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New Applications of Fuzzy Set Concept in the Geometric Theory of Analytic Functions

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Abstract: Zadeh's fuzzy set theory offers a logical, adaptable solution to the challenge of defining, assessing and contrasting various sustainability scenarios. The results presented in this paper use the fuzzy set concept embedded into the theories of differential subordination and superordination established and developed in geometric function theory. As an extension of the classical concept of differential subordination, fuzzy differential subordination was first introduced in geometric function theory in 2011. In order to generalize the idea of fuzzy differential superordination, the dual notion of fuzzy differential superordination was developed later, in 2017. The two dual concepts are applied in this article making use of the previously introduced operator defined as the convolution product of the generalized Sălgean operator and the Ruscheweyh derivative. Using this operator, a new subclass of functions, normalized analytic in U, is defined and investigated. It is proved that this class is convex, and new fuzzy differential subordinations are established by applying known lemmas and using the functions from the new class and the aforementioned operator. When possible, the fuzzy best dominants are also indicated for the fuzzy differential subordinations. Furthermore, dual results involving the theory of fuzzy differential superordinations and the convolution operator are established for which the best subordinants are also given. Certain corollaries obtained by using particular convex functions as fuzzy best dominants or fuzzy best subordinants in the proved theorems and the numerous examples constructed both for the fuzzy differential subordinations and for the fuzzy differential superordinations prove the applicability of the new theoretical results presented in this study.

Keywords: fuzzy set theory; differential operator; convex function; fuzzy differential subordination; fuzzy best dominant; fuzzy differential superordination; fuzzy best subordinant; Hadamard product; Ruscheweyh derivative; generalized Sălăgean operator

MSC: 30C45; 30A20; 34A40

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1. Introduction

Being based on current economic, ecological and social problems and facts, sustainability implies a continual dynamic evolution that is motivated by human hopes about potential future prospects. The fuzzy set notion, which Lotfi A. Zadeh first proposed in 1965 [1], has numerous applications in science and technology. Fuzzy mathematical models are created in this research employing fuzzy set theory to evaluate sustainable development regarding the socio-scientific environment. Fuzzy set theory connects human expectations for development as stated in language concepts to numerical facts reflected in measurements of sustainability indicators, despite the fact that decision-making regarding sustainable development is subjective.

An intuitionistic fuzzy set is applied in order to introduce a new extension to the multi-criteria decision-making model for sustainable supplier selection based on sustainable supply chain management practices in Reference [2], taking into account the idea that choosing a suitable supplier is the key element of contemporary businesses from a sustainability perspective. One of the generalized forms of orthopairs uses intuitionistic fuzzy sets. The study presented in Reference [3] focuses on introducing and analyzing several basic aspects of a generalized frame for orthopair fuzzy sets called "(m, n)-Fuzzy sets". Supply chain sustainability is considered in the fuzzy context for the steel industry in Reference [4], and a model for sustainable energy usage in the textile sector based on intuitionistic fuzzy sets is introduced in Reference [5]. The study proposed in Reference [6] using nonlinear integrated fuzzy modeling can help in predicting how comfortable an office building will be and how that will affect people's health for optimized sustainability. The healthcare system is of the utmost importance, and optimization models have been investigated using generalizations of the fuzzy set concept in recent studies, for example, by proposing an updated multi-criteria integrated decision-making approach involving interval-valued intuitionistic fuzzy sets in Reference [7] or a flexible optimization model based on bipolar interval-valued neutrosophic sets in Reference [8].

The use of the notion of a fuzzy set in studies has led to the development of extensions for many fields of mathematics. In the review papers [9,10], different applications of this notion in mathematical domains are presented. In geometric function theory, the introduction of the concept of fuzzy subordination used the notion of a fuzzy set in 2011 [11], and the theory of fuzzy differential subordination has been in development since 2012 [12] when Miller and Mocanu's classical theory of differential subordination [13] started to be adapted by involving fuzzy theory aspects. In 2017, the dual notion was introduced, namely fuzzy differential superordination [14]. Since then, numerous researchers have studied different properties of differential operators involving fuzzy differential subordinations and superordinations, such as the Wanas operator [15,16], generalized Noor-Sălăgean operator [17], Sălăgean and Ruscheweyh operators [18] or a linear operator [19]. Univalence criteria were also derived using fuzzy differential subordination theory [20].

There is no indication up to this point as to how the concepts can be further used in real life or in other branches of research. For now, this is simply a new line of research which is developing nicely as part of geometric function theory. The connection between fuzzy sets theory and the branch of complex analysis that studies analytic functions in view of their geometric properties is also shown in Reference [20]. The confluent hypergeometric function's fractional integral is studied using classical theories of differential subordination and superordination in Reference [21] and the fuzzy corresponding theories in References [22,23]. In a recent paper [24], an operator is introduced and studied involving the fuzzy differential subordination theory [25] and is used for obtaining results involving the classical theory of differential subordination. This shows that both approaches produce interesting results and that investigations from the fuzzy point of view do not exclude the nice outcome obtained when classical theories of differential subordination and superordination are implemented on the same topics. Many papers have been published regarding the study of analytic functions via fuzzy concepts at this moment, and in them all the aspects of the classical notions of geometric function theory are given this new fuzzy perspective. For instance, meromorphic functions are investigated in the fuzzy context in Reference [26], and strong Janowski functions are approached using fuzzy differential subordinations in Reference [27]. Fuzzy α-convex functions are considered for study in Reference [28] and are associated with quantum calculus aspects in Reference [29] and with Hadamard product in Reference [30]. q-analogue operators, which have been thoroughly investigated using classical methods concerning analytic functions, are now also considered in the fuzzy context, as can be seen in References [31–33]. Spiral-like functions are considered in terms of the fuzzy differential subordination theory in Reference [34].

In this article, we derive certain fuzzy differential subordinations and fuzzy differential superordinations for an operator defined as a Hadamard product between the Ruscheweyh derivative and the generalized Sălăgean operator introduced in Reference [35].

In order to obtain the results of the article, we used the notions and results exposed below:

$\mathcal{H}(U)$ contains all the holomorphic functions in $U = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc, and we studied the geometric properties of the functions from its subclasses.

$$\mathcal{A} = \{f(z) = z + \sum_{j=2}^{\infty} a_j z^j\} \subset \mathcal{H}(U),$$

and

$$\mathcal{H}[a,n] = \{f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \subset\}\mathcal{H}(U),$$

where $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

We denote $\mathbb{C}^* = \mathbb{C} - 0$.

Definition 1 ([11]). *A fuzzy collection of a set $\mathcal{X}$ is a family $(\mathcal{F}_A)_A$ indexed by subsets of $\mathcal{X}$, where for each $A \subset \mathcal{X}$, $\mathcal{F}_A$ is a function $\mathcal{F}_A : \mathcal{X} \to [0,1]$ such that $A = \{z \in \mathcal{X} : 0 < \mathcal{F}_A(z) \leq 1\}$. Each $\mathcal{F}_A$ is called a fuzzy set of $\mathcal{X}$, and A is called the support of $\mathcal{F}_A$.*

Definition 2 ([11]). *Fix a fuzzy collection $\mathcal{F} = (\mathcal{F}_A)_{A\subset\mathbb{C}}$ of $\mathbb{C}$. $f \in \mathcal{H}(D)$ is a function fuzzy subordinate to $g \in \mathcal{H}(D)$, denoted $f \prec_{\mathcal{F}} g$, where $D \subset \mathbb{C}$, when:*
(1) for $z_0 \in D$ a fixed point, we have $f(z_0) = g(z_0)$,
(2) for any $z \in D$, $\mathcal{F}_{f(D)} f(z) \leq \mathcal{F}_{g(D)} g(z)$.

Definition 3 ([12], Definition 2.2). *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and the function h, univalent in U such that $h(0) = \psi(a, 0; 0) = a$. When the analytic function p in U, having the property $p(0) = a$ verifies for any $z \in U$ the fuzzy differential subordination*

$$\mathcal{F}_{\psi(\mathbb{C}^3\times U)} \psi(p(z), zp'(z), z^2 p''(z); z) \leq \mathcal{F}_{h(U)} h(z), \tag{1}$$

then the fuzzy differential subordination has p as fuzzy solution. A fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination represents q, a univalent function with the property $\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{q(U)} q(z)$, for all p verifying (1) and for any $z \in U$. The fuzzy best dominant represents a fuzzy dominant $\widetilde{q}$ with the property $\mathcal{F}_{\widetilde{q}(U)} \widetilde{q}(z) \leq \mathcal{F}_{q(U)} q(z)$, for all fuzzy dominants q of (1) and any $z \in U$.

Definition 4 ([14]). *Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and the function h analytic in U. When the univalent functions p and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ verifies for any $z \in U$ the fuzzy differential superordination*

$$\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{\varphi(\mathbb{C}^3\times U)} \varphi(p(z), zp'(z), z^2 p''(z); z), \tag{2}$$

then the fuzzy differential superordination has p as a fuzzy solution. A fuzzy subordinant of the fuzzy differential superordination represents q, an analytic function with the property

$$\mathcal{F}_{q(U)} q(z) \leq \mathcal{F}_{p(U)} p(z),$$

for all p verifying (2) and for any $z \in U$. The fuzzy best subordinant represents a univalent fuzzy subordinant $\widetilde{q}$ such that $\mathcal{F}_{q(U)} q \leq \mathcal{F}_{q(U)} \widetilde{q}$ for all fuzzy subordinants q of (2) and any $z \in U$.

Definition 5 ([12]). *Q contains all injective and analytic functions f on $\overline{U} \backslash E(f)$, with the property $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$, and $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$.*

We used the lemmas presented below to obtain our fuzzy inequalities:

Lemma 1 ([36]). *Set for any $z \in U$,*

$$h(z) = n\alpha z g'(z) + g(z),$$

for g a convex function in U, α > 0 and n a positive integer. Denote for any z ∈ U,

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots$$

If p verifies for any z ∈ U the fuzzy differential subordination

$$\mathcal{F}_{p(U)}(\alpha z p'(z) + p(z)) \leq \mathcal{F}_{h(U)} h(z),$$

and it is holomorphic in U, then yields the sharp fuzzy differential subordination

$$\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{g(U)} g(z).$$

Lemma 2 ([36])**.** *Consider $\alpha \in \mathbb{C}^*$ such that Re $\alpha \geq 0$ and h is a convex function with the property $h(0) = a$. If $p \in \mathcal{H}[a, n]$ verifies for any $z \in U$ the fuzzy differential subordination*

$$\mathcal{F}_{p(U)}\left(\frac{zp'(z)}{\alpha} + p(z)\right) \leq \mathcal{F}_{h(U)} h(z),$$

then the fuzzy differential subordinations

$$\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{h(U)} h(z), \quad z \in U,$$

is satisfied by the function

$$g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t) t^{\frac{\alpha}{n}-1} dt, \quad z \in U.$$

Lemma 3 ([13], Corollary 2.6g.2, p. 66)**.** *Consider $\alpha \in \mathbb{C}^*$ such that Re $\alpha \geq 0$ and h a convex function with the property $h(0) = a$. If $p \in Q \cap \mathcal{H}[a, n]$ and $\frac{zp'(z)}{\alpha} + p(z)$ verifies for any $z \in U$ the fuzzy differential superordination*

$$\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{p(U)}\left(\frac{zp'(z)}{\alpha} + p(z)\right),$$

and it is univalent in U, then the fuzzy differential superordination

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{p(U)} p(z), \; z \in U,$$

is satisfied for any $z \in U$ by the convex function $g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t) t^{\frac{\alpha}{n}-1} dt$, which is the fuzzy best subordinant.

Lemma 4 ([13], Corollary 2.6g.2, p. 66)**.** *Set for any $z \in U$ the function*

$$h(z) = \frac{zg'(z)}{\alpha} + g(z),$$

for g a convex function in U, $\alpha \in \mathbb{C}^*$ such that Re $\alpha \geq 0$. When $p \in Q \cap \mathcal{H}[a, n]$ and $\frac{zp'(z)}{\alpha} + p(z)$ verifies for any $z \in U$ the fuzzy differential superordination

$$\mathcal{F}_{g(U)}\left(\frac{zg'(z)}{\alpha} + g(z)\right) \leq \mathcal{F}_{p(U)}\left(\frac{zp'(z)}{\alpha} + p(z)\right),$$

and it is univalent in U, then the fuzzy differential superordination

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{p(U)} p(z),$$

is satisfies for any $z \in U$ by the function $g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t) t^{\frac{\alpha}{n}-1} dt$, which is the fuzzy best subordinant.

We remind the definition of the Hadamard (convolution) product of the Ruscheweyh derivative and the generalized Sălăgean differential operator:

Definition 6 ([35])**.** *Consider $n \in \mathbb{N}$ and $\lambda \geq 0$. The operator $DR_\lambda^n : \mathcal{A} \to \mathcal{A}$ is defined for each nonnegative integer n and for any $z \in U$ as the Hadamard product between the Ruscheweyh derivative R^n and the generalized Sălăgean operator D_λ^n:*

$$DR_\lambda^n f(z) = (D_\lambda^n * R^n) f(z).$$

Remark 1. *For $f \in \mathcal{A}$, the operator has the following form*
$DR_\lambda^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n [1 + (j-1)\lambda]^n a_j^2 z^j$*, for* $z \in U$.

The generalized Sălăgean differential operator introduced by Al Oboudi [37] $D_\lambda^n : \mathcal{A} \to \mathcal{A}$ is defined by the following relations:

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= \lambda z f'(z) + (1-\lambda) f(z) = D_\lambda f(z) \\ &\dots \\ D_\lambda^n f(z) &= \lambda z (D_\lambda^n f(z))' + (1-\lambda) D_\lambda^{n-1} f(z) = D_\lambda \left(D_\lambda^{n-1} f(z) \right), \end{aligned}$$

for $n \in \mathbb{N}$, $\lambda \geq 0$, $z \in U$ and $f \in \mathcal{A}$. For $f \in \mathcal{A}$, the operator has the following form $D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n a_j z^j$, for any $z \in U$.

The Ruscheweyh derivative [38] $R^n : \mathcal{A} \to \mathcal{A}$ is defined by the following relations:

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\dots \\ (n+1) R^{n+1} f(z) &= n R^n f(z) + z (R^n f(z))', \quad z \in U. \end{aligned}$$

for $n \in \mathbb{N}$ and $f \in \mathcal{A}$.

For $f \in \mathcal{A}$, the operator has the following form $R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j$, for any $z \in U$.

Using the operator DR_λ^n resulting from the convolution product of the generalized Sălăgean differential operator and the Ruscheweyh derivative given in Definition 6, a new subclass of normalized analytic functions in the open unit disc U is introduced in Definition 7 of Section 2. Fuzzy subordination results are investigated in the theorems of Section 2 using the convexity property and involving the operator DR_λ^n and functions from the newly introduced class. Moreover, examples are provided to show how the findings might be applied. In Section 3, fuzzy differential superordinations regarding the operator DR_λ^n are considered for which the best subordinants are also found. The results' relevance is also illustrated with examples.

2. Fuzzy Differential Subordination

Using the operator $DR_\lambda^n f$ from Definition 6, we introduce the class $\mathcal{DR}_{\lambda,n}^{\mathcal{F}}(\alpha)$ and we establish fuzzy differential subordinations for the functions belonging to this class.

Definition 7. *The class $\mathcal{DR}_{\lambda,n}^{\mathcal{F}}(\alpha)$ contains all the functions $f \in \mathcal{A}$ which verify for any $z \in U$ the inequality*

$$\mathcal{F}_{(DR_\lambda^n f)'(U)} (DR_\lambda^n f(z))' > \alpha, \tag{3}$$

where $n \in \mathbb{N}$ and $\alpha \in [0,1)$.

Theorem 1. *Consider g a convex function in U and $m > 0$ and define the function $h(z) = g(z) + \frac{1}{m+2} z g'(z)$, $z \in U$. If $f \in \mathcal{DR}^{\mathcal{F}}_{\lambda,n}(\alpha)$ and $I_m(f)(z) = \frac{m+2}{z^{m+1}} \int_0^z t^m f(t)dt$, for any $z \in U$, then*

$$\mathcal{F}_{DR^n_\lambda f(U)}(DR^n_\lambda f(z))' \leq \mathcal{F}_{h(U)} h(z), \tag{4}$$

implies

$$\mathcal{F}_{DR^n_\lambda I_m(f)(U)}(DR^n_\lambda I_m(f)(z))' \leq \mathcal{F}_{g(U)} g(z),$$

for any $z \in U$ and this result is sharp.

Proof. The function $I_m(f)$, with m a positive real number, satisfies the relation $z^{m+1} I_m(f)(z) = (m+2) \int_0^z t^m f(t)dt$, and differentiating it with respect to z, we get

$$z(I_m(f))'(z) + (m+1) I_m(f)(z) = (m+2) f(z),$$

and applying the operator DR^n_λ, yields for any $z \in U$

$$z(DR^n_\lambda I_m(f)(z))' + (m+1) DR^n_\lambda I_m(f)(z) = (m+2) DR^n_\lambda f(z). \tag{5}$$

Differentiating the relation (5) with respect to z, we get for any $z \in U$

$$\frac{1}{m+2} z(DR^n_\lambda I_m(f)(z))'' + (DR^n_\lambda I_m(f)(z))' = (DR^n_\lambda I_m(f)(z))'.$$

Using the last relation, the fuzzy differential subordination (4) will be

$$\mathcal{F}_{DR^n_\lambda I_m(f)(U)}\left(\frac{1}{m+2} z(DR^n_\lambda I_m f(z))'' + (DR^n_\lambda I_m(f)(z))'\right) \leq \mathcal{F}_{g(U)}\left(\frac{1}{m+2} z g'(z) + g(z)\right). \tag{6}$$

Denoting

$$p(z) = (DR^n_\lambda I_m(f)(z))', \tag{7}$$

we find that $p \in \mathcal{H}[1,1]$.

Using the notation, the relation (6) becomes for any $z \in U$ the fuzzy differential subordination

$$\mathcal{F}_{p(U)}\left(\frac{1}{m+2} z p'(z) + p(z)\right) \leq \mathcal{F}_{g(U)}\left(\frac{1}{m+2} z g'(z) + g(z)\right).$$

Applying Lemma 1, we get $\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{g(U)} g(z)$, for any $z \in U$, written in the following form $\mathcal{F}_{DR^n_\lambda I_m(f)(U)}\left(DR^n_\lambda I_m(f)(z)\right)' \leq \mathcal{F}_{g(U)} g(z)$, where the sharpness is given by the fact that g is the fuzzy best dominant. □

Theorem 2. *Consider $h(z) = \frac{(2a-1)z+1}{z+1}$, $a \in [0,1)$. For I_m given by Theorem 1, with $m > 0$, the following inclusion holds*

$$I_m\left[\mathcal{DR}^{\mathcal{F}}_{\lambda,n}(\alpha)\right] \subset \mathcal{DR}^{\mathcal{F}}_{\lambda,n}(\alpha^*), \tag{8}$$

where $\alpha^* = 2(1-a)(m+2) \int_0^1 \frac{t^{m+1}}{t+1} dt + 2a - 1$.

Proof. Following the same ideas as the proof of Theorem 1 regarding the convex function h and taking into account the conditions from Theorem 2, we obtain $\mathcal{F}_{p(U)}\left(\frac{1}{m+2} z p'(z) + p(z)\right) \leq \mathcal{F}_{h(U)} h(z)$, with the function p defined by (7).

Applying Lemma 2, it yields $\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{h(U)} h(z)$, written in the following form $\mathcal{F}_{DR^n_\lambda I_m(f)(U)}\left(DR^n_\lambda I_m(f)(z)\right)' \leq F_{g(U)} g(z) \leq F_{h(U)} h(z)$, where $g(z) = \frac{m+2}{z^{m+2}} \int_0^z t^{m+1} \frac{(2a-1)t+1}{t+1} dt = \frac{(m+2)(2-2a)}{z^{m+2}} \int_0^z \frac{t^{m+1}}{t+1} dt + 2a - 1$.

The function g being convex and taking into account that $g(U)$ is symmetric with respect to the real axis, we establish

$$\mathcal{F}_{DR_\lambda^n I_m(f)(U)}(DR_\lambda^n I_m(f)(z))' \geq \min_{|z|=1} \mathcal{F}_{g(U)} g(z) = \mathcal{F}_{g(U)} g(1) \tag{9}$$

and $a^* = g(1) = 2(1-a)(m+2)\int_0^1 \frac{t^{m+1}}{t+1}dt + 2a - 1$. □

Theorem 3. *Consider g a convex function with the property $g(0) = 1$ and the function $h(z) = zg'(z) + g(z)$, for any $z \in U$. If $f \in \mathcal{A}$ verifies the fuzzy differential subordination, for any $z \in U$,*

$$\mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))' \leq \mathcal{F}_{h(U)} h(z), \tag{10}$$

then it yields for any $z \in U$ the sharp subordination

$$\mathcal{F}_{DR_\lambda^n f(U)} \frac{DR_\lambda^n f(z)}{z} \leq \mathcal{F}_{g(U)} g(z).$$

Proof. Denoting $p(z) = \frac{DR_\lambda^n f(z)}{z}$, we deduce for any $z \in U$ that $zp'(z) + p(z) = \left(DR_\lambda^n f(z)\right)'$. The fuzzy differential subordination

$$\mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))' \leq \mathcal{F}_{h(U)} h(z),$$

with $z \in U$, can be written using the notation made above in the following form

$$\mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right) \leq \mathcal{F}_{h(U)} h(z) = \mathcal{F}_{g(U)}\left(zg'(z) + g(z)\right),$$

for $z \in U$.

Applying Lemma 1, we get

$$\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{g(U)} g(z),$$

written as

$$\mathcal{F}_{DR_\lambda^n f(U)} \frac{DR_\lambda^n f(z)}{z} \leq \mathcal{F}_{g(U)} g(z),$$

for any $z \in U$.

The sharpness is given by the fact that g is the fuzzy best dominant. □

Theorem 4. *Consider $f \in \mathcal{A}$ and a function h convex with the property $h(0) = 1$. When the fuzzy differential subordination*

$$\mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))' \leq \mathcal{F}_{h(U)} h(z), \tag{11}$$

is satisfied for any $z \in U$, we get the fuzzy differential subordination

$$\mathcal{F}_{DR_\lambda^n f(U)} \frac{DR_\lambda^n f(z)}{z} \leq \mathcal{F}_{g(U)} g(z), \quad z \in U,$$

where the fuzzy best dominant is the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$.

Proof. Denote $p(z) = \frac{DR_\lambda^n f(z)}{z} \in \mathcal{H}[1,1]$ and differentiating it with respect to z, we deduce for any $z \in U$ that

$$(DR_\lambda^n f(z))' = zp'(z) + p(z)$$

and the fuzzy differential subordination (11) becomes

$$\mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right) \leq \mathcal{F}_{h(U)} h(z).$$

Using Lemma 2, we obtain
$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{g(U)}g(z),$$
for any $z \in U$, written taking into account the notation made above
$$\mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^n f(z)}{z} \leq \mathcal{F}_{g(U)}g(z),$$
for any $z \in U$, and
$$g(z) = \frac{1}{z}\int_0^z h(t)dt$$
is a convex function that satisfies the differential equation associated with the fuzzy differential subordination (11),
$$zg'(z) + g(z) = h(z),$$
therefore it is the fuzzy best dominant. □

Corollary 1. *Considering the function* $h(z) = \frac{(2a-1)z+1}{z+1}$ *is convex in* U, $0 \leq a < 1$, *when* $f \in \mathcal{A}$ *satisfies for any* $z \in U$ *the fuzzy differential subordination*
$$\mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))' \leq \mathcal{F}_{h(U)}h(z), \tag{12}$$
then
$$\mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^n f(z)}{z} \leq \mathcal{F}_{g(U)}g(z), \ z \in U,$$
where the function $g(z) = \frac{2(1-a)}{z}\ln(z+1) + 2a - 1$, $z \in U$, *is convex and it is the fuzzy best dominant.*

Proof. From Theorem 4 setting $p(z) = \frac{DR_\lambda^n f(z)}{z}$, the fuzzy differential subordination (12) takes the following form
$$\mathcal{F}_{p(U)}\big(zp'(z) + p(z)\big) \leq \mathcal{F}_{h(U)}h(z),$$
for any $z \in U$, and applying Lemma 2, we deduce
$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{g(U)}g(z),$$
written as
$$\mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^n f(z)}{z} \leq \mathcal{F}_{g(U)}g(z)$$
and
$$g(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{(2a-1)t+1}{t+1}dt = \frac{2(1-a)}{z}\ln(1+z) + 2a - 1, \ z \in U,$$
is the fuzzy best dominant. □

Example 1. *Let the function* $h(z) = \frac{1-z}{z+1}$ *is convex in* U *with* $h(0) = 1$ *and* $f(z) = z^2 + z$, $z \in U$.

For $n = 1$ *and* $\lambda = \frac{1}{2}$, *we obtain* $DR_{\frac{1}{2}}^1 f(z) = 3z^2 + z$ *and* $\left(DR_{\frac{1}{2}}^1 f(z)\right)' = 6z + 1$ *and* $\frac{DR_{\frac{1}{2}}^1 f(z)}{z} = 3z + 1$.

We deduce that $g(z) = \frac{1}{z}\int_0^z \frac{1-t}{t+1}dt = \frac{2\ln(z+1)}{z} - 1$.

Using Theorem 4 it yields
$$\mathcal{F}_U(6z+1) \leq \mathcal{F}_U\left(\frac{1-z}{z+1}\right), z \in U,$$

imply

$$\mathcal{F}_U(3z+1) \le \mathcal{F}_U\left(\frac{2\ln(z+1)}{z} - 1\right),\ z \in U.$$

Theorem 5. *Set the function $h(z) = zg'(z) + g(z)$ for any $z \in U$ and a function g convex with the property $g(0) = 1$. If $f \in \mathcal{A}$ verifies for any $z \in U$ the fuzzy differential subordination*

$$\mathcal{F}_{DR_\lambda^n f(U)}\left(\frac{zDR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)}\right)' \le \mathcal{F}_{h(U)} h(z), \tag{13}$$

then it yields for any $z \in U$ the sharp subordination

$$\mathcal{F}_{DR_\lambda^n f(U)} \frac{DR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)} \le \mathcal{F}_{g(U)} g(z).$$

Proof. Denote $p(z) = \frac{DR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)}$ and differentiating this relation, we get

$$p'(z) = \frac{\left(DR_\lambda^{n+1} f(z)\right)'}{DR_\lambda^n f(z)} - p(z) \cdot \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)},$$

written as $zp'(z) + p(z) = \left(\frac{zDR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)}\right)'$.

The fuzzy differential subordination (13) takes the following form using the notation above

$$\mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right) \le \mathcal{F}_{h(U)} h(z) = \mathcal{F}_{g(U)}\left(zg'(z) + g(z)\right),$$

for any $z \in U$, and by applying Lemma 1, we obtain

$$\mathcal{F}_{p(U)} p(z) \le \mathcal{F}_{g(U)} g(z),$$

written in the following form

$$\mathcal{F}_{DR_\lambda^n f(U)} \frac{DR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)} \le \mathcal{F}_{g(U)} g(z),$$

for any $z \in U$.

The sharpness is given by the fact that g is the fuzzy best dominant. □

Theorem 6. *Set for any $z \in U$ and $n \in \mathbb{N}$ the function $h(z) = \frac{n\lambda}{n\lambda+1} zg'(z) + g(z)$, for function g is a convex with the property $g(0) = 1$. If $f \in \mathcal{A}$ and the fuzzy differential subordination*

$$\mathcal{F}_{DR_\lambda^n f(U)}\left(\frac{n+1}{(n\lambda+1)z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z} DR_\lambda^n f(z)\right) \le \mathcal{F}_{h(U)} h(z), \tag{14}$$

holds for any $z \in U$, then the sharp subordination

$$\mathcal{F}_{DR_\lambda^n f(U)}\left(DR_\lambda^n f(z)\right)' \le \mathcal{F}_{g(U)} g(z),$$

holds too, for $z \in U$.

Proof. Denoting $p(z) = \left(DR_\lambda^n f(z)\right)'$, we have $p(0) = 1$, and differentiating the relation we deduce

$$\frac{\lambda}{n\lambda+1} zp'(z) + p(z) = \frac{n+1}{(n\lambda+1)z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z} DR_\lambda^n f(z).$$

The fuzzy differential subordination from the hypothesis takes the following form

$$\mathcal{F}_{p(U)}\left(\frac{\lambda}{n\lambda+1}zp'(z)+p(z)\right) \leq F_{h(U)}h(z) = F_{g(U)}\left(\frac{n\lambda}{n\lambda+1}zg'(z)+g(z)\right),$$

for $z \in U$. By applying Lemma 1, it yields

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{g(U)}g(z),$$

written as

$$\mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))' \leq \mathcal{F}_{g(U)}g(z),$$

for any $z \in U$, and the subordination is sharp because the function g is the fuzzy best dominant. □

Theorem 7. *Consider $f \in \mathcal{A}$ and h is a convex function with the property $h(0) = 1$. If the fuzzy differential subordination is verifies for any $z \in U$,*

$$\mathcal{F}_{DR_\lambda^n f(U)}\left(\frac{n+1}{(n\lambda+1)z}DR_\lambda^{n+1}f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z}DR_\lambda^n f(z)\right) \leq \mathcal{F}_{h(U)}h(z), \tag{15}$$

then the subordination

$$\mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))' \leq \mathcal{F}_{g(U)}g(z),$$

is verified for the fuzzy best dominant $g(z) = \frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda}}}\int_0^z h(t)t^{\frac{n\lambda+1}{\lambda}-1}dt$, a convex function.

Proof. Denoting $p(z) = \left(DR_\lambda^n f(z)\right)'$ and making a simple calculus regarding the operator DR_λ^n we deduce for any $z \in U$ that

$$\frac{n+1}{(n\lambda+1)z}DR_\lambda^{n+1}f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z}DR_\lambda^n f(z) = \frac{\lambda}{n\lambda+1}zp'(z)+p(z).$$

In these conditions, the fuzzy differential subordination (15) becomes

$$\mathcal{F}_{p(U)}\left(\frac{\lambda}{n\lambda+1}zp'(z)+p(z)\right) \leq \mathcal{F}_{h(U)}h(z).$$

Applying Lemma 2, we obtain

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{g(U)}g(z),$$

for $z \in U$, where

$$g(z) = \frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda n}}}\int_0^z h(t)t^{\frac{n\lambda+1}{\lambda}-1}dt, \ \ z \in U,$$

equivalent with

$$\mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))' \leq \mathcal{F}_{g(U)}g(z), \ \ z \in U.$$

The function g is a convex and satisfies the differential equation

$$\frac{\lambda}{n\lambda+1}zg'(z)+g(z) = h(z)$$

for the fuzzy differential subordination (15), so it is the fuzzy best dominant. □

Corollary 2. *Considering $h(z) = \frac{(2a-1)z+1}{z+1}$ convex in U, $0 \leq a < 1$, and $f \in \mathcal{A}$ which verifies for any $z \in U$ the fuzzy differential subordination*

$$\mathcal{F}_{DR_{\lambda}^{n}f(U)}\left(\frac{n+1}{(n\lambda+1)z}DR_{\lambda}^{n+1}f(z)-\frac{n(1-\lambda)}{(n\lambda+1)z}DR_{\lambda}^{n}f(z)\right)\leq\mathcal{F}_{h(U)}h(z),\qquad(16)$$

then the fuzzy subordination

$$\mathcal{F}_{DR_{\lambda}^{n}f(U)}(DR_{\lambda}^{n}f(z))'\leq\mathcal{F}_{g(U)}g(z),\ \ z\in U,$$

is satisfied for the fuzzy best dominant $g(z)=\frac{2(1-a)(n\lambda+1)}{\lambda z^{\frac{n\lambda+1}{\lambda}}}\int_0^z\frac{t^{\frac{n\lambda+1}{\lambda}-1}}{t+1}dt+2a-1$, *with* $z\in U$, *which is a convex function.*

Proof. Denoting $p(z)=\left(DR_{\lambda}^{n}f(z)\right)'$ and taking into account the Theorem 7, the fuzzy differential subordination (16) is written in the following form

$$\mathcal{F}_{p(U)}\left(\frac{\lambda}{n\lambda+1}zp'(z)+p(z)\right)\leq\mathcal{F}_{h(U)}h(z),\ \ z\in U.$$

Applying Lemma 2, we get

$$\mathcal{F}_{p(U)}p(z)\leq\mathcal{F}_{g(U)}g(z),$$

written as

$$\mathcal{F}_{DR_{\lambda}^{n}f(U)}(DR_{\lambda}^{n}f(z))'\leq\mathcal{F}_{g(U)}g(z)$$

and

$$g(z)=\frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda}}}\int_0^z h(t)t^{\frac{n\lambda+1}{\lambda}-1}dt=\frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda}}}\int_0^z t^{\frac{n\lambda+1}{\lambda}-1}\frac{(2a-1)t+1}{t+1}dt$$
$$=\frac{2(1-a)(n\lambda+1)}{\lambda z^{\frac{n\lambda+1}{\lambda}}}\int_0^z\frac{t^{\frac{n\lambda+1}{\lambda}-1}}{t+1}dt+2a-1,\ \ z\in U,$$

is the fuzzy best dominant. □

Example 2. *Let* $h(z)=\frac{1-z}{z+1}$ *and* $f(z)=z^2+z$, $z\in U$, *as in the Example 1.*
For $n=1$, $\lambda=\frac{1}{2}$,*we obtain* $DR_{\frac{1}{2}}^{1}f(z)=3z^2+z$. *Then* $\left(DR_{\frac{1}{2}}^{1}(z)\right)'=6z+1$.
We obtain also $\frac{n+1}{(n\lambda+1)z}DR_{\lambda}^{n+1}f(z)-\frac{n(1-\lambda)}{(n\lambda+1)z}DR_{\lambda}^{n}f(z)=\frac{4}{3}DR_{\frac{1}{2}}^{2}f(z)-\frac{1}{3z}DR_{\frac{1}{2}}^{1}f(z)=8z+1$, *where* $DR_{\frac{1}{2}}^{2}f(z)=\frac{27}{4}z^2+z$.
We have $g(z)=\frac{3}{z^3}\int_0^z\frac{1-t}{t+1}t^2dt=\frac{6\ln(1+z)}{z^3}-\frac{6}{z^2}+\frac{3}{z}-1$.
Using Theorem 7, we deduce

$$\mathcal{F}_{U}(8z+1)\leq\mathcal{F}_{U}\left(\frac{1-z}{z+1}\right),\ \ z\in U,$$

generates

$$\mathcal{F}_{U}(6z+1)\leq\mathcal{F}_{U}\left(\frac{6\ln(1+z)}{z^3}-\frac{6}{z^2}+\frac{3}{z}-1\right),\ \ z\in U.$$

3. Fuzzy Differential Superordination

In this section, using the fuzzy differential superordinations, we deduce interesting properties of the studied differential operator DR_{λ}^{n}.

Theorem 8. *Considering* h *is a convex function in* U *with the property* $h(0)=1$, *for* $f\in\mathcal{A}$ *suppose that* $\left(DR_{\lambda}^{n}f(z)\right)'$ *is univalent in* U, $\left(DR_{\lambda}^{n}I_m(f)(z)\right)'\in Q\cap\mathcal{H}[1,1]$ *and the fuzzy superordination*

$$\mathcal{F}_{h(U)}h(z)\leq\mathcal{F}_{DR_{\lambda}^{n}f(U)}(DR_{\lambda}^{n}f(z))',\qquad(17)$$

holds for any $z \in U$, then the fuzzy superordination

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{DR^n_\lambda I_m(f)(U)}(DR^n_\lambda I_m(f)(z))'$$

is verified for any $z \in U$ by the fuzzy best subordinant $g(z) = \frac{m+2}{z^{m+2}} \int_0^z h(t)t^{m+1}dt$, which is convex.

Proof. The function $I_m(f)$ satisfies the relation $z^{m+1}I_m(f)(z) = (m+2)\int_0^z t^m f(t)dt$, and differentiating it, we get

$$z(I_m(f))'(z) + (m+1)I_m(f)(z) = (m+2)f(z)$$

and applying the operator DR^n_λ we get

$$z(DR^n_\lambda I_m(f)(z))' + (m+1)DR^n_\lambda I_m(f)(z) = (m+2)DR^n_\lambda f(z),\ z \in U. \tag{18}$$

Differentiating relation (18) we obtain

$$\frac{1}{m+2}z(DR^n_\lambda I_m(f)(z))'' + (DR^n_\lambda I_m(f)(z))' = (DR^n_\lambda I_m(f)(z))', z \in U.$$

Using the last relation, the fuzzy differential superordination (17) becomes

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{DR^n_\lambda I_m(f)(U)}\left(\frac{1}{m+2}z(DR^n_\lambda I_m f(z))'' + (DR^n_\lambda I_m(f)(z))'\right). \tag{19}$$

Denoting

$$p(z) = (DR^n_\lambda I_m(f)(z))',\ \ z \in U, \tag{20}$$

the fuzzy differential superordination (19) takes the following form

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{p(U)}\left(\frac{1}{m+2}zp'(z) + p(z)\right),\quad z \in U.$$

Applying Lemma 3 we deduce

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z),\ \ z \in U,$$

written as

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{DR^n_\lambda I_m(f)(U)}(DR^n_\lambda I_m(f)(z))',\ \ z \in U,$$

and the fuzzy best subordinant is the convex function $g(z) = \frac{m+2}{z^{m+2}} \int_0^z h(t)t^{m+1}dt$. □

Corollary 3. *Considering $h(z) = \frac{(2a-1)z+1}{z+1}$, with $a \in [0,1)$, for $f \in \mathcal{A}$, assume that $\left(DR^n_\lambda f(z)\right)'$ is univalent in U, $\left(DR^n_\lambda I_m(f)(z)\right)' \in Q \cap \mathcal{H}[1,1]$ and*

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{DR^n_\lambda f(U)}(DR^n_\lambda f(z))',\ z \in U, \tag{21}$$

then

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{DR^n_\lambda I_m(f)(U)}(DR^n_\lambda I_m(f)(z))',\quad z \in U,$$

with the fuzzy best subordinant is the convex function $g(z) = \frac{2(1-a)(m+2)}{z^{m+2}} \int_0^z \frac{t^{m+1}}{t+1}dt + 2a - 1$, $z \in U$.

Proof. From Theorem 8, taking $p(z) = \left(DR^n_\lambda I_m(f)(z)\right)'$, the fuzzy differential superordination (21) becomes

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{p(U)}\left(\frac{1}{m+2}zp'(z) + p(z)\right),\quad z \in U.$$

Applying Lemma 3, we get $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z)$, written as

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{DR_{\lambda}^{n}I_m(f)(U)}(DR_{\lambda}^{n}I_m(f)(z))', \quad z \in U,$$

and

$$g(z) = \frac{m+2}{z^{m+2}}\int_0^z h(t)t^{m+1}dt = \frac{m+2}{z^{m+2}}\int_0^z \frac{(2a-1)t+1}{t+1}t^{m+1}dt$$
$$= \frac{2(1-a)(m+2)}{z^{m+2}}\int_0^z \frac{t^{m+1}}{t+1}dt + 2a - 1, \quad z \in U,$$

is convex and it is the fuzzy best subordinant. □

Example 3. *Let* $h(z) = \frac{1-z}{z+1}$ *and* $f(z) = z^2 + z$, $z \in U$, *as in Example 1. For* $n = 1$, $\lambda = \frac{1}{2}$, *we have* $DR_{\frac{1}{2}}^{1}f(z) = 3z^2 + z$ *and* $\left(DR_{\frac{1}{2}}^{1}f(z)\right)' = 6z + z$ *univalent in* U.

For $m = 3$ *we get* $I_3(f)(z) = \frac{5}{z^4}\int_0^z t^3(t^2+t)dt = \frac{5}{6}z^2 + z$ *and* $R^1 I_3(f)(z) = z(I_3(f))'(z) = \frac{5}{3}z^2 + z$, $D_{\frac{1}{2}}^{1}I_3(f)(z) = \frac{1}{2}I_3(f)(z) + \frac{1}{2}z(I_3(f))'(z) = \frac{5}{4}z^2 + z$, *so* $DR_{\frac{1}{2}}^{1}I_3(f)(z) = \frac{25}{12}z^2 + z$ *and* $\left(DR_{\frac{1}{2}}^{1}I_3(f)(z)\right)' = \frac{25}{6}z + 1 \in Q \cap \mathcal{H}[1,1]$.

We deduce $g(z) = \frac{5}{z^5}\int_0^z \frac{1-t}{t+1}t^4 dt = \frac{10\ln(z+1)}{z^5} - \frac{10}{z^4} + \frac{5}{z^3} - \frac{10}{3z^2} + \frac{5}{2z} - 1$.

Applying Theorem 8, we get

$$\mathcal{F}_U\left(\frac{1-z}{z+1}\right) \leq \mathcal{F}_U(6z+1), \quad z \in U,$$

induces

$$\mathcal{F}_U\left(\frac{10\ln(z+1)}{z^5} - \frac{10}{z^4} + \frac{5}{z^3} - \frac{10}{3z^2} + \frac{5}{2z} - 1\right) \leq \mathcal{F}_U\left(\frac{25}{6}z + 1\right), \quad z \in U.$$

Theorem 9. *Set for any* $z \in U$ *and* m *a complex number with* $\mathrm{Re}\, m > -2$, *the function* $h(z) = \frac{1}{m+2}zg'(z) + g(z)$, *for a function* g *convex in* U. *For* $f \in \mathcal{A}$ *consider that* $\left(DR_{\lambda}^{n}f(z)\right)'$ *is univalent in* U, $\left(DR_{\lambda}^{n}I_m(f)(z)\right)' \in Q \cap \mathcal{H}[1,1]$ *and it is verified for any* $z \in U$ *the fuzzy superordination*

$$\mathcal{F}_{h(U)}h(z) \leq F_{DR_{\lambda}^{n}f(U)}(DR_{\lambda}^{n}f(z))', \tag{22}$$

then the fuzzy superordination

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{DR_{\lambda}^{n}I_m(f)(U)}(DR_{\lambda}^{n}I_m(f)(z))',$$

is verified by the fuzzy best subordinant $g(z) = \frac{m+2}{z^{m+2}}\int_0^z h(t)t^{m+1}dt$, *for* $z \in U$.

Proof. Denoting $p(z) = \left(DR_{\lambda}^{n}I_m(f)(z)\right)'$, with $z \in U$, and following the ideas as in the proof of Theorem 8, the fuzzy differential superordination (22) will be written as

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{p(U)}\left(\frac{1}{m+2}zp'(z) + p(z)\right), \quad z \in U.$$

Applying Lemma 4, we deduce

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z), \quad z \in U,$$

written as

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{DR_{\lambda}^{n}I_m(f)(U)}(DR_{\lambda}^{n}I_m(f)(z))', \quad z \in U,$$

and the function $g(z) = \frac{m+2}{z^{m+2}}\int_0^z h(t)t^{m+1}dt$ is the fuzzy best subordinant. □

Theorem 10. *Consider a function h convex such that $h(0) = 1$, $f \in \mathcal{A}$ and assume that $\left(DR_\lambda^n f(z)\right)'$ is univalent and $\frac{DR_\lambda^n f(z)}{z} \in Q \cap \mathcal{H}[1,1]$. If the fuzzy superordination*

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\left(DR_\lambda^n f(z)\right)', \tag{23}$$

holds for any $z \in U$, then the fuzzy superordination

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^n f(z)}{z},$$

is satisfied by the fuzzy best subordinant $g(z) = \frac{1}{z}\int_0^z h(t)dt$, which is a convex function, for any $z \in U$.

Proof. Denote $p(z) = \frac{DR_\lambda^n f(z)}{z} \in \mathcal{H}[1,1]$ and differentiating it, we have $zp'(z) + p(z) = \left(DR_\lambda^n f(z)\right)'$, for any $z \in U$.

Then the fuzzy differential superordination (23) becomes

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right), \quad z \in U.$$

Applying Lemma 3, we get

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z), \ z \in U,$$

written as

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^n f(z)}{z}, \quad z \in U,$$

and the fuzzy best subordinant is the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$. □

Corollary 4. *Considering $h(z) = \frac{(2a-1)z+1}{z+1}$ a convex function in U, with $0 \le a < 1$, for $f \in \mathcal{A}$ assume that $\left(DR_\lambda^n f(z)\right)'$ is univalent and $\frac{DR_\lambda^n f(z)}{z} \in Q \cap \mathcal{H}[1,1]$. If the fuzzy superordination*

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\left(DR_\lambda^n f(z)\right)', \tag{24}$$

is satisfies for any $z \in U$, then the following fuzzy superordination

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^n f(z)}{z}, \quad z \in U,$$

is satisfied by the fuzzy best subordinant $g(z) = \frac{2(1-a)}{z}\ln(z+1) + 2a - 1$,, convex function for $z \in U$.

Proof. From Theorem 10 for $p(z) = \frac{DR_\lambda^n f(z)}{z}$, the fuzzy differential superordination (24) takes the following form

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right), \quad z \in U.$$

Applying Lemma 3, we have $\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z)$, written as

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^n f(z)}{z}, \quad z \in U,$$

and

$$g(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{(2a-1)t+1}{t+1}dt$$
$$= \frac{2(1-a)}{z}\ln(z+1) + 2a - 1, \quad z \in U.$$

The function g is convex and it is the fuzzy best subordinant. □

Example 4. *Let* $h(z) = \frac{1-z}{z+1}$ *and* $f(z) = z^2 + z$, $z \in U$, *as in Example 1. For* $n = 1$, $\lambda = \frac{1}{2}$, *we obtain* $DR^1_{\frac{1}{2}} f(z) = 3z^2 + z$ *and* $\left(DR^1_{\frac{1}{2}} f(z)\right)' = 6z + 1$ *univalent in* U, $\frac{DR^1_{\frac{1}{2}} f(z)}{z} = 3z + 1 \in Q \cap \mathcal{H}[1, 1]$.

We get $g(z) = \frac{1}{z}\int_0^z \frac{1-t}{t+1} dt = \frac{2\ln(z+1)}{z} - 1$.
Using Theorem 10, we deduce

$$\mathcal{F}_U\left(\frac{1-z}{z+1}\right) \le \mathcal{F}_U(6z+1), \quad z \in U,$$

imply

$$\mathcal{F}_U\left(\frac{2\ln(z+1)}{z} - 1\right) \le \mathcal{F}_U(3z+1), \ z \in U.$$

Theorem 11. *Consider* g *a convex function in* U *and set the function* $h(z) = zg'(z) + g(z)$*. If* $f \in \mathcal{A}$, $\frac{DR^n_\lambda f(z)}{z} \in Q \cap \mathcal{H}[1, 1]$, *the function* $(DR^n f(z))'$ *is univalent and the fuzzy differential superordination*

$$\mathcal{F}_{h(U)} h(z) \le \mathcal{F}_{DR^n_\lambda f(U)}(DR^n_\lambda f(z))', \tag{25}$$

is verified for any $z \in U$, *then the fuzzy superordination*

$$\mathcal{F}_{g(U)} g(z) \le \mathcal{F}_{DR^n_\lambda f(U)} \frac{DR^n_\lambda f(z)}{z}, \quad z \in U,$$

holds and the fuzzy best subordinant is the function $g(z) = \frac{1}{z}\int_0^z h(t)dt$.

Proof. Denoting $p(z) = \frac{DR^n_\lambda f(z)}{z} \in \mathcal{H}[1, 1]$, differentiating it, we get for $z \in U$ that $zp'(z) + p(z) = \left(DR^n_\lambda f(z)\right)'$, and the fuzzy differential superordination (25) becomes

$$\mathcal{F}_{g(U)}\left(zg'(z) + g(z)\right) \le \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right), \quad z \in U.$$

Applying Lemma 4, it yields

$$\mathcal{F}_{g(U)} g(z) \le \mathcal{F}_{p(U)} p(z), \ z \in U,$$

written as

$$\mathcal{F}_{g(U)} g(z) \le \mathcal{F}_{DR^n_\lambda f(U)} \frac{DR^n_\lambda f(z)}{z}, \quad z \in U,$$

and $g(z) = \frac{1}{z}\int_0^z h(t)dt$ is the best subordinant. □

Theorem 12. *Let a convex function* h *with the property* $h(0) = 1$, *for* $f \in \mathcal{A}$ *assume that* $\left(\frac{zDR^{n+1}_\lambda f(z)}{DR^n_\lambda f(z)}\right)'$ *is univalent and* $\frac{DR^{n+1}_\lambda f(z)}{DR^n_\lambda f(z)} \in Q \cap \mathcal{H}[1, 1]$*. If*

$$\mathcal{F}_{h(U)} h(z) \le \mathcal{F}_{DR^n_\lambda f(U)} \left(\frac{zDR^{n+1}_\lambda f(z)}{DR^n_\lambda f(z)}\right)', \quad z \in U, \tag{26}$$

then

$$\mathcal{F}_{g(U)} g(z) \le \mathcal{F}_{DR^n_\lambda f(U)} \frac{DR^{n+1}_\lambda f(z)}{DR^n_\lambda f(z)}, \quad z \in U,$$

and the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$ *is the fuzzy best subordinant.*

Proof. Denote $p(z) = \frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)} \in \mathcal{H}[1,n]$, after differentiating it, we have $p'(z) = \frac{\left(DR_{\lambda}^{n+1}f(z)\right)'}{DR_{\lambda}^{n}f(z)} - p(z) \cdot \frac{\left(DR_{\lambda}^{n}f(z)\right)'}{DR_{\lambda}^{n}f(z)}$ and $zp'(z) + p(z) = \left(\frac{zDR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}\right)'$. In these conditions, the fuzzy differential superordination (26) takes the following form

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right), \quad z \in U,$$

and applying Lemma 3, we get $\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z)$, $z \in U$, written as

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_{\lambda}^{n}f(U)}\frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}, \quad z \in U,$$

and the fuzzy best subordinant is the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$. □

Corollary 5. *Considering the convex function* $h(z) = \frac{(2a-1)z+1}{z+1}$ *in* U, $0 \le a < 1$, *for* $f \in \mathcal{A}$ *assume that* $\left(\frac{zDR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}\right)'$ *is univalent and* $\frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)} \in Q \cap \mathcal{H}[1,1]$. *If*

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{DR_{\lambda}^{n}f(U)}\left(\frac{zDR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}\right)', \quad z \in U, \tag{27}$$

then

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_{\lambda}^{n}f(U)}\frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}, \quad z \in U,$$

and the fuzzy best subordinant is the convex function $g(z) = \frac{2(1-a)}{z}\ln(z+1) + 2a - 1$, $z \in U$.

Proof. From Theorem 12 for $p(z) = \frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}$, the fuzzy differential superordination (27) becomes

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right), \quad z \in U,$$

and from Lemma 3, we have $\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z)$, i.e.,

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_{\lambda}^{n}f(U)}\frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}, \quad z \in U,$$

and

$$\begin{aligned} g(z) &= \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{(2a-1)t+1}{t+1}dt \\ &= \frac{2(1-a)}{z}\ln(z+1) + 2a - 1. \end{aligned}$$

The function q is convex and becomes the fuzzy best subordinant. □

Theorem 13. *Taking a function* g *convex in* U, *define* $h(z) = zg'(z) + g(z)$, *for any* $z \in U$. *For* $f \in \mathcal{A}$, *assume that* $\left(\frac{zDR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}\right)'$ *is univalent,* $\frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)} \in Q \cap \mathcal{H}[1,1]$ *and satisfies the fuzzy differential superordination*

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{DR_{\lambda}^{n}f(U)}\left(\frac{zDR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}\right)', \quad z \in U, \tag{28}$$

then

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_{\lambda}^{n}f(U)}\frac{DR_{\lambda}^{n+1}f(z)}{DR_{\lambda}^{n}f(z)}, \quad z \in U,$$

and the fuzzy best subordinant is the function $g(z) = \frac{1}{z}\int_0^z h(t)dt$.

Proof. Denote $p(z) = \frac{DR_\lambda^{n+1}f(z)}{DR_\lambda^n f(z)} \in \mathcal{H}[1,1]$; differentiating this relation, we get $zp'(z) + p(z) = \left(\frac{zDR_\lambda^{n+1}f(z)}{DR_\lambda^n f(z)}\right)'$, $z \in U$. With this notation, the fuzzy differential superordination (28) becomes

$$\mathcal{F}_{g(U)}(zg'(z) + g(z)) \le \mathcal{F}_{p(U)}(zp'(z) + p(z)), \quad z \in U.$$

Using Lemma 4, it yields $\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z)$, $z \in U$, written as

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\frac{DR_\lambda^{n+1}f(z)}{DR_\lambda^n f(z)}, \quad z \in U,$$

where $g(z) = \frac{1}{z}\int_0^z h(t)dt$ is the fuzzy best subordinant. □

Theorem 14. *For a function h convex such that* $h(0) = 1$ *and for* $f \in \mathcal{A}$, *consider that* $\frac{n+1}{(n\lambda+1)z}DR_\lambda^{n+1}f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z}DR_\lambda^n f(z)$ *is univalent and* $\left(DR_\lambda^n f(z)\right)' \in Q \cap \mathcal{H}[1,1]$. *If*

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\left(\frac{n+1}{(n\lambda+1)z}DR_\lambda^{n+1}f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z}DR_\lambda^n f(z)\right), \quad z \in U, \tag{29}$$

then

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))', \quad z \in U,$$

and the fuzzy best subordinant represents the convex function $g(z) = \frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda}}}\int_0^z h(t)t^{\frac{n\lambda+1}{\lambda}-1}dt$.

Proof. Denoting $p(z) = \left(DR_\lambda^n f(z)\right)' \in \mathcal{H}[1,1]$, with $p(0) = 1$, we obtain after differentiating this relation that $zp'(z) + p(z) = \frac{n+1}{\lambda z}DR_\lambda^{n+1}f(z) - \left(n - 1 + \frac{1}{\lambda}\right)\left(DR_\lambda^n f(z)\right)' - \frac{n(1-\lambda)}{\lambda z}DR_\lambda^n f(z)$ and $\frac{\lambda}{n\lambda+1}zp'(z) + p(z) = \frac{n+1}{(n\lambda+1)z}DR_\lambda^{n+1}f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z}DR_\lambda^n f(z)$.

With the notation above, the fuzzy differential superordination (29) becomes

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{p(U)}\left(\frac{\lambda}{n\lambda+1}zp'(z) + p(z)\right), \quad z \in U,$$

and by using Lemma 3, we deduce $\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z)$, $z \in U$, which implies

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))', \quad z \in U,$$

and the fuzzy best subordinant becomes the convex function $g(z) = \frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda}}}\int_0^z h(t)t^{\frac{n\lambda+1}{\lambda}-1}dt$. □

Corollary 6. *Considering the function* $h(z) = \frac{(2a-1)z+1}{z+1}$ *convex in U,* $0 \le a < 1$, *for* $f \in \mathcal{A}$ *suppose that* $\frac{n+1}{(n\lambda+1)z}DR_\lambda^{n+1}f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z}DR_\lambda^n f(z)$ *is univalent and* $\left(DR_\lambda^n f(z)\right)' \in Q \cap \mathcal{H}[1,1]$. *If*

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{DR_\lambda^n f(U)}\left(\frac{n+1}{(n\lambda+1)z}DR_\lambda^{n+1}f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z}DR_\lambda^n f(z)\right), \quad z \in U, \tag{30}$$

then

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))', \quad z \in U,$$

and the fuzzy best subordinant becomes the convex function $g(z) = \frac{2(1-a)(n\lambda+1)}{\lambda z^{\frac{n\lambda+1}{\lambda}}} \int_0^z \frac{t^{\frac{n\lambda+1}{\lambda}-1}}{t+1} dt + 2a - 1$.

Proof. From Theorem 14 for $p(z) = \left(DR_\lambda^n f(z)\right)'$, the fuzzy differential superordination (30) becomes

$$\mathcal{F}_{h(U)} h(z) \le \mathcal{F}_{p(U)} \left(zp'(z) + p(z)\right), \quad z \in U,$$

and applying Lemma 3, we have $\mathcal{F}_{g(U)} g(z) \le \mathcal{F}_{p(U)} p(z), z \in U$, equivalent with

$$\mathcal{F}_{g(U)} g(z) \le \mathcal{F}_{DR_\lambda^n f(U)} (DR_\lambda^n f(z))', \quad z \in U,$$

and

$$g(z) = \frac{n\lambda + 1}{\lambda z^{\frac{n\lambda+1}{\lambda}}} \int_0^z h(t) t^{\frac{n\lambda+1}{\lambda}-1} dt = \frac{n\lambda + 1}{\lambda z^{\frac{n\lambda+1}{\lambda}}} \int_0^z \frac{(2a-1)t + 1}{t+1} t^{\frac{n\lambda+1}{\lambda}-1} dt$$
$$= \frac{2(1-a)(n\lambda+1)}{\lambda z^{\frac{n\lambda+1}{\lambda}}} \int_0^z \frac{t^{\frac{n\lambda+1}{\lambda}-1}}{t+1} dt + 2a - 1.$$

The function q is convex and it is the fuzzy best subordinant. □

Example 5. *Let* $h(z) = \frac{1-z}{z+1}$ *convex in* U *such that* $h(0) = 1$ *and* $f(z) = z^2 + z$, $z \in U$. *For* $n = 1$, $\lambda = \frac{1}{2}$, *we get* $DR_{\frac{1}{2}}^1 f(z) = 3z^2 + z$ *and* $\left(DR_{\frac{1}{2}}^1 f(z)\right)' = 6z + 1 \in Q \cap \mathcal{H}[1,1]$. *Assume that function* $\frac{n+1}{(n\lambda+1)z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z} DR_\lambda^n f(z) = \frac{4}{3} DR_{\frac{1}{2}}^2 f(z) - \frac{1}{3z} DR_{\frac{1}{2}}^1 f(z) = 8z + 1$ *is univalent in* U, *where* $R^2 f(z) = \frac{z}{2}\left(R^1 f(z)\right)' + \frac{1}{2} R^1 f(z) = 3z^2 + z$,
$D_{\frac{1}{2}}^2 f(z) = \frac{1}{2} D_{\frac{1}{2}}^1 f(z) + \frac{z}{2}\left(D_{\frac{1}{2}}^1 f(z)\right)' = \frac{9}{4} z^2 + z$,
$DR_{\frac{1}{2}}^2 f(z) = \frac{27}{4} z^2 + z$.
We obtain $g(z) = \frac{3}{z^3} \int_0^z \frac{1-t}{t+1} t^2 dt = \frac{6\ln(z+1)}{z^3} - \frac{6}{z^2} + \frac{3}{z} - 1$.
Using Theorem 14 we obtain

$$\mathcal{F}_U\left(\frac{1-z}{z+1}\right) \le \mathcal{F}_U(8z+1), \quad z \in U,$$

induce

$$\mathcal{F}_U\left(\frac{6\ln(z+1)}{z^3} - \frac{6}{z^2} + \frac{3}{z} - 1\right) \le \mathcal{F}_U(6z+1), \ \ z \in U.$$

Theorem 15. *Set the function* $h(z) = \frac{\lambda}{n\lambda+1} zg'(z) + g(z)$ *for a function* g *convex in* U. *For* $f \in \mathcal{A}$ *consider that* $\frac{n+1}{(n\lambda+1)z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z} DR_\lambda^n f(z)$ *is univalent and* $\left(DR_\lambda^n f(z)\right)' \in Q \cap \mathcal{H}[1,1]$ *and verifies the fuzzy differential superordination*

$$\mathcal{F}_{g(U)}\left(\frac{\lambda}{n\lambda+1} zg'(z) + g(z)\right) \le \mathcal{F}_{DR_\lambda^n f(U)}\left(\frac{n+1}{(n\lambda+1)z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z} DR_\lambda^n f(z)\right), \tag{31}$$

$z \in U$, *then*

$$\mathcal{F}_{g(U)} g(z) \le \mathcal{F}_{DR_\lambda^n f(U)} (DR_\lambda^n f(z))', \quad z \in U,$$

where $g(z) = \frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda}}} \int_0^z h(t) t^{\frac{n\lambda+1}{\lambda}-1} dt$ *is the fuzzy best subordinant.*

Proof. Denoting $p(z) = \left(DR_\lambda^n f(z)\right)'$, differentiating it and making some calculus, we obtain $\frac{\lambda}{m\lambda+1} zp'(z) + p(z) = \frac{n+1}{(n\lambda+1)z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{(n\lambda+1)z} DR_\lambda^n f(z)$, $z \in U$. With this notation the fuzzy differential superordination (31) takes the following form

$$\mathcal{F}_{g(U)}\left(\frac{\lambda}{n\lambda+1}zg'(z)+g(z)\right) \leq \mathcal{F}_{p(U)}\left(\frac{\lambda}{n\lambda+1}zp'(z)+p(z)\right), \quad z \in U.$$

Applying Lemma 4, we deduce $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z), z \in U$, equivalent with

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{DR_\lambda^n f(U)}(DR_\lambda^n f(z))', \quad z \in U,$$

and $g(z) = \frac{n\lambda+1}{\lambda z^{\frac{n\lambda+1}{\lambda}}} \int_0^z h(t)t^{\frac{n\lambda+1}{\lambda}-1}dt$ is the fuzzy best subordinant. □

4. Conclusions

The relationship between fuzzy sets theory and the geometric theory of analytic functions is undeniably solid and long-lasting, and it is clear that applying the ideas from the theories of differential subordination and superordination to fuzzy sets theory works and produces results that are intriguing for complex analysis researchers who want to extend their area of study. The primary goal of the study described in this paper is to present new results concerning fuzzy aspects introduced in the geometric theory of analytic functions in the hope that it will be useful in future research just as numerous other applications of the fuzzy set concept have prompted the creation of sustainability models in a variety of economic, environmental and social activities.

As future ideas for research on the operator DR_λ^n studied in this paper, the definition and investigation of additional classes of univalent functions using it could be accomplished. The operator DR_λ^n could be used for obtaining higher-order fuzzy differential subordinations since the classical theory of differential subordination already presents third-order differential subordination results as seen in Reference [24], for example, but also in many other studies. Fourth order is also considered for classical differential subordination; hence, it is possible to extend the results obtained here in this direction, too. Furthermore, the operator DR_λ^n could be adapted to quantum calculus and obtain differential subordinations and superordinations for it involving q-fractional calculus, as seen in Reference [39]. Conditions for univalence can be derived for the class introduced in Definition 7 as obtained in Reference [23]. Coefficient studies could be conducted regarding the new class given in Definition 7, such as estimations for Hankel determinants of different orders, Toeplitz determinants or the Fekete–Szegő problem.

Hopefully, the new fuzzy results presented here will find applications in future studies concerning real life contexts.

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