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Fuzzy Differential Inequalities for Convolution Product of Ruscheweyh Derivative and Multiplier Transformation

Alina Alb Lupaş

Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania; dalb@uoradea.ro or alblupas@gmail.com

Abstract: In this paper, the author combines the geometric theory of analytic function regarding differential superordination and subordination with fuzzy theory for the convolution product of Ruscheweyh derivative and multiplier transformation. Interesting fuzzy inequalities are obtained by the author.

Keywords: fuzzy set theory; convex function; differential operator; fuzzy best dominant; fuzzy differential subordination; fuzzy best subordinant; fuzzy differential superordination; convolution; multiplier transformation; Ruscheweyh derivative

MSC: 30C45; 30A20; 34A40

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Citation: Alb Lupaş, A. Fuzzy Differential Inequalities for Convolution Product of Ruscheweyh Derivative and Multiplier Transformation. *Axioms* **2023**, *12*, 470. https://doi.org/10.3390/axioms12050470

Academic Editors: Zhenyu Zhang, Vladimir Simic and Jing Li

Received: 13 April 2023
Revised: 10 May 2023
Accepted: 11 May 2023
Published: 13 May 2023

1. Introduction and Preliminaries

Based on current economic, ecological, and social problems and facts, sustainability implies a continual dynamic evolution that is motivated by human hopes about potential future prospects. The fuzzy set notion, which Lotfi A. Zadeh first proposed in 1965 [1], has multiple applications in science and technology. Fuzzy mathematical models are created in the research by employing the fuzzy set theory to evaluate the sustainable development regarding a socio-scientific environment. Fuzzy set theory connects human expectations for development stated in language concepts to numerical facts, which are reflected in measurements of sustainability indicators, despite the fact that decision-making regarding sustainable development is subjective.

Intuitionistic fuzzy set is applied to introduce a new extension to the multi-criteria decision-making model for sustainable supplier selection based on sustainable supply chain management practices in Ref. [2], taking into account the idea that choosing a suitable supplier is the key element of contemporary businesses from a sustainability perspective. Supply chain sustainability is considered in the fuzzy context for steel industry in Ref. [3] and a model for sustainable energy usage in the textile sector based on intuitionistic fuzzy sets is introduced in Ref. [4]. The study proposed in Ref. [5] using nonlinear integrated fuzzy modeling can help to predict how comfortable an office building will be and how that will affect people's health for optimized sustainability. Healthcare system is of outermost importance and optimization models are investigated using generalizations of the fuzzy set concept in recent studies proposing an updated multi-criteria integrated decision-making approach involving interval-valued intuitionistic fuzzy sets in Ref. [6] or a flexible optimization model based on bipolar interval-valued neutrosophic sets in Ref. [7]. Another application of the fuzzy theory to integro-differential equations domain is presented in Ref. [8].

The introduction of the notion of a fuzzy set into the studies has led to the development of extensions for many domains of mathematics. Refs. [9,10] exposed different applications in mathematical domains of this notion. In geometric function theory, the introduction of the notion of fuzzy subordination used the notion of fuzzy set in 2011 [11] and the theory of fuzzy differential subordination has developed since 2012 [12], which is when Miller and Mocanu's classical theory of differential subordination [13] started to be adapted by involving fuzzy theory aspects. The dual notion, namely fuzzy differential superordination, was introduced in 2017 [14]. Since then, numerous researchers have studied different properties of differential operators involving fuzzy differential subordinations and superordinations: Wanas operator [15,16], generalized Noor-Sălăgean operator [17], Ruscheweyh and Sălăgean operators [18] or a linear operator [19]. Univalence criteria were also derived using fuzzy differential subordination theory [20].

It is obvious that applying the fuzzy context to the theories of differential subordination and superordination generates outcomes that are interesting for complex analysis researchers who want to broaden their area of study. For example, Confluent Hypergeometric Function's fractional integral was used for obtaining classical differential subordinations and superordinations in Ref. [21] and also to develop fuzzy differential subordinations and superordinations in Refs. [22,23]. This demonstrates that both methodologies yield intriguing findings and that studies from a fuzzy perspective are not incompatible with the interesting results attained when applying the traditional theories of differential subordination and superordination to the same subjects.

Considering this idea, in this article, the operator previously introduced in Ref. [24] as the convolution of the generalized Sălăgean operator and the Ruscheweyh derivative is used to apply the dual theories of fuzzy differential subordination and superordination. A novel class of normalized analytic functions in U is introduced via this operator and examined in the fuzzy context created in geometric function theory by embedding the concept of fuzzy set connected with analytic functions. Certain inclusion relations involving the class parameters are proved. Furthermore, interesting fuzzy differential subordinations are developed by using frequently referred to lemmas, the functions from the new class and the previously mentioned operator. When feasible, the fuzzy best dominants are also shown. Additionally, dual findings consisting of new fuzzy differential superordinations emerge, involving the convolution operator, ensuring that the best subordinants are also provided. The significance of the new theoretical findings presented in this study is demonstrated by the numerous examples generated for results obtained regarding the two dual theories, as well as by specific corollaries obtained, implying the appropriate convex functions as the fuzzy best dominants or fuzzy best subordinants within the established theorems.

To obtain the results of the article, we need the notions and results presented below:

$\mathcal{H}(U)$ contains all holomorphic functions in $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc, and we worked on the particular subclasses

$$\mathcal{A} := \{\mathfrak{f}(z) = z + \sum_{j=2}^{\infty} \mathfrak{a}_j z^j \in \mathcal{H}(\mathfrak{U})\},$$

and

$$\mathcal{H}[\mathfrak{a}, \mathfrak{n}] := \{\mathfrak{f}(z) = \mathfrak{a} + \mathfrak{a}_n z^{\mathfrak{n}} + \mathfrak{a}_{n+1} z^{\mathfrak{n}+1} + \cdots \in \mathcal{H}(\mathfrak{U})\},$$

with $\mathfrak{a} \in \mathbb{C}$, $\mathfrak{n} \in \mathbb{N}$.

Definition 1. *([11]) The pair* $(\mathfrak{A}, \mathcal{F}_{\mathfrak{A}})$ *is the fuzzy subset of* $\mathcal{X}$, *where* $\mathcal{F}_{\mathfrak{A}} : \mathcal{X} \to [0,1]$ *and* $\mathfrak{A} = \{x : 0 < \mathcal{F}_{\mathfrak{A}}(x) \leq 1\}$. *The set* $\mathfrak{A}$ *represents the support of the fuzzy set* $(\mathfrak{A}, \mathcal{F}_{\mathfrak{A}})$ *and* $\mathcal{F}_{\mathfrak{A}}$ *represents the membership function of* $(\mathfrak{A}, \mathcal{F}_{\mathfrak{A}})$, *we denote* $\mathfrak{A} = \mathrm{supp}(\mathfrak{A}, \mathcal{F}_{\mathfrak{A}})$.

Definition 2. *([11]) The function* $\mathfrak{f} \in \mathcal{H}(\mathfrak{D})$ *is the fuzzy subordinate to the function* $\mathfrak{g} \in \mathcal{H}(\mathfrak{D})$, *denoted* $\mathfrak{f} \prec_{\mathcal{F}} \mathfrak{g}$, *where* $\mathfrak{D} \subset \mathbb{C}$, *when*

(1) $\mathfrak{f}(z_0) = \mathfrak{g}(z_0)$, *for* $z_0 \in \mathfrak{D}$ *a fixed point*

(2) $\mathcal{F}_{\mathfrak{f}(\mathfrak{D})}\mathfrak{f}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{D})}\mathfrak{g}(z)$, $z \in \mathfrak{D}$.

Definition 3. *([12] Definition 2.2) Let $\psi : \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ and $\mathfrak{h}$ is a univalent function in $\mathfrak{U}$ such that $\mathfrak{h}(0) = \psi(\mathfrak{a}, 0; 0) = \mathfrak{a}$. If the analytic function $\mathfrak{p}$ in $\mathfrak{U}$ with the property $\mathfrak{p}(0) = \mathfrak{a}$ verifies the fuzzy subordination*

$$\mathcal{F}_{\psi(\mathbb{C}^3\times U)}\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z); z) \leq \mathcal{F}_{\mathfrak{h}(U)}\mathfrak{h}(z), \quad z \in \mathfrak{U}, \tag{1}$$

then the fuzzy differential subordination has as fuzzy solution, which is the function $\mathfrak{p}$. A fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination is the univalent function $\mathfrak{q}$ if $\mathcal{F}_{\mathfrak{p}(U)}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{q}(U)}\mathfrak{q}(z)$, $z \in \mathfrak{U}$, for all $\mathfrak{p}$ verifying (1). The fuzzy best dominant is a fuzzy dominant $\tilde{\mathfrak{q}}$ with the property $\mathcal{F}_{\tilde{\mathfrak{q}}(U)}\tilde{\mathfrak{q}}(z) \leq \mathcal{F}_{\mathfrak{q}(U)}\mathfrak{q}(z)$, $z \in \mathfrak{U}$, for all fuzzy dominants $\mathfrak{q}$ of (1).

Definition 4. *([14]) Let $\varphi : \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ and $\mathfrak{h}$ an analytic function in $\mathfrak{U}$.*
If $\mathfrak{p}$ and $\varphi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z); z)$ are univalent functions in $\mathfrak{U}$ and the fuzzy differential superordination holds

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{\varphi(\mathbb{C}^3\times\mathfrak{U})}\varphi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z); z), \quad z \in \mathfrak{U}, \tag{2}$$

then p represents a fuzzy solution for the fuzzy differential superordination. A fuzzy subordinant for the fuzzy differential superordination is an analytic function $\mathfrak{q}$ with the property

$$\mathcal{F}_{\mathfrak{q}(\mathfrak{U})}\mathfrak{q}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z), \ \ z \in \mathfrak{U},$$

for all $\mathfrak{p}$ verifying (2). The fuzzy best subordinate is a univalent fuzzy subordination $\tilde{\mathfrak{q}}$ with property $\mathcal{F}_{\mathfrak{q}(\mathfrak{U})}\mathfrak{q} \leq \mathcal{F}_{\tilde{\mathfrak{q}}(\mathfrak{U})}\tilde{\mathfrak{q}}$, for all fuzzy subordinate $\mathfrak{q}$ of (2).

Definition 5. *([12]) $\mathfrak{Q}$ denotes the set of all injective and analytic functions $\mathfrak{f}$ on $\overline{\mathfrak{U}}\backslash E(\mathfrak{f})$, with the property $\mathfrak{f}'(\zeta) \neq 0$ for $\zeta \in \partial\mathfrak{U}\backslash E(\mathfrak{f})$, and $E(\mathfrak{f}) = \{\zeta \in \partial\mathfrak{U} : \lim_{z\to\zeta}\mathfrak{f}(z) = \infty\}$.*

We use the lemmas presented below to show our fuzzy inequalities:

Lemma 1. *([25]) Let $\mathfrak{g}$ be a convex function in $\mathfrak{U}$ and consider the function*

$$\mathfrak{h}(z) = \mathfrak{n}\alpha z\mathfrak{g}'(z) + \mathfrak{g}(z),$$

with $z \in \mathfrak{U}$, $\mathfrak{n} \in \mathbb{N}$ and $\alpha > 0$.
For the holomorphic function

$$\mathfrak{g}(0) + \mathfrak{p}_n z^{\mathfrak{n}} + \mathfrak{p}_{\mathfrak{n}+1}z^{\mathfrak{n}+1} + \cdots = \mathfrak{p}(z), \quad z \in \mathfrak{U},$$

which satisfies the fuzzy differential subordination

$$\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\big(\alpha z\mathfrak{p}'(z) + \mathfrak{p}(z)\big) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U},$$

the sharp fuzzy differential subordination

$$\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$$

is satisfied.

Lemma 2. *([25]) Consider $\alpha \in \mathbb{C}^*$ with Re $\alpha \geq 0$ and $\mathfrak{h}$ a convex function with the property $\mathfrak{h}(0) = \mathfrak{a}$. If $\mathfrak{p} \in \mathcal{H}[\mathfrak{a}, \mathfrak{n}]$ satisfies the fuzzy differential subordination*

$$\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{z\mathfrak{p}'(z)}{\alpha} + \mathfrak{p}(z)\right) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U},$$

then the fuzzy differential subordinations

$$\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U},$$

is satisfied and

$$\mathfrak{g}(z) = \frac{\alpha}{\mathfrak{n}z^{\frac{\alpha}{\mathfrak{n}}}} \int_0^z \mathfrak{h}(t)t^{\frac{\alpha}{\mathfrak{n}}-1}dt, \quad z \in \mathfrak{U}.$$

Lemma 3. *([13] [Corollary 2.6g.2, p. 66]) Consider $\alpha \in \mathbb{C}^*$ with $Re\ \alpha \geq 0$ and $\mathfrak{h}$ a convex function with the property $\mathfrak{h}(0) = \mathfrak{a}$. If $\mathfrak{p} \in \mathfrak{Q} \cap \mathcal{H}[\mathfrak{a}, \mathfrak{n}]$, the function $\frac{z\mathfrak{p}'(z)}{\alpha} + \mathfrak{p}(z)$ is univalent in $\mathfrak{U}$ and satisfies the fuzzy differential superordination*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{z\mathfrak{p}'(z)}{\alpha} + \mathfrak{p}(z)\right), \ z \in \mathfrak{U},$$

then the fuzzy differential superordination

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z), \ z \in \mathfrak{U},$$

is satisfied and the convex function $\mathfrak{g}(z) = \frac{\alpha}{\mathfrak{n}z^{\frac{\alpha}{\mathfrak{n}}}} \int_0^z \mathfrak{h}(t)t^{\frac{\alpha}{\mathfrak{n}}-1}dt$, $z \in \mathfrak{U}$ represents the fuzzy best subordinant.

Lemma 4. *([13] [Corollary 2.6g.2, p. 66]) Taking $\alpha \in \mathbb{C}^*$, with $Re\ \alpha \geq 0$ and $\mathfrak{g}$ a convex function in $\mathfrak{U}$, we define the function*

$$\mathfrak{h}(z) = \frac{z\mathfrak{g}'(z)}{\alpha} + \mathfrak{g}(z), \ z \in \mathfrak{U}.$$

If $\mathfrak{p} \in \mathfrak{Q} \cap \mathcal{H}[\mathfrak{a}, \mathfrak{n}]$ and the univalent function $\frac{z\mathfrak{p}'(z)}{\alpha} + \mathfrak{p}(z)$ in $\mathfrak{U}$ satisfies the fuzzy differential superordination

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\left(\frac{z\mathfrak{g}'(z)}{\alpha} + \mathfrak{g}(z)\right) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{z\mathfrak{p}'(z)}{\alpha} + \mathfrak{p}(z)\right), \ z \in \mathfrak{U},$$

then the fuzzy differential superordination

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z), \ z \in \mathfrak{U},$$

is satisfied and $\mathfrak{g}(z) = \frac{\alpha}{\mathfrak{n}z^{\frac{\alpha}{\mathfrak{n}}}} \int_0^z \mathfrak{h}(t)t^{\frac{\alpha}{\mathfrak{n}}-1}dt$, $z \in \mathfrak{U}$ represents the fuzzy best subordinant.

We remind the definition of the convolution product between Ruscheweyh derivative and the multiplier transformation:

Definition 6. *([24]) Let $\mathfrak{n}, l, \lambda \in \mathbb{N} \cup \{0\}$. The operator denoted by $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ is defined as the convolution product between the multiplier transformation $I(\mathfrak{n}, \lambda, \mathfrak{l})$ and the Ruscheweyh derivative $R^{\mathfrak{n}}$, $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}} : \mathcal{A} \to \mathcal{A}$,*

$$IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) := (I(\mathfrak{n}, \lambda, \mathfrak{l}) * R^{\mathfrak{n}})\mathfrak{f}(z).$$

Remark 1. *For $\mathfrak{f}(z) = z + \sum_{j=2}^{\infty} \mathfrak{a}_j z^j \in \mathcal{A}$, the operator has the following form $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) = z + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}} \frac{\Gamma(\mathfrak{n}+j)}{\Gamma(\mathfrak{n}+1)\Gamma(j)} \mathfrak{a}_j^2 z^j$, $z \in \mathfrak{U}$, where Γ is the Gamma function.*

We remind also the definition of the multiplier transformation [26]:

For $\mathfrak{n} \in \mathbb{N}$, $\mathfrak{l}, \lambda \geq 0$ and $\mathfrak{f} \in \mathcal{A}$, the operator $I(\mathfrak{n}, \lambda, \mathfrak{l})\mathfrak{f}(z)$ is defined by relation

$$I(\mathfrak{n}, \lambda, \mathfrak{l})\mathfrak{f}(z) := z + \sum_{j=2}^{\infty}\left(\frac{\lambda(j-1) + \mathfrak{l} + 1}{\mathfrak{l} + 1}\right)^{\mathfrak{n}} \mathfrak{a}_j z^j,$$

and has the properties:

$$I(0,\lambda,\mathfrak{l})\mathfrak{f}(z) = \mathfrak{f}(z),$$

$$(\mathfrak{l}+1)I(\mathfrak{n}+1,\lambda,\mathfrak{l})\mathfrak{f}(z) = \lambda z(I(\mathfrak{n},\lambda,\mathfrak{l})\mathfrak{f}(z))' + (\mathfrak{l}+1-\lambda)I(\mathfrak{n},\lambda,\mathfrak{l})\mathfrak{f}(z), \quad z \in \mathfrak{U}.$$

The definition of Ruscheweyh derivative [27] follows:

For $\mathfrak{n} \in \mathbb{N}$ and $\mathfrak{f} \in \mathcal{A}$, the Ruscheweyh derivative $R^{\mathfrak{n}} : \mathcal{A} \to \mathcal{A}$ is introduced by relations

$$\begin{aligned} R^0\mathfrak{f}(z) &= \mathfrak{f}(z) \\ R^1\mathfrak{f}(z) &= z\mathfrak{f}'(z) \\ &\ldots \\ (\mathfrak{n}+1)R^{\mathfrak{n}+1}\mathfrak{f}(z) &= \mathfrak{n}R^{\mathfrak{n}}\mathfrak{f}(z) + z(R^{\mathfrak{n}}\mathfrak{f}(z))', \quad z \in \mathfrak{U}. \end{aligned}$$

For $\mathfrak{f}(z) = z + \sum_{j=2}^{\infty} \mathfrak{a}_j z^j \in \mathcal{A}$, the operator has the following form $R^{\mathfrak{n}}\mathfrak{f}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(\mathfrak{n}+j)}{\Gamma(\mathfrak{n}+1)\Gamma(j)} \mathfrak{a}_j z^j$, $z \in \mathfrak{U}$.

Using the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ introduced in Definition 6, a new subclass of functions, the normalized analytic in $\mathfrak{U}$, is defined in Section 2 of this article and it shows the convexity of this class. Using Lemmas 1 and 2, we obtain fuzzy inequalities regarding differential subordination, implying the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$. In Section 3, we obtain new fuzzy inequalities regarding differential superordinations involving the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ by using Lemmas 3 and 4.

2. Fuzzy Differential Subordination

Using the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}$ from Definition 6, we introduce the class $\mathcal{IR}^{\mathcal{F}}_{\mathfrak{n},\lambda,\mathfrak{l}}(\alpha)$ following the pattern set in Ref. [18] and we study the fuzzy inequalities regarding differential subordinations.

Definition 7. *Consider $\alpha \in [0,1)$, $\mathfrak{n} \in \mathbb{N}$ and $\mathfrak{l}, \lambda \geq 0$. The class $\mathcal{IR}^{\mathcal{F}}_{\mathfrak{n},\lambda,\mathfrak{l}}(\alpha)$ contains the functions $\mathfrak{f} \in \mathcal{A}$ for which the inequality*

$$\operatorname{Re}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' > \alpha, \quad z \in \mathfrak{U}, \tag{3}$$

is satisfied.

Theorem 1. *Taking a function $\mathfrak{g}$ convex in $\mathfrak{U}$, we define $\mathfrak{h}(z) = \mathfrak{g}(z) + \frac{1}{\mathfrak{m}+2}z\mathfrak{g}'(z)$, $z \in \mathfrak{U}$, where $\mathfrak{m} > 0$. If $\mathfrak{f} \in \mathcal{IR}^{\mathcal{F}}_{\mathfrak{n},\lambda,\mathfrak{l}}(\alpha)$ and $\mathfrak{I}_{\mathfrak{m}}(\mathfrak{f})(z) = \frac{\mathfrak{m}+2}{z^{\mathfrak{m}+1}}\int_0^z t^{\mathfrak{m}}\mathfrak{f}(t)dt$, $z \in \mathfrak{U}$, then*

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U}, \tag{4}$$

implies the sharp inequality

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), \quad z \in \mathfrak{U}.$$

Proof. The function $\mathfrak{I}_m(\mathfrak{f})$ satisfies the relation $z^{\mathfrak{m}+1}\mathfrak{I}_m(\mathfrak{f})(z) = (\mathfrak{m}+2)\int_0^z t^{\mathfrak{m}}\mathfrak{f}(t)dt$, and after differentiation operation to apply for it, we get

$$z(\mathfrak{I}_m(\mathfrak{f}))'(z) + (\mathfrak{m}+1)\mathfrak{I}_m(\mathfrak{f})(z) = (\mathfrak{m}+2)\mathfrak{f}(z),$$

and applying the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$, we have

$$z\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' + (\mathfrak{m}+1)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z) = (\mathfrak{m}+2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z), \ z \in \mathfrak{U}. \tag{5}$$

Applying the differentiation operation to the relation (5), we obtain

$$\frac{1}{\mathfrak{m}+2}z\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'' + \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', z \in \mathfrak{U}.$$

Using the last relation, the fuzzy inequality (4) becomes

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m\mathfrak{f}(z)\right)'' + \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'\right) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\mathfrak{g}'(z) + \mathfrak{g}(z)\right). \quad (6)$$

Denoting

$$\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)', \quad (7)$$

we find that $\mathfrak{p} \in \mathcal{H}[1,1]$.

In these conditions, the fuzzy inequality (6) can be written as

$$\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\mathfrak{p}'(z) + \mathfrak{p}(z)\right) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\mathfrak{g}'(z) + \mathfrak{g}(z)\right), \quad z \in \mathfrak{U}.$$

Using Lemma 1, we get $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), z \in \mathfrak{U}$, written in the following form $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), z \in \mathfrak{U}$, where the sharpness is given by the fact that $\mathfrak{g}$ is the fuzzy best dominant. □

Theorem 2. *Consider* $\mathfrak{h}(z) = \frac{(2a-1)z+1}{z+1}$, $a \in [0,1)$. *For* $\mathfrak{I}_m$ *given by Theorem 1, with* $\mathfrak{m} > 0$, *the following inclusion holds*

$$\mathfrak{I}_m\left[\mathcal{IR}^{\mathcal{F}}_{\mathfrak{n},\lambda,\mathfrak{l}}(\alpha)\right] \subset \mathcal{IR}^{\mathcal{F}}_{\mathfrak{n},\lambda,\mathfrak{l}}(\alpha^*), \quad (8)$$

where $\alpha^* = 2(1-a)(\mathfrak{m}+2)\int_0^1 \frac{t^{\mathfrak{m}+1}}{t+1}dt + 2a - 1$.

Proof. Following the same ideas as in the proof of Theorem 1 regarding the convex function $\mathfrak{h}$, taking account the conditions from Theorem 2, we obtain $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\mathfrak{p}'(z) + \mathfrak{p}(z)\right) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)$, where the function $\mathfrak{p}$ is defined by (7).

Applying Lemma 2, we have $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)$, written in the following form $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_{\mathfrak{m}}(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_{\mathfrak{m}}(\mathfrak{f})(z)\right)' \leq F_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq F_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)$, and
$\mathfrak{g}(z) = \frac{\mathfrak{m}+2}{z^{\mathfrak{m}+2}}\int_0^z t^{\mathfrak{m}+1}\frac{(2a-1)t+1}{t+1}dt = \frac{(\mathfrak{m}+2)(2-2a)}{z^{\mathfrak{m}+2}}\int_0^z \frac{t^{\mathfrak{m}+1}}{t+1}dt + 2a - 1$. The function $\mathfrak{g}$ is convex, and considering that $\mathfrak{g}(\mathfrak{U})$ is symmetric with respect to the real axis, we get

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_{\mathfrak{m}}(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_{\mathfrak{m}}(\mathfrak{f})(z)\right)' \geq \min_{|z|=1}\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) = \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(1) \quad (9)$$

and $a^* = \mathfrak{g}(1) = 2(1-a)(\mathfrak{m}+2)\int_0^1 \frac{t^{\mathfrak{m}+1}}{t+1}dt + 2a - 1$. □

Theorem 3. *For a function* $\mathfrak{g}$ *convex such that* $\mathfrak{g}(0) = 1$, *we consider the function* $\mathfrak{h}(z) = z\mathfrak{g}'(z) + \mathfrak{g}(z), z \in \mathfrak{U}$. *For* $\mathfrak{f} \in \mathcal{A}$, *which satisfies the fuzzy inequality*

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U}, \quad (10)$$

then the sharp inequality holds

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), \quad z \in \mathfrak{U}.$$

Proof. Denoting $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}$, we obtain $z\mathfrak{p}'(z) + \mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', z \in \mathfrak{U}$. The fuzzy inequality $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \ z \in \mathfrak{U}$, using the notation made above, can be written in the following form $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}(z\mathfrak{p}'(z) + \mathfrak{p}(z)) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) = \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}(z\mathfrak{g}'(z) + \mathfrak{g}(z))$,

$z \in \mathfrak{U}$. Applying Lemma 1, we obtain $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$, written as $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$. The sharpness is given by the fact that $\mathfrak{g}$ is the fuzzy best dominant. □

Theorem 4. *When $\mathfrak{h}$ is a convex function such that $\mathfrak{h}(0) = 1$, and $f \in \mathcal{A}$ satisfies the fuzzy inequality*

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U}, \tag{11}$$

we get the fuzzy inequality as a differential subordination

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), \quad z \in \mathfrak{U},$$

and the fuzzy best dominant is the convex function $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$.

Proof. Take $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \in \mathcal{H}[1,1]$ and after differentiation operator applying for it, yields $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' = z\mathfrak{p}'(z) + \mathfrak{p}(z)$, $z \in \mathfrak{U}$, and the fuzzy inequality (11) takes the form $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}(z\mathfrak{p}'(z) + \mathfrak{p}(z)) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)$, $z \in \mathfrak{U}$. Applying Lemma 2, we obtain $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$, written by considering the notation made above $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$, and $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$ is a convex function that verifies the differential equation associated to the fuzzy differential subordination (11) $z\mathfrak{g}'(z) + \mathfrak{g}(z) = \mathfrak{h}(z)$, therefore it is the fuzzy best dominant. □

Corollary 1. *For the convex function $\mathfrak{h}(z) = \frac{(2a-1)z+1}{z+1}$ in $\mathfrak{U}$, $0 \leq a < 1$, when $\mathfrak{f} \in \mathcal{A}$ satisfies the fuzzy inequality*

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U}, \tag{12}$$

then

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(U)}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), \quad z \in \mathfrak{U},$$

where the function $\mathfrak{g}(z) = \frac{2(1-a)}{z}\ln(z+1) + 2a - 1$, $z \in U$, is the convex fuzzy best dominant.

Proof. From Theorem 4 considering $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}$, the fuzzy inequality (12) takes the following form $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}(z\mathfrak{p}'(z) + \mathfrak{p}(z)) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)$, $z \in \mathfrak{U}$, and applying Lemma 2, we deduce $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, written as $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$ and $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt = \frac{1}{z}\int_0^z \frac{(2a-1)t+1}{t+1}dt = \frac{2(1-a)}{z}\ln(z+1) + 2a - 1$, $z \in \mathfrak{U}$, is the fuzzy best dominant. □

Example 1. *Let the convex function $\mathfrak{h}(z) = \frac{1-z}{z+1}$ in $\mathfrak{U}$ having the property $\mathfrak{h}(0) = 1$ and $\mathfrak{f}(z) = z^2 + z$, $z \in \mathfrak{U}$. For $\mathfrak{n} = 1$, $\lambda = 1$, $\mathfrak{l} = 2$, we get $IR^1_{1,2}\mathfrak{f}(z) = \frac{8}{3}z^2 + z$ and $\left(IR^1_{1,2}\mathfrak{f}(z)\right)' = \frac{16}{3}z + 1$ and $\frac{IR^1_{1,2}\mathfrak{f}(z)}{z} = \frac{8}{3}z + 1$. We have $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \frac{1-t}{t+1}dt = \frac{2\ln(z+1)}{z} - 1$.*

Using Theorem 4 we get $\mathcal{F}_{\mathfrak{U}}\left(\frac{16}{3}z + 1\right) \leq \mathcal{F}_{\mathfrak{U}}\left(\frac{1-z}{z+1}\right)$, $z \in \mathfrak{U}$, imply $\mathcal{F}_{\mathfrak{U}}\left(\frac{8}{3}z + 1\right) \leq \mathcal{F}_{\mathfrak{U}}\left(\frac{2\ln(z+1)}{z} - 1\right)$, $z \in \mathfrak{U}$.

Theorem 5. *Taking a function $\mathfrak{g}$ convex with the property $\mathfrak{g}(0) = 1$, consider the function $\mathfrak{h}(z) = z\mathfrak{g}'(z) + \mathfrak{g}(z)$, $z \in U$. If $\mathfrak{f} \in \mathcal{A}$ satisfies the fuzzy inequality*

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)' \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad z \in \mathfrak{U}, \tag{13}$$

then the sharp inequality holds

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z),\ \ z \in \mathfrak{U}.$$

Proof. Denote $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}$ and differentiating this relation, we get $\mathfrak{p}'(z) = \frac{\left(IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} - \mathfrak{p}(z)\cdot\frac{\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}$, written as $z\mathfrak{p}'(z) + \mathfrak{p}(z) = \left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)'$. The fuzzy differential subordination (13) takes the following form using the notation above $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}(z\mathfrak{p}'(z) + \mathfrak{p}(z)) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}h(z) = \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}(z\mathfrak{g}'(z) + \mathfrak{g}(z))$, $z \in \mathfrak{U}$, and by applying Lemma 1, we get $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$, written as $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$. The sharpness is given by the fact that $\mathfrak{g}$ is the fuzzy best dominant. □

Theorem 6. *Taking a function* $\mathfrak{g}$ *convex with the property* $\mathfrak{g}(0) = 1$, *consider the function* $\mathfrak{h}(z) = \frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{g}'(z) + \mathfrak{g}(z)$, $z \in \mathfrak{U}$, *with* $\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} > 0$. *If* $\mathfrak{f} \in \mathcal{A}$ *meets the fuzzy inequality*

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(\frac{1}{z}\left(\frac{(\mathfrak{n}+1)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - \frac{(\mathfrak{n}-2)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)+\right.$$

$$\left.\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-2(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt\right) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad (14)$$

$z \in \mathfrak{U}$, *then the sharp inequality holds*

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z),\ \ z \in \mathfrak{U}.$$

Proof. Consider $\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$, with $\mathfrak{p}(0) = 1$, and differentiating the relation we deduce for $\mathfrak{f}(z) = z + \sum_{j=\mathfrak{n}+1}^{\infty}\mathfrak{a}_j z^j$, that $\mathfrak{p}(z) + z\mathfrak{p}'(z) =$
$1 + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}j\mathfrak{a}_j^2 z^{j-1} + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}j(j-1)\mathfrak{a}_j^2 z^{j-1} =$
$1 + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}j^2\mathfrak{a}_j^2 z^{j-1} =$
$\frac{1}{z}\left(z + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}+1}C^{\mathfrak{n}+1}_{\mathfrak{n}+j}\frac{\mathfrak{n}+1}{\lambda}\mathfrak{a}_j^2 z^j - \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}\frac{\lambda(\mathfrak{n}-1)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}j\mathfrak{a}_j^2 z^j -\right.$
$\left.\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}\frac{\mathfrak{n}-2}{\lambda}\mathfrak{a}_j^2 z^j - \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}\frac{1}{j-1}\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}+1)}\mathfrak{a}_j^2 z^j\right) =$
$\frac{1}{z}\left[\frac{\mathfrak{n}+1}{\lambda}\left(z + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}+1}C^{\mathfrak{n}+1}_{\mathfrak{n}+j}\mathfrak{a}_j^2 z^j\right) - \frac{\mathfrak{n}-2}{\lambda}\left(z + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}\mathfrak{a}_j^2 z^j\right)\right] +$
$\left(1 - \frac{\mathfrak{n}+1}{\lambda} - \frac{\mathfrak{n}-2}{\lambda}\right) + \left(1 + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}\mathfrak{a}_j^2 jz^{j-1}\right)\frac{\lambda(\mathfrak{n}-1)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)} - \frac{\lambda(\mathfrak{n}-1)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)} -$
$\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}\frac{1}{j-1}\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}+1)}\mathfrak{a}_j^2 z^{j-1} =$
$\frac{1}{z}\left(\frac{\mathfrak{n}+1}{\lambda}IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - \frac{\mathfrak{n}-2}{\lambda}IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right) + \frac{\lambda(\mathfrak{n}-1)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' + \frac{\lambda\mathfrak{l}-\lambda\mathfrak{n}+2\lambda-2\mathfrak{l}-2}{\lambda(\mathfrak{l}+1)} -$
$\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}+1)}\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+\mathfrak{l}}{\mathfrak{l}+1}\right)^{\mathfrak{n}}C^{\mathfrak{n}}_{\mathfrak{n}+j-1}\frac{1}{j-1}\mathfrak{a}_j^2 z^{j-1} =$
$\frac{1}{z}\left(\frac{\mathfrak{n}+1}{\lambda}IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - \frac{\mathfrak{n}-2}{\lambda}IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right) + \frac{\lambda(\mathfrak{n}-1)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}\mathfrak{p}(z) + \left(1 - \frac{\mathfrak{n}-1}{\mathfrak{l}+1} - \frac{2}{\lambda}\right) -$
$\frac{2(l+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}+1)}\int_0^z\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt.$

Therefore $\mathfrak{p}(z) + \frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z) =$
$\frac{1}{z}\left(\frac{(\mathfrak{n}+1)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - \frac{(\mathfrak{n}-2)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right) + \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-2(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} -$

$\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(t)-t}{t^2}dt.$

The fuzzy differential subordination from the hypothesis takes the form $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z)+\mathfrak{p}(z)\right) \leq F_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) = F_{\mathfrak{g}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{g}'(z)+\mathfrak{g}(z)\right)$, $z \in \mathfrak{U}$. By applying Lemma 1, we get $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$, written as $\mathcal{F}_{IR_{\lambda,\mathfrak{l}}^{n}\mathfrak{f}(\mathfrak{U})}\left(IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$, and this result is sharp because the function g is the fuzzy best dominant. □

Theorem 7. *For a function $\mathfrak{h}$ convex such that $\mathfrak{h}(0) = 1$, and for $\mathfrak{f} \in \mathcal{A}$, which meets the fuzzy differential subordination*

$$\mathcal{F}_{IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{U})}\left(\frac{1}{z}\left(\frac{(\mathfrak{n}+1)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}+1}\mathfrak{f}(z)-\frac{(\mathfrak{n}-2)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(z)\right)+\right.$$

$$\left.\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-2(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}-\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z\frac{IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(t)-t}{t^2}dt\right) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z), \quad (15)$$

$z \in \mathfrak{U}$, then

$$\mathcal{F}_{IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{U})}\left(IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), \ \ z \in \mathfrak{U},$$

and the fuzzy best dominant is the convex function $\mathfrak{g}(z) = \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \mathfrak{h}(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt.$

Proof. Taking $\mathfrak{p}(z) = \left(IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(z)\right)'$ and using the properties of the operator $IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}$ and the calculus made in the proof of Theorem 6, we deduce

$$\frac{1}{z}\left(\frac{(\mathfrak{n}+1)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}+1}\mathfrak{f}(z)-\frac{(\mathfrak{n}-2)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(z)\right)+$$

$$\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-2(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}-\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z\frac{IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(t)-t}{t^2}dt =$$
$$\mathfrak{p}(z)+\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z), \quad z \in \mathfrak{U}.$$

In these conditions, the fuzzy inequality (15) becomes $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z)+\mathfrak{p}(z)\right) \leq \mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)$, $z \in \mathfrak{U}$. Applying Lemma 2, we obtain $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z) \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, $z \in \mathfrak{U}$, where $\mathfrak{g}(z) = \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \mathfrak{h}(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt$, $z \in \mathfrak{U}$, equivalent with

$$\mathcal{F}_{IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{U})}\left(IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(z)\right)' \leq \mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z), z \in \mathfrak{U}.$$

The convex function $\mathfrak{g}$ satisfies the differential equation of the fuzzy subordination (15) $\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{g}'(z)+\mathfrak{g}(z) = \mathfrak{h}(z)$, therefore it represents the fuzzy best dominant. □

Corollary 2. *Taking the convex function $\mathfrak{h}(z) = \frac{(2a-1)z+1}{z+1}$ in $\mathfrak{U}$, $0 \leq a < 1$, and $\mathfrak{f} \in \mathcal{A}$ which satisfies the fuzzy inequality*

$$\mathcal{F}_{IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{U})}\left(\frac{1}{z}\left(\frac{(\mathfrak{n}+1)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}+1}\mathfrak{f}(z)-\frac{(\mathfrak{n}-2)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR_{\lambda,\mathfrak{l}}^{\mathfrak{n}}\mathfrak{f}(z)\right)+\right.$$

$$\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-2(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}-\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt\Bigg)\leq\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z),\quad(16)$$

$z\in\mathfrak{U}$, then

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'\leq\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z),\ \ z\in\mathfrak{U},$$

and the fuzzy best dominant represents the convex function

$\mathfrak{g}(z)=(2a-1)+\frac{2(1-a)[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z\frac{t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}{t+1}dt,\ z\in\mathfrak{U}.$

Proof. Taking $\mathfrak{p}(z)=\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$ and by Theorem 7, we can write the fuzzy inequality (16) as $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z)+\mathfrak{p}(z)\right)\leq\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)$, $z\in\mathfrak{U}$.

Using Lemma 2, we have $\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)\leq\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$, written as $\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'\leq\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)$ and $\mathfrak{g}(z)=\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z\mathfrak{h}(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt=$ $\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}\frac{(2a-1)t+1}{t+1}dt=(2a-1)+\frac{2(1-a)[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z\frac{t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}{t+1}dt,z\in$ $\mathfrak{U}$, is the fuzzy best dominant. □

Example 2. *Let $\mathfrak{h}(z)=\frac{1-z}{z+1}$ and $\mathfrak{f}(z)=z^2+z$, $z\in\mathfrak{U}$, as in the Example 1. For $\mathfrak{n}=1$, $\mathfrak{l}=1$, $\lambda=2$, we have $IR^1_{2,1}\mathfrak{f}(z)=4z^2+z$. Then $\left(IR^1_{2,1}\mathfrak{f}(z)\right)'=8z+1$. We obtain also* $\frac{1}{z}\left(\frac{(\mathfrak{n}+1)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)-\frac{(\mathfrak{n}-2)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)+\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-2(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}-$ $\frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt=\frac{1}{z}\left(2IR^2_{2,1}\mathfrak{f}(z)+IR^1_{2,1}\mathfrak{f}(z)\right)+2\int_0^z\frac{IR^1_{2,1}\mathfrak{f}(t)-t}{t^2}dt=36z+3$, *where $IR^2_{2,1}\mathfrak{f}(z)=12z^2+z$. We have $\mathfrak{g}(z)=\frac{1}{2z^{\frac{1}{2}}}\int_0^z\frac{1-t}{t+1}t^{\frac{1}{2}}dt=2-\frac{z}{3}-\frac{2arctg\sqrt{z}}{\sqrt{z}}$.*

Using Theorem 7 we deduce $\mathcal{F}_{\mathfrak{U}}(36z+3)\leq\mathcal{F}_{\mathfrak{U}}\left(\frac{1-z}{z+1}\right)$, $z\in\mathfrak{U}$, generates $\mathcal{F}_{\mathfrak{U}}(8z+1)\leq\mathcal{F}_{\mathfrak{U}}\left(2-\frac{z}{3}-\frac{2arctg\sqrt{z}}{\sqrt{z}}\right)$, $z\in\mathfrak{U}$.

3. Fuzzy Differential Superordination

In this section we deduce interesting properties of the studied differential operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ by using the fuzzy differential superordinations.

Theorem 8. *Considering a function $\mathfrak{h}$ convex in $\mathfrak{U}$ such that $\mathfrak{h}(0)=1$, for $\mathfrak{f}\in\mathcal{A}$ suppose that $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$ is univalent in $\mathfrak{U}$, $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'\in\mathfrak{Q}\cap\mathcal{H}[1,1]$, where $m>0$, and*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z)\leq\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)',\ \ z\in\mathfrak{U},\quad(17)$$

then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z)\leq\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)',\ \ z\in\mathfrak{U},$$

and the fuzzy best subordinant represents the convex function $\mathfrak{g}(z)=\frac{\mathfrak{m}+2}{z^{\mathfrak{m}+2}}\int_0^z\mathfrak{h}(t)t^{\mathfrak{m}+1}dt$.

Proof. The function $\mathfrak{I}_m(\mathfrak{f})$ satisfies the relation $z^{\mathfrak{m}+1}\mathfrak{I}_m(\mathfrak{f})(z)=(\mathfrak{m}+2)\int_0^z t^{\mathfrak{m}}\mathfrak{f}(t)dt$, and applying on it the operation of differentiating, we get

$$z(\mathfrak{I}_m(\mathfrak{f}))'(z)+(\mathfrak{m}+1)\mathfrak{I}_m(\mathfrak{f})(z)=(\mathfrak{m}+2)\mathfrak{f}(z)$$

and applying the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ we get

$$z\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' + (\mathfrak{m}+1)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z) = (\mathfrak{m}+2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z),\ z \in \mathfrak{U}. \tag{18}$$

By applying the differentiation operation to relation (18) again, we obtain

$$\frac{1}{\mathfrak{m}+2}z\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'' + \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', z \in \mathfrak{U}.$$

In this condition, the fuzzy inequality involving differential superordination (17) becomes

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'' + \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'\right). \tag{19}$$

Denoting

$$\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)',\ z \in \mathfrak{U}, \tag{20}$$

the fuzzy inequality (19) takes the following form

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U}.$$

From Lemma 3, we deduce

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z),\ z \in \mathfrak{U},$$

written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)',\ z \in \mathfrak{U},$$

and the fuzzy best subordinant represents the convex function $\mathfrak{g}(z) = \frac{\mathfrak{m}+2}{z^{\mathfrak{m}+2}}\int_0^z \mathfrak{h}(t)t^{\mathfrak{m}+1}dt$. □

Corollary 3. *Considering* $\mathfrak{h}(z) = \frac{(2a-1)z+1}{z+1}$, *with* $a \in [0,1)$, *for* $\mathfrak{f} \in \mathcal{A}$, *assume that* $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$ *is univalent in* $\mathfrak{U}$, $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' \in \mathfrak{Q} \cap \mathcal{H}[1,1]$ *and*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)',\ z \in \mathfrak{U}, \tag{21}$$

then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)', \quad z \in \mathfrak{U},$$

and the fuzzy best subordinant represents the convex function $\mathfrak{g}(z) = \frac{2(1-a)(\mathfrak{m}+2)}{z^{\mathfrak{m}+2}}\int_0^z \frac{t^{\mathfrak{m}+1}}{t+1}dt + 2a-1$, $z \in \mathfrak{U}$.

Proof. From Theorem 8, denoting $\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'$, the fuzzy inequality (21) becomes

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U}.$$

From Lemma 3, we obtain $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)', \quad z \in \mathfrak{U},$$

and

$$g(z) = \frac{m+2}{z^{m+2}}\int_0^z h(t)t^{m+1}dt = \frac{m+2}{z^{m+2}}\int_0^z \frac{(2a-1)t+1}{t+1}t^{m+1}dt$$

$$= \frac{2(1-a)(m+2)}{z^{m+2}} \int_0^z \frac{t^{m+1}}{t+1} dt + 2a - 1, \quad z \in U,$$

is the convex fuzzy best subordinant. □

Example 3. *Let $\mathfrak{h}(z) = \frac{1-z}{z+1}$ and $\mathfrak{f}(z) = z^2 + z$, $z \in \mathfrak{U}$, as in Example 1. For $\mathfrak{n} = 1$, $\mathfrak{l} = 2$, $\lambda = 1$, we have $IR^1_{1,2}\mathfrak{f}(z) = \frac{8}{3}z^2 + z$ and $\left(IR^1_{1,2}\mathfrak{f}(z)\right)' = \frac{16}{3}z + z$ univalent functions in $\mathfrak{U}$.*

For $\mathfrak{m} = 3$ we get $\mathfrak{I}_3(\mathfrak{f})(z) = \frac{5}{z^4}\int_0^z t^3(t^2+t)dt = \frac{5}{6}z^2 + z$ and $R^1\mathfrak{I}_3(\mathfrak{f})(z) = z(\mathfrak{I}_3(\mathfrak{f}))'(z) = \frac{5}{3}z^2 + z$, $I(1,1,2)\mathfrak{I}_3(z) = \frac{2}{3}\mathfrak{I}_3(z) + \frac{1}{3}z(\mathfrak{I}_3)'(z) = \frac{10}{9}z^2 + z$ and $IR^1_{1,2}\mathfrak{I}_3(\mathfrak{f})(z) = \frac{50}{27}z^2 + z$, so $\left(IR^1_{1,2}\mathfrak{I}_3(f)(z)\right)' = \frac{100}{27}z + 1 \in \mathfrak{Q} \cap \mathcal{H}[1,1]$.

We deduce $\mathfrak{g}(z) = \frac{5}{z^5}\int_0^z \frac{1-t}{t+1}t^4 dt = \frac{10\ln(z+1)}{z^5} - \frac{10}{z^4} + \frac{5}{z^3} - \frac{10}{3z^2} + \frac{5}{2z} - 1$.

Applying Theorem 8, we get

$$\mathcal{F}_{\mathfrak{U}}\left(\frac{1-z}{z+1}\right) \le \mathcal{F}_{\mathfrak{U}}\left(\frac{16}{3}z + 1\right), \quad z \in \mathfrak{U},$$

induce

$$\mathcal{F}_{\mathfrak{U}}\left(\frac{10\ln(z+1)}{z^5} - \frac{10}{z^4} + \frac{5}{z^3} - \frac{10}{3z^2} + \frac{5}{2z} - 1\right) \le \mathcal{F}_{\mathfrak{U}}\left(\frac{100}{27}z + 1\right), \quad z \in \mathfrak{U}.$$

Theorem 9. *For a convex function $\mathfrak{g}$ in $\mathfrak{U}$, consider $\mathfrak{h}(z) = \frac{1}{\mathfrak{m}+2}z\mathfrak{g}'(z) + \mathfrak{g}(z)$, where $z \in \mathfrak{U}$, Re $\mathfrak{m} > -2$. For $\mathfrak{f} \in \mathcal{A}$ assume that $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$ is univalent in $\mathfrak{U}$, $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)' \in \mathfrak{Q} \cap \mathcal{H}[1,1]$ and*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le F_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U}, \tag{22}$$

then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)', \quad z \in \mathfrak{U},$$

and the fuzzy best subordinant is $\mathfrak{g}(z) = \frac{\mathfrak{m}+2}{z^{\mathfrak{m}+2}}\int_0^z \mathfrak{h}(t)t^{\mathfrak{m}+1}dt$.

Proof. Taking $\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)'$, $z \in \mathfrak{U}$, and following the ideas from the proof of Theorem 8, the fuzzy inequality (22) takes the form

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{1}{\mathfrak{m}+2}z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U},$$

and from Lemma 4, we deduce

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z), \quad z \in \mathfrak{U},$$

written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{I}_m(\mathfrak{f})(z)\right)', \quad z \in \mathfrak{U},$$

and the function $\mathfrak{g}(z) = \frac{\mathfrak{m}+2}{z^{\mathfrak{m}+2}}\int_0^z \mathfrak{h}(t)t^{\mathfrak{m}+1}dt$ represents the fuzzy best subordinant. □

Theorem 10. *For a function $\mathfrak{h}$ convex such that $\mathfrak{h}(0) = 1$ and $\mathfrak{f} \in \mathcal{A}$ assume that $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$ is univalent and $\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \in \mathfrak{Q} \cap \mathcal{H}[1,1]$. The fuzzy inequality*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U}, \tag{23}$$

implies the fuzzy inequality

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}, \quad z \in \mathfrak{U},$$

and the convex function $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$ *represents the fuzzy best subordinant.*

Proof. Set $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \in \mathcal{H}[1,1]$ and applying differentiation operation, we get $z\mathfrak{p}'(z) + \mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$, $z \in \mathfrak{U}$.

Then the fuzzy differential superordination (23) becomes

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}(z\mathfrak{p}'(z) + \mathfrak{p}(z)), \quad z \in \mathfrak{U}.$$

Applying Lemma 3, we get

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z), \ \ z \in \mathfrak{U},$$

written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}, \quad z \in \mathfrak{U},$$

and the fuzzy best subordinant represents the convex function $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$. □

Corollary 4. *Considering the function* $\mathfrak{h}(z) = \frac{(2a-1)z+1}{z+1}$ *convex in* $\mathfrak{U}$, $0 \leq a < 1$, *for* $\mathfrak{f} \in \mathcal{A}$ *suppose that* $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$ *is univalent and* $\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \in \mathfrak{Q} \cap \mathcal{H}[1,1]$. *If the fuzzy inequality holds*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U}, \tag{24}$$

then the fuzzy inequality holds

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}, \quad z \in \mathfrak{U},$$

and the convex function $\mathfrak{g}(z) = \frac{2(1-a)}{z}\ln(z+1) + 2a - 1$, $z \in \mathfrak{U}$, *represents the fuzzy best subordinant.*

Proof. From Theorem 10 denoting $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}$, the fuzzy differential superordination (24) becomes

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}(z\mathfrak{p}'(z) + \mathfrak{p}(z)), \quad z \in \mathfrak{U}.$$

Using Lemma 3, we get $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}, \quad z \in \mathfrak{U},$$

with

$$\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt = \frac{1}{z}\int_0^z \frac{(2a-1)t+1}{t+1}dt$$
$$= \frac{2(1-a)}{z}\ln(z+1) + 2a - 1, \quad z \in \mathfrak{U},$$

and $\mathfrak{g}$ is the convex fuzzy best subordinant. □

Example 4. *Let* $\mathfrak{h}(z) = \frac{1-z}{z+1}$ *and* $\mathfrak{f}(z) = z^2 + z$, $z \in \mathfrak{U}$. *For* $\mathfrak{n} = 1$, $\mathfrak{l} = 2$, $\lambda = 1$, *as in Example 3, we obtain* $IR^1_{1,2}\mathfrak{f}(z) = \frac{8}{3}z^2 + z$ *and* $\left(IR^1_{1,2}\mathfrak{f}(z)\right)' = \frac{16}{3}z + 1$ *univalent in* $\mathfrak{U}$, $\frac{IR^1_{1,2}\mathfrak{f}(z)}{z} = \frac{8}{3}z + 1 \in \mathfrak{Q} \cap \mathcal{H}[1,1]$.

We get $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \frac{1-t}{t+1}dt = \frac{2\ln(z+1)}{z} - 1$.

From Theorem 10, we get

$$\mathcal{F}_{\mathfrak{U}}\left(\frac{1-z}{z+1}\right) \leq \mathcal{F}_{\mathfrak{U}}\left(\frac{16}{3}z + 1\right), \quad z \in \mathfrak{U},$$

imply

$$\mathcal{F}_{\mathfrak{U}}\left(\frac{2\ln(z+1)}{z} - 1\right) \leq \mathcal{F}_{\mathfrak{U}}\left(\frac{8}{3}z + 1\right), \quad z \in \mathfrak{U}.$$

Theorem 11. *Let a function g convex in* $\mathfrak{U}$ *and take the function* $\mathfrak{h}(z) = z\mathfrak{g}'(z) + \mathfrak{g}(z)$. *Let* $\mathfrak{f} \in \mathcal{A}$ *and assume that* $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$ *is univalent,* $\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \in \mathfrak{Q} \cap \mathcal{H}[1,1]$ *and the fuzzy inequality involving superordination*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U}, \tag{25}$$

holds, then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}, \quad z \in \mathfrak{U},$$

and the function $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$ *represents the fuzzy best subordinant.*

Proof. Denoting $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z} \in \mathcal{H}[1,1]$, applying the differentiation operation on it, we get $z\mathfrak{p}'(z) + \mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$, $z \in \mathfrak{U}$, and the fuzzy inequality (25) is

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\left(z\mathfrak{g}'(z) + \mathfrak{g}(z)\right) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U}.$$

By Lemma 4, we derive

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z), \quad z \in \mathfrak{U},$$

written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{z}, \quad z \in \mathfrak{U},$$

and the best subordinate is $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$. □

Theorem 12. *Considering a function* $\mathfrak{h}$ *convex such that* $\mathfrak{h}(0) = 1$, *for* $\mathfrak{f} \in \mathcal{A}$ *suppose that* $\left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)'$ *is univalent and* $\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} \in \mathfrak{Q} \cap \mathcal{H}[1,1]$. *When the fuzzy inequality holds*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)', \quad z \in \mathfrak{U}, \tag{26}$$

then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}, \quad z \in \mathfrak{U},$$

and the convex function $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$ *represents the fuzzy best subordinant.*

Proof. Denote $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} \in \mathcal{H}[1,\mathfrak{n}]$, applying on it the differentiation operation, we derive $\mathfrak{p}'(z) = \frac{\left(IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} - \mathfrak{p}(z) \cdot \frac{\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}$ and $z\mathfrak{p}'(z) + \mathfrak{p}(z) = \left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)'$. In this condition, the fuzzy inequality (26) takes the following form

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U},$$

and applying Lemma 3, we get $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, $z \in \mathfrak{U}$, written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}, \quad z \in \mathfrak{U},$$

and the fuzzy best subordinant becomes the convex function $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$. □

Corollary 5. *Considering the function* $\mathfrak{h}(z) = \frac{(2a-1)z+1}{z+1}$ *convex in* $\mathfrak{U}$, $0 \leq a < 1$, *for* $\mathfrak{f} \in \mathcal{A}$ *assume that* $\left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)'$ *is univalent and* $\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} \in \mathfrak{Q} \cap \mathcal{H}[1,1]$. *When the fuzzy inequality*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)', \quad z \in \mathfrak{U}, \tag{27}$$

holds, then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}, \quad z \in \mathfrak{U},$$

and the convex function $\mathfrak{g}(z) = \frac{2(1-a)}{z}\ln(z+1) + 2a - 1$, $z \in \mathfrak{U}$ *represents the fuzzy best subordinant.*

Proof. Applying Theorem 12 for $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}$, the fuzzy inequality (27) has the form

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U},$$

and from Lemma 3, we derive $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, i.e.,

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}, \quad z \in \mathfrak{U},$$

and the function

$$\begin{aligned}\mathfrak{g}(z) &= \frac{1}{z}\int_0^z \mathfrak{h}(t)dt = \frac{1}{z}\int_0^z \frac{(2a-1)t+1}{t+1}dt \\ &= \frac{2(1-a)}{z}\ln(z+1) + 2a - 1.\end{aligned}$$

$\mathfrak{g}$ becomes the convex fuzzy best subordinant. □

Theorem 13. *Considering a function* $\mathfrak{g}$ *convex in* $\mathfrak{U}$, *define* $\mathfrak{h}(z) = z\mathfrak{g}'(z) + \mathfrak{g}(z)$, $z \in \mathfrak{U}$. *For* $\mathfrak{f} \in \mathcal{A}$, *assume that* $\left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)'$ *is univalent,* $\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} \in \mathfrak{Q} \cap \mathcal{H}[1,1]$ *and satisfies the fuzzy inequality involving superordination*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)', \quad z \in \mathfrak{U}, \tag{28}$$

then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}, \quad z \in \mathfrak{U},$$

and the fuzzy best subordinant represents the function $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$.

Proof. Denote $\mathfrak{p}(z) = \frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)} \in \mathcal{H}[1,1]$ and differentiating this relation, we derive $z\mathfrak{p}'(z) + \mathfrak{p}(z) = \left(\frac{zIR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}\right)'$, $z \in \mathfrak{U}$. With this notation, the fuzzy inequality (28) becomes

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\left(z\mathfrak{g}'(z) + \mathfrak{g}(z)\right) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U}.$$

From Lemma 4, we obtain $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, $z \in \mathfrak{U}$, written as

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\frac{IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)}, \quad z \in \mathfrak{U},$$

where $\mathfrak{g}(z) = \frac{1}{z}\int_0^z \mathfrak{h}(t)dt$ is the fuzzy best subordinant. □

Theorem 14. *For a function* $\mathfrak{h}$ *convex such that* $\mathfrak{h}(0) = 1$ *and for* $\mathfrak{f} \in \mathcal{A}$ *assume that* $\frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \left(1 - \frac{\mathfrak{l}+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt$ *is univalent and* $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \in \mathfrak{Q} \cap \mathcal{H}[1,1]$. *When the fuzzy inequality*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(\frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \left(1 - \frac{\mathfrak{l}+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt\right), \tag{29}$$

for $z \in \mathfrak{U}$ *holds, then*

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U},$$

and the convex function $\mathfrak{g}(z) = \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \mathfrak{h}(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt$ *is the fuzzy best subordinant.*

Proof. Set $\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \in \mathcal{H}[1,1]$, with $\mathfrak{p}(0) = 1$, we obtain after differentiating this relation that

$p(z) + \frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z) = \frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \left(1 - \frac{\mathfrak{l}+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt.$

With the notation above, the fuzzy differential superordination (29) becomes

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U},$$

and from Lemma 3, we deduce $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, $z \in \mathfrak{U}$, which implies

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U},$$

and the convex function $\mathfrak{g}(z) = \frac{\lambda(\mathfrak{l}-n+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z h(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt$ is the fuzzy best subordinant. □

Corollary 6. *Considering the function* $\mathfrak{h}(z) = \frac{(2a-1)z+1}{z+1}$ *convex in* $\mathfrak{U}$, $0 \le a < 1$, *for* $\mathfrak{f} \in \mathcal{A}$ *assume that* $\frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \left(1 - \frac{\mathfrak{l}+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt$ *is univalent and* $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \in \mathfrak{Q}\cap\mathcal{H}[1,1]$. *When the fuzzy inequality involving superordination*

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\Bigg(\frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \left(1 - \frac{\mathfrak{l}+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt\Bigg), \tag{30}$$

for $z \in \mathfrak{U}$ *holds, then*

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U},$$

and the convex function $\mathfrak{g}(z) = 2(1-a)\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \frac{t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}{1+t}dt + 2a - 1$ *is the fuzzy best subordinant.*

Proof. From Theorem 14 for $\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$, the fuzzy superordination (30) is

$$\mathcal{F}_{\mathfrak{h}(\mathfrak{U})}\mathfrak{h}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U},$$

and from Lemma 3, we get $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, $z \in \mathfrak{U}$, equivalent with

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \le \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U},$$

and the convex function

$$\mathfrak{g}(z) = \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \mathfrak{h}(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt =$$

$$\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \frac{(2a-1)t+1}{t+1}t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt =$$

$$2(1-a)\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \frac{t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}{1+t}dt + 2a - 1$$

represents the fuzzy best subordinant. □

Example 5. *Let the convex function* $\mathfrak{h}(z) = \frac{1-z}{z+1}$ *in* $\mathfrak{U}$, $\mathfrak{h}(0) = 1$ *and* $\mathfrak{f}(z) = z^2 + z$, $z \in \mathfrak{U}$. *For* $\mathfrak{n} = 1$, $\mathfrak{l} = 1$, $\lambda = 2$, *as in Example 2 we get* $IR^1_{2,1}\mathfrak{f}(z) = 4z^2 + z$ *and* $\left(IR^1_{2,1}\mathfrak{f}(z)\right)' = 8z + 1 \in \mathfrak{Q}\cap\mathcal{H}[1,1]$.

Assume that function $\frac{1}{z}\left(\frac{(\mathfrak{n}+1)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - \frac{(\mathfrak{n}-2)(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right) + \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-2(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt = \frac{1}{z}\left(2IR^2_{2,1}\mathfrak{f}(z) + IR^1_{2,1}\mathfrak{f}(z)\right)$
$+2\int_0^z \frac{IR^1_{2,1}f(t)-t}{t^2}dt = 36z + 3$ *is univalent in* $\mathfrak{U}$, *where* $R^2 f(z) = \frac{z}{2}\left(R^1\mathfrak{f}(z)\right)' + \frac{1}{2}R^1\mathfrak{f}(z) = 3z^2 + z$,

$I(2,2,1)\mathfrak{f}(z) = z(I(1,2,1)\mathfrak{f}(z))' = 4z^2 + z$,

$IR^2_{2,1}\mathfrak{f}(z) = 12z^2 + z$.

We deduce $\mathfrak{g}(z) = \frac{1}{2z^{\frac{1}{2}}}\int_0^z \frac{1-t}{1+t}t^{\frac{1}{2}}dt = 2 - \frac{z}{3} - \frac{2arctg\sqrt{z}}{\sqrt{z}}$.

Using Theorem 14, we get

$$\mathcal{F}_{\mathfrak{U}}\left(\frac{1-z}{z+1}\right) \leq \mathcal{F}_{\mathfrak{U}}(36z+3), \quad z \in \mathfrak{U},$$

imply

$$\mathcal{F}_{U}\left(2-\frac{z}{3}-\frac{2arctg\sqrt{z}}{\sqrt{z}}\right) \leq \mathcal{F}_{\mathfrak{U}}(8z+1), \ \ z \in \mathfrak{U}.$$

Theorem 15. *Setting the function $\mathfrak{g}$ convex in $\mathfrak{U}$, consider $\mathfrak{h}(z) = \frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} z\mathfrak{g}'(z) + \mathfrak{g}(z)$. Assume that $\frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \left(1 - \frac{\mathfrak{l}+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt$ is univalent for $\mathfrak{f} \in \mathcal{A}$ and $\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)' \in \mathfrak{Q} \cap \mathcal{H}[1,1]$ for which the fuzzy superordination*

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} z\mathfrak{g}'(z) + \mathfrak{g}(z)\right) \leq \tag{31}$$

$$\mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(\frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \right.$$

$$\left.\left(1 - \frac{l+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt\right),$$

holds for $z \in \mathfrak{U}$, then

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U},$$

where $\mathfrak{g}(z) = \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \mathfrak{h}(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt$ represents the fuzzy best subordinant.

Proof. Denoting $\mathfrak{p}(z) = \left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)'$, differentiating it and making some calculus, we obtain $\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} z\mathfrak{p}'(z) + \mathfrak{p}(z) = \frac{\mathfrak{l}+1}{[\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)]z}\left[(\mathfrak{n}+1)IR^{\mathfrak{n}+1}_{\lambda,\mathfrak{l}}\mathfrak{f}(z) - (\mathfrak{n}-2)IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right] + \left(1 - \frac{\mathfrak{l}+1}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\right) - \frac{2(\mathfrak{l}+1)(\mathfrak{n}-1)-2\lambda\mathfrak{n}}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}\int_0^z \frac{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(t)-t}{t^2}dt$, $z \in \mathfrak{U}$. With this notation the fuzzy differential superordination (31) takes the following form

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} z\mathfrak{g}'(z) + \mathfrak{g}(z)\right) \leq$$

$$\mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\left(\frac{\lambda(\mathfrak{l}+1)}{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)} z\mathfrak{p}'(z) + \mathfrak{p}(z)\right), \quad z \in \mathfrak{U}.$$

From Lemma 4, we deduce $\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{\mathfrak{p}(\mathfrak{U})}\mathfrak{p}(z)$, $z \in \mathfrak{U}$, equivalent with

$$\mathcal{F}_{\mathfrak{g}(\mathfrak{U})}\mathfrak{g}(z) \leq \mathcal{F}_{IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(\mathfrak{U})}\left(IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}\mathfrak{f}(z)\right)', \quad z \in \mathfrak{U},$$

and $\mathfrak{g}(z) = \frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)z^{\frac{\lambda(\mathfrak{l}-\mathfrak{n}+2)-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}}\int_0^z \mathfrak{h}(t)t^{\frac{\lambda(1-\mathfrak{n})-(\mathfrak{l}+1)}{\lambda(\mathfrak{l}+1)}}dt$ represents the fuzzy best subordinant. □

4. Conclusions

The primary goal of the study described in this paper is to present new results concerning fuzzy aspects introduced in the geometric theory of analytic functions in the hope that it will be useful in future research on sustainability, similar to how numerous other applications of the fuzzy set concept have prompted the creation of sustainability models in a variety of economic, environmental, and social activities.

The operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ resulted from the convolution product of the Ruscheweyh derivative and multiplier transformation from Definition 6. In Definition 7 of Section 2 we introduced a new subclass of functions in $\mathfrak{U}$. Fuzzy inequalities involving subordinations are studied in the theorems of Section 2 using the convexity property and involving the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ and functions from the newly introduced class. Moreover, examples are provided to establish how the findings might be applied. In Section 3, fuzzy inequalities involving superordinations regarding the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ are established and the best subordinants are given. The relevance of the results is also illustrated using examples.

As future research, the operator $IR^{\mathfrak{n}}_{\lambda,\mathfrak{l}}$ studied in this paper could be adapted to quantum calculus and obtain differential subordinations and superordinations for it by involving q-fractional calculus, as seen in Ref. [28]. In addition, coefficient studies can be done regarding the new class introduced in Definition 7 such as estimations for Hankel determinants of different orders and Toeplitz determinants or the Fekete–Szegö problem. Hopefully, the new fuzzy results presented here will find applications in future studies concerning real life contexts.

Funding: The publication of this research was partially supported by University of Oradea.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflicts of interest.

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