axioms

Article

Norm of Hilbert Operator's Commutants

Hadi Roopaei

Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 2Y2, Canada; h.roopaei@gmail.com

Abstract: In this study, we obtain the ℓ_p-norms of six classes of operators that commute with the infinite Hilbert operators.

Keywords: commutators; Hilbert matrix; Cesàro matrix; norm

MSC: 46B45; 46A45; 40G05; 47B37

1. Introduction

We use the notation $\|T\|_p$ for the norm of a linear operator from the sequence space ℓ_p to itself. Several references have addressed the problem of finding the norm and lower bound of operators on matrix domains [1–7]. Our study considers infinite matrices $[A]_{j,k}$, where all the indices j and k are non-negative.

Definition 1 (Hilbert matrix). *If n is a non-negative integer, we define the Hilbert matrix of order n, H_n, as follows:*

$$[H_n]_{j,k} = \frac{1}{j+k+n+1} \qquad j,k = 0,1,\ldots.$$

In the case of $n = 0$, $H_0 = H$ is the well-known Hilbert matrix, which was introduced by David Hilbert in 1894. According to [8] theorem 323, the Hilbert matrix is a bounded operator on ℓ_p and

$$\|H\|_p = \Gamma(1/p)\Gamma(1/p^*) = \pi\csc(\pi/p),$$

where p^* is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.

Definition 2 (Hausdorff matrices). *One of the best examples of summability matrices is H_μ, which is defined as*

$$[H_\mu]_{j,k} = \begin{cases} \int_0^1 \binom{j}{k}\theta^k(1-\theta)^{j-k}d\mu(\theta) & 0 \le k \le j, \\ 0 & \text{otherwise}. \end{cases}$$

where μ is a probability measure on $[0,1]$. Even though it is a difficult task to obtain the ℓ_p-norm of operators, the Hausdorff matrices can be computed using Hardy's formula [9], Theorem 216, which states that this matrix is a bounded operator on ℓ_p, if and only if

$$\int_0^1 \theta^{\frac{-1}{p}}\, d\mu(\theta) < \infty, \qquad 1 \le p < \infty.$$

check for updates

Citation: Roopaei, H. Norm of Hilbert Operator's Commutants. *Axioms* **2023**, *12*, 422. https://doi.org/10.3390/axioms12050422

Academic Editor: Palle E.T. Jorgensen

Received: 16 March 2023
Revised: 21 April 2023
Accepted: 24 April 2023
Published: 26 April 2023

In fact,

$$\|H_\mu\|_p = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta). \tag{1}$$

Hausdorff operators have the interesting norm-separating property.

Theorem 1 ([7], Theorem 9)**.** *Let $p \geq 1$ and H_μ, H_φ and H_ν be Hausdorff matrices such that $H_\mu = H_\varphi H_\nu$. Then, H_μ is bounded on ℓ_p if and only if both H_φ and H_ν are bounded on ℓ_p. Moreover, we have*

$$\|H_\mu\|_p = \|H_\varphi\|_p \|H_\nu\|_p.$$

For comprehensive information about the Hausdorff matrices, the enthusiastic reader can refer to [10,11].

Several famous matrices have been derived from the Hausdorff matrix. For positive integer n, the following are the two classes.

Definition 3 (Cesàro matrix)**.** *The measure $d\mu(\theta) = n(1-\theta)^{n-1}d\theta$ gives the Cesàro matrix of order n, C_n, for which*

$$[C_n]_{j,k} = \begin{cases} \dfrac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \leq k \leq j, \\ 0 & \textit{otherwise}. \end{cases}$$

Note that $C_0 = I$, where I is the identity matrix, and $C_1 = C$ is the classical Cesàro matrix. According to (1), C_n has the ℓ_p-norm

$$\|C_n\|_p = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)},$$

which for the famous Cesàro matrix that is $\|C\|_p = \frac{p}{p-1}$.

Definition 4 (Gamma matrix)**.** *The measure $d\mu(\theta) = n\theta^{n-1}d\theta$ gives the Gamma matrix of order n, G_n, for which*

$$[G_n]_{j,k} = \begin{cases} \frac{\binom{n+k-1}{k}}{\binom{n+j}{j}} & 0 \leq k \leq j, \\ 0 & \textit{otherwise}. \end{cases}$$

Hence, by Hardy's formula, G_n has the ℓ_p-norm

$$\|G_n\|_p = \frac{np}{np-1}.$$

You should note that G_1 is the classical Cesàro matrix C.

A well-known property of Hausdorff means is that products are determined by the diagonal elements. Specifically, if A, B, and C are Hausdorff means and $[A]_{j,j}[B]_{j,j} = [C]_{j,j}$ for all j, then $AB = C$. (This is proved in [9], Section 11.3, though in different notation.) The following result is also known from [9]:

Theorem 2. *$C_{n-1}G_n = C_n$, hence $C_n = S_{n,m}C_m$ for $m < n$, where $S_{n,m} = G_{m+1} \cdots G_n$.*

Proof. The diagonal elements of $C_{n-1}G_n$ are

$$[C_{n-1}G_n]_{j,j} = \frac{j!}{n(n+1)\cdots(n+j-1)}\frac{n}{n+j} = [C_n]_{j,j}.$$

Hence, the stated identity. □

The following result is known as the Hellinger–Toeplitz theorem.

Theorem 3 ([1], Proposition 7.2)**.** *Let $1 < p, q < \infty$. The matrix M maps ℓ_p into ℓ_q if and only if the transposed matrix, M^t, maps ℓ_{q^*} into ℓ_{p^*}. Then, we have*

$$\|M\|_{\ell_p \to \ell_q} = \|M^t\|_{\ell_{q^*} \to \ell_{p^*}}.$$

As an example of the Hellinger–Toeplitz theorem, the transposed Cesàro matrix of order n has the ℓ_p-norm

$$\|C_n^t\|_p = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}.$$

Motivation. Hilbert operators are used in a wide range of fields including approximation theory, cryptography, image processing, functional analysis, representation theory, and noncommutative geometry. The estimates of the norm of this operator and the study of its properties in various spaces are of considerable interest and have a long history. Recently, ref. [12] has introduced some classes of Hilbert's commutators mostly based on Cesàro and Gamma matrices. In this study, we establish the ℓ_p norm of these operators.

For non-negative integers n, j and k, let us define the matrix B_n by

$$[B_n]_{j,k} = \binom{n+k}{k}\beta(j+k+1, n+1) = \frac{(k+1)\cdots(k+n)}{(j+k+1)\cdots(j+k+n+1)},$$

where the β function is

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dz \qquad (m, n = 1, 2, \ldots).$$

$B_0 = H$ where H represents Hilbert's matrix.

We need the following lemma before we can discuss the Hilbert operator's commutants, which reveals the relationship between the Hilbert operator and the Cesàro and Gamma matrices.

Lemma 1 (Lemmas 2.3 and 3.1 of [13,14])**.** *Hilbert matrices satisfy the following identities for positive integer n:*

- $H = B_n C_n$
- $H_n = C_n B_n$
- $H_n C_n = C_n H$
- $H_n G_n = G_n H_{n-1}$
- *B_n is a bounded operator that has the ℓ_p-norm*

$$\|B_n\|_p = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}.$$

- $\|H\|_p = \|B_n\|_p \|C_n\|_p$,

where C_n and G_n are the Cesàro and Gamma matrices of order n and B_n is the matrix, which was defined earlier.

Commutants of the infinite Hilbert operator. Assume that n is a non-negative integer, and define the symmetric matrix as follows:

$$\Phi_n^b = B_n^t B_n \qquad \Psi_n^b = B_n B_n^t$$

$$\Phi_n^c = C_n^t C_n \qquad \Psi_n^c = C_n C_n^t$$

and for $n \geq 1$

$$\Phi_n^g = G_n^t G_n \qquad \Psi_n^g = G_n G_n^t,$$

Note that for $n = 1$,

$$\Psi := \Psi_1^c = \Psi_1^g = CC^t \quad \textit{and} \quad \Phi := \Phi_1^c = \Phi_1^g = C^t C.$$

In [12] Theorems 11.2.2 and 11.2.4, the author has proved that the above matrices are commutants of Hilbert operators. We present those theorems with their proofs.

Theorem 4. *The operators Φ_n^c and Ψ_n^b are commutants of H.*

Proof. By applying Lemma 1 twice, we have

$$\begin{aligned} \Phi_n^c H &= C_n^t H_n C_n = (H_n C_n)^t C_n \\ &= (C_n H)^t C_n = H C_n^t C_n = H \Phi_n^c. \end{aligned}$$

It can easily be seen from Lemma 1 that $HB_n = B_n H_n$. Now,

$$\begin{aligned} \Psi_n^b H &= B_n (HB_n)^t = B_n (B_n H_n)^t \\ &= B_n H_n B_n^t = HB_n B_n^t = H\Psi_n^b. \end{aligned}$$

□

Theorem 5. *The operators Φ_n^b, Φ_{n+1}^g, Ψ_n^c and Ψ_n^g are commutants of the Hilbert operator of order n.*

Proof. By applying Lemma 1 twice, we have

$$\begin{aligned} \Psi_n^c H_n &= C_n (H_n C_n)^t = C_n (C_n H)^t \\ &= C_n H C_n^t = H_n C_n C_n^t = H_n \Psi_n^c. \end{aligned}$$

Additionally, applying Lemma 1 results in

$$\begin{aligned} \Psi_n^g H_n &= G_n (H_n G_n)^t = G_n (G_n H_{n-1})^t \\ &= G_n H_{n-1} G_n^t = H_n G_n G_n^t = H_n \Psi_n^g. \end{aligned}$$

The proof of the other items is similar. □

2. Main Results

For non-negative integers m and n, let us define the following matrices:

$$\Phi_{m,n}^b = B_m^t B_n \qquad \Psi_{m,n}^b = B_m B_n^t$$

$$\Phi_{m,n}^c = C_m^t C_n \qquad \Psi_{m,n}^c = C_m C_n^t$$

and for $m, n \geq 1$

$$\Phi^{g}_{m,n} = G^{t}_{m} G_{n} \qquad \Psi^{g}_{m,n} = G_{m} G^{t}_{n}.$$

Note that for $m = n$, all the above matrices are reduced to the Hilbert operator's commutators that we introduced earlier. Through this section, we will prove the norm-separating property for the Cesàro and Gamma matrices of the form:

$$\|C_m C^t_n\|_p = \|C_m\|_p \|C^t_n\|_p,$$

$$\|C^t_m C_n\|_p = \|C^t_m\|_p \|C_n\|_p,$$

$$\|G_m G^t_n\|_p = \|G_m\|_p \|G^t_n\|_p,$$

$$\|G^t_m G_n\|_p = \|G^t_m\|_p \|G_n\|_p.$$

Theorem 6. *For non-negative integers m and n, matrices $\Psi^c_{m,n}$ and $\Phi^c_{m,n}$ are bounded operators on ℓ_p and*

$$\|\Psi^c_{m,n}\|_p = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+1/p^*)\Gamma(n+1/p)} \pi \csc(\pi/p)$$

$$\|\Phi^c_{m,n}\|_p = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+1/p)\Gamma(n+1/p^*)} \pi \csc(\pi/p).$$

In particular, the matrices Ψ^c_n and Φ^c_n are bounded operators on ℓ_p and

$$\|\Psi^c_n\|_p = \|\Phi^c_n\|_p = \frac{\Gamma^2(n+1)}{\Gamma(n+1/p)\Gamma(n+1/p^*)} \pi \csc(\pi/p).$$

Theorem 7. *For positive integers m and n, matrices $\Psi^g_{m,n}$ and $\Phi^g_{m,n}$ are bounded operators on ℓ_p and*

$$\|\Psi^g_{m,n}\|_p = \frac{mnpp^*}{(mp-1)(np^*-1)}$$

$$\|\Phi^g_{m,n}\|_p = \frac{mnpp^*}{(mp^*-1)(np-1)}.$$

In particular, the matrices Ψ^g_n and Φ^g_n are bounded operators on ℓ_p and

$$\|\Psi^g_n\|_p = \|\Phi^g_n\|_p = \frac{n^2 pp^*}{(np-1)(np^*-1)}.$$

Theorem 8. *For non-negative integers m and n, the matrices $\Psi^b_{m,n}$ and $\Phi^b_{m,n}$ are bounded operators on ℓ_p and*

$$\|\Psi^b_{m,n}\|_p = \frac{\Gamma(m+1/p^*)\Gamma(n+1/p)}{\Gamma(m+1)\Gamma(n+1)} \pi \csc(\pi/p)$$

$$\|\Phi^b_{m,n}\|_p = \frac{\Gamma(m+1/p)\Gamma(n+1/p^*)}{\Gamma(m+1)\Gamma(n+1)} \pi \csc(\pi/p).$$

In particular, the matrices Ψ_n^b and Φ_n^b are bounded operators on ℓ_p and

$$\|\Psi_n^b\|_p = \|\Phi_n^b\|_p = \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)}{\Gamma^2(n+1)}\pi\csc(\pi/p).$$

3. Proof of Theorems

In this section, we focus on proving our claims, but first, we need the following lemmas.

Lemma 2. *For the Hilbert operator, we have $\|H^2\|_p = \|H\|_p^2$.*

Proof. Let H be the Hilbert operator with matrix entries $1/(j+k)(j,k \geq 1)$, and write $M_r = \pi/\sin(r\pi)$. It is well known that $\|H\|_p \leq M_{1/p}$ for $p > 1$. Here, we show that $\|H\|_p \geq M_{1/p}$ and $\|H^2\|_p \geq M_{1/p}^2$ (so that equality holds in both cases). The same statements hold for the alternative Hilbert operator with matrix entries $\frac{1}{j+k-1}$.

Choose r with $rp > 1$, and let $x_k = 1/k^r$ for $k \geq 1$. Let $y = Hx$ and $z = Hy$. Then,

$$y_j = \sum_{k=1}^{\infty} \frac{1}{(j+k)k^r} \geq \int_1^{\infty} \frac{1}{(t+j)t^r}dt.$$

Now,

$$\int_0^{\infty} \frac{1}{(t+j)t^r}dt = \frac{M_r}{j^r},$$

and

$$\int_0^1 \frac{1}{(t+j)t^r}dt \leq \int_0^1 \frac{1}{jt^r}dt = \frac{1}{(1-r)j}.$$

so

$$y_j \geq \frac{M_r}{j^r} - \frac{1}{(1-r)j}. \tag{2}$$

Informally, y_j is approximately $M_r x_j$, so $\|y\|_{\ell_p}$ is approximately $M_r\|x\|_{\ell_p}$. For $0 < x < a$, we have $(1-\frac{x}{a})^p \geq 1 - \frac{px}{a}$, hence $(a-x)^p \geq a^p - pa^{p-1}x$. Hence,

$$y_j^p \geq \frac{M_r^p}{j^{rp}} - \frac{p}{1-r}\frac{M_r^{p-1}}{j^{rp-r+1}},$$

so,

$$\sum_{j=1}^{\infty} y_j^p \geq M_r^p\zeta(rp) - \frac{p}{1-r}M_r^{p-1}\zeta(rp-r+1),$$

while $\sum_{k=1}^{\infty} x_k^p = \zeta(rp)$. Now, let $r \to 1/p$ from above. Then, $\zeta(rp) \to \infty$, while $\zeta(rp-r+1) \to \zeta(2-1/p)$. Hence, $\frac{\|y\|_{\ell_p}}{\|x\|_{\ell_p}}$ tends to $M_{1/p}$.

We now turn to H^2. We require the following:

Let $u_k = 1/k$ for $k \geq 1$. Then,

$$\begin{aligned}(Hu)_j &= \sum_{k=1}^{\infty} \frac{1}{(j+k)k} = \frac{1}{j}\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{j+k}\right) \\ &= \frac{1}{j}\left(1 + \frac{1}{2} + \cdots + \frac{1}{j}\right) = L_j/j,\end{aligned}$$

where $L_j = \sum_{i=1}^{j} \frac{1}{i}$. By (2), $y \geq M_r x - u/(1-r)$, so

$$z \geq M_r(Hx) - \frac{Hu}{1-r} = M_r y - \frac{Hu}{1-r}.$$

So, again by (2),

$$z_j \geq \frac{M_r^2}{j^r} - \frac{M_r}{(1-r)j} - \frac{L_j}{(1-r)j}.$$

Hence,

$$z_j^p \geq \frac{M_r^{2p}}{j^{rp}} - p\frac{M_r^{2p-2}}{j^{r(p-1)}} \frac{M_r + L_j}{(1-r)j}.$$

Write $\eta(s) = \sum_{j=1}^{\infty} \frac{L_j}{j^s}$: this is convergent for $s > 1$. Then,

$$\sum_{j=1}^{\infty} z_j^p \geq M_r^{2p}\zeta(rp) - \frac{p}{(1-r)} M_r^{2p-2}(M_r\zeta(rp-r+1) + \eta(rp-r+1)).$$

When $r \to 1/p$ from above, $\eta(rp-r+1)$ tends to the finite limit $\eta(2-1/p)$. So $\|z\|_{\ell_p}/\|x\|_{\ell_p}$ tends to $M_{1/p}^2$. □

Lemma 3. *For the Hilbert operator of order n, we have $\|H_n^2\|_p = \|H_n\|_p^2$.*

Proof. With x and z as defined in the previous lemma, it shows that, given $\epsilon > 0$, there exists $\delta > 0$ such that if $r < \frac{1}{p} + \delta$, then $\|z\|_{\ell_p} \geq (M_{1/p}^2 - \epsilon)\|x\|_{\ell_p}$. Now, let $u = (H - H_n)x$ and $w = (H^2 - H_n^2)x = (H + H_n)u$. Then, $\|w\|_{\ell_p} \leq 2M_{1/p}\|u\|_{\ell_p}$. We show that for r close enough to $\frac{1}{p}$, $\|u\|_{\ell_p} \leq \epsilon\|x\|_{\ell_p}$. Now, for any $r > \frac{1}{p}$,

$$u_j = \sum_{k=1}^{\infty}\left(\frac{1}{j+k} - \frac{1}{j+k+n}\right)\frac{1}{k^r} < \sum_{k=1}^{\infty} \frac{n}{(j+k)^2 k^{1/p}}.$$

Hence,

$$\|u\|_{\ell_p} \leq \|u\|_{\ell_1} = \sum_{j=1}^{\infty} u_j < n\sum_{k=1}^{\infty}\frac{1}{k^{1/p}}\sum_{j=1}^{\infty}\frac{1}{(j+k)^2} < n\zeta(1+\frac{1}{p}),$$

since $\sum_{j=1}^{\infty} \frac{1}{(j+k)^2} < \frac{1}{k}$. Meanwhile, $\|u\|_{\ell_p} = \zeta(rp) \to \infty$ as $r \to \frac{1}{p}$. Hence, for r close enough to $\frac{1}{p}$, $\|u\|_{\ell_p} \leq \epsilon\|x\|_{\ell_p}$, as required. □

Proof of Theorem 6. We first compute the ℓ_p-norm of $\Psi_{m,n}^c$. Obviously,

$$\|\Psi_{m,n}^c\|_p \quad \leq \quad \|C_m\|_p\|C_n^t\|_p.$$

According to the Lemma 2, we also have $\|H\|_p = \|B_n^t\|_p\|C_n^t\|_p$, which results in

$$\|H\|_p^2 = \|B_m\|_p\|C_m\|_p\|C_n^t\|_p\|B_n^t\|_p.$$

Now, regarding the identity $H^2 = B_m\Psi_{m,n}^c B_n^t$ and Lemma 1,

$$\|H\|_p^2 = \|H^2\|_p \leq \|B_m\|_p\|\Psi_{m,n}^c\|_p\|B_n^t\|_p.$$

Hence,

$$\|\Psi^c_{m,n}\|_p \geq \|C_m\|_p \|C^t_n\|_p,$$

which completes the proof.

For computing the norm of $\Phi^c_{m,n}$, we suppose that $m \geq n$. The other case $m < n$ has a similar proof. In this case, regarding Lemma 1 and Theorem 2, we have

$$H^2_m = (C_m B_m)^t C_m B_m = B^t_m C^t_m C_n S_{m,n} B_m = B^t_m \Phi^c_{m,n} S_{m,n} B_m.$$

However, by applying Lemma 3,

$$\begin{aligned} \|H_m\|^2_p &= \|B^t_m\|_p \|C^t_m\|_p \|C_m\|_p \|B_m\|_p \\ &= \|B^t_m\|_p \|C^t_m\|_p \|C_n\|_p \|S_{m,n}\|_p \|B_m\|_p \\ &= \|H^2_m\|_p \leq \|B^t_m\|_p \|\Phi^c_{m,n}\|_p \|S_{m,n}\|_p \|B_m\|_p, \end{aligned}$$

which shows

$$\|\Phi^c_{m,n}\|_p \geq \|C^t_m\|_p \|C_n\|_p.$$

The other side of the above inequality is obvious, so the proof is complete. □

Proof of Theorem 7. First, $\|\Psi^g_{m,n}\|_p \leq \|G_m\|_p \|G^t_n\|_p$. From the definition and relation $C_n = C_{n-1} G_n = G_n C_{n-1}$, we have

$$\Psi^c_{m,n} = C_{m-1} G_m (C_{n-1} G_n)^t = C_{m-1} \Psi^g_{m,n} C^t_{n-1},$$

so $\|\Psi^c_{m,n}\|_p \leq \|G_{m-1}\|_p \|\Psi^g_{m,n}\|_p \|C^t_{n-1}\|_p$. By Theorem 6 and $\|C_m\|_p = \|C_{m-1}\|_p \|G_m\|_p$

$$\|\Psi^c_{m,n}\|_p = \|C_m\|_p \|C^t_n\|_p = \|C_{m-1}\|_p \|G_m\|_p \|G^t_n\|_p \|C^t_{n-1}\|_p,$$

hence $\|\Psi^g_{m,n}\|_p \geq \|G_m\|_p \|G^t_n\|_p$.

Similarly, using $C^t_m C_n = C^t_{m-1} \Phi^g_{m,n} C_{n-1}$ and Theorem 6. □

Proof of Theorem 8. Using the identities $H = B_n C_n = C^t_m B^t_m$, we obtain $H^2 = C^t_m \Phi^b_{m,n} C_n$. Reasoning as in the proof of Theorem 6, we obtain

$$\|\Phi^b_{m,n}\|_p \geq \|B^t_m\|_p \|B^t_m\|_p,$$

hence equality. Similarly, $\|\Psi^b_{m,n}\|_p = \|B_m\|_p \|B^t_n\|_p$. □

4. Conclusions

The author, in his previous work, has introduced the following symmetric matrices:

$$\Phi^b_n = B^t_n B_n \qquad \Psi^b_n = B_n B^t_n$$

$$\Phi^c_n = C^t_n C_n \qquad \Psi^c_n = C_n C^t_n$$

$$\Phi^g_n = G^t_n G_n \qquad \Psi^g_n = G_n G^t_n,$$

as the Hilbert operators commutants. In this study, we have obtained the ℓ_p-norm of these operators as:

- $\|\Psi^b_n\|_p = \|\Phi^b_n\|_p = \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)}{\Gamma^2(n+1)} \pi \csc(\pi/p)$

- $\|\Psi_n^c\|_p = \|\Phi_n^c\|_p = \frac{\Gamma^2(n+1)}{\Gamma(n+1/p)\Gamma(n+1/p^*)}\pi\csc(\pi/p)$
- $\|\Psi_n^g\|_p = \|\Phi_n^g\|_p = \frac{n^2pp^*}{(np-1)(np^*-1)}$

Through this research, we have also proved the norm-separating property for the Cesàro and Gamma matrices of the form:

$$\|C_mC_n^t\|_p = \|C_m\|_p\|C_n^t\|_p$$

$$\|C_m^tC_n\|_p = \|C_m^t\|_p\|C_n\|_p$$

$$\|G_mG_n^t\|_p = \|G_m\|_p\|G_n^t\|_p$$

$$\|G_m^tG_n\|_p = \|G_m^t\|_p\|G_n\|_p$$

Funding: There is no funding for this research.

Data Availability Statement: No data have been used in this study.

Acknowledgments: I am grateful to G.J.O. Jameson for several useful suggestions, in particular the proof of Lemmas 2 and 3.

Conflicts of Interest: The author declares that there are no conflict of interest.

References

1. Bennett, G. *Factorizing the Classical Inequalities*; American Mathematical Society: Providence, RI, USA, 1996; Memoirs of the American Mathematical Society; Volume 576.
2. Bennett, G. Lower bounds for matrices. *Linear Algebra Appl.* **1986**, *82*, 81–98. [CrossRef]
3. Chen, C.P.; Luor, D.C.; Ou, Z.Y. Extensions of Hardy inequality. *J. Math. Anal. Appl.* **2002**, *273*, 160–171. [CrossRef]
4. Bouthat, L.; Mashreghi, J. The norm of an infinite L-matrix. *Oper. Matrices* **2021**, *15*, 47–58. [CrossRef]
5. Jameson, G.J.O.; Lashkaripour, R. Norms of certain operators on weighted ℓ_p spaces and Lorentz sequence spaces. *J. Ineq. Pure Appl. Math.* **2002**, *3*, 1–17.
6. Gao, P. A note on ℓ_p norms of weighted mean matrices. *J. Inequal. Appl.* **2012**, *2012*, 110. [CrossRef]
7. Bennett, G. Lower bounds for matrices II. *Canad. J. Math.* **1992**, *44*, 54–74. [CrossRef]
8. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*, 2nd ed.; Cambridge University Press: Cambridge, MA, USA, 2001.
9. Hardy, G.H. *Divergent Series*; Oxford University Press: Oxford, UK, 1973.
10. Jameson, G.J.O. Hausdorff Means. Available online: http://www.maths.lancs.ac.uk/jameson/ (accessed on 20 April 2023).
11. Hardy, G.H. An inequality for Hausdorff means. *J. Lond. Math. Soc.* **1943**, *18*, 46–50. [CrossRef]
12. Roopaei, H. Commutants of the Infinite Hilbert operators. In *Approximation Theory, Sequence Spaces and Applications*; Springer: Berlin/Heidelberg, Germany, 2022; Chapter 11; pp. 203–219.
13. Roopaei, H. Norm of Hilbert operator on sequence spaces. *J. Inequal. Appl.* **2020**, *117*, 2020. [CrossRef]
14. Roopaei, H. Bounds of operators on the Hilbert sequence space. *Concr. Oper.* **2020**, *7*, 155–165. [CrossRef]