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Joint Discrete Approximation of Analytic Functions by Shifts of the Riemann Zeta Function Twisted by Gram Points II

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Abstract: In this paper, a theorem is obtained on the approximation in short intervals of a collection of analytic functions by shifts $(\zeta(s+it_k^{\alpha_1}),\dots,\zeta(s+it_k^{\alpha_r}))$ of the Riemann zeta function. Here, $\{t_k : k\in\mathbb{N}\}$ is the sequence of Gram numbers, and $\alpha_1,\dots,\alpha_r$ are different positive numbers not exceeding 1. It is proved that the above set of shifts in the interval $[N, N+M]$, here $M=o(N)$ as $N\to\infty$, has a positive lower density. For the proof, a joint limit theorem in short intervals for weakly convergent probability measures is applied.

Keywords: Gram numbers; Riemann zeta-function; universality; weak convergence

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1. Introduction

Let $s=\sigma+it$ be a complex variable, and $\mathbb{P}$ the set of all prime numbers. The Riemann zeta function $\zeta(s)$ is defined, in the half-plane $\sigma>1$, by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

or by the infinite Euler product

$$\zeta(s)=\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)^{-1},$$

and has analytic continuation to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1.

The function $\zeta(s)$ is a significant analytic object which is used not only in many branches of mathematics, but also in solving problems in other natural sciences, see, for example, [1–3]. This is due to deep connections between $\zeta(s)$ and objects of arithmetic, analytic and probabilistic character. It is not surprising that the function $\zeta(s)$ has a certain link to the famous mathematician and philosopher Pythagoras, who was not only a geometer but also the founder of mathematical philosophy. He saw mathematics everywhere, and said that all things are numbers, and began to use mathematics in astronomy and even music. Since the function $\zeta(s)$ is the main tool for the investigation of numbers, and has unexpected results even in cosmology and music (tuning problem), the theory of $\zeta(s)$ supports and develops the Pythagorean philosophy. Last time, applications of $\zeta(s)$ crossed the threshold of numbers, and the function $\zeta(s)$ became universal among functions. This paper is devoted to approximation problems of analytic functions by shifts $\zeta(s+i\tau)$, and is a continuation of the works in [4,5]. We recall that a possibility of the approximation of a class of functions by shifts of one and the same function is called universality, and was found by S.M. Voronin in [6], see also [7,8]. The discrete variant of the Voronin theorem was proposed by A. Reich in [9]. Let $\mathfrak{D}=\{s\in\mathbb{C}: 1/2<\sigma<1\}$, and let $\mathfrak{K}$ stand for the set of compact sets lying in $\mathfrak{D}$ and having connected complements. Moreover, denote by $H_0(K)$, $K \in \mathfrak{K}$ the set of non-vanishing continuous functions on $K \in \mathfrak{K}$, which are analytic inside of K. Let $\#A$ stand for the cardinality of the set A, and let $N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, the last discrete version of the Voronin universality theorem is the following statement [9]. For all $K \in \mathfrak{K}$, $f(s) \in H_0(K)$, and positive h and ε,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |f(s)| - \zeta(s+ikh)| < \varepsilon \right\} > 0.$$

In [10], we began to consider the joint approximation of analytic functions by shifts $(\zeta(s+it_k^{\alpha_1}), \dots, \zeta(s+it_k^{\alpha_r}))$, where $\{t_k\}$ is the sequence of Gram numbers. Let $\Gamma(s)$ be the gamma function. Then, the function $\zeta(s)$ has the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \quad s \in \mathbb{C}.$$

Let $\Theta(t)$, $t > 0$ be the increment of the argument of the product $\pi^{-s/2}\Gamma(s/2)$ along the segment which connect the points $s_1 = 1/2$ and $s_2 = 1/2 + it$. Since the function $\Theta(t)$ increases and is unbounded from above for $t \geqslant t^* = 6.2\dots$, the equation

$$\Theta(t) = \pi(n-1), \quad n \in \mathbb{N}_0, \tag{1}$$

has the unique solution t_n for $t \geqslant t^*$. J.P. Gram was the first to consider the points t_n in connection to non-trivial zeros $1/2 + i\gamma_n$ of the function $\zeta(s)$ [11], therefore they are called Gram points. More information on Gram points can be found in [12–14]. Equation (1) is also considered with arbitrary real $\tau \geqslant 0$ in place of n. In this case, we have the Gram function t_τ.

In [10], we obtained a joint universality theorem on the approximation of analytic functions by shifts $(\zeta(s+it_\tau^{\alpha_1}), \dots, \zeta(s+it_\tau^{\alpha_r}))$ with different fixed positive numbers $\alpha_1, \dots, \alpha_r$. The latter theorem was extended to short intervals in [4]. The paper in [5] is devoted to the discrete version of the results of [10].

Theorem 1 ([5])**.** *Let $\alpha_1, \dots, \alpha_r$ be fixed different positive numbers not exceeding 1, and for $j = 1, \dots, r$, $K_j \in \mathfrak{K}$ and $f_j(s) \in H_0(K_j)$. Then, for any $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+it_k^{\alpha_j}) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, the lower limit can be replaced by the limit for all but at most countably many $\varepsilon > 0$.

Keeping in mind the effectivization of Theorem 1, we prove in this paper a version of Theorem 1 in short intervals. Without a loss of generality, we may suppose that $\alpha_1 < \alpha_2 < \dots < \alpha_r$. For brevity, we set

$$\Psi_\alpha(\tau) = (t_\tau^\alpha)^{1/3} (\log t_\tau^\alpha)^{26/15},$$

where $\alpha = \alpha_1$. Moreover, we use the notation $(t_N^\alpha)' = (t_\tau^\alpha)'_{\tau=N}$. The objective of the paper is to prove that the set of approximating shifts $(\zeta(s+it_k^{\alpha_1}), \dots, \zeta(s+it_k^{\alpha_r}))$ in Theorem 1 has a positive lower density for every $\varepsilon > 0$ (and a positive density for all but at most countably many $\varepsilon > 0$) for k in the interval $[\Psi_\alpha(N)((t_N^\alpha)')^{-1}, N-1]$.

The paper is organized in the following way: In Section 2, some mean square estimates for the function $\zeta(s)$ in short intervals are obtained. Section 3 is devoted to a joint discrete limit theorem in short intervals on weakly convergent probability measures in r-dimensional space of analytic functions. Finally, in Section 4, we prove the main theorem.

2. Some Estimates

It is well known that mean square estimates occupy a central place in the proofs of universality theorems on the approximation of analytic functions by zeta functions. This, in a more complicated form, also takes place in the case of short intervals. Recall that the notation $x \ll_\delta y$, $x \in \mathbb{C}$, $y > 0$ means the existence of a constant $c = c(\delta) > 0$ such that $|x| \leqslant cy$.

We start with recalling a mean square estimate for the function $\zeta(s)$ in short intervals.

Lemma 1. *Suppose that* $\sigma \in (1/2, 13/22]$ *is fixed, and* $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$*. Then, uniformly in* H*,*

$$\int\limits_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \,\mathrm{d}t \ll_\sigma H.$$

Proof. Suppose that the exponential pair (κ, λ) and σ are connected by the inequality $1 + \lambda - \kappa \geqslant 2\sigma$, and $T^{(\kappa+\lambda+1)/(\kappa+1)}(\log T)^{(2+\kappa)/(\kappa+1)} \leqslant H \leqslant T$. Then, Theorem 7.1 of [15] asserts that, uniformly in H,

$$\int\limits_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \,\mathrm{d}t \ll_\sigma H.$$

Therefore, the exponential pair $(4/11, 6/11)$ gives the lemma. □

For the proof of a discrete limit theorem for the function $\zeta(s)$ twisted by Gram points in short intervals, we need the corresponding mean square estimate. Unfortunately, we do not know a discrete version of Lemma 1. Therefore, we will derive the desired estimate from a continuous one which is contained in the next lemma.

Lemma 2. *Suppose that* $0 < \alpha \leqslant 1$ *and* $\sigma \in (1/2, 13/22]$ *are fixed,* $\Psi_\alpha(T)((t_T^\alpha)')^{-1} \leqslant H \leqslant T$*, and* $t \in \mathbb{R}$*. Then,*

$$\int\limits_{T-H}^{T+H} |\zeta(\sigma+it+it_\tau^\alpha)|^2 \,\mathrm{d}\tau \ll_\sigma H(1+|t|).$$

Proof. It is known [12] that, for $\tau \to \infty$,

$$t_\tau = \frac{2\pi\tau}{\log\tau}(1+o(1)) \tag{2}$$

and

$$t'_\tau = \frac{2\pi}{\log\tau}(1+o(1)). \tag{3}$$

Thus, since $0 < \alpha \leqslant 1$, the function $(t_\tau^\alpha)'$ is decreasing for large τ. Hence,

$$\begin{aligned}
I \;&\overset{\text{def}}{=} \int\limits_{T}^{T+H} |\zeta(\sigma+it_\tau^\alpha+it)|^2 \,\mathrm{d}\tau = \int\limits_{T}^{T+H} \frac{1}{(t_\tau^\alpha)'} |\zeta(\sigma+it_\tau^\alpha+it)|^2 \,\mathrm{d}(t_\tau^\alpha) \\
&= \int\limits_{T}^{T+H} \frac{1}{(t_\tau^\alpha)'} \,\mathrm{d}\left(\int\limits_{T}^{t_\tau^\alpha+t} |\zeta(\sigma+iu)|^2 \,\mathrm{d}u \right) = \frac{1}{(t_{T+H}^\alpha)'} \int\limits_{T}^{T+H} \mathrm{d}\left(\int\limits_{T}^{t_{T+H}^\alpha+t} |\zeta(\sigma+iu)|^2 \,\mathrm{d}u \right) \\
&= \frac{1}{(t_{T+H}^\alpha)'} \int\limits_{t_\zeta^\alpha+t}^{t_{T+H}^\alpha+t} |\zeta(\sigma+iu)|^2 \,\mathrm{d}u \leqslant \frac{1}{(t_{2T}^\alpha)'} \int\limits_{t_T^\alpha-|t|}^{t_{T+H}^\alpha+|t|} |\zeta(\sigma+iu)|^2 \,\mathrm{d}u \\
&\leqslant \frac{1}{(t_{2T}^\alpha)'} \int\limits_{t_T^\alpha-H(t_T^\alpha)'-|t|}^{t_T^\alpha+H(t_T^\alpha)'+|t|} |\zeta(\sigma+iu)|^2 \,\mathrm{d}u,
\end{aligned} \tag{4}$$

where $T \leqslant \xi \leqslant T+H$. In view of the hypotheses for H,

$$H(t_T^\alpha)' + |t| \geqslant H(t_T^\alpha)' \geqslant \Psi_\alpha(T).$$

Therefore, by Lemma 1, and if $H(t_T^\alpha)' + |t| \leqslant t_T^\alpha$, then, by (4),

$$I \ll_\sigma \frac{H(t_T^\alpha)' + |t|}{(t_{2T}^\alpha)'} \ll_{\sigma,\alpha} H + \frac{|t|}{(t_{2T}^\alpha)'} \ll_{\sigma,\alpha} H(1+|t|). \tag{5}$$

It is well known that, for $1/2 < \sigma < 1$,

$$\int_{-T}^{T} |\zeta(\sigma+it)|^2 \,\mathrm{d}t \ll_\sigma T. \tag{6}$$

If $H(t_T^\alpha)' + |t| > t_T^\alpha$, then $t_T^\alpha + H(t_T^\alpha)' + |t| < 2(H(t_T^\alpha)' + |t|)$, and $t_T^\alpha - H(t_T^\alpha)' - |t| > -2(H(t_T^\alpha)' + |t|)$. Thus, by (6), if $H(t_T^\alpha)' + |t| > t_T^\alpha$, then

$$I \ll \frac{1}{(t_{2T}^\alpha)'} \int_{-2(H(t_T^\alpha)'+|t|)}^{2(H(t_T^\alpha)'+|t|)} |\zeta(\sigma+iu)|^2 \,\mathrm{d}u \ll_\sigma \frac{H(t_T^\alpha)' + |t|}{(t_{2T}^\alpha)'} \ll_{\sigma,\alpha} H(1+|t|).$$

This and (5) prove the lemma. □

The next lemma (Gallagher lemma) together with Lemma 2 will imply the bound for the discrete mean square.

Lemma 3 ([16])**.** *Suppose that $T_1, T_2 \geqslant \eta > 0$, $\mathcal{A}$ is a finite non-empty set in the interval $[T_1+\eta/2, T_1+T_2-\eta/2]$, and*

$$\mathcal{Z}_\eta(x) = \sum_{\substack{t\in\mathcal{A} \\ |t-x|<\eta}} 1.$$

Let a complex valued function $F(t)$ be continuous on $[T_1, T_1+T_2]$ and have a continuous derivative on (T_1, T_1+T_2). Then,

$$\sum_{t\in\mathcal{A}} \mathcal{Z}_\eta^{-1}(t)|F(t)|^2 \leqslant \frac{1}{\eta} \int_{T_1}^{T_1+T_2} |F(t)|^2 \,\mathrm{d}t + \left(\int_{T_1}^{T_1+T_2} |F(t)|^2 \,\mathrm{d}t \int_{T_1}^{T_1+T_2} |F'(t)|^2 \,\mathrm{d}t \right)^{1/2}.$$

Unfortunately, an application of Lemma 3 requires the restriction $\alpha \leqslant 1$.

Lemma 4. *Suppose that $0 < \alpha \leqslant 1$ and $\sigma \in (1/2, 13/22]$ are fixed, $\Psi_\alpha(N)((t_N^\alpha)')^{-1} \leqslant M \leqslant N-1$, and $t \in \mathbb{R}$. Then,*

$$\sum_{k=N}^{N+M} |\zeta(\sigma + it_k^\alpha + it)|^2 \ll_{\sigma,\alpha} M(1+|t|).$$

Proof. We take in Lemma 3 $\eta = 1$, $T_1 = N-1$, $T_2 = M+2$ and $\mathcal{A} = \{N, N+1, \dots, N+M\}$. Obviously, $\mathcal{Z}_\eta(x) = 1$. Therefore, an application of Lemma 3 with the function $F(\tau) = \zeta(\sigma + it_\tau^\alpha + it)$ yields

$$\begin{aligned} \textstyle\sum_{k=N}^{N+M} |\zeta(\sigma+it_k^\alpha+it)|^2 &\leqslant \int_{N-1}^{N+M+1} |\zeta(\sigma+it_\tau^\alpha+it)|^2 \,\mathrm{d}\tau \\ &+ \left(\int_{N-1}^{N+M+1} |\zeta(\sigma+it_\tau^\alpha+it)|^2 \,\mathrm{d}\tau \int_{N-1}^{N+M+1} |\zeta'(\sigma+it_\tau^\alpha+it)|^2 \,\mathrm{d}\tau \right)^{1/2}. \end{aligned} \tag{7}$$

By Lemma 2, we have

$$\int_{N-1}^{N+M+1} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \, d\tau \ll_{\sigma,\alpha} M(1 + |t|). \tag{8}$$

The Cauchy integral formula gives

$$\zeta'(s + it_\tau^\alpha + it) = \frac{1}{2\pi i} \int_L \frac{\zeta(s + it_\tau^\alpha + it)}{(z - \sigma)^2} \, dz,$$

where L is a circle $|z - \sigma| = r$ lying in the strip $1/2 < \sigma \leqslant 13/22$. Therefore,

$$|\zeta'(\sigma + it_\tau^\alpha + it)|^2 \ll \int_L \frac{|dz|}{|z - \sigma|^4} \int_L |\zeta(\sigma + it_\tau^\alpha + it)|^2 \, |dz|.$$

Hence,

$$\int_{N-1}^{N+M+1} |\zeta'(\sigma + it_\tau^\alpha + it)|^2 \, d\tau \ll_{\sigma,L} \int_L |dz| \int_{N-1}^{N+M+1} |\zeta(\mathrm{Re} z + it_\tau^\alpha + it + i\mathrm{Im} z)|^2 \, d\tau.$$

Thus, in view of (8),

$$\int_{N-1}^{N+M+1} |\zeta'(\sigma + it_\tau^\alpha + it)|^2 \, d\tau \ll_{L,\sigma,\alpha} M(1 + |t|).$$

The latter estimate, and (7) and (8) prove the lemma. □

Now we are ready to approximate $\zeta(s + it_k^\alpha)$ by an absolutely convergent Dirichlet series. Let $\kappa > 1/2$ be a fixed number, and

$$v_n(m;\kappa) = \exp\left\{-\left(\frac{m}{n}\right)^\kappa\right\}, \quad m, n \in \mathbb{N}.$$

Then the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m;\kappa)}{m^s}$$

is absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0.

Lemma 5. *Suppose that $K \subset \mathfrak{D}$ is a compact set, and $\Psi_\alpha(N)((t_N^\alpha)')^{-1} \leqslant M \leqslant N - 1$. Then,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K} |\zeta(s + it_k^\alpha) - \zeta_n(s + it_k^\alpha)| = 0.$$

Proof. The Mellin formula

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s} \, ds = e^{-a}, \quad a, b > 0,$$

implies the integral representation, see, for example, [17],

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \zeta(s+z) l_n(z,\kappa) \, dz, \tag{9}$$

where $l_n(z;\kappa) = 1/\kappa\Gamma(z/\kappa)n^z$. There exists $0 < \delta \leqslant 1/11$ such that $1/2 + 2\delta \leqslant \sigma \leqslant 1 - \delta$ for $s = \sigma + it \in K$. The integrand in (9) has simple poles at $z = 0$ and $z = 1 - s$. Therefore, taking $\kappa = 1/2 + \delta$ and $\kappa_1 = 1/2 + \delta - \sigma < 0$, we find by the residue theorem

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int\limits_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} \zeta(s+z) l_n(z;\kappa)\,\mathrm{d}z + l_n(1-s;\kappa).$$

Thus, for all $s \in K$,

$$\begin{aligned}
&\zeta_n(s + it_k^\alpha) - \zeta(s + it_k^\alpha) \\
&= \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \delta + it + it_k^\alpha + i\tau\right) l_n\left(\frac{1}{2} + \delta - \sigma + i\tau;\kappa\right) \mathrm{d}\tau + l_n(1 - s - it_k^\alpha;\kappa) \\
&\ll \int\limits_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + \delta + it_k^\alpha + i\tau\right)\right| \sup_{s\in K}\left| l_n\left(\frac{1}{2} + \delta - s + i\tau;\kappa\right)\right| \mathrm{d}\tau + \sup_{s\in K}|l_n(1 - s - it_k^\alpha;\kappa)|
\end{aligned}$$

in virtue of a shift $t + \tau \to \tau$. Hence,

$$\begin{aligned}
&\frac{1}{M+1}\sum_{k=N}^{N+M} \sup_{s\in K}|\zeta_n(s + it_k^\alpha) - \zeta(s + it_k^\alpha)| \\
&\quad\ll \int\limits_{-\infty}^{\infty} \left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\frac{1}{2} + \delta + it_k^\alpha + i\tau\right)\right|\right) \sup_{s\in K}\left| l_n\left(\frac{1}{2} + \delta - s + i\tau;\kappa\right)\right| \mathrm{d}\tau \\
&\qquad + \frac{1}{M+1}\sum_{k=N}^{N+M} \sup_{s\in K}|l_n(1 - s - it_k^\alpha;\kappa)| \stackrel{\text{def}}{=} S_1 + S_2.
\end{aligned} \tag{10}$$

In view of Lemma 4,

$$\begin{aligned}
\tfrac{1}{M+1}\textstyle\sum_{k=N}^{N+M}\left|\zeta\left(\tfrac{1}{2} + \delta + it_k^\alpha + i\tau\right)\right| &\leqslant \left(\tfrac{1}{M+1}\textstyle\sum_{k=N}^{N+M}\left|\zeta\left(\tfrac{1}{2} + \delta + it_k^\alpha + i\tau\right)\right|^2\right)^{1/2} \\
&\ll_{\delta,\alpha} (1 + |\tau|)^{1/2}.
\end{aligned} \tag{11}$$

It is well known that, for large $|t|$, the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\} \tag{12}$$

with $c > 0$, uniformly in σ in any interval $[\sigma_1, \sigma_2]$, $\sigma_1 < \sigma_2$, is valid. Therefore, for all $s \in K$,

$$l_n\left(\frac{1}{2} + \delta - s + i\tau\right) \ll_\kappa n^{1/2+\delta-\sigma} \exp\left\{-\frac{c}{\kappa}|\tau - t|\right\} \ll_{\kappa,K} n^{-\delta} \exp\{-c_1|\tau|\}$$

with $c_1 > 0$. Thus, by (11),

$$S_1 \ll_{\delta,\alpha,\kappa,K} n^{-\delta} \int\limits_{-\infty}^{\infty} (1 + |\tau|)^{1/2} \exp\{-c_1|\tau|\}\,\mathrm{d}\tau \ll_{\delta,\alpha,\kappa,K} n^{-\delta}. \tag{13}$$

To estimate S_2, we observe that (12), for all $s \in K$, implies the bound

$$l_n(1 - s - it_k^\alpha) \ll_\kappa n^{1-\sigma} \exp\left\{-\frac{c}{\kappa}|t_k^\alpha + t|\right\} \ll_{\kappa,K} n^{1/2-2\delta} \exp\{-c_2 t_k^\alpha\}$$

with $c_2 > 0$. Hence, in view of (2),

$$S_2 \ll_{\kappa,K} n^{1/2-2\delta} \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\{-c_2 t_k^{\alpha}\} \ll_{\kappa,K} n^{1/2-2\delta} \sum_{k=N}^{N+M} \exp\left\{-c_3 \left(\frac{k}{\log k}\right)^{\alpha}\right\}$$
$$\ll_{\theta,K} n^{1/2-2\varepsilon} \exp\left\{-c_4 \left(\frac{N}{\log N}\right)^{\alpha}\right\}$$

with positive c_3 and c_4. This, (13) and (10) show that

$$\frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K} |\zeta(s+it_k^{\alpha}) - \zeta_n(s+it_k^{\alpha})| \ll_{\delta,\alpha,\kappa,K} n^{-\delta} + n^{1/2-2\delta} \exp\left\{-c_4 \left(\frac{N}{\log N}\right)^{\alpha}\right\}.$$

Now, taking $N \to \infty$, and then $n \to \infty$, we obtain the equality of the lemma. □

3. Weak Convergence

Let $\mathcal{X}$ be a certain topological space with the Borel σ-field $\mathcal{B}(\mathcal{X})$, and P and P_n, $n \in \mathbb{N}$, probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. By the definition, P_n converges weakly to P as $n \to \infty$, ($P_n \xrightarrow[n\to\infty]{\mathrm{W}} P$) if

$$\lim_{n\to\infty} \int_{\mathcal{X}} g \, \mathrm{d}P_n = \int_{\mathcal{X}} g \, \mathrm{d}P$$

for every real bounded continuous function g on $\mathcal{X}$. In this section, we will obtain the weak convergence for some measures in the space of analytic functions. Denote by $\mathcal{H}(\mathfrak{D})$ the space of analytic on $\mathfrak{D}$ functions endowed with the topology of uniform convergence on compacta, and set

$$\mathcal{H}^r(\mathfrak{D}) = \underbrace{\mathcal{H}(\mathfrak{D}) \times \cdots \times \mathcal{H}(\mathfrak{D})}_{r}.$$

For $A \in \mathcal{B}(\mathcal{H}^r(\mathfrak{D}))$, define

$$P_{N,M,\underline{\alpha}}(A) = \frac{1}{M+1} \#\left\{N \leqslant k \leqslant N+M : \underline{\zeta}(s+i\underline{t}_k^{\alpha}) \in A\right\},$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{t}_k^{\alpha} = (t_k^{\alpha_1}, \dots, t_k^{\alpha_r})$, and

$$\underline{\zeta}(s+i\underline{t}_k^{\alpha}) = (\zeta(s+it_k^{\alpha_1}), \dots, \zeta(s+it_k^{\alpha_r})).$$

We consider the weak convergence of $P_{N,M,\underline{\alpha}}$ as $N \to \infty$, where $M = o(N)$.

For the definition of the limit measure, we need some notation. Define the Cartesian product

$$\Omega = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

The infinite-dimensional torus Ω equipped with the product topology and operation of pointwise multiplication becomes a compact topological Abelian group; therefore,

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$, again is a compact topological group. Hence, the probability Haar measure μ_H on the space $(\Omega^r, \mathcal{B}(\Omega^r))$ exits, and we arrive to the probability space $(\Omega^r, \mathcal{B}(\Omega^r), \mu_H)$. For $j = 1, \dots, r$, let $\omega_j = \{\omega_j(p) : p \in \mathbb{P}\} \in \Omega_j$ and $\omega = (\omega_1, \dots, \omega_r)$ be the element of Ω^r. Now, on the above probability space, define the $\mathcal{H}^r(\mathfrak{D})$-valued random element

$$\underline{\zeta}(s,\omega) = \left(\prod_{p\in\mathbb{P}} \left(1 - \frac{\omega_1(p)}{p^s}\right)^{-1}, \dots, \prod_{p\in\mathbb{P}} \left(1 - \frac{\omega_r(p)}{p^s}\right)^{-1}\right).$$

The latter infinite products, for almost all ω_j, converge uniformly on compact sets of the strip $\mathfrak{D}$ [17]. Let

$$\mathcal{P}_{\underline{\zeta}}(A) = \mu_H\Big\{\omega \in \Omega^r : \underline{\zeta}(s,\omega) \in A\Big\}, \quad A \in \mathcal{B}(\Omega^r),$$

i.e., $\mathcal{P}_{\underline{\zeta}}$ is the distribution of $\underline{\zeta}(s,\omega)$. Now we state a limit theorem for $P_{N,M,\underline{\alpha}}$.

Theorem 2. *Suppose that* $0 < \alpha_1 < \cdots < \alpha_r \leqslant 1$, *and* $\Psi_\alpha(N)((t_N^\alpha)')^{-1} \leqslant M \leqslant N-1$. *Then,* $P_{N,M,\underline{\alpha}} \xrightarrow[N\to\infty]{\mathrm{W}} \mathcal{P}_{\underline{\zeta}}$.

Before the proof of Theorem 2, we prove several separate lemmas. First of them is devoted to the space Ω^r. For $A \in \mathcal{B}(\Omega^r)$, set

$$Q_{N,M,\underline{\alpha}}(A) = \frac{1}{M+1}\#\Big\{N \leqslant k \leqslant N+M : \Big(\Big(p^{-it_k^{\alpha_1}} : p \in \mathbb{P}\Big), \ldots, \Big(p^{-it_k^{\alpha_r}} : p \in \mathbb{P}\Big)\Big) \in A\Big\}.$$

Lemma 6. *Suppose that* $0 < \alpha_1 < \cdots < \alpha_r \leqslant 1$, *and* $\Psi_\alpha(N)((t_N^\alpha)')^{-1} \leqslant M \leqslant N-1$. *Then* $Q_{N,M,\underline{\alpha}} \xrightarrow[N\to\infty]{\mathrm{W}} \mu_H$.

Proof. On groups, it is convenient to apply the Fourier transform method. Let $\mathcal{F}_{N,M,\underline{\alpha}}(\underline{k}_1,\ldots,\underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z},\ p \in \mathbb{P})$, $j = 1,\ldots,r$, be the Fourier transform of $Q_{N,M,\underline{\alpha}}$. It is well known that

$$\mathcal{F}_{N,M,\underline{\alpha}}(\underline{k}_1,\ldots,\underline{k}_r) = \int\limits_{\Omega^r} \prod_{j=1}^r \prod_{p\in\mathbb{P}}{}^{*}\, \omega_j^{k_{jp}}(p)\, \mathrm{d}Q_{N,M,\underline{\alpha}},$$

where the star “ $*$ ” shows that only a finite number of integers k_{jp} are distinct from zero. Thus, the definition of $Q_{N,M,\underline{\alpha}}$ implies

$$\begin{aligned} \mathcal{F}_{N,M,\underline{\alpha}}(\underline{k}_1,\ldots,\underline{k}_r) &= \tfrac{1}{M+1}\textstyle\sum_{k=N}^{N+M} \prod_{j=1}^r \prod_{p\in\mathbb{P}}{}^{*}\, p^{-ik_{jp}t_k^{\alpha_j}} \\ &= \tfrac{1}{M+1}\textstyle\sum_{k=N}^{N+M} \exp\Big\{-i\sum_{j=1}^r t_k^{\alpha_j} \sum_{p\in\mathbb{P}}{}^{*}\, k_{jp}\log p\Big\}. \end{aligned} \tag{14}$$

Clearly,

$$\mathcal{F}_{N,M,\underline{\alpha}}(\underline{0},\ldots,\underline{0}) = 1, \tag{15}$$

where $\underline{0} = (0,\ldots,0,\ldots)$.

Now suppose that $(\underline{k}_1,\ldots,\underline{k}_r) \neq (\underline{0},\ldots,\underline{0})$. Then there exists at least one $j \in \{1,\ldots,r\}$, such that $\underline{k}_j \neq \underline{0}$. Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers,

$$a_j \stackrel{\mathrm{def}}{=} \sum_{p\in\mathbb{P}}{}^{*}\, k_{jp}\log p \neq 0$$

for such j. Let j_0 be the largest of j with $a_j \neq 0$. Hence,

$$A_{\underline{\alpha}}(\tau) \stackrel{\mathrm{def}}{=} \sum_{j=1}^r a_j t_\tau^{\alpha_j} = \sum_{j=1}^{j_0} a_j t_\tau^{\alpha_j},$$

and

$$\begin{aligned} A'_{\underline{\alpha}}(\tau) &= \textstyle\sum_{j=1}^{j_0} a_j (t_\tau^{\alpha_j})' = \sum_{j=1}^{j_0} a_j\alpha_j t_\tau^{\alpha_j-1} t'_\tau(1+o(1)) \\ &= 2\pi a_{j_0}\alpha_{j_0}\tfrac{(2\pi\tau)^{\alpha_{j_0}-1}}{(\log\tau)^{\alpha^{j_0}}}(1+o(1)), \quad \tau\to\infty, \end{aligned} \tag{16}$$

in view of (2) and (3). For the estimation the sum (14), we apply a representation of trigonometric sums by integrals, see, for example, [18]. Suppose that the real-valued function $g(x)$ has a monotonic derivative on $[a,b]$, such that $|g'(x)| \leqslant \xi < 1$. Then,

$$\sum_{a\leqslant m\leqslant b} \mathrm{e}^{2\pi i g(m)} = \int_a^b \mathrm{e}^{2\pi i g(x)}\,\mathrm{d}x + O\left(\frac{1}{1-\xi}\right). \tag{17}$$

Relation (16) shows that the function $A_{\underline{\alpha}}(\tau)$, for sufficiently large N, satisfies the above requirements on $[N, N+M]$. Thus, by (14) and (17),

$$\mathcal{F}_{N,M,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r) = \frac{1}{M+1}\int_N^{N+M} \exp\{-iA_{\underline{\alpha}}(\tau)\}\,\mathrm{d}\tau + O\left(\frac{1}{(M+1)(1-\delta_N)}\right), \tag{18}$$

with

$$\delta_N = 4\pi a_{j_0}\alpha_{j_0}\frac{(2\pi N)^{\alpha_{j_0}-1}}{(\log N)^{\alpha_{j_0}}}.$$

Moreover, by the mean value theorem and (16),

$$\begin{aligned}\int_N^{N+M} \cos(A_{\underline{\alpha}}(\tau))\,\mathrm{d}\tau &= \int_N^{N+M}\frac{1}{A'_{\underline{\alpha}}(\tau)}\cos(A_{\underline{\alpha}}(\tau))\,\mathrm{d}A_{\underline{\alpha}}(\tau) \ll_{\underline{\alpha}} (A'_{\underline{\alpha}}(N+M))\\ &\ll_{\underline{\alpha}} \frac{(\log(N+M))^{\alpha_{j_0}}}{(N+M)^{\alpha_{j_0}-1}}\end{aligned}$$

and

$$\int_N^{N+M} \sin(A_{\underline{\alpha}}(\tau))\,\mathrm{d}\tau \ll_{\underline{\alpha}} \frac{(\log(N+M))^{\alpha_{j_0}}}{(N+M)^{\alpha_{j_0}-1}}$$

Therefore, by (18), for sufficiently large N,

$$\mathcal{F}_{N,M,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r) \ll_{\underline{\alpha}} \frac{1}{M}N^{1-\alpha_{j_0}}(\log N)^{\alpha_{j_0}} \ll_{\underline{\alpha}} \frac{1}{M}N^{1-\alpha_1}\log N. \tag{19}$$

By the hypotheses for M and (2) and (3),

$$M \gg_{\underline{\alpha}} N^{\alpha_1/3+1-\alpha}\left(\log\frac{N}{\log N}\right)^{26/15}(\log N)^{-\alpha/3+\alpha} \gg_{\underline{\alpha}} N^{1-(2\alpha/3)}(\log N)^b, \quad b>0.$$

Hence, in view of (19),

$$\mathcal{F}_{N,M,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r) \ll_{\underline{\alpha}} N^{-\alpha/3}(\log N)^{1-b}.$$

This, together with (15), shows that

$$\lim_{N\to\infty}\mathcal{F}_{N,M,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1,\dots,\underline{k}_r) = (\underline{0},\dots,\underline{0}),\\ 0 & \text{if} \quad (\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0}),\end{cases}$$

and the lemma is proved because the right-hand side of the latter equality is the Fourier transform of the measure μ_H. □

Lemma 6 implies the weak convergence for

$$P_{N,M,n,\underline{\alpha}}(A) = \frac{1}{M+1}\#\left\{N\leqslant k\leqslant N+M : \underline{\zeta}_n(s+it_{\underline{k}}^{\underline{\alpha}}) \in A\right\}, \quad A\in\mathcal{B}(\mathcal{H}^r(\mathfrak{D})),$$

as $N \to \infty$, where

$$\underline{\zeta}_n(s + it_k^{\underline{\alpha}}) = (\zeta_n(s + it_k^{\alpha_1}), \dots, \zeta_n(s + it_k^{\alpha_r})).$$

Define the mapping $u_n : \Omega^r \to \mathcal{H}^r(\mathfrak{D})$ by

$$u_n(\omega) = \underline{\zeta}_n(s, \omega)$$

with

$$\underline{\zeta}_n(s, \omega) = (\zeta_n(s, \omega_1), \dots, \zeta(s, \omega_r))$$

and

$$\zeta_n(s, \omega_j) = \sum_{m=1}^{\infty} \frac{\omega_j(m) v_n(m; \kappa)}{m^s}, \quad j = 1, \dots, r.$$

Then the measure μ_H on $(\Omega^r, \mathcal{B}(\Omega^r))$ defines the unique probability measure $U_n = \mu_H u_n^{-1}$ on $(\mathcal{H}^r(\mathfrak{D}), \mathcal{B}(\mathcal{H}^r(\mathfrak{D})))$, where

$$\mu_H u_n^{-1}(A) = \mu_H(u_n^{-1} A), \quad A \in \mathcal{B}(\mathcal{H}^r(\mathfrak{D})).$$

Lemma 7. *Under hypotheses of Lemma 6,* $P_{N,M,n,\underline{\alpha}} \xrightarrow[N \to \infty]{\mathrm{W}} U_n$.

Proof. Since the series for $\zeta_n(s, \omega_j)$ converges absolutely for $\sigma \geqslant 1/2$, the mapping u_n is continuous. Moreover,

$$u_n\Big(\Big(p^{-it_k^{\alpha_1}} : p \in \mathbb{P}\Big), \dots, \Big(p^{-it_k^{\alpha_r}} : p \in \mathbb{P}\Big)\Big) = \underline{\zeta}_n(s + it_k^{\underline{\alpha}}).$$

Therefore,

$$\begin{aligned} &P_{N,M,n,\underline{\alpha}}(A) \\ &\quad = \frac{1}{M+1} \# \Big\{ N \leqslant k \leqslant N+M : \Big(\Big(p^{-it_k^{\alpha_1}} : p \in \mathbb{P}\Big), \dots, \Big(p^{-it_k^{\alpha_r}} : p \in \mathbb{P}\Big)\Big) \in u_n^{-1} A \Big\} \\ &\quad = Q_{N,M,\underline{\alpha}}(u_n^{-1} A), \quad A \in \mathcal{B}(\mathcal{H}^r(\mathfrak{D})). \end{aligned}$$

Thus, we have

$$P_{N,M,n,\underline{\alpha}} = Q_{N,M,\underline{\alpha}} u_n^{-1}.$$

The latter equality, Lemma 6, the continuity of u_n and the well-known property of preservation of the weak convergence, see, for example, Theorem 2.7 of [19], prove the lemma. □

The weak convergence of the measure U_n is very important for that of $P_{N,M,\underline{\alpha}}$. The following statement is true:

Lemma 8 ([10])**.** *The relation* $U_n \xrightarrow[n \to \infty]{\mathrm{W}} \mathcal{P}_{\underline{\zeta}}$ *holds.*

To prove Theorem 2, we need one statement on convergence in distribution of random elements. Let X_n, $n \in \mathbb{N}$, and X be $\mathcal{X}$-valued random elements. Recall that X_n as $n \to \infty$ converges to X in distribution ($X_n \xrightarrow[n \to \infty]{\mathcal{D}} X$) if the distribution of X_n converges weakly to the distribution of X.

Lemma 9 ([19])**.** *Suppose that the space* $(\mathcal{X}, d)$ *is separable, and the* $\mathcal{X}$*-valued random elements* X_{mn} *and* Y_n, $m, n \in \mathbb{N}$ *are defined on the same probability space* $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), P)$*. If*

$$X_{mn} \xrightarrow[n \to \infty]{\mathcal{D}} X_m, \qquad X_m \xrightarrow[m \to \infty]{\mathcal{D}} X,$$

and for every $\delta > 0$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mu\{d(X_{mn},Y_n)\geqslant\delta\}=0,$$

then also $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Before the proof of Theorem 2, recall the metric in the space $\mathcal{H}^r(\mathfrak{D})$. There exists a sequence $\{K_l : l\in\mathbb{N}\}\subset\mathfrak{D}$ of embedded compact subsets such that the union of the sets K_l is the region $\mathfrak{D}$, and, for every compact set $K\subset\mathfrak{D}$, there exists K_l, $K\subset K_l$. Taking

$$d(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|},\quad g_1,g_2\in\mathcal{H}(\mathfrak{D}),$$

gives a metric in $\mathcal{H}(\mathfrak{D})$ inducing the topology of uniform convergence on compacta. Then,

$$\underline{d}(\underline{g}_1,\underline{g}_2)=\max_{1\leqslant j\leqslant r}d(g_{1j},g_{2j}),\quad \underline{g}_k=(g_{k1},\dots,g_{kr})\in\mathcal{H}^r(\mathfrak{D}),\ k=1,2,$$

is a metric in $\mathcal{H}^r(\mathfrak{D})$ which induces its product topology.

Proof of Theorem 2. Denote by X_n the $\mathcal{H}^r(\mathfrak{D})$-valued random element with distribution U_n. Then, by Lemma 8, we have

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{P}_{\underline{\zeta}}. \tag{20}$$

Let $\xi_{N,M}$ be a random variable on a certain probability space with measure P with the distribution

$$P\{\xi_{N,M}=k\}=\frac{1}{M+1},\quad k=N,\dots,N+M.$$

Define two $\mathcal{H}^r(\mathfrak{D})$-valued random elements

$$X_{N,M,n,\underline{\alpha}}=X_{N,M,n,\underline{\alpha}}(s)=\underline{\zeta}_n(s+i\underline{t}^{\underline{\alpha}}_{\xi_{N,M}})$$

and

$$X_{N,M,\underline{\alpha}}=X_{N,M,\underline{\alpha}}(s)=\underline{\zeta}(s+i\underline{t}^{\underline{\alpha}}_{\xi_{N,M}}).$$

Lemma 7 implies the relation

$$X_{N,M,n,\underline{\alpha}} \xrightarrow[N\to\infty]{\mathcal{D}} X_n. \tag{21}$$

From the definitions of the metrics d and $\underline{d}$, and Lemma 5, it follows that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{M+1}\sum_{k=N}^{N+M}\underline{\rho}\Big(\underline{\zeta}(s+i\underline{t}^{\underline{\alpha}}_k),\underline{\zeta}_n(s+i\underline{t}^{\underline{\alpha}}_k)\Big)=0.$$

Therefore, the definitions of $X_{N,M,n,\underline{\alpha}}$ and $X_{N,M,\underline{\alpha}}$, together with Chebyshev's type inequality, give, for every $\delta>0$,

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{N\to\infty}\mu\{\underline{d}(X_{N,M,\underline{\alpha}},X_{N,M,n,\underline{\alpha}})\geqslant\delta\}\\
&\quad=\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{M+1}\#\Big\{N\leqslant k\leqslant N+M:\underline{d}(\underline{\zeta}(s+i\underline{t}^{\underline{\alpha}}_k),\underline{\zeta}_n(s+i\underline{t}^{\underline{\alpha}}_k))\geqslant\delta\Big\}\\
&\quad\leqslant\frac{1}{(M+1)\delta}\sum_{k=N}^{N+M}\underline{d}(\underline{\zeta}(s+i\underline{t}^{\underline{\alpha}}_k),\underline{\zeta}_n(s+i\underline{t}^{\underline{\alpha}}_k))=0.
\end{aligned}$$

This, (20) and (21) show that all hypotheses of Lemma 9 are satisfied. Thus,

$$X_{N,M,\underline{\alpha}} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{P}_{\underline{\zeta}},$$

and the theorem is proved. □

4. Main Theorem

The main result of the paper is the following theorem:

Theorem 3. *Suppose that $\alpha_1, \dots, \alpha_r$ are different fixed positive numbers not exceeding 1, and $\Psi_\alpha(N)((t_N^\alpha)')^{-1} \leqslant M \leqslant N-1$. For $j = 1, \dots, r$, let $K_j \in \mathfrak{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{M+1} \# \left\{ N \leqslant k \leqslant N+M : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + it_k^{\alpha_j}) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, the lower limit can be replaced by the limit for all but at most countably many $\varepsilon > 0$.

Theorem 3 easily follows from Theorem 2 and the Mergelyan theorem on the approximation of analytic functions by polynomials [20].

Let P be a probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, and the space $\mathcal{X}$ is separable. Recall that the support of the measure P is a minimal closed set $S_P \subset \mathcal{X}$ such that $P(S_P) = 1$. The set S_P consists of all $x \in \mathcal{X}$ such that, for every open neighbourhood G of x, the inequality $P(G) > 0$ is satisfied. Let $S = \{g \in \mathcal{H}(\mathfrak{D}) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$, and

$$S^r = \underbrace{S \times \cdots \times S}_{r}.$$

Lemma 10 ([10])**.** *The support of the measure $P_{\underline{\zeta}}$ is the set S^r.*

Proof of Theorem 3. By the Mergelyan theorem, there exist polynomials $q_1(s), \dots, q_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| f_j(s) - \mathrm{e}^{q_j(s)} \right| < \frac{\varepsilon}{2}. \tag{22}$$

Consider the set

$$G_\varepsilon = \left\{ g_1, \dots, g_r \in \mathcal{H}(\mathfrak{D}) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{q_j(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

In view of Lemma 10, $(\mathrm{e}^{q_1(s)}, \dots, \mathrm{e}^{q_r(s)})$ is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore, G_ε is an open neighbourhood of an element of the support, hence, we have

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{23}$$

From this, using Theorem 2 and the equivalent of weak convergence in terms of open sets, see, for example, Theorem 2.1 of [19], we find

$$\liminf_{N\to\infty} P_{N,M,\underline{\alpha}}(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

This, the definitions of $P_{N,M,\underline{\alpha}}$ and G_ε, and inequality (22) prove the first assertion of the theorem.

To prove the second assertion of the theorem, define one more set

$$\widehat{G}_\varepsilon = \left\{ (g_1, \ldots, g_r) \in \mathcal{H}^r(\mathfrak{D}) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}.$$

Then the boundaries $\partial \widehat{G}_{\varepsilon_1}$ and $\partial \widehat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Hence, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$, i.e., $P_{\underline{\zeta}}(\partial \widehat{G}_\varepsilon) = 0$, for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 2 and the equivalent of weak convergence in terms of continuity set, see, for example, Theorem 2.1 of [19], we have

$$\lim_{N \to \infty} P_{N,M,\underline{\alpha}}(\widehat{G}_\varepsilon) = P_{\underline{\zeta}}(\widehat{G}_\varepsilon) \tag{24}$$

for all but at most countably many $\varepsilon > 0$. In view of (22), the inclusion $G_\varepsilon \subset \widehat{G}_\varepsilon$ holds. Therefore, by (23), the inequality $P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0$ is valid. This, (24) and the definitions of $P_{N,M,\underline{\alpha}}$ and $\widehat{G}_\varepsilon$ prove the second assertion of the theorem. $\square$

Theorem 3 is stronger than Theorem 1 because the numbers k for which $(\zeta(s + it_k^{\alpha_1}), \ldots, \zeta(s + it_k^{\alpha_r}))$ has the approximating property of analytic functions lie in the interval of length M, which may be taken $M = o(N)$ as $N \to \infty$.

5. Conclusions

Let $\{t_n\}$ be a sequence of Gram points, and $0 < \alpha_1 < \cdots < \alpha_r \leqslant 1$ fixed numbers. In this paper, it is obtained that the interval $[\Psi_{\alpha_1}(N)((t_N^{\alpha_1})')^{-1}, N - 1]$ with $\Psi_{\alpha_1}(N) = (t_N^{\alpha_1})^{1/3}(\log t_N^{\alpha_1})^{26/15}$ contains infinitely many $k \in \mathbb{N}$, such that the shifts $(\zeta(s + it_k^{\alpha_1}), \ldots, \zeta(s + it_k^{\alpha_r}))$ approximate every collection $(f_1(s), \ldots, f_r(s))$ of analytics in $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ non-vanishing functions.

The problems for future studies are the following:

1° To remove the requirement $\alpha_r \leqslant 1$;

2° To decrease the lower bound for $\Psi_{\alpha_1}(N)$.

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