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Article

Norms of a Product of Integral and Composition Operators between Some Bloch-Type Spaces

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Abstract: We present some formulas for the norm, as well as the essential norm, of a product of composition and an integral operator between some Bloch-type spaces of analytic functions on the unit ball, in terms of given symbols and weights.

Keywords: operator norm; essential norm; composition operator; integral operator; Bloch-type space

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1. Introduction

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$, with the scalar product $\langle z, w\rangle = \sum_{k=1}^n z_k\overline{w}_k$ and the norm $|z| = \sqrt{\langle z, z\rangle}$ (here, as usual, $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$, and $\overline{z} = (\overline{z}_1, \dots, \overline{z}_n)$). We denote the space of analytic functions on $\mathbb{B}$ by $H(\mathbb{B})$, whereas we denote the class of analytic self-maps of $\mathbb{B}$ by $S(\mathbb{B})$ [1,2]. The linear operator $\Re f(z) = \sum_{j=1}^n z_j D_j f(z)$, where $D_j f = \frac{\partial f}{\partial z_j}$, $j = \overline{1, n}$, is called a radial derivative.

We denote the set of all positive and continuous functions on $\mathbb{B}$ by $W(\mathbb{B})$. A $w \in W(\mathbb{B})$ is called a weight. Let $\mu \in W(\mathbb{B})$. Then,

$$H_\mu^\infty(\mathbb{B}) = \{f \in H(\mathbb{B}) : \|f\|_{H_\mu^\infty} := \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < +\infty\}$$

is called a weighted-type space. This space with the norm $\|\cdot\|_{H_\mu^\infty}$ is a Banach space. A little weighted-type space consists of $f \in H_\mu^\infty(\mathbb{B})$ such that $\lim_{|z|\to 1} \mu(z)|f(z)| = 0$. These spaces have been studied for a long time (see, e.g., [3–9]), as well as the operators acting on them (see, e.g., [10–17] and the references therein). If μ is a nonzero constant, we obtain the space $H^\infty(\mathbb{B})$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$ (bounded analytic functions).

Let $\mu \in W(\mathbb{B})$. Then, the space

$$\mathcal{B}_\mu(\mathbb{B}) = \{f \in H(\mathbb{B}) : b_\mu(f) := \sup_{z\in\mathbb{B}} \mu(z)|\Re f(z)| < +\infty\},$$

is called a Bloch-type space. With the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$, it is a Banach space. A little Bloch-type space consists of $f \in \mathcal{B}_\mu(\mathbb{B})$ such that $\lim_{|z|\to 1} \mu(z)|\Re f(z)| = 0$. We obtain the Bloch space $\mathcal{B}$ and little Bloch space $\mathcal{B}_0$ for $\mu(z) = 1 - |z|^2$, whereas for $\mu(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, we obtain the α-Bloch space $\mathcal{B}^\alpha$ and the little α-Bloch space $\mathcal{B}_0^\alpha$. For

$$\mu(z) = \mu_{\log_k}(z) = (1 - |z|^2)\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1 - |z|^2},$$

where $k \in \mathbb{N}$, $e^{[1]} = e$, $e^{[l]} = e^{e^{[l-1]}}$, $l \in \mathbb{N} \setminus \{1\}$ and

$$\ln^{[j]} z = \underbrace{\ln \cdots \ln}_{j \text{ times}} z,$$

we obtain the iterated logarithmic Bloch space $\mathcal{B}_{\log_k}(\mathbb{B}) = \mathcal{B}_{\log_k}$, which for $k = 1$, reduces to $\mathcal{B}_{\log_1} = \mathcal{B}_{\log}$. The quantity

$$\|f\|'_{\mathcal{B}_{\log_k}} = |f(0)| + \sup_{z\in\mathbb{B}} \mu_{\log_k}(z)|\nabla f(z)|, \tag{1}$$

is a norm on $\mathcal{B}_{\log_k}(\mathbb{B})$. From $|\Re f(z)| \le |\nabla f(z)|$ and a known theorem ([18–20]), it follows that (1) is equivalent to the norm $\|f\|_{\mathcal{B}_{\log_k}} = |f(0)| + \sup_{z\in\mathbb{B}} \mu_{\log_k}(z)|\Re f(z)|$ on $\mathcal{B}_{\log_k}$.

Suppose $a \in [e^{[k]}, +\infty)$. Then, for every $z \in \mathbb{B}$, we have

$$\begin{aligned}
&(1-|z|)\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-|z|} = (1-|z|)\prod_{j=1}^{k}\ln^{[j]} e^{[k]}\frac{a(1+|z|)}{e^{[k]}(1-|z|^2)}\\
\le&(1-|z|^2)\prod_{j=1}^{k}\ln^{[j]}\left(\frac{2a}{e^{[k]}}\frac{e^{[k]}}{1-|z|^2}\right) = (1-|z|^2)\prod_{j=1}^{k}\ln^{[j-1]}\left(\ln\frac{e^{[k]}}{1-|z|^2}+\ln\frac{2a}{e^{[k]}}\right)\\
\le&(1-|z|^2)\prod_{j=1}^{k}\ln^{[j-1]}\left(\left(1+\ln\frac{2a}{e^{[k]}}\right)\ln\frac{e^{[k]}}{1-|z|^2}\right)\\
=&(1-|z|^2)\left(1+\ln\frac{2a}{e^{[k]}}\right)\ln\frac{e^{[k]}}{1-|z|^2}\prod_{j=2}^{k}\ln^{[j-2]}\left(\ln\left(1+\ln\frac{2a}{e^{[k]}}\right)+\ln^{[2]}\frac{e^{[k]}}{1-|z|^2}\right)\\
\le&(1-|z|^2)\left(1+\ln\frac{2a}{e^{[k]}}\right)\ln\frac{e^{[k]}}{1-|z|^2}\prod_{j=2}^{k}\ln^{[j-2]}\left(\left(1+\ln\left(1+\ln\frac{2a}{e^{[k]}}\right)\right)\ln^{[2]}\frac{e^{[k]}}{1-|z|^2}\right)\\
=&(1-|z|^2)\left(1+\ln\frac{2a}{e^{[k]}}\right)\ln\frac{e^{[k]}}{1-|z|^2}\left(1+\ln\left(1+\ln\frac{2a}{e^{[k]}}\right)\right)\ln^{[2]}\frac{e^{[k]}}{1-|z|^2}\\
&\times\prod_{j=3}^{k}\ln^{[j-2]}\left(\left(1+\ln\left(1+\ln\frac{2a}{e^{[k]}}\right)\right)\ln^{[2]}\frac{e^{[k]}}{1-|z|^2}\right)\\
&\qquad\vdots\\
\le&(1-|z|^2)\left(1+\ln\frac{2a}{e^{[k]}}\right)\ln\frac{e^{[k]}}{1-|z|^2}\left(1+\ln\left(1+\ln\frac{2a}{e^{[k]}}\right)\right)\ln^{[2]}\frac{e^{[k]}}{1-|z|^2}\\
&\cdots\left(1+\ln\left(1+\cdots+\ln\left(1+\ln\left(1+\ln\frac{2a}{e^{[k]}}\right)\right)\cdots\right)\right)\ln^{[k]}\frac{e^{[k]}}{1-|z|^2}\\
=&c_a(1-|z|^2)\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\\
\le&2c_a(1-|z|)\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-|z|}.
\end{aligned}\tag{2}$$

The consideration leading to (2) implies that, for $a \in [e^{[k]}, +\infty)$, the quantity

$$\|f\|^{(a)}_{\mathcal{B}_{\log_k}} = |f(0)| + b^{(a)}_{\log_k}(f) := |f(0)| + \sup_{z\in\mathbb{B}}(1-|z|)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-|z|}\right)|\nabla f(z)|, \tag{3}$$

presents another equivalent norm on $\mathcal{B}_{\log_k}$.

We define the corresponding little iterated logarithmic Bloch space $\mathcal{B}_{\log_k,0}(\mathbb{B}) = \mathcal{B}_{\log_k,0}$ as the set of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1}(1-|z|)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-|z|}\right)|\nabla f(z)| = 0.$$

For some facts on logarithmic-type spaces, see, e.g., [10,14,21–23].

The product of the composition operator $C_\varphi f(z) = f(\varphi(z))$ and an equivalent form of the integral operator in [24,25]

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad z \in \mathbb{B}, \tag{4}$$

where $g \in H(\mathbb{B})$, $g(0) = 0$ and $\varphi \in S(\mathbb{B})$, was studied, e.g., in [22,26]. The introduction of the operators in [24,25] was motivated by some special cases mentioned therein (see also [27]). Many facts about this topic can be found in [28]. Operator (4), as well as some related ones, has been considerably studied (see, e.g., [29–34] and the cited references therein). Beside this product-type operator, many others have been studied during the last two decades. One can consult the following references: [10,14,15,35,36].

The essential norm of a linear operator $L : X \to Y$, where X and Y are Banach spaces and $\|\cdot\|_{X\to Y}$ denotes the operator norm, is the quantity

$$\|L\|_{e,X\to Y} = \inf\Big\{\|L+K\|_{X\to Y} \ : \ K : X \to Y,\ K \text{ is compact}\Big\}.$$

One of the most popular topics in studying concrete linear operators is characterization of their operator-theoretic properties in terms of the induced symbols. One of the basic problems is the calculation of their norms and essential norms [18–20,37–39]. Some recent formulas for the norms can be found in [11–14,23,26,31].

Let $M_u(f)(z) = u(z)f(z)$, where $u \in H(\mathbb{B})$. The following result was proved in [11].

Theorem 1. *Let $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$ and $M_uC_\varphi : X \to H_\mu^\infty$ be bounded, where $X \in \{\mathcal{B}, \mathcal{B}_0\}$. Then,*

$$\|M_uC_\varphi\|_{X\to H_\mu^\infty} = \max\left\{\|u\|_{H_\mu^\infty}, \frac{1}{2}\sup_{z\in B}\mu(z)|u(z)|\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\}, \tag{5}$$

where the norm on $\mathcal{B}$ is given by $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{B}}(1-|z|^2)|\nabla f(z)|$.

One can try to calculate the norm of $M_uC_\varphi : \mathcal{B}^\alpha \to H_\mu^\infty$. To solve it, in [13], we had to change the weight $(1-|z|^2)^\alpha$. The method also works in some other situations [23]. Here, we employ this idea to calculate the norm of $P_\varphi^g : \mathcal{B}_{\log_k}$ (or $\mathcal{B}_{\log_k,0}$) $\to \mathcal{B}_\mu$ (or $\mathcal{B}_{\mu,0}$). Beside this, we present a formula for its essential norm, extending the results in [23]. We use some of the methods and ideas in [13,14,23,26].

2. Auxiliary Results

Our first auxiliary result is a nontrivial technical lemma.

Lemma 1. *Assume that $k \in \mathbb{N}$, $a \in [e^{[k]}, +\infty)$. Then,*

$$h_k(x) = x\prod_{j=1}^{k}\ln^{[j]}\frac{a}{x}, \tag{6}$$

is a nonnegative and increasing function on $(0, \frac{a}{e^{[k]}}]$.

Proof. The case $k = 1$ is simple [23]. So, assume $k \in \mathbb{N} \setminus \{1\}$. We have

$$h_k(x) = h_{k-1}(x) \ln^{[k]}\Big(\frac{a}{x}\Big). \tag{7}$$

From (7), it follows that

$$h'_k(x) = h'_{k-1}(x) \ln^{[k]}\Big(\frac{a}{x}\Big) - 1. \tag{8}$$

The recursive relation in (8) implies

$$h'_k(x) = \Big(\Big(\cdots\Big(\Big(\ln\Big(\frac{a}{x}\Big) - 1\Big) \ln^{[2]}\Big(\frac{a}{x}\Big) - 1\Big) \cdots\Big) \ln^{[k-1]}\Big(\frac{a}{x}\Big) - 1\Big) \cdot \ln^{[k]}\Big(\frac{a}{x}\Big) - 1. \tag{9}$$

From (9), it follows that $h'_k(x)$ is decreasing on the interval $(0, \frac{a}{e^{[k-1]}})$ (here, we regard that $e^{[0]} = 1$). Hence,

$$\begin{aligned} h'_k(x) \geq & h'_k\Big(\frac{a}{e^{[k]}}\Big) \\ =& \Big(\Big(\cdots\Big(\Big(\ln e^{[k]} - 1\Big) \ln^{[2]} e^{[k]} - 1\Big) \cdots\Big) \ln^{[k-1]} e^{[k]} - 1\Big) \cdot \ln^{[k]} e^{[k]} - 1 \\ =& \Big(\Big(\cdots\Big(\Big(e^{[k-1]} - 1\Big) e^{[k-2]} - 1\Big) \cdots\Big) e^{[1]} - 1\Big) - 1 > 0, \end{aligned}$$

for $x \in (0, \frac{a}{e^{[k]}}]$, from which the lemma follows. □

Now, we present some point evaluation estimates for the functions in $\mathcal{B}_{\log_k}(\mathbb{B})$.

Lemma 2. *Assume that $k \in \mathbb{N}$, $a \in [e^{[k]}, +\infty)$, $f \in \mathcal{B}_{\log_k}(\mathbb{B})$, $z \in \mathbb{B}$, and $r \in [0, 1)$. Then,*

$$|f(z) - f(rz)| \leq b^{(a)}_{\log_k}(f)\Big(\ln^{[k+1]} \frac{a}{1 - |z|} - \ln^{[k+1]} \frac{a}{1 - r|z|}\Big), \tag{10}$$

and

$$|f(z)| \leq \|f\|^{(a)}_{\mathcal{B}_{\log_k}} \max\Big\{1, \ln^{[k+1]} \frac{a}{1 - |z|} - \ln^{[k+1]} a\Big\}. \tag{11}$$

Proof. Let $\nabla f = (D_1 f, \ldots, D_n f)$. Then,

$$\begin{aligned} |f(z) - f(rz)| =& \Big|\int_r^1 \langle \nabla f(tz), \bar{z}\rangle\, dt\Big| \\ \leq& b^{(a)}_{\log_k}(f) \int_r^1 \frac{|z| dt}{(1 - |z|t) \prod_{j=1}^k \ln^{[j]} \frac{a}{1-|z|t}} \\ =& b^{(a)}_{\log_k}(f)\Big(\ln^{[k+1]} \frac{a}{1 - |z|} - \ln^{[k+1]} \frac{a}{1 - r|z|}\Big). \end{aligned} \tag{12}$$

From (12), for $r = 0$, it follows that

$$|f(z) - f(0)| \leq b^{(a)}_{\log_k}(f)\Big(\ln^{[k+1]} \frac{a}{1 - |z|} - \ln^{[k+1]} a\Big). \tag{13}$$

Relation (13), along with the definition of $\|\cdot\|^{(a)}_{\mathcal{B}_{\log_k}}$ and the triangle inequality for numbers, implies (11). □

For the next lemma, see [22].

Lemma 3. *Let $f, g \in H(\mathbb{B})$ and $g(0) = 0$. Then,*

$$\Re P_\varphi^g(f)(z) = f(\varphi(z))g(z), \quad z \in \mathbb{B}. \tag{14}$$

The following result is closely related to the corresponding one in [40], because of which the proof is omitted.

Lemma 4. *Assume that $g \in H(\mathbb{B})$, $g(0) = 0$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, $P_\varphi^g : \mathcal{B}_{\log_k}$ (or $\mathcal{B}_{\log_k,0}$) $\to \mathcal{B}_\mu$ is compact if and only if it is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{B}_{\log_k}$ (or $\mathcal{B}_{\log_k,0}$) converging to zero uniformly on compacts of $\mathbb{B}$, we have $\lim_{k\to+\infty} \|P_\varphi^g f_k\|_{\mathcal{B}_\mu} = 0$.*

3. Main Results

Now, we are in a position to state and prove our main results.

Theorem 2. *Suppose that $k \in \mathbb{N}$, $a \in [2e^{[k]}, +\infty)$, $g \in H(\mathbb{B})$, $g(0) = 0$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$ and that $P_\varphi^g : X \to \mathcal{B}_\mu$ is bounded, where $X \in \{\mathcal{B}_{\log_k}, \mathcal{B}_{\log_k,0}\}$. Then,*

$$\|P_\varphi^g\|_{X\to\mathcal{B}_\mu} = \max\left\{\|g\|_{H_\mu^\infty}, \sup_{z\in\mathbb{B}} \mu(z)|g(z)|\left(\ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)\right\}. \tag{15}$$

Proof. From (14) and (11), it follows that, for $f \in \mathcal{B}_{\log_k}$, we have

$$\begin{aligned}\|P_\varphi^g f\|_{\mathcal{B}_\mu} =& \sup_{z\in\mathbb{B}} \mu(z)|g(z)f(\varphi(z))| \\ \le& \|f\|_{\mathcal{B}_{\log_k}}^{(a)} \sup_{z\in\mathbb{B}} \mu(z)|g(z)| \max\left\{1, \ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right\},\end{aligned} \tag{16}$$

Hence,

$$\|P_\varphi^g\|_{X\to\mathcal{B}_\mu} \le \max\left\{\|g\|_{H_\mu^\infty}, \sup_{z\in\mathbb{B}} \mu(z)|g(z)|\left(\ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)\right\}. \tag{17}$$

If $P_\varphi^g : X \to \mathcal{B}_\mu$ is bounded, then for $f_0(z) \equiv 1 \in \mathcal{B}_{\log_k,0}$, we have $\|f_0\|_{\mathcal{B}_{\log_k}} = 1$, from which together with the boundedness, it follows that

$$\|P_\varphi^g\|_{X\to\mathcal{B}_\mu} \ge \|P_\varphi^g f_0\|_{\mathcal{B}_\mu} = \sup_{z\in\mathbb{B}} \mu(z)|g(z)|. \tag{18}$$

Let

$$h_w(z) = \ln^{[k+1]} \frac{a}{1-\langle z, w\rangle} - \ln^{[k+1]} a, \tag{19}$$

and $w \in \mathbb{B}$.

Then,

$$1 - |z| \le |1 - \langle z, w\rangle| < 2. \tag{20}$$

for $z, w \in \mathbb{B}$. Relation (20) together with Lemma 1 implies

$$(1-|z|)\left(\prod_{j=1}^{k} \ln^{[j]} \frac{a}{1-|z|}\right)|\nabla h_w(z)| = \frac{|w|(1-|z|)\prod_{j=1}^{k} \ln^{[j]} \frac{a}{1-|z|}}{|1-\langle z,w\rangle|\left|\prod_{j=1}^{k} \ln^{[j]} \frac{a}{1-\langle z,w\rangle}\right|} \tag{21}$$

$$\leq \frac{|w|(1-|z|)\prod_{j=1}^{k} \ln^{[j]} \frac{a}{1-|z|}}{|1-\langle z,w\rangle|\prod_{j=1}^{k} \ln^{[j]} \frac{a}{|1-\langle z,w\rangle|}} < 1. \tag{22}$$

Inequality (22) along with the fact that $h_w(0) = 0$ implies

$$\sup_{w\in\mathbb{B}} \|h_w\|^{(a)}_{\mathcal{B}_{\log_k}} \leq 1. \tag{23}$$

Let $|z| \to 1$ in (21); then, we have $h_w \in \mathcal{B}_{\log_k,0}$, $w \in \mathbb{B}$.

If $\varphi(w) \neq 0$ and $t \in (0,1)$, then from the boundedness of $P^g_\varphi : X \to \mathcal{B}_\mu$ and (23), we have

$$\begin{aligned}\|P^g_\varphi\|_{X\to\mathcal{B}_\mu} \geq& \|P^g_\varphi h_{t\varphi(w)/|\varphi(w)|}\|_{\mathcal{B}_\mu} \\ =& \sup_{z\in\mathbb{B}} \mu(z)|g(z)|\left|\ln^{[k+1]} \frac{a}{1-t\langle\varphi(z),\varphi(w)/|\varphi(w)|\rangle} - \ln^{[k+1]} a\right| \\ \geq& \mu(w)|g(w)|\left(\ln^{[k+1]} \frac{a}{1-t|\varphi(w)|} - \ln^{[k+1]} a\right).\end{aligned} \tag{24}$$

Note that (24) also holds when $\varphi(w) = 0$.

Let $t \to 1^-$ in (24), and taking the supremum over $\mathbb{B}$, we obtain

$$\|P^g_\varphi\|_{X\to\mathcal{B}_\mu} \geq \sup_{z\in\mathbb{B}} \mu(z)|g(z)|\left(\ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right). \tag{25}$$

Relations (18) and (25) imply

$$\|P^g_\varphi\|_{X\to\mathcal{B}_\mu} \geq \max\left\{\|g\|_{H^\infty_\mu}, \sup_{z\in\mathbb{B}} \mu(z)|g(z)|\left(\ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)\right\}. \tag{26}$$

Combining the inequalities in (17) and (26), the formula in (15) immediately follows. □

Using the test function $f_0(z) \equiv 1$ and the fact that the set of polynomials is dense in $\mathcal{B}_{\log_k,0}$, the following theorem is easily proved. We omit the standard proof.

Theorem 3. *Suppose that $k \in \mathbb{N}$, $g \in H(\mathbb{B})$, $g(0) = 0$, $\varphi \in S(\mathbb{B})$, and $\mu \in W(\mathbb{B})$. Then, $P^g_\varphi : \mathcal{B}_{\log_k,0} \to \mathcal{B}_{\mu,0}$ is bounded if and only if $P^g_\varphi : \mathcal{B}_{\log_k,0} \to \mathcal{B}_\mu$ is bounded and $g \in H^\infty_{\mu,0}$.*

The following result is a consequence of the previous two theorems.

Corollary 1. *Suppose that $k \in \mathbb{N}$, $g \in H(\mathbb{B})$, $g(0) = 0$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$ and that $P^g_\varphi : \mathcal{B}_{\log_k,0} \to \mathcal{B}_{\mu,0}$ is bounded. Then,*

$$\|P^g_\varphi\|_{\mathcal{B}_{\log_k,0}\to\mathcal{B}_{\mu,0}} = \max\left\{\|g\|_{H^\infty_\mu}, \sup_{z\in\mathbb{B}} \mu(z)|g(z)|\left(\ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)\right\}.$$

Theorem 4. *Suppose that $k \in \mathbb{N}$, $a \in [2e^{[k]}, +\infty)$, $g \in H(\mathbb{B})$, $g(0) = 0$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$ and $P^g_\varphi : X \to \mathcal{B}_\mu$ is bounded, where $X \in \{\mathcal{B}_{\log_k}, \mathcal{B}_{\log_k,0}\}$. Then,*

(a) *If $\|\varphi\|_\infty = 1$, we have*

$$\|P_\varphi^g\|_{e,X\to\mathcal{B}_\mu} = \limsup_{|\varphi(z)|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right); \tag{27}$$

(b) *If $\|\varphi\|_\infty < 1$, we have*

$$\|P_\varphi^g\|_{e,X\to\mathcal{B}_\mu} = 0. \tag{28}$$

Proof. *(a)* Let $\varepsilon > 0$ and $w \in \mathbb{B}\setminus\{0\}$ be fixed, and

$$h_{w,\varepsilon}(z) = \left(\ln^{[k+1]}\frac{a}{1-|w|} - \ln^{[k+1]} a\right)^{-\varepsilon}\left(\ln^{[k+1]}\frac{a(1+|w|)}{1-\langle z,w\rangle} - \ln^{[k+1]} a\right)^{\varepsilon+1}, \; z\in\mathbb{B}.$$

Then,

$$\begin{aligned}
&(1-|z|)\left(\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-|z|}\right)|\nabla h_{w,\varepsilon}(z)| \\
=&(\varepsilon+1)\frac{|w|(1-|z|)\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-|z|}}{|1-\langle z,w\rangle|\left|\prod_{j=1}^{k}\ln^{[j]}\frac{a(1+|w|)}{1-\langle z,w\rangle}\right|} \\
&\times\left|\ln^{[k+1]}\frac{a(1+|w|)}{1-\langle z,w\rangle} - \ln^{[k+1]} a\right|^{\varepsilon}\left(\ln^{[k+1]}\frac{a}{1-|w|} - \ln^{[k+1]} a\right)^{-\varepsilon} \qquad (29)\\
\le&(\varepsilon+1)\frac{|w|(1-|z|)\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-|z|}}{|1-\langle z,w\rangle|\left|\prod_{j=1}^{k}\ln^{[j]}\frac{a(1+|w|)}{1-\langle z,w\rangle}\right|}\left(\ln^{[k+1]}\frac{a}{1-|w|} - \ln^{[k+1]} a\right)^{-\varepsilon} \\
&\times\left(\ln\left(\ln\left(\cdots\left(\ln\frac{a(1+|w|)}{|1-\langle z,w\rangle|}+2\pi\right)\cdots\right)+2\pi\right)+2\pi-\ln^{[k+1]} a\right)^{\varepsilon} \\
\le&(\varepsilon+1)|w|\left(\ln^{[k+1]}\frac{a}{1-|w|} - \ln^{[k+1]} a\right)^{-\varepsilon} \\
&\left(\ln\left(\ln\left(\cdots\left(\ln\frac{a(1+|w|)}{1-|w|}+2\pi\right)\cdots\right)+2\pi\right)+2\pi-\ln^{[k+1]} a\right)^{\varepsilon}. \qquad (30)
\end{aligned}$$

Relation (29) implies $h_{w,\varepsilon}\in\mathcal{B}_{\log_k,0}$, for $w\in\mathbb{B}\setminus\{0\}$, while by taking limit in relation (30), we obtain

$$\limsup_{|w|\to 1} b_{\log_k}^{(a)}(h_{w,\varepsilon}) \le \varepsilon+1. \tag{31}$$

Note also that

$$\lim_{|w|\to 1}|h_{w,\varepsilon}(0)| = 0. \tag{32}$$

From (31) and (32), it follows that

$$\limsup_{|w|\to 1}\|h_{w,\varepsilon}\|_{\mathcal{B}_{\log_k}}^{(a)} \le \varepsilon+1. \tag{33}$$

If $(\varphi(z_k))_{k\in\mathbb{N}}\subset\mathbb{B}$ satisfies the condition $|\varphi(z_k)|\to 1$ as $k\to+\infty$, then (33) for

$$f_k(z) := h_{\varphi(z_k),\varepsilon}(z), \quad k\in\mathbb{N},$$

implies

$$\limsup_{k\to+\infty}\|f_k\|^{(a)}_{\mathcal{B}_{\log_k}} \le \varepsilon+1. \tag{34}$$

The assumption $f_k \to 0$ on compacts of $\mathbb{B}$ implies that $f_k \to 0$ weakly in $\mathcal{B}_{\log_k,0}$ as $k\to+\infty$. Indeed, the operator $L(f)=f'$ is an isometric isomorphism between $\mathcal{B}_{\log_k,0}/\mathbb{C}$ and $H^\infty_{\log_k,0}$. On the other hand, a bounded sequence converges weakly to zero in $H^\infty_{\log_k,0}$ if and only if it converges to zero uniformly on compacts of $\mathbb{B}$ (see, e.g., some reasoning in [3] and the estimate in (11), and note that the unit ball in $H^\infty_{\log_k,0}$ is a normal family).

Hence, if $K:\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu$ is compact, then $\lim_{k\to+\infty}\|Kf_k\|_{\mathcal{B}_\mu}=0$. This fact, (34), and the estimate

$$\begin{aligned}\|f_k\|^{(a)}_{\mathcal{B}_{\log_k}}\|P^g_\varphi+K\|_{\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu} \ge&\|(P^g_\varphi+K)(f_k)\|_{\mathcal{B}_\mu}\\ \ge&\|P^g_\varphi f_k\|_{\mathcal{B}_\mu}-\|Kf_k\|_{\mathcal{B}_\mu},\end{aligned}$$

imply

$$\begin{aligned}\frac{\|P^g_\varphi+K\|_{\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu}}{(\varepsilon+1)^{-1}} \ge&\limsup_{k\to\infty}\|f_k\|^{(a)}_{\mathcal{B}_{\log_k}}\|P^g_\varphi+K\|_{\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu}\\ \ge&\limsup_{k\to\infty}(\|P^g_\varphi f_k\|_{\mathcal{B}_\mu}-\|Kf_k\|_{\mathcal{B}_\mu})\\ =&\limsup_{k\to\infty}\|P^g_\varphi f_k\|_{\mathcal{B}_\mu}\\ =&\limsup_{k\to\infty}\sup_{z\in\mathbb{B}}\mu(z)|g(z)||f_k(\varphi(z))|\\ \ge&\limsup_{k\to\infty}\mu(z_k)|g(z_k)f_k(\varphi(z_k))|\\ =&\limsup_{k\to\infty}\mu(z_k)|g(z_k)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z_k)|}-\ln^{[k+1]}a\right). \end{aligned} \tag{35}$$

From (35) and since $K:\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu$ is an arbitrary compact operator, by letting $\varepsilon\to+0$, we have

$$\|P^g_\varphi\|_{e,\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu}\ge\limsup_{k\to\infty}\mu(z_k)|g(z_k)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z_k)|}-\ln^{[k+1]}a\right).$$

Hence,

$$\|P^g_\varphi\|_{e,\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu}\ge\limsup_{|\varphi(z)|\to1}\mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|}-\ln^{[k+1]}a\right). \tag{36}$$

Let $\rho_m\in(0,1)$, $m\in\mathbb{N}$, $\rho_m\nearrow1$ as $m\to+\infty$, and

$$P^g_{\rho_m\varphi}(f)(z)=\int_0^1 f(\rho_m\varphi(tz))g(tz)\frac{dt}{t},\quad m\in\mathbb{N}. \tag{37}$$

Suppose that $(h_k)_{k\in\mathbb{N}}\subset X$ is bounded and $h_k\to0$ uniformly on compacts of $\mathbb{B}$. We have $P^g_\varphi(f_0)=g\in H^\infty_\mu$, so

$$\mu(z)|\Re P^g_{\rho_m\varphi}(h_k)(z)|=\mu(z)|g(z)h_k(\rho_m\varphi(z))|\le\|g\|_{H^\infty_\mu}\sup_{|w|\le\rho_m}|h_k(w)|\to0,$$

as $k\to+\infty$.

Thus, Lemma 4 implies the compactness of $P^g_{\rho_m\varphi}:X\to\mathcal{B}_\mu$, for each $m\in\mathbb{N}$.

Since $g\in H^\infty_\mu$, by Lemmas 2 and 3, we have that, for $r\in(0,1)$,

$$
\begin{aligned}
\|P_{\varphi}^{g}-P_{\rho_m\varphi}^{g}\|_{\mathcal{B}_{\log_k}\to\mathcal{B}_\mu} = & \sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\sup_{z\in\mathbb{B}}\mu(z)|g(z)||f(\varphi(z))-f(\rho_m\varphi(z))| \\
\le & \sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\sup_{|\varphi(z)|\le r}\mu(z)|g(z)||f(\varphi(z))-f(\rho_m\varphi(z))| \\
& +\sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\sup_{|\varphi(z)|>r}\mu(z)|g(z)||f(\varphi(z))-f(\rho_m\varphi(z))| \\
\le & \|g\|_{H_\mu^\infty}\sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\sup_{|\varphi(z)|\le r}|f(\varphi(z))-f(\rho_m\varphi(z))| \\
& +\sup_{|\varphi(z)|>r}\mu(z)|g(z)|\Big(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|}-\ln^{[k+1]}\frac{a}{1-\rho_m|\varphi(z)|}\Big) \\
\le & \|g\|_{H_\mu^\infty}\sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\sup_{|\varphi(z)|\le r}|f(\varphi(z))-f(\rho_m\varphi(z))| \\
& +\sup_{|\varphi(z)|>r}\mu(z)|g(z)|\Big(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|}-\ln^{[k+1]}a\Big).
\end{aligned}
\tag{38}
$$

Furthermore,

$$
\begin{aligned}
& \limsup_{m\to+\infty}\sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\sup_{|\varphi(z)|\le r}|f(\varphi(z))-f(\rho_m\varphi(z))| \\
\le & \limsup_{m\to+\infty}\sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\sup_{|\varphi(z)|\le r}(1-\rho_m)|\varphi(z)|\sup_{|w|\le r}|\nabla f(w)| \\
\le & \limsup_{m\to+\infty}\frac{(1-\rho_m)r}{(1-r)\prod_{j=1}^{k}\ln^{[j]}\frac{a}{1-r}}\sup_{\|f\|_{\mathcal{B}_{\log_k}}^{(a)}\le 1}\|f\|_{\mathcal{B}_{\log_k}}^{(a)}=0.
\end{aligned}
\tag{39}
$$

Letting $m\to+\infty$ in (38), using (39), then letting $r\to 1$, it follows that

$$
\|P_{\varphi}^{g}\|_{e,\mathcal{B}_{\log_k}\to\mathcal{B}_\mu}\le\limsup_{|\varphi(z)|\to 1}\mu(z)|g(z)|\Big(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|}-\ln^{[k+1]}a\Big).
\tag{40}
$$

Relations (36), (40), and the obvious inequality

$$
\|P_{\varphi}^{g}\|_{e,\mathcal{B}_{\log_k}\to\mathcal{B}_\mu}\ge\|P_{\varphi}^{g}\|_{e,\mathcal{B}_{\log_k,0}\to\mathcal{B}_\mu},
$$

imply (27).

(b) From this assumption, the compactness of $P_{\varphi}^{g}:X\to\mathcal{B}_\mu$ follows, similar to the operator in (37). So, (28) holds. □

Theorem 5. *Suppose that $k\in\mathbb{N}$, $a\in[2e^{[k]},+\infty)$, $g\in H(\mathbb{B})$, $g(0)=0$, $\varphi\in S(\mathbb{B})$, $\mu\in W(\mathbb{B})$, and $P_{\varphi}^{g}:X\to\mathcal{B}_{\mu,0}$ is bounded, where $X\in\{\mathcal{B}_{\log_k},\mathcal{B}_{\log_k,0}\}$. Then,*

$$
\|P_{\varphi}^{g}\|_{e,X\to\mathcal{B}_{\mu,0}}=\limsup_{|z|\to 1}\mu(z)|g(z)|\Big(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|}-\ln^{[k+1]}a\Big).
\tag{41}
$$

Proof. Since $P_{\varphi}^{g}:X\to\mathcal{B}_{\mu,0}$ is bounded, we have $P_{\varphi}^{g}f_0=g\in H_{\mu,0}^{\infty}$.

Assume that $\|\varphi\|_\infty = 1$. Then,

$$\limsup_{|z|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)$$
$$\geq \limsup_{|\varphi(z)|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right). \tag{42}$$

Choose $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ so that the following relation holds

$$\begin{aligned}&\limsup_{|z|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)\\ &= \lim_{k\to\infty} \mu(z_k)|g(z_k)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z_k)|} - \ln^{[k+1]} a\right).\end{aligned} \tag{43}$$

If $\sup_{k\in\mathbb{N}} |\varphi(z_k)| < 1$, then the fact that $g \in H^\infty_{\mu,0}$, implies

$$\lim_{k\to\infty} \mu(z_k)|g(z_k)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z_k)|} - \ln^{[k+1]} a\right) = 0.$$

Thus, (42) and (43) imply

$$\limsup_{|\varphi(z)|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right) = 0.$$

If $\sup_{k\in\mathbb{N}} |\varphi(z_k)| = 1$, then $|\varphi(z_{k_m})| \to 1$ as $m \to +\infty$, for a subsequence $(\varphi(z_{k_m}))_{m\in\mathbb{N}}$. Hence,

$$\limsup_{|z|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)$$
$$= \limsup_{|\varphi(z)|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right).$$

This, along with Theorem 4, implies the theorem in this case.

If $\|\varphi\|_\infty < 1$, then $P^g_\varphi : X \to \mathcal{B}_{\mu,0}$ is compact, so that $\|P^g_\varphi\|_{e,X\to\mathcal{B}_{\mu,0}} = 0$. Since $g \in H^\infty_{\mu,0}$, we have

$$\limsup_{|z|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)$$
$$\leq \left(\ln^{[k+1]}\frac{a}{1-\|\varphi\|_\infty} - \ln^{[k+1]} a\right) \lim_{|z|\to 1} \mu(z)|g(z)| = 0.$$

Hence, in this case, (41) holds. □

Corollary 2. *Suppose that $k \in \mathbb{N}$, $a \in [2e^{[k]}, +\infty)$, $g \in H(\mathbb{B})$, $g(0) = 0$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$, and $X \in \{\mathcal{B}_{\log_k}, \mathcal{B}_{\log_k,0}\}$. Then, the following claims hold.*

(a) *$P^g_\varphi : X \to \mathcal{B}_\mu$ is bounded if and only if*

$$\max\left\{\|g\|_{H^\infty_\mu}, \sup_{z\in\mathbb{B}} \mu(z)|g(z)|\left(\ln^{[k+1]}\frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right)\right\} < +\infty.$$

(b) *If* $P_\varphi^g : X \to \mathcal{B}_\mu$ *is bounded, then* $P_\varphi^g : X \to \mathcal{B}_\mu$ *is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right) = 0.$$

(c) *If* $P_\varphi^g : X \to \mathcal{B}_{\mu,0}$ *is bounded, then* $P_\varphi^g : X \to \mathcal{B}_{\mu,0}$ *is compact if and only if*

$$\lim_{|z|\to 1} \mu(z)|g(z)|\left(\ln^{[k+1]} \frac{a}{1-|\varphi(z)|} - \ln^{[k+1]} a\right) = 0.$$

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References

1. Rudin, W. *Function Theory in Polydiscs*; W. A. Benjamin, Inc.: New York, NY, USA; Amsterdam, The Netherlands, 1969.
2. Rudin, W. *Function Theory in the Unit Ball of* $\mathbb{C}^n$; Grundlehren der Mathematischen Wissenschaften: New York, NY, USA; Springer: Berlin, Germany, 1980.
3. Bierstedt, K.D.; Summers, W.H. Biduals of weighted Banach spaces of analytic functions. *J. Aust. Math. Soc. A* **1993**, *54*, 70–79. [CrossRef]
4. Lusky, W. On the structure of $H_v^0(D)$ and $h_v^0(D)$. *Math. Nachr.* **1992**, *159*, 279–289. [CrossRef]
5. Lusky, W. On weighted spaces of harmonic and holomorphic functions. *J. Lond. Math. Soc.* **1995**, *59*, 309–320. [CrossRef]
6. Lusky, W. On the isomorphic classification of weighted spaces of holomorphic functions. *Acta Univ. Carolin. Math. Phys.* **2000**, *41*, 51–60.
7. Lusky, W. On the isomorphism classes of weighted spaces of harmonic and holomorphic functions. *Stud. Math.* **2006**, *175*, 19–45. [CrossRef]
8. Rubel, L.A.; Shields, A.L. The second duals of certain spaces of analytic functions. *J. Aust. Math. Soc.* **1970**, *11*, 276–280. [CrossRef]
9. Shields, A.L.; Williams, D.L. Bounded projections, duality and multipliers in spaces of analytic functions. *Trans. Am. Math. Soc.* **1971**, *162*, 287–302. [CrossRef]
10. Liu, Y.M.; Yu, Y.Y. Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball. *J. Math. Anal. Appl.* **2015**, *423*, 76–93. [CrossRef]
11. Stević, S. Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball. *Ars Combin.* **2008**, *88*, 125–127.
12. Stević, S. Norms of some operators from Bergman spaces to weighted and Bloch-type space. *Util. Math.* **2008**, *76*, 59–64.
13. Stević, S. Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces. *Appl. Math. Comput.* **2009**, *215*, 818–820. [CrossRef]
14. Stević, S. Norm of some operators from logarithmic Bloch-type spaces to weighted-type spaces. *Appl. Math. Comput.* **2012**, *218*, 11163–11170. [CrossRef]
15. Yang, W.; Yan, W. Generalized weighted composition operators from area Nevanlinna spaces to weighted-type spaces. *Bull. Korean Math. Soc.* **2011**, *48*, 1195–1205.
16. Zhu, X. Weighted composition cperators from $F(p,q,s)$ spaces to H_μ^∞ spaces. *Abst. Appl. Anal.* **2009**, *2009*, 290978. [CrossRef]
17. Zhu, X. Weighted composition operators from weighted Hardy spaces to weighted-type spaces. *Demonstr. Math.* **2013**, *46*, 335–344.
18. Dunford, N.; Schwartz, J.T. *Linear Operators I*; Interscience Publishers, Jon Willey and Sons: New York, NY, USA, 1958. [CrossRef]
19. Rudin, W. *Functional Analysis*, 2nd ed.; McGraw-Hill, Inc.: New York, NY, USA, 1991.
20. Trenogin, V.A. *Funktsional'niy Analiz*; Nauka: Moskva, Russia, 1980. (In Russian)
21. Malavé-Malavé, R.J.; Ramos-Fernández, J.C. Superposition operators between logarithmic Bloch spaces. *Rend. Circ. Mat. Palermo II Ser.* **2019**, *68*, 105–121.
22. Stević, S. On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on unit ball. *Appl. Math. Comput.* **2010**, *215*, 4248–4255. [CrossRef]
23. Stević, S. Norm and essential norm of an integral-type operator from the logarithmic Bloch space to the Bloch-type space on the unit ball. *Math. Methods Appl. Sci.* **2022**, *45*, 11905–11915. [CrossRef]
24. Hu, Z.J. Extended Cesàro operators on mixed norm spaces. *Proc. Am. Math. Soc.* **2003**, *131*, 2171–2179. [CrossRef]
25. Hu, Z.J. Extended Cesàro operators on the Bloch space in the unit ball of $\mathbb{C}^n$. *Acta Math. Sci. Ser. B Engl. Ed.* **2003**, *23*, 561–566. [CrossRef]

26. Stević, S. On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball. *J. Math. Anal. Appl.* **2009**, *354*, 426–434. [CrossRef]
27. Benke, G.; Chang, D.C. A note on weighted Bergman spaces and the Cesáro operator. *Nagoya Math. J.* **2000**, *159*, 25–43. [CrossRef]
28. Chang, D.C.; Li, S.; Stević, S. On some integral operators on the unit polydisk and the unit ball. *Taiwan J. Math.* **2007**, *11*, 1251–1286. [CrossRef]
29. Du, J.; Zhu, X. Essential norm of an integral-type operator from ω-Bloch spaces to μ-Zygmund spaces on the unit ball. *Opusc. Math.* **2018**, *38*, 829–839.
30. Li, H.; Guo, Z. Note on a Li-Stević integral-type operator from mixed-norm spaces to nth weighted spaces. *J. Math. Inequal.* **2017**, *11*, 77–85. [CrossRef]
31. Li, H.; Li, S. Norm of an integral operator on some analytic function spaces on the unit disk. *J. Inequal. Math.* **2013**, *2013*, 342. [CrossRef]
32. Li, S. An integral-type operator from Bloch spaces to Q_p spaces in the unit ball. *Math. Inequal. Appl.* **2012**, *15*, 959–972. [CrossRef]
33. Li, S. On an integral-type operator from the Bloch space into the $Q_K(p,q)$ space. *Filomat* **2012**, *26*, 331–339. [CrossRef]
34. Pan, C. On an integral-type operator from $Q_k(p,q)$ spaces to α-Bloch spaces. *Filomat* **2011**, *25*, 163–173.
35. Guo, Z.; Liu, L.; Shu, Y. On Stević-Sharma operators from the mixed-norm spaces to Zygmund-type spaces. *Math. Inequal. Appl.* **2021**, *24*, 445–461. [CrossRef]
36. Zhu, X. Generalized weighted composition operators on weighted Bergman spaces. *Numer. Funct. Anal. Opt.* **2009**, *30*, 881–893.
37. Pelczynski, A. Norms of classical operators in function spaces. *Astérisque* **1985**, *131*, 137–162.
38. Rudin, W. *Real and Complex Analysis*, 3rd ed.; Higher Mathematics Series; McGraw-Hill Education: London, UK; New York, NY, USA; Sidney, Australia, 1976.
39. Trenogin, V.A.; Pisarevskiy, B.M.; Soboleva, T.S. *Zadachi i Uprazhneniya po Funktsional'nomu Analizu*; Nauka: Moskva, Russia, 1984. (In Russian)
40. Schwartz, H.J. Composition Operators on H^p. Ph.D. Thesis, University of Toledo, Toledo, OH, USA, 1969.