




## Article

# On an Indefinite Metric on a Four-Dimensional Riemannian Manifold

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**Abstract:** Our research focuses on the tangent space of a point on a four-dimensional Riemannian manifold. Besides having a positive definite metric, the manifold is endowed with an additional tensor structure of type  $(1, 1)$ , whose fourth power is minus the identity. The additional structure is skew-circulant and compatible with the metric, such that an isometry is induced in every tangent space on the manifold. Both structures define an indefinite metric. With the help of the indefinite metric, we determine circles in different two-planes in the tangent space on the manifold. We also calculate the length and area of the circles. On a smooth closed curve, such as a circle, we define a vector force field. Further, we obtain the circulation of the vector force field along the curve, as well as the flux of the curl of this vector force field across the curve. Finally, we find a relation between these two values, which is an analog of the well-known Green's formula in the Euclidean space.

**Keywords:** Riemannian manifold; indefinite metric tensor; length; area; Green's formula

**MSC:** 53B30; 53A04; 26B15; 26B20



**Citation:** Razpopov, D.; Dzhelepov, G.; Dokuzova, I. On an Indefinite Metric on a Four-Dimensional Riemannian Manifold. *Axioms* **2023**, *12*, 432. <https://doi.org/10.3390/axioms12050432>

Academic Editors: Zhigang Wang, Yanlin Li, Juan De Dios Pérez and Emil Saucan

Received: 11 April 2023

Accepted: 25 April 2023

Published: 27 April 2023



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## 1. Introduction

Riemannian manifolds with additional tensor structures are extensively studied in modern differential geometry. Riemannian almost product manifolds and almost Hermitian manifolds are examples of such manifolds. Four-dimensional Riemannian manifolds with circulant structures are associated with Riemannian almost product manifolds [1], and four-dimensional Riemannian manifolds with skew-circulant structures are associated with Hermitian manifolds [2].

In this paper, we continue our research, previously addressed in [2,3], on Riemannian manifolds with an additional tensor structure, whose fourth power is minus the identity. The additional structure is skew-circulant, i.e., its components form a skew-circulant matrix. The properties and some applications of such matrices can be found in [4–9]. We define an indefinite metric on the manifold using the Riemannian metric and the additional structure, and obtain some useful formulae with respect to this metric, which are analogs of well-known formulae (such as length and area of a circle, circulation of a vector force field, and Green's formula) in the Euclidean case. We can find more motivations for our work from several papers (see [10–12]).

If  $k$  is a simple closed curve in a plane, then it surrounds some region in the plane. Green's theorem transforms the line integral around  $k$  into a double integral over the region inside  $k$ . In physics, this provides the relationship between the circulation  $C = \oint_k F \cdot ds$  of the vector force field  $F$  around the path  $k$  and the flux, done by the curl of  $F$ , across the region inside  $k$ .

Green’s theorem is a special case of Stokes’ theorem. Both theorems are widely used in the study of electric and magnetic fields. The modern approach to these theorems on manifolds using differential forms is exhibited, for example, in [13–18].

We consider a four-dimensional Riemannian manifold  $M$  with an additional tensor field  $S$  of type  $(1, 1)$ , whose fourth power is minus the identity. The structure  $S$  is compatible with the metric  $g$ , such that an isometry is induced in every tangent space  $T_pM$  on  $M$ . Both structures,  $g$  and  $S$ , define an indefinite metric  $\tilde{g}$  [2]. The metric  $\tilde{g}$  determines space-like, isotropic and time-like vectors in  $T_pM$ . We consider circles  $k_i$ , with respect to  $\tilde{g}$ , in special two-planes  $\beta_i$  of  $T_pM$ , constructed on space-like or time-like vectors. We calculate their lengths and areas (with respect to  $\tilde{g}$ ), which, in some cases, are imaginary or negative numbers. We note that some problems related to circles, concerning their lengths or areas considered in terms of indefinite metrics, are addressed in [19–23]. Finally, we obtain analogs of Green’s theorem that provide a relation between the circulation of the vector force field  $F$  around a closed curve (in particular, a circle)  $k_i$  in  $\beta_i$  and the flux, done by the curl of  $F$ , across the region inside  $k_i$ .

The paper is organized as follows. In Section 2, we provide some facts, definitions and statements, which are necessary for the present considerations. In Section 3, we introduce a special two-plane  $\beta_1$  of  $T_pM$  and determine an equation of a circle  $k_1$  in  $\beta_1$  with respect to  $\tilde{g}$ . In Sections 3.1 and 3.2 we calculate the length and area of  $k_1$ . In Section 3.3, we find the circulation of a vector force field  $F$  around the smooth closed curve,  $k_1$ , and the flux, done by the curl of  $F$ , across the region inside  $k_1$ . In Section 4, we introduce a two-plane,  $\beta_2$ , of  $T_pM$  and determine an equation of a circle  $k_2$  in  $\beta_2$  with respect to  $\tilde{g}$ . Further, we calculate the length and area of  $k_2$ . We derive the circulation of a vector force field  $F$  around a smooth closed curve  $k_2$  and the flux, done by the curl of  $F$ , across the region inside  $k_2$ . All values obtained in Sections 3 and 4 are calculated with respect to  $\tilde{g}$ .

## 2. Preliminaries

In this paper, we study a four-dimensional Riemannian manifold, equipped with tensor structures whose component matrices are right skew-circulant. Thus, we recall the definition of such matrices. They are Toeplitz matrices, and were addressed in [4,6].

The real right skew-circulant matrix with the first row  $(a_1, a_2, a_3, a_4) \in \mathbf{R}^4$  is a square matrix of the form:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_4 & a_1 & a_2 & a_3 \\ -a_3 & -a_4 & a_1 & a_2 \\ -a_2 & -a_3 & -a_4 & a_1 \end{pmatrix}.$$

The skew-circulant matrices form a vector space with the following basis:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Further, we use the matrix  $S = E_2$ . We note that  $E$  is the identity matrix and  $S^2 = E_3$ ,  $S^3 = E_4$ . These matrices are also a part of our considerations. The square of  $S$  gives  $J = E_3$ , which is an example of the well-known complex structure.

We equip a four-dimensional differentiable manifold  $M$  with a tensor structure  $S$  of type  $(1, 1)$ , such that  $S$  satisfies

$$S^4 = -\text{id}. \tag{1}$$

We suppose that, at each point on  $M$ , the component matrix of  $S$ , with respect to a basis in the tangent space  $T_pM$ , is skew-circulant.

The matrix  $E_2$  is one solution of the equation  $S^4 = -E$ . Then, we consider the skew-circulant structure  $S$  whose component matrix, with respect to a basis in  $T_pM$ , is

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Obviously,  $S$  satisfies (1) and  $J = S^2$  is a complex structure.

Let  $g$  be a positive definite metric on  $M$ , which is compatible with  $S$ , i.e.,

$$g(Su, Sv) = g(u, v). \tag{3}$$

Here, and anywhere in this work,  $u, v, w, e_1, e_2$  stand for arbitrary smooth vector fields on  $M$  or arbitrary vectors in the tangent space  $T_pM$ ,  $p \in M$ .

The conditions (2) and (3) imply that the matrix of  $g$ , with respect to the same basis, has the following form:

$$(g_{ij}) = \begin{pmatrix} A & B & 0 & -B \\ B & A & B & 0 \\ 0 & B & A & B \\ -B & 0 & B & A \end{pmatrix},$$

where  $A = A(p)$  and  $B = B(p)$  are smooth functions of an arbitrary point  $p$  on  $M$ . Moreover,  $A(p) > \sqrt{2}B(p) > 0$  in order for  $g$  to be positive definite.

Such a manifold  $(M, g, S)$  was introduced in [2].

If  $u$  is a nonzero vector on  $(M, g, S)$ , then, according to (2), we have  $Su \neq \pm u$ . Thus, the angle  $\varphi$  between  $u$  and  $Su$  belongs to the interval  $(0, \pi)$ . The vectors  $u, Su, S^2u$  and  $S^3u$  determine six angles, which satisfy equalities [2]:

$$\begin{aligned} \angle(u, Su) &= \angle(Su, S^2u) = \angle(S^2u, S^3u) = \varphi, \\ \angle(S^3u, u) &= \pi - \varphi, \quad \angle(u, S^2u) = \angle(Su, S^3u) = \frac{\pi}{2}. \end{aligned} \tag{4}$$

**Definition 1.** A basis of type  $\{S^3u, S^2u, Su, u\}$  of  $T_pM$  is called an  $S$ -basis. In this case, we say that the vector  $u$  induces an  $S$ -basis of  $T_pM$ .

In [2], the conditions under which such a basis exists are described, as well as the following statement:

**Lemma 1.** Let a vector  $u$  induce an  $S$ -basis  $\{S^3u, S^2u, Su, u\}$  in  $T_pM$ . Then, the angle  $\varphi = \angle(u, Su)$  satisfies inequalities

$$\frac{\pi}{4} < \varphi < \frac{3\pi}{4}. \tag{5}$$

The associated metric  $\tilde{g}$  on  $(M, g, S)$  is determined by

$$\tilde{g}(u, v) = g(u, Sv) + g(Su, v). \tag{6}$$

The matrix of its components is

$$(\tilde{g}_{ij}) = \begin{pmatrix} 2B & A & 0 & -A \\ A & 2B & A & 0 \\ 0 & A & 2B & A \\ -A & 0 & A & 2B \end{pmatrix}.$$

Two of the eigenvalues of  $\tilde{g}_{ij}$  are negative, and the other two are positive. So,  $\tilde{g}$  has signature (2, 2) and it is an indefinite metric [2].

According to (6), for an arbitrary vector  $v$  the following is valid:

$$\tilde{g}(v, v) = 2g(v, Sv) = R^2, \quad R^2 \in \mathbb{R}. \tag{7}$$

The norm of every vector  $u$  and the cosine of  $\varphi$  are given by the following equalities:

$$\|u\| = \sqrt{g(u, u)}, \quad \cos \varphi = \frac{g(u, Su)}{g(u, u)}. \tag{8}$$

In the rest of the paper, we assume that  $\|u\| = 1$  and, using (8), we obtain

$$\cos \varphi = g(u, Su). \tag{9}$$

Due to (3), (6), (8) and (9) we state that the normalized  $S$ -basis  $\{S^3u, S^2u, Su, u\}$  satisfies the following equalities:

$$\begin{aligned} \tilde{g}(u, u) &= \tilde{g}(Su, Su) = \tilde{g}(S^2u, S^2u) = \tilde{g}(S^3u, S^3u) = 2 \cos \varphi, \\ \tilde{g}(u, Su) &= \tilde{g}(Su, S^2u) = \tilde{g}(S^2u, S^3u) = -\tilde{g}(S^3u, u) = 1. \\ \tilde{g}(u, S^2u) &= \tilde{g}(Su, S^3u) = 0. \end{aligned} \tag{10}$$

A circle  $k$  in a two-plane of  $T_pM$  of a radius  $R$  centered at the origin  $p \in T_pM$ , with respect to the associated metric  $\tilde{g}$  on  $(M, g, S)$ , is determined by (7), where  $v$  is the radius vector of an arbitrary point on  $k$ .

Further, we consider circles  $k_1$  and  $k_2$ , and the regions  $D_1$  and  $D_2$  inside them, in two different subspaces  $\beta_1$  and  $\beta_2$  of  $T_pM$ , spanned by two-planes  $\{u, S^2u\}$  and  $\{u, Su\}$ , respectively. According to (4), (5) and (10), the vectors  $u, Su, S^2u$  and  $S^3u$  are space-like if, and only if,  $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2})$ , and they are time-like if, and only if,  $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{4})$ . Therefore, the two-planes  $\beta_1$  and  $\beta_2$  are space-like when  $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2})$ , and they are time-like when  $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{4})$ .

**Remark 1.** The rest of the two-planes  $\{u, S^3u\}$ ,  $\{Su, S^2u\}$ ,  $\{Su, S^3u\}$ ,  $\{S^2u, S^3u\}$  constructed with the basis vectors of  $\{S^3u, S^2u, Su, u\}$  in  $T_pM$  have the same properties as  $\beta_1$  or  $\beta_2$ .

We note that the two-plane  $\beta_1$  also belongs to the tangent space of the associated Hermitian manifold  $(M, g, J = S^2)$  with the complex structure  $J$ .

### 3. Circles in the Two-Plane $\beta_1$

Due to (4), it is true that the vectors  $u$  and  $S^2u$  form an orthonormal basis of  $\beta_1$ . The coordinate system  $p_{xy}$  on  $\beta_1$ , wherein  $u$  is on the axis  $p_x$  and  $S^2u$  is on the axis  $p_y$ , is an orthonormal coordinate system of  $\beta_1$ .

A circle  $k_1$  in  $\beta_1$  centered at the origin  $p$ , with respect to  $\tilde{g}$  on  $(M, g, S)$ , is defined by (7). The equation of  $k_1$  with respect to  $p_{xy}$  is obtained as follows:

**Theorem 1 ([3]).** Let  $\tilde{g}$  be the associated metric on  $(M, g, S)$  and let  $\beta_1$  be a two-plane in  $T_pM$  with a basis  $\{u, S^2u\}$ . If  $p_{xy}$  is a coordinate system such that  $u \in p_x, S^2u \in p_y$ , then, the equation of the circle (7) in  $\beta_1$  is given by:

$$2 \cos \varphi x^2 + 2 \cos \varphi y^2 = R^2. \tag{11}$$

The only closed curve  $k_1$ , determined by (11), is a circle in terms of  $g$ , with the following parameters:

**Case (A)**  $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2})$  and  $R^2 > 0$ ;

**Case (B)**  $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{4})$  and  $R^2 < 0$ .

The two-plane  $\beta_1$  is constructed on space-like vectors in Case (A), and  $\beta_1$  is constructed on time-like vectors in Case (B).

3.1. Length of a Circle with Respect to  $\tilde{g}$

Firstly, we consider Case (A). The circle (7) has a radius  $R > 0$  and the angle  $\varphi$  satisfies

$$\frac{\pi}{4} < \varphi < \frac{\pi}{2}. \tag{12}$$

**Theorem 2.** The circle  $k_1$  with (12) and a radius  $R > 0$  has length

$$L = 2\pi R. \tag{13}$$

**Proof.** Let  $v = xu + yS^2u$  be a radius vector of an arbitrary point on the circle  $k_1$ . Then,  $dv = dxu + dyS^2u$  is a tangent vector on  $k_1$ . The length  $L$  of  $k_1$  with respect to  $\tilde{g}$  is determined, as usual, by:

$$dL = \sqrt{\tilde{g}(dv, dv)}.$$

Then, using (10) and (11), we obtain:

$$L = \oint_{k_1} \sqrt{2 \cos \varphi dx^2 + 2 \cos \varphi dy^2}. \tag{14}$$

We substitute

$$x = \frac{R}{\sqrt{2 \cos \varphi}} \cos t, \quad y = \frac{R}{\sqrt{2 \cos \varphi}} \sin t, \quad t \in [0, 2\pi],$$

into (14) and find (13).  $\square$

Now, we consider Case (B). The circle  $k_1$  has a radius  $R = ri, r > 0, i^2 = -1$  and the angle  $\varphi$  satisfies

$$\frac{\pi}{2} < \varphi < \frac{3\pi}{4}. \tag{15}$$

Therefore, the Equation (11) transforms into

$$2 \cos \varphi x^2 + 2 \cos \varphi y^2 = -r^2. \tag{16}$$

By calculations similar to those of Case (A) we find the integral (14) over  $k_1$  with (16). Using the substitutions

$$x = \frac{r}{\sqrt{-2 \cos \varphi}} \cos t, \quad y = \frac{r}{\sqrt{-2 \cos \varphi}} \sin t, \quad t \in [0, 2\pi],$$

we obtain

**Proposition 1.** The circle  $k_1$  with (15) and a radius  $R = ri, r > 0, i^2 = -1$  has an imaginary length

$$L = 2\pi R.$$

3.2. Area of a Circle with Respect to  $\tilde{g}$

For Case (A) we state the following:

**Theorem 3.** The area  $A''$  of the circle  $k_1$  with (12) with a radius  $R > 0$  is

$$A'' = \pi R^2. \tag{17}$$

**Proof.** We denote, by  $\widetilde{\cos}\angle(u, S^2u)$  and  $\widetilde{\sin}\angle(u, S^2u)$ , the cosine and the sine of the angle  $\angle(u, S^2u)$  with respect to  $\tilde{g}$ . Considering  $\tilde{g}(u, S^2u) = 0$  (presented in (10)), we have

$$\widetilde{\cos}\angle(u, S^2u) = 0,$$

and, hence,

$$\widetilde{\sin}\angle(u, S^2u) = 1. \tag{18}$$

In the coordinate plane  $p_{xy}$ , we construct a parallelogram with locus vectors  $dxu$  and  $dySu$ . For its area  $A''$  with respect to  $\tilde{g}$  we obtain

$$dA'' = \sqrt{\tilde{g}(dxu, dxu)}\sqrt{\tilde{g}(dySu, dySu)}\widetilde{\sin}\angle(u, S^2u).$$

We apply (10) and (18) in the latter equality and find

$$dA'' = 2 \cos \varphi dx dy. \tag{19}$$

We integrate (19) over the region  $D_1$  inside  $k_1$  and calculate

$$A'' = 2 \cos \varphi \int \int_{D_1} dx dy, \tag{20}$$

with

$$D_1 : 2 \cos \varphi x^2 + 2 \cos \varphi y^2 \leq R^2.$$

We substitute

$$x = \frac{R}{\sqrt{2 \cos \varphi}} \rho \cos t, \quad y = \frac{R}{\sqrt{2 \cos \varphi}} \rho \sin t, \quad t \in [0, 2\pi], \quad \rho \in [0, 1],$$

and Jacobian  $\Delta = \frac{R^2}{2 \cos \varphi} \rho$  into the integral (20) and obtain (17).  $\square$

Now, we consider Case (B).

**Proposition 2.** *The area  $A''$  of the circle  $k_1$  with (15) and a radius  $R = ri$ ,  $r > 0$ ,  $i^2 = -1$  has a negative value*

$$A'' = \pi R^2. \tag{21}$$

**Proof.** The circle  $k_1$  has an Equation (16) with conditions (15) and a radius  $R = ri$ , where  $r > 0$ ,  $i^2 = -1$ . By calculations similar to those of Case (A), we find that the area of  $k_1$  is given by

$$A'' = 2 \cos \varphi \int \int_{D_1} dx dy, \tag{22}$$

with

$$D_1 : -2 \cos \varphi x^2 - 2 \cos \varphi y^2 \leq r^2.$$

We substitute

$$x = \frac{r}{\sqrt{-2 \cos \varphi}} \rho \cos t, \quad y = \frac{r}{\sqrt{-2 \cos \varphi}} \rho \sin t, \quad t \in [0, 2\pi], \quad \rho \in [0, 1],$$

and Jacobian  $\Delta = \frac{r^2}{2 \cos \varphi} \rho$  into the integral (22) and obtain  $A'' = -\pi r^2$ , which implies (21).  $\square$

### 3.3. Circulation and Flux with Respect to $\tilde{g}$

We consider a closed curve  $k_1$  in  $\beta_1$ , given by

$$x = x(t), \quad y = y(t), \quad t \in [t_1, t_2], \tag{23}$$

where  $x(t_1) = x(t_2), y(t_1) = y(t_2)$ .

Let

$$F(x, y) = P(x, y)u + Q(x, y)S^2u \tag{24}$$

be a vector force field on the curve  $k_1$ .

For the circulation  $C$  of a vector field  $F$  along a curve  $k$  we assume the following definition:

$$C = \oint_k \tilde{g}(F, dv), \tag{25}$$

where  $v$  is the radius vector of a point on  $k$ .

We denote, by  $D_1$ , the region inside  $k_1$ . For both cases (A) and (B) of circle (11) the following statements hold.

**Theorem 4.** *The circulation  $C$ , done by the force (24) along the curve (23), is expressed by*

$$C = 2 \cos \varphi \oint_{k_1} (P(x, y)dx + Q(x, y)dy), \tag{26}$$

where  $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{4})$ .

**Proof.** Let  $v = xu + yS^2u$  be the radius vector of a point on  $k_1$ . By virtue of (10) and (24), and bearing in mind  $dv = dxu + dyS^2u$ , we obtain

$$\tilde{g}(F, dv) = 2 \cos \varphi (P(x, y)dx + Q(x, y)dy). \tag{27}$$

Obviously, (26) follows from (23), (25) and (27).  $\square$

We determine a vector  $w$  in  $T_pM$  by the equality

$$w = \frac{1}{\sqrt{1 - 2 \cos^2 \varphi}} (\cos \varphi u - Su + \cos \varphi S^2u), \tag{28}$$

where  $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{4})$ . By using (1), (3) and (9) it is easy to verify that

$$g(w, u) = g(w, S^2u) = 0, \quad g(w, w) = 1.$$

We construct an orthonormal coordinate system  $Oxyz$ , such that  $u \in Ox, S^2u \in Oy, w \in Oz$ .

We suppose that the curl of  $F$ , determined by (24), with respect to  $Oxyz$ , is

$$\text{curl } F = (Q_x - P_y)w.$$

The flux  $T$  of the vector field  $\text{curl } F$  across the region  $D_1$  inside the curve  $k_1$  is given by

$$T = \iint_{D_1} \tilde{g}(\text{curl } F, w) dA''. \tag{29}$$

With the help of (10) and (28) we obtain  $\tilde{g}(w, w) = -2 \cos \varphi$ . Then, from (19) and (29) we state the following.

**Theorem 5.** *The flux  $T$  of the vector field  $\text{curl } F$  across the region  $D_1$  inside the curve (23) is expressed by*

$$T = -4 \cos^2 \varphi \iint_{D_1} (Q_x - P_y) dx dy, \tag{30}$$

where  $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{4})$ .

On the other hand, due to Green’s formula, we have

$$\int \int_D (Q_x - P_y) dx dy = \oint_k (P dx + Q dy).$$

Bearing in mind the above formula we obtain the statements that follow.

**Theorem 6.** *The relation between the circulation (26) and the flux (30) is determined by*

$$T = -2 \cos \varphi C.$$

**Corollary 1.** *The relation between the circulation C and the flux T is*

- (a)  $T = -C$ , in case  $\varphi = \frac{\pi}{3}$ ;
- (b)  $T = C$ , in case  $\varphi = \frac{2\pi}{3}$ .

#### 4. Circles in the Two-Plane $\beta_2$

**Lemma 2 ([3]).** *Let  $\beta_2$  be the 2-plane spanned by unit vectors  $u$  and  $Su$ . The system of vectors  $\{e_1, e_2\}$ , determined by the equalities*

$$e_1 = \frac{1}{\sqrt{2(1 + \cos \varphi)}}(u + Su), \quad e_2 = \frac{1}{\sqrt{2(1 - \cos \varphi)}}(-u + Su), \tag{31}$$

*is an orthonormal basis of  $\beta_2$  with respect to  $g$ .*

The coordinate system  $p_{xy}$  on  $\beta_2$ , such that  $e_1$  is on the axis  $p_x$  and  $e_2$  is on the axis  $p_y$ , is orthonormal. Due to (10), we ascertain that the system  $\{e_1, e_2\}$  satisfies the following equalities:

$$\tilde{g}(e_1, e_1) = \frac{2 \cos \varphi + 1}{1 + \cos \varphi}, \quad \tilde{g}(e_2, e_2) = \frac{2 \cos \varphi - 1}{1 - \cos \varphi}, \quad \tilde{g}(e_1, e_2) = 0. \tag{32}$$

A circle  $k_2$  in  $\beta_2$  centered at the origin  $p$ , with respect to  $\tilde{g}$  on  $(M, g, S)$ , is defined by (7). The equation of  $k_2$  with respect to  $p_{xy}$  is obtained as below.

**Theorem 7 ([3]).** *Let  $\tilde{g}$  be the associated metric on  $(M, g, S)$  and let  $\beta_2 = \{u, Su\}$  be a 2-plane in  $T_p M$  with an orthonormal basis (31). If  $p_{xy}$  is a coordinate system, such that  $e_1 \in p_x, e_2 \in p_y$ , then the equation of the circle (7) in  $\beta_2$  is given by*

$$\frac{2 \cos \varphi + 1}{1 + \cos \varphi} x^2 + \frac{2 \cos \varphi - 1}{1 - \cos \varphi} y^2 = R^2. \tag{33}$$

The only closed curve  $k_2$ , determined by (33), is an ellipse in terms of  $g$  with parameters:

**Case (A)**  $\varphi \in (\frac{\pi}{4}, \frac{\pi}{3})$  and  $R^2 > 0$ ;

**Case (B)**  $\varphi \in (\frac{2\pi}{3}, \frac{3\pi}{4})$  and  $R^2 < 0$ .

The two-plane  $\beta_2$  is constructed on space-like vectors in Case (A), and  $\beta_2$  is constructed on time-like vectors in Case (B).

##### 4.1. Length of a Circle with Respect to $\tilde{g}$

Firstly, we consider Case (A). The circle (7) has a radius  $R > 0$  and  $\varphi$  satisfies

$$\frac{\pi}{4} < \varphi < \frac{\pi}{3}. \tag{34}$$

**Theorem 8.** *The circle  $k_2$  with (34) and a radius  $R > 0$  has length*

$$L = 2\pi R. \tag{35}$$



**Proof.** The radius vector  $v$  of an arbitrary point on the curve  $k_2$  is  $v = xe_1 + ye_2$ . Then,  $dv = dx e_1 + dy e_2$  is a tangent vector on  $k_2$ . The length  $L$  of  $k_2$ , with respect to  $\tilde{g}$ , is

$$dL = \sqrt{\tilde{g}(dv, dv)}.$$

From (32), we find

$$\tilde{g}(dv, dv) = \sqrt{\frac{2 \cos \varphi + 1}{1 + \cos \varphi} dx^2 + \frac{2 \cos \varphi - 1}{1 - \cos \varphi} dy^2}.$$

Then, we obtain

$$L = \oint_{k_2} \sqrt{\frac{2 \cos \varphi + 1}{1 + \cos \varphi} dx^2 + \frac{2 \cos \varphi - 1}{1 - \cos \varphi} dy^2}. \tag{36}$$

We substitute

$$x = R \sqrt{\frac{1 + \cos \varphi}{2 \cos \varphi + 1}} \cos t, \quad y = R \sqrt{\frac{1 - \cos \varphi}{2 \cos \varphi - 1}} \sin t, \quad t \in [0, 2\pi]$$

into (36) and obtain

$$L = \int_0^{2\pi} \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} dt,$$

which implies (35).  $\square$

Now, we consider Case (B). The circle  $k_2$  has a radius  $R = ri$ ,  $r > 0$ ,  $i^2 = -1$  and the angle  $\varphi$  satisfies

$$\frac{2\pi}{3} < \varphi < \frac{3\pi}{4}. \tag{37}$$

The Equation (33) transforms into

$$\frac{2 \cos \varphi + 1}{1 + \cos \varphi} x^2 + \frac{2 \cos \varphi - 1}{1 - \cos \varphi} y^2 = -r^2. \tag{38}$$

By calculations similar to those of Case (A), we find the integral (36) over  $k_2$  with (38). Using the substitutions

$$x = \sqrt{-\frac{1 + \cos \varphi}{2 \cos \varphi + 1}} r \cos t, \quad y = \sqrt{-\frac{1 - \cos \varphi}{2 \cos \varphi - 1}} r \sin t, \quad t \in [0, 2\pi],$$

we calculate the length of  $k_2$  and formulate the next proposition.

**Proposition 3.** *The circle  $k_2$  with (37) and a radius  $R = ri$ ,  $r > 0$ ,  $i^2 = -1$  has an imaginary length*

$$L = 2\pi R.$$

#### 4.2. Area of a Circle with Respect to $\tilde{g}$

For Case (A) we state the following.

**Theorem 9.** *The area  $A''$  of the circle  $k_2$  with (34) and a radius  $R > 0$  is*

$$A'' = \pi R^2. \tag{39}$$

**Proof.** Let us denote, by  $\widetilde{\cos}\theta$ , the cosine of  $\theta = \angle(e_1, e_2)$  with respect to  $\tilde{g}$ , which is

$$\widetilde{\cos}\theta = \frac{\tilde{g}(e_1, e_2)}{\sqrt{\tilde{g}(e_1, e_1)}\sqrt{\tilde{g}(e_2, e_2)}}.$$

Then, using (32), we derive  $\widetilde{\cos}\theta = 0$ , which implies

$$\widetilde{\sin}\theta = 1. \tag{40}$$

In the coordinate plane  $p_{xy}$ , we construct a parallelogram with locus vectors  $dx e_1$  and  $dy e_2$ . For its area  $A''$  with respect to  $\tilde{g}$  we obtain

$$dA'' = \sqrt{\tilde{g}(dx e_1, dx e_1)}\sqrt{\tilde{g}(dy e_2, dy e_2)}\widetilde{\sin}\theta.$$

We apply (32) and (40) in the above equality and obtain

$$dA'' = \frac{\sqrt{4 \cos^2 \varphi - 1}}{\sin \varphi} dx dy. \tag{41}$$

We integrate (41) over the region  $D_2$  inside  $k_2$  and calculate

$$A'' = \frac{\sqrt{4 \cos^2 \varphi - 1}}{\sin \varphi} \int \int_{D_2} dx dy, \tag{42}$$

with

$$D_2 : \frac{2 \cos \varphi + 1}{1 + \cos \varphi} x^2 + \frac{2 \cos \varphi - 1}{1 - \cos \varphi} y^2 \leq R^2.$$

We substitute

$$x = \sqrt{\frac{1 + \cos \varphi}{2 \cos \varphi + 1}} R \rho \cos t, \quad y = \sqrt{\frac{1 - \cos \varphi}{2 \cos \varphi - 1}} R \rho \sin t, \quad t \in [0, 2\pi], \quad \rho \in [0, 1],$$

and Jacobian  $\Delta = \frac{\sin \varphi}{\sqrt{4 \cos^2 \varphi - 1}} R^2 \rho$  into (42). Finally we get (39).  $\square$

Now, we consider Case (B).

**Proposition 4.** The area  $A''$  of the circle  $k_2$  with (34) and a radius  $R = ri$ ,  $r > 0$ ,  $i^2 = -1$  has a negative value

$$A'' = \pi R^2. \tag{43}$$

**Proof.** The circle  $k_2$  has Equation (38), with conditions (37) and a radius  $R = ri$ , where  $r > 0$ ,  $i^2 = -1$ . By calculations analogous to the previous case, we find that the area of  $k_2$  is given by

$$A'' = \frac{\sqrt{4 \cos^2 \varphi - 1}}{\sin \varphi} \int \int_{D_2} dx dy, \tag{44}$$

with

$$D_2 : -\frac{2 \cos \varphi + 1}{1 + \cos \varphi} x^2 - \frac{2 \cos \varphi - 1}{1 - \cos \varphi} y^2 \leq r^2.$$

We substitute

$$x = \sqrt{-\frac{1 + \cos \varphi}{2 \cos \varphi + 1}} r \rho \cos t, \quad y = \sqrt{-\frac{1 - \cos \varphi}{2 \cos \varphi - 1}} r \rho \sin t, \quad t \in [0, 2\pi], \quad \rho \in [0, 1],$$

and Jacobian  $\Delta = \frac{\sin \varphi}{\sqrt{4 \cos^2 \varphi - 1}} r^2 \rho$  into the integral (44) and obtain  $A'' = -\pi r^2$ , which implies (43).  $\square$

4.3. Circulation and Flux with Respect to  $\tilde{g}$

We consider a closed curve  $k_2$  in  $\beta_2$ , given by

$$x = x(t), \quad y = y(t), \quad t \in [t_1, t_2], \tag{45}$$

where  $x(t_1) = x(t_2), y(t_1) = y(t_2)$ .

Let

$$F(x, y) = P(x, y)e_1 + Q(x, y)e_2 \tag{46}$$

be a vector force field on the curve  $k_2$ .

We denote, by  $D_2$ , the region inside  $k_2$ . For both cases (A) and (B) of ellipse (33) the following statements hold.

**Theorem 10.** The circulation  $C$  of the force (46) along the curve (45) is expressed by

$$C = \oint_{k_2} \left( \frac{2 \cos \varphi + 1}{1 + \cos \varphi} P(x, y) dx + \frac{2 \cos \varphi - 1}{1 - \cos \varphi} Q(x, y) dy \right), \tag{47}$$

where  $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$ .

**Proof.** For the circulation  $C$  of a vector force field  $F$  acting along the curve (45) we use (25), where  $v = xe_1 + ye_2$  is the radius vector of a point on  $k_2$ . Therefore, we have

$$C = \oint_{k_2} \tilde{g}(F, dv), \tag{48}$$

with a tangent vector  $dv = dx e_1 + dy e_2$  on  $k_2$ . Then, by virtue of (32) and (46), we obtain

$$\tilde{g}(F, dv) = \left( \frac{2 \cos \varphi + 1}{1 + \cos \varphi} P(x, y) dx + \frac{2 \cos \varphi - 1}{1 - \cos \varphi} Q(x, y) dy \right). \tag{49}$$

Hence, (45), (48) and (49) imply (47).  $\square$

**Theorem 11.** The flux  $T$  of the vector field  $\text{curl } F$  across the region  $D_2$  inside the curve (45) is expressed by

$$T = -2 \cot^3 \varphi \sqrt{4 \cos^2 \varphi - 1} \int \int_{D_2} (Q_x - P_y) dx dy, \tag{50}$$

where  $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$ .

**Proof.** We determine a vector  $w$  in  $T_p M$  by the equality

$$w = \frac{1}{\sin \varphi \sqrt{1 - 2 \cos^2 \varphi}} \left( \cos^2 \varphi u - (\cos \varphi) S u + \sin^2 \varphi S^2 u \right), \tag{51}$$

where  $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$ . Then, using (1), (3), (9) and (31), we verify that

$$g(w, e_1) = g(w, e_2) = 0, \quad g(w, w) = 1.$$

The coordinate system  $Oxyz$ , such that  $e_1 \in Ox, e_2 \in Oy, w \in Oz$ , is orthonormal.

We obtain the curl of  $F$ , determined by (46), by the equality  $\text{curl } F = (Q_x - P_y)w$ . For the flux  $T$  of the vector field  $\text{curl } F$  across the region  $D_2$  inside the curve (45) we have

$$T = \int \int_{D_2} \tilde{g}(\text{curl } F, w) dA''. \tag{52}$$

Now, with the help of (32) and (51), we calculate

$$\tilde{g}(w, w) = -\frac{2 \cos^3 \varphi}{\sin^2 \varphi}.$$

Then, from (41) and (52), it follows (50).  $\square$

We introduce the following notations:

$$c_1 = \frac{2 \cos \varphi + 1}{1 + \cos \varphi} \oint_{k_2} P dx, \quad c_2 = \frac{2 \cos \varphi - 1}{1 - \cos \varphi} \oint_{k_2} Q dy. \tag{53}$$

On the other hand, due to Green’s formula, we have

$$\int \int_D P_y dx dy = - \oint_k P dx, \quad \int \int_D Q_x dx dy = \oint_k Q dy.$$

Bearing in mind the latter equalities we derive the next statement.

**Theorem 12.** *The relation between the circulation (47) and the flux (50) is determined by*

$$T = -2 \cot^3 \varphi \left( (1 + \cos \varphi) \sqrt{\frac{2 \cos \varphi - 1}{2 \cos \varphi + 1}} c_1 + (1 - \cos \varphi) \sqrt{\frac{2 \cos \varphi + 1}{2 \cos \varphi - 1}} c_2 \right),$$

where  $c_1$  and  $c_2$  are given in (53).

### 5. Conclusions

In this paper, we investigated the properties of special two-planes  $\beta_1$  and  $\beta_2$  of  $T_p M$  of a four-dimensional Riemannian manifold  $(M, g, S)$ , equipped with an additional indefinite metric  $\tilde{g}(\cdot, \cdot) = g(\cdot, S \cdot) + g(S \cdot, \cdot)$ . In these two-planes, circles, with respect to  $\tilde{g}$ , are transformed into closed curves in terms of  $g$ . Therefore, we can consider analogs of well-known formulae, such as circulation of a vector force field along the curve and flux of the curl of a vector force field across the curve. It turns out that the length and area, calculated with respect to the indefinite metric, of the circles in  $\beta_1$  and  $\beta_2$  are the same as in the Euclidean space.

**Author Contributions:** Conceptualization, D.R., G.D. and I.D.; methodology, D.R., G.D. and I.D.; investigation, D.R., G.D. and I.D.; writing—original draft preparation, D.R., G.D. and I.D.; writing—review and editing, D.R., G.D. and I.D.; funding acquisition, D.R. and G.D. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was partially funded by project 17-12 “Support for publishing activities”, Agricultural University of Plovdiv, Bulgaria.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** We are grateful to the editors and reviewers for their helpful comments.

**Conflicts of Interest:** The authors declare no conflict of interest.

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