon V(G)} (\omega(v) deg(v))^2 + \sum_{e \epsilon E(G)} (\varrho(e) deg(e))^2$
$= \sum_{v \epsilon V(G) - o} (\omega(v)\omega(o))^2 + (\omega(o)(p-1)\omega(o))^2 + \sum_{e \epsilon E(G)} (\omega(o)(p-2)\omega(o))^2$
$= (\omega(o))^2 (\sum_{v \epsilon V(G) - o} \omega(v)^2 + \omega(o)^2 (p-1)(p^2 - 3p + 3))$.
This shows that $M_1^z(G) = (\omega(o))^2 (\sum_{v \epsilon V(G) - o} \omega(v)^2 + \omega(o)^2 (p-1)(p^2 - 3p + 3))$. □

Theorem 13. *If $G = S(a,b)$ is a double star, c_1 is the center of the first star and c_2 is the center of the second star and satisfies the condition $\omega(c_1) = \omega(c_2) \leq \omega(v)$ for all $v\epsilon V(G)$, then $M_1^z(G) = ZF_1G + \omega(c_1)^4(a^3 + b^3 + (a+b)^2)$.*

Proof. Since $\omega(c_1) = \omega(c_2) \leq \omega(v)$ for all $v\epsilon V(G)$,
$\varrho(c_1u) = \omega(c_1), \varrho(c_2v) = \omega(c_2) = \omega(c_1)$.
Additionally, $deg(u) = \omega(c_1), deg(v) = \omega(c_2) = \omega(c_1)$,
$deg(c_1) = (a+1)\omega(c_1), deg(c_2) = (b+1)\omega(c_2) = (b+1)\omega(c_1)$,
$deg(uc_1) = a\omega(c_1), deg(uc_2) = b\omega(c_2) = b\omega(c_1), deg(c_1c_2) = (a+b)\omega(c_1)$.
Now, $M_1^z = \sum\limits_{x\epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e\epsilon E(G)} \big(\varrho(e)d(e)\big)^2$
$= ZF_1(G) + \sum\limits_{e\epsilon E(G_1)} \big(\varrho(e)deg(e)\big)^2 + \sum\limits_{e\epsilon E(G_2)} \big(\varrho(e)deg(e)\big)^2 + \varrho(c_1c_2)deg(c_1c_2)^2$
$= ZF_1(G) + \sum\limits_{e\epsilon E(G_1)} \big(\omega(c_1)a\omega(c_1)\big)^2 + \sum\limits_{e\epsilon E(G_2)} \big(\omega(c_1)b\omega(c_1)\big)^2 + \big(\omega(c_1)(a+b)\omega(c_1)\big)^2$
$= ZF_1(G) + \omega(c_1)^4(a^3 + b^3 + (a+b)^2)$
$M_1^z(G) = ZF_1G + \omega(c_1)^4(a^3 + b^3 + (a+b)^2)$. □

Theorem 14. *For a firefly graph $F_{s,t,p-2s-2t-1}$, if the MV of each vertex as well as the edge is one, then the value of the FEZI is $M_1^z(G) = p + 10s + 4t - 1 + (p-t-1)^2 + (p-t-1)(p-t-2)^2$.*

Proof. Since the MV of each vertex and each edge is one, the number of vertices with degree 1 is $p - 2s - 2t - 1 + t = p - 2s - t - 1$,
the number of vertices with degree 2 is $2s + t$,
and $deg(c) = 2(p+s-1) - (p-2s-t-1) - 2(2s+t) = p - t - 1$,
the number of edges with degree 1 is t,
the number of edges with degree 2 is s,
the remaining (p-t-1) edges have degree $(p-t-2)$.
Then, the FEZI is $M_1^z(G) = \sum\limits_{x\epsilon V(G)} \big(\omega(x)d(x)\big)^2 + \sum\limits_{e\epsilon E(G)} \big(\varrho(e)d(e)\big)^2$
$= (p-2s-t-1) + 4(2s+t) + (p-t-1)^2 + t + 4s + (p-t-1)(p-t-2)^2 = p + 10s + 4t - 1 + (p-t-1)^2 + (p-t-1)(p-t-2)^2$. □

Corollary 2. *If $s = t = 0$, then for a firefly graph,*

$$M_1^z(G) = (\omega(c))^2\Big(\sum_{v\epsilon V(G)-c} \omega(v)^2 + \omega(c)^2(n-1)(n^2-3n+3)\Big).$$

If we put $s = t = 0$ in $F_{s,t,n-2s-2t-1}$, then the graph is a star similar to $K_{1,n-1}$
So, $M_1^z(G) = M_1^z(K_{1,n-1}) = (\omega(c))^2\Big(\sum\limits_{v\epsilon V(G)-c} \omega(v)^2 + \omega(c)^2(n-1)(n^2-3n+3)\Big)$.

4. Application of FEZI for Fuzzy Graphs to Find out the State Which Require More Development for Internet System

4.1. Model Construction

In the modern day, the internet is the most important part of our regular life. Here in this paper, we analyzed the Reliance Jio infocomm Ltd internet system in India. The data of Reliance Jio infocomm Ltd internet users are given in Table 1. These data were taken from https://dot.gov.in/sites/default/files/2022, accessed on 15 December 2022 . Then, we constructed a Reliance Jio infocomm Ltd internet system graph (see Figure 10). Here, the whole graph is similar to a star where Reliance Jio infocomm Ltd (C) is the center of the star and each state is a pendent vertex of the star.

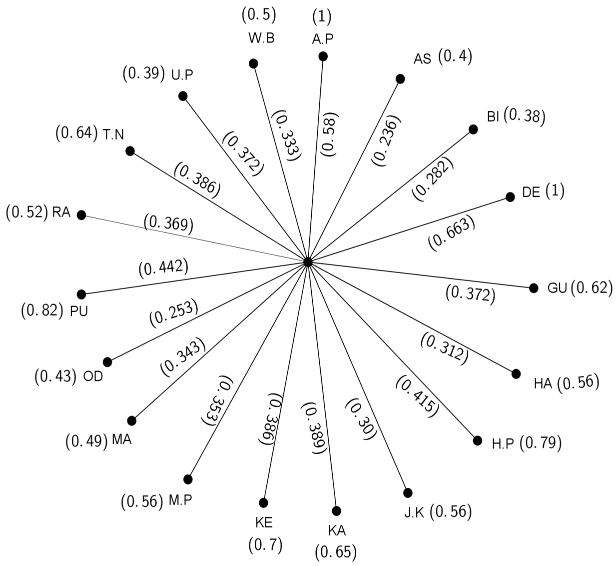

Figure 10. Fuzzy graph of internet network of Reliance jio.

Table 1. Data of internet users for all states.

State	Internet Users (in Million)	Total Population (in Million)	Population Percentage of the State
Andhra Pradesh (A.P)	56.06	52.883	4
Assam (AS)	14.14	35.4	2.6
Bihar (BI)	48.11	125.1	9.2
Delhi (DE)	38.89	31.2	2.3
Gujrat (GU)	43.68	70.7	5.2
Haryana (HA)	16.74	29.9	2.2
Himachal Pradesh (H.P)	5.89	7.45	0.50
Jammu and Kashmir (J.K)	7.55	13.6	1.0
Karnataka (KA)	43.68	67.3	4.9
Kerala (KE)	24.92	35.4	2.6
Madhya Pradesh (M.P)	47.78	85.6	6.3
Maharashtra (MA)	61.12	125.5	9.3
Odisha (OD)	19.02	44.2	3.3
Punjab (PU)	25.1	30.6	2.2
Rajasthan (RA)	41.75	80.2	5.9
Tamil Nadu (T.N)	49.17	76.7	5.6
Uttar Pradesh (U.P)	91.35	233.4	17.2
West Bengal (W.B)	49.1	98.7	7.3

4.2. Representation of Membership Values

Now, the MV of a vertex is denoted as $\omega(S)$ according to the formula below:
$\wedge \{1, \frac{Total\ internet\ users\ in\ the\ state}{Total\ population\ in\ the\ state}\}$.
Here we see that $\omega(S) \in [0,1]$, since there is an edge between center C and a state S. The MV of this edge is denoted as $\varrho(CS)$ and is defined by the following formula:
$\wedge\{1, \frac{Total\ Jio\ internet\ users\ in\ this\ state\ S}{Total\ population\ in\ this\ state\ S} + \frac{Population\ persentage\ in\ this\ state\ S}{100}\}$.
Here, we see that $\varrho(CS) \in [0,1]$.

The membership values are given in the Tables 2 and 3.

Table 2. Some values with respect to internet users.

State	Jio Internet Users (Million)	$\frac{Total\ internet\ users}{Total\ population}$	$\frac{Jio\ internet\ users}{Total\ population}$	$\frac{Jio\ internet\ users}{Total\ population} + \frac{population\ percentage}{100}$
A.P	28.9	1.06	0.54	0.58
AS	7.3	0.4	0.21	0.236
BI	24.82	0.38	0.19	0.282
DE	20.06	1.25	0.64	0.663
GU	22.53	0.62	0.32	0.372
HA	8.64	0.56	0.29	0.312
H.P	3.03	0.79	0.41	0.415
J.K	3.89	0.56	0.29	0.30
KA	22.53	0.65	0.34	0.389
KE	12.86	0.70	0.36	0.386
M.P	24.65	0.56	0.29	0.353
MA	31.54	0.49	0.25	0.343
OD	9.81	0.43	0.22	0.253
PU	12.95	0.82	0.42	0.442
RA	21.54	0.52	0.27	0.369
T.N	25.37	0.64	0.33	0.386
U.P	47.13	0.39	0.20	0.372
W.B	25.34	0.5	0.26	0.333

Then the value of the FEZI is given by

$$M_1^z(G) = \sum_{v \epsilon V(G)} \big(\omega(v)d(v)\big)^2 + \sum_{e \epsilon E(G)} \big(\varrho(e)d(e)\big)^2 = 112.02.$$

The FEZI of states (vertex) are given in the Table 4, and they are calculated by the formula $M_1^z(State) = M_1^z(G) - M_1^z(G_{State})$.

4.3. Decision Making

From the Table 4, we have $M_1^z(AS) = M_1^z(BI) = M_1^z(OD) < M_1^z(U.P) < M_1^z(HA) = M_1^z(J.K) = M_1^z(MA) = M_1^z(W.B) < M_1^z(M.P) = M_1^z(RA) < M_1^z(GU) < M_1^z(KA) = M_1^z(TN) < M_1^z(KE) < M_1^z(H.P) < M_1^z(PU) < M_1^z(A.P) < M_1^z(DE)$.

Now, the least FEZI of a vertex indicates that the vertex is most crucial for the development of internet network systems. Here, the first entire Zagreb index of two or more states is equal. In this situation, we will find the population percentage of these states who cannot use the internet. Then, the state having the largest population percentage that does not use the internet becomes the first to develop an internet network system.

Here, the percentages of the population who do not use the internet (PP) of the states are as follows: PP(AS) = 21.26, PP(BI) = 25.18, PP(OD) = 76.99, PP(HA) = 13.16, PP(J.K) = 6.05, PP(MA) = 64.38, PP(W.B) = 49.6, PP(M.P) = 37.82, PP(RA) = 38.45, PP(KA) = 23.62, PP(T.N) = 27.53. Then we can order the states as follows, and needing more development:

Odisha, Bihar, Assam, Uttar Pradesh, Maharashtra, West Bengal, Haryana, Jammu and Kashmir, Rajasthan, Madhya Pradesh, Gujrat, Tamil Nadu, Karnataka, Kerala, Himachal Pradesh, Punjab, Andhra Pradesh, Delhi.

Table 3. Membership values and degrees of the FG of Figure 10.

State	MV of the State (Vertex)	Degree of the State (Vertex) = MV of the Edge	Degree of an Edge between State and Center of the Star
A.P	1	0.58	6.060
AS	0.4	0.236	6.550
BI	0.38	0.282	6.504
DE	1	0.663	6.123
GU	0.62	0.372	6.414
HA	0.56	0.312	6.474
H.P	0.79	0.415	6.371
J.K	0.56	0.30	6.486
KA	0.65	0.389	6.397
KE	0.70	0.386	6.400
M.P	0.56	0.353	6.433
MA	0.49	0.343	6.443
OD	0.43	0.253	6.533
PU	0.82	0.442	6.344
RA	0.52	0.369	6.417
T.N	0.64	0.386	6.400
U.P	0.39	0.372	6.414
W.B	0.5	0.333	6.453

Table 4. Value of first entire Zagreb index for all states.

State	$M_1^z(G - State)$	$M_1^z(State)$
A.P	111.68	0.34
AS	112.01	0.01
BI	112.01	0.01
DE	111.58	0.44
GU	111.97	0.05
HA	111.99	0.03
H.P	111.91	0.11
J.K	111.99	0.03
KA	111.96	0.06
KE	111.95	0.07
M.P	111.98	0.04
MA	111.99	0.03
OD	112.01	0.01
PU	111.89	0.13
RA	111.98	0.04
T.N	111.96	0.06
U.P	112.00	0.02
W.B	111.99	0.03

5. Conclusions

In this article, the FEZI (first entire Zagreb index) was introduced as a graph parameter to quantify the structural characteristics of a graph. This study introduced several results and established relationships between various isomorphic graphs and α-cut fuzzy graphs. Additionally, the paper provided bounds for some fuzzy graphs and applied these results to real-life problems in the field of internet system development. The precise values or boundaries of FEZI with regards to graphs such as the firefly graph, star, path, cycle, and others, were explored in this study. The relationship between the value of the FEZI of a graph and its subgraph, two isomorphic graphs, and the first Zagreb index and FEZI were also described here. To analyze the Reliance Jio infocomm Ltd. internet system in India, this paper constructed an internet system graph. In this graph, the least FEZI of a vertex indicates that the vertex is most crucial for the development of the internet network system, and the first entire Zagreb index of two or more states is equal. According to this paper, the state with the highest proportion of people who do not use the internet is the first to develop an internet network system. The first entire Zagreb index is also important in biochemistry, chemical graph theory, spectral graph theory, etc.

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