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Article

Coefficient Estimation Utilizing the Faber Polynomial for a Subfamily of Bi-Univalent Functions

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Abstract: The paper introduces a new family of analytic bi-univalent functions that are injective and possess analytic inverses, by employing a q-analogue of the derivative operator. Moreover, the article establishes the upper bounds of the Taylor–Maclaurin coefficients of these functions, which can aid in approximating the accuracy of approximations using a finite number of terms. The upper bounds are obtained by approximating analytic functions using Faber polynomial expansions. These bounds apply to both the initial few coefficients and all coefficients in the series, making them general and early, respectively.

Keywords: analytic function; univalent functions; bi-univalent functions; Faber polynomial; q-derivative operator; quantum calculus

MSC: 30C45; 30C50

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Citation: Alsoboh, A.; Amourah, A.; Sakar, F.M.; Ogilat, O.; Gharib, M.G.; Zomot, N. Coefficient Estimation Utilizing the Faber Polynomial for a Subfamily of Bi-Univalent Functions. *Axioms* **2023**, *12*, 512. https://doi.org/10.3390/axioms12060512

Academic Editor: Georgia Irina Oros

Received: 9 April 2023
Revised: 18 May 2023
Accepted: 19 May 2023
Published: 24 May 2023

1. Introduction

A Faber polynomial is a sequence of polynomials used to approximate an analytic function on a compact set. It is named after the German mathematician Georg Faber, who introduced the Faber polynomials in 1903 [1]. The Faber polynomial of degree n for a given analytic function f is defined as the unique polynomial $P_n(z)$ of degree n that interpolates f at its first $n+1$ distinct zeros, counting multiplicities, on the compact set. The sequence of Faber polynomials is known to converge uniformly to f on the compact set, and the convergence rate is related to the smoothness of f. Faber polynomial expansions are often used to obtain upper bounds on the Taylor–Maclaurin coefficients of analytic functions.

Q-calculus is a branch of mathematics that generalizes and extends calculus by introducing a new parameter q, which is a complex number or a variable. Jackson [2] pioneered and systematically developed the application of q-calculus. It has applications in various fields of mathematics and physics, such as number theory, combinatorics, quantum mechanics, and statistical mechanics. In q-calculus, basic concepts, such as derivatives, integrals, and functions, are modified to incorporate the parameter q. For instance, the q-derivative is defined as the difference quotient involving q-analogs of the usual derivatives. Similarly, the q-integral is defined as the q-analog of the Riemann integral. Q-calculus also includes q-special functions, such as q-binomial coefficients, q-factorials, and q-hypergeometric functions, which play significant roles in various areas of mathematics and physics. Overall,

q-calculus provides a powerful tool for studying and solving problems involving discrete and quantum systems.

Fractional calculus operators have found extensive use in the description and resolution of problems in applied sciences, as well as in geometric functions, as noted in [3]. Fractional q-calculus is an extension of ordinary fractional calculus and has been applied in a range of areas, including optimal control problems, solving q-difference and q-integral equations, and ordinary fractional calculus. To learn more about this topic, one can refer to [4] and recent papers, such as [5–7].

2. Preliminaries

Let $\mathbf{A}$ denote the set of analytic functions that can be expressed in the following form

$$\Phi(\eta) = \eta + \sum_{\mathbb{k}=2}^{\infty} \varrho_{\mathbb{k}} \eta^{\mathbb{k}}, \quad (\varrho_{\mathbb{k}} \in \mathbb{C}), \tag{1}$$

and are defined in the open unit disk $\nabla = \{\eta \in \mathbb{C} : |\eta| < 1\}$. Within $\mathbf{A}$, there is a subfamily $\mathcal{S}$ that consists of univalent functions in ∇. Additionally, let P denote the subclass of analytic functions in ∇ that satisfy the inequality $Re(\varphi(\eta)) > 0$ and are of the form

$$\varphi(\eta) = 1 + \sum_{\mathbb{k}=1}^{\infty} \varphi_{\mathbb{k}} \eta^{\mathbb{k}}, \tag{2}$$

where $|\varphi_{\mathbb{k}}| < 2$. Caratheodory's Lemma (refer to [8]).

In the context of analytic functions defined in the open unit disk ∇, we can define a relationship between two of such functions, Φ_1 and Φ_2, known as "subordination". We note that Φ_1 is subordinate to Φ_2, denoted by $\Phi_1 \prec \Phi_2$ $(\eta \in \nabla)$, if there exists a Schwarz function:

$$\psi(\eta) = \sum_{\mathbb{k}=1}^{\infty} m_{\mathbb{k}} \eta^{\mathbb{k}}, \qquad \text{with } \psi(0) = 0 \quad \text{and} \quad \psi(1) = 1,$$

such that

$$\Phi_1(\eta) = \Phi_2(\psi(\eta)), \quad \text{for } (\eta \in \nabla).$$

In other words, Φ_1 can be expressed as a composition of Φ_2 with a certain conformal mapping $\psi(\eta)$, where $\psi(\eta)$ maps the unit disk to itself and satisfies certain conditions. This notion of subordination is described in [9].

Koebe's one-quarter theorem, named after Paul Koebe, is a result of the complex analysis, which states that if a biholomorphic mapping f maps the unit disk ∇ onto a domain D in the complex plane, then the image of each tangent disk to ∇ under f contains a disk of radius $\frac{1}{4}$ of the radius of the tangent disk. In other words, if $z_0 \in \nabla$ and $r > 0$ is such that the disk $B(z_0, r)$ is tangent to ∇ at some point, then $f(B(z_0, r))$ contains a disk of radius $\frac{1}{4}r$.

It is a well-known fact that, as per Koebe's theorem, for any $\Phi \in \mathcal{S}$, the image of the open unit disk under Φ satisfies $\Phi(\nabla) \geq \frac{1}{4}$. Moreover, every univalent function Φ has an inverse function Φ^{-1}, which is defined by the following properties:

1. For $\eta \in \nabla$, $\Phi^{-1}(\Phi(\eta)) = \eta$.

2. For $\rho \in \mathbb{D}(0, r_0)$, where $r_0 \geq \frac{1}{4}$ is a positive constant. Then, $\Phi(\Phi^{-1}(\rho)) = \rho$. Here, $\mathbb{D}(0, r_0)$ denotes the open disk centered at the origin with radius r_0.

The inverse function Φ^{-1} can be expressed as a power series of the form:

$$h(\rho) = \Phi^{-1}(\rho) = \rho - \varrho_2 \rho^2 + (2\varrho_2^2 - \varrho_3)\rho^3 - (5\varrho_2^3 - 5\varrho_2\varrho_3 + \varrho_4)\rho^4 + \dots . \tag{3}$$

Here, $\varrho_{\Bbbk}$'s are the Taylor coefficients of Φ in the power series expansion of $\Phi(\eta)$, which is given by (1), and $h(\rho)$ is the inverse function evaluated at ρ.

Netanyahu [10] improved this bound to $|\varrho_2| \leq \frac{4}{3}$. On the other hand, Brannan and Clunie [11] improved Lewin's [12] result and they showed that $|\varrho_2| \leq \sqrt{2}$.

Some examples of functions in the class Σ are

$$\Phi_1(\varrho) = \frac{\eta}{1-\eta}, \quad \Phi_2(\varrho) = -\log(1-\eta) \quad \text{and} \quad \Phi_3(\varrho) = \frac{1}{2}\log\left(\frac{1+\eta}{1-\eta}\right).$$

The inverse functions that correspond to these:

$$\Phi_1^{-1}(\rho) = \frac{\rho}{1+\rho}, \quad \Phi_2^{-1}(\rho) = \frac{e^{2\rho}-1}{e^{2\rho}+1} \quad \text{and} \quad \Phi_3^{-1}(\rho) = \frac{e^{\rho}-1}{e^{\rho}}.$$

are also univalent functions. Thus, the functions $\Phi_1(\varrho), \Phi_2(\varrho)$, and $\Phi_3(\varrho)$ are bi-univalent functions.

However, it is well-known that the Koebe function of the form

$$\Phi(\eta) = \frac{\varrho}{(1-\varrho)^2},$$

is not in the class Σ. For more details, we refer to [13].

We emphasize that, as in the class $\mathcal{S}$ of normalized univalent functions, the convex combination of two functions of class Σ need not be bi-univalent. For example, the functions

$$\varphi_1(\eta) = \frac{\eta}{1-\eta} \quad \text{and} \quad \varphi_2(\eta) = \frac{\eta}{1+i\eta}$$

are bi-univalent but their sum $\varphi_1 + \varphi_2$ is not even univalent, as its derivative vanishes at $\frac{1}{2}(1+i)$. However, the class Σ is preserved under a number of elementary transformations.

Several subclasses of bi-univalent functions have been investigated and introduced by various authors, including Srivastava [13]. The class of bi-univalent functions in ∇ given by (1) is denoted by Σ. Other different subclasses of Σ have also been studied by many authors (see, for example, [14–35].

The significance of Faber polynomials in geometric function theory was demonstrated by Schiffer [36]. However, there are only a few articles in the literature that use the Faber polynomial expansion to determine the early and general coefficient bounds $|\varrho_{\Bbbk}|$ for bi-univalent functions. Consequently, very little is known about the general coefficient bounds $\varrho_{\Bbbk}$ for $\Bbbk \geq 4$, due to the unpredictable nature of the coefficients of both Φ and Φ^{-1} when bi-univalency is required (see, for instance, [37–44]).

The coefficient of $h(\rho) = \Phi^{-1}(\rho)$ of the form (3), can be expressed using the Faber polynomial expansion as:

$$h(\rho) = \Phi^{-1}(\rho) = \rho + \sum_{\pounds=2}^{\infty} \frac{1}{\pounds} K_{\pounds-1}^{-\pounds}(\varrho_2, \varrho_3, \varrho_4, \ldots)\rho^{\pounds},$$

where

$$\begin{aligned} K_{\pounds-1}^{-\pounds} = {} & \frac{(-\pounds)!}{(-2\pounds+1)!(\pounds-1)!}\varrho_2^{\pounds-1} + \frac{(-\pounds)!}{[2(-\pounds+1)]!(\pounds-3)!}\varrho_2^{\pounds-3}\varrho_3 \\ & + \frac{(-\pounds)!}{(-2\pounds+3)!(\pounds-4)!}\varrho_2^{\pounds-4}\varrho_4 + \frac{(-\pounds)!}{[2(-\pounds+2)]!(\pounds-5)!}\varrho_2^{\pounds-5}[\varrho_5 + (-\pounds+2)\varrho_3^2] \\ & + \frac{(-\pounds)!}{(-2\pounds+5)!(\pounds-6)!}\varrho_2^{\pounds-6}[\varrho_6 + (-2\pounds+5)\varrho_3\varrho_4] + \sum_{\aleph\geq 7}^{\infty} \varrho_2^{\pounds-\aleph} V_{\aleph}, \end{aligned} \quad (4)$$

where $V_\aleph$ denotes a function such that $7 \leq \aleph \leq \pounds$. It can be expressed as a homogeneous polynomial of degree $\aleph$ in the variables $\varrho_2, \varrho_3, \ldots, \varrho_\pounds$. All of the pertinent details can be found in [41].

In particular, the first three terms of $K_{\pounds-1}^{-\pounds}$ are given below:

$$\frac{1}{2}K_1^{-2} = -\varrho_2$$

$$\frac{1}{3}K_2^{-3} = 2\varrho_2^2 - \varrho_3$$

$$\frac{1}{4}K_3^{-4} = -(5\varrho_2^3 - 5\varrho_2\varrho_3 + \varrho_4).$$

In general, for any $\pounds \in \mathbb{N}$, an expansion of the Faber polynomial is given by [45],

$$K_{\kappa-1}^{\pounds} = \pounds\varrho_\kappa + \frac{\pounds(\pounds-1)}{2}E_\kappa^2 + \frac{\pounds!}{(\pounds-3)!(3)!}E_\kappa^3 + \ldots + \frac{\pounds!}{(\pounds-\kappa+1)!(\kappa-1)!}E_{\kappa-1}^{\kappa-1} \tag{5}$$

where $E_{\kappa-1}^{\pounds} = E_{\kappa-1}^{\pounds}(\varrho_2, \varrho_3, \ldots)$, and by [45],

$$E_{\kappa-1}^{\pounds}(\varrho_2, \varrho_3, \ldots, \varrho_\kappa) = \sum_{\kappa=2}^{\infty} \frac{m!(\varrho_2)^{\mu_2}(\varrho_3)^{\mu_3}\ldots(\varrho_\kappa)^{\mu_{\kappa-1}}}{\mu_1!\,\mu_2!\,\ldots\,\mu_{\kappa-1}!}, \qquad (\pounds \leq \kappa),$$

while $\varrho_1 = 1$, the sum is taken over all nonnegative integers $\mu_1\,\mu_2\,\ldots\,\mu_\kappa$, satisfying

$$\mu_1 + \mu_2 + \ldots + \mu_\kappa = m$$

$$\mu_1 + 2\mu_2 + \ldots + (\kappa-1)\mu_{\kappa-1} = \kappa - 1.$$

Evidently,

$$E_{\kappa-1}^{\kappa-1}(\varrho_2, \varrho_3, \ldots, \varrho_\kappa) = \varrho_2^{\kappa-1},$$

or, equivalently, by [46]

$$E_{\kappa}^{\pounds}(\varrho_2, \varrho_3, \ldots, \varrho_\kappa) = \sum_{\kappa=2}^{\infty} \frac{m!(\varrho_2)^{\mu_2}(\varrho_3)^{\mu_3}\ldots(\varrho_\kappa)^{\mu_\kappa}}{\mu_1!\,\mu_2!\,\ldots\,\mu_\kappa!}, \qquad (\pounds \leq \kappa),$$

while $\varrho_1 = 1$, the sum is taken over all nonnegative integers $\mu_1\,\mu_2\,\ldots\,\mu_\kappa$, satisfying

$$\mu_1 + \mu_2 + \ldots + \mu_\kappa = m$$

$$\mu_1 + 2\mu_2 + \ldots + (\kappa-1)\mu_{\kappa-1} + (\kappa)\mu_\kappa = \kappa.$$

It is clear that

$$E_\kappa^\kappa(\varrho_2, \varrho_3, \ldots, \varrho_\kappa) = E_1^\kappa$$

where the first and last polynomials are

$$E_\kappa^\kappa = \varrho_1^\kappa \text{ and } E_\kappa^1 = \varrho_\kappa.$$

The concept of q-calculus was first introduced by Jackson in a systematic way, and it has since been studied by many mathematicians [47–50]. In this article, we introduce some key concepts and definitions of q-calculus, assuming $0 < q < 1$. Some of these concepts include:

Definition 1. *The* $[\varkappa]_q$ *denotes the basic (or* $q-$*) number, where* $0 < q < 1$ *is defined as follows:*

$$[\varkappa]_q = \begin{cases} (1-q^{\varkappa})(1-q)^{-1} & , \ \varkappa \in \mathbb{C}\backslash\{0\} \\ 0 & , \ \varkappa = 0 \\ q^{\Bbbk-1} + q^{\Bbbk-2} + \cdots + q^2 + q + 1 = \sum_{i=0}^{\Bbbk-1} q^i & , \ \varkappa = \Bbbk \in \mathbb{N}. \end{cases}$$

It is obvious from Definition 1 that $\lim_{q\to 1^-} [\Bbbk]_q = \lim_{q\to 1^-} \frac{1-q^{\Bbbk}}{1-q} = \Bbbk.$

Definition 2 ([51])**.** *The q-difference operator (or q-derivative) of a function* f *is defined by*

$$\partial_q(\Phi(\eta)) = \begin{cases} \frac{\Phi(\eta)-\Phi(q\eta)}{\eta - q\eta} & \eta \in \mathbb{C}\backslash\{0\} \\ 1 & \eta = 0. \end{cases}$$

We note that $\lim_{q\to 1^-} \partial_q \Phi(\eta) = \Phi'(\eta)$ *if* Φ *is differentiable for all* $\eta \in \mathbb{C}$.

One can easily see that

$$\partial_q \left\{ \sum_{\Bbbk=2}^{\infty} \varrho_{\Bbbk} \eta^{\Bbbk} \right\} = \sum_{\Bbbk=2}^{\infty} [\Bbbk]_q \varrho_{\Bbbk} \eta^{\Bbbk-1}, \qquad (\Bbbk \in \mathbb{N}, \eta \in \nabla), \tag{6}$$

and

$$\partial_q^{\kappa} \left\{ \sum_{\Bbbk=2}^{\infty} \varrho_{\Bbbk} \eta^{\Bbbk} \right\} = \partial_q \left(\partial_q^{\kappa-1} \left\{ \sum_{\Bbbk=2}^{\infty} \varrho_{\Bbbk} \eta^{\Bbbk} \right\} \right) = \varrho_{\Bbbk} [\Bbbk]_q!, \qquad (\Bbbk \in \mathbb{N}). \tag{7}$$

In 2019, Alsoboh and Darus [48] introduced the q-derivative operator $\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta} : \mathbf{A} \to \mathbf{A}$, as:

$$\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta} \Phi(\eta) = z + \sum_{\Bbbk=2}^{\infty} \Omega^{\ell}_{\Bbbk} \varrho_{\Bbbk} \eta^{\Bbbk}, \tag{8}$$

where

$$\Omega^{\ell}_{\Bbbk} = \left[(\gamma - \delta)(\beta - \mu)([\Bbbk]_q - 1) + 1 \right]^{\ell}, \tag{9}$$

and $(\mu, \beta, \gamma, \delta \geq 0, \gamma > \delta, \beta > \mu, \ell \in \mathbb{N}_0, \eta \in \nabla)$.

Lemma 1 ([52])**.** *Let the Schwarz function* $\omega(\eta)$ *be given by*

$$\omega(\eta) = \omega_1 \eta + \omega_2 \eta^2 + \omega_3 \eta^3 + \ldots + \omega_{\Bbbk} \eta^{\Bbbk} + \ldots \qquad (\eta \in \nabla);$$

then

$$\begin{aligned} &|\omega_1| \leq 1, \\ &|\omega_2| \leq 1 - |\omega_1|^2, \\ &|\omega_2 - t\omega_1^2| \leq 1 + (|t| - 1)|\omega_1|^2. \end{aligned} \tag{10}$$

3. Class $D^{\ell}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi)$

In this section, we define and study a new subclass of bi-univalent functions in an open unit disk that has symmetry, using the derivative operator $\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta)$ of the form (8) and the principle of subordination, as follows:

Definition 3. *For $\mu, \beta, \gamma, \delta \geq 0$, $\gamma > \delta, \beta > \mu$ and $k \in \mathbb{N}_0$, a bi-univalent function Φ of the form (1) is in the class $D^{\ell}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi)$ if it satisfies the following subordination conditions:*

$$(1-\chi)\frac{\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta)}{\eta} + \chi\partial_q\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta) \prec \varphi(\eta), \quad (\eta \in \nabla; \chi \geq 1), \tag{11}$$

and

$$(1-\chi)\frac{\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}h(\rho)}{\rho} + \chi\partial_q\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}h(\rho) \prec \varphi(\rho), \quad (\rho \in \nabla; \chi \geq 1), \tag{12}$$

where $h(\rho)$ and $\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta)$ are defined by (3) and (8), respectively.

Example 1. *A bi-univalent function f of the form (1) is referred to as being in the class $D^{0}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi) = D_{\Sigma}(q; \chi, \varphi)$, if the following conditions of subordination are met:*

$$(1-\chi)\frac{\Phi(\eta)}{\eta} + \chi\partial_q\Phi(\eta) \prec \varphi(\eta), \quad (\eta \in \nabla),$$

and

$$(1-\chi)\frac{h(\rho)}{\rho} + \chi\partial_q h(\rho) \prec \varphi(\rho), \quad (\rho \in \nabla),$$

where $h(\rho)$ is defined by (3). This class was introduced by Altınkaya and Yalçın [37].

Example 2. *A bi-univalent function f of the form (1) is referred to as being in the class $\lim_{q\to 1^-} D^{0}_{\Sigma_q}$ $(\chi, \delta, \gamma, \mu; \varphi) = R_{\sigma}(\chi, \varphi)$, if the following conditions of subordination are met:*

$$(1-\chi)\frac{\Phi(\eta)}{\eta} + \chi(\Phi(\eta))' \prec \varphi(\eta), \quad (\eta \in \nabla),$$

and

$$(1-\chi)\frac{h(\rho)}{\rho} + \chi(h(\rho))' \prec \varphi(\rho), \quad (\rho \in \nabla),$$

where $h(\rho)$ is defined by (3). This class was introduced by Kumar et al. [53].

4. Coefficient Bounds of the Class $D^{\ell}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi)$

The following theorem provides an estimate of the bounds of the coefficients for functions of class $D^{\ell}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi)$. The theorem provides an estimate of the coefficients $\varrho_{\Bbbk}$ for $\Bbbk \geq \ell + 2$ in terms of the parameters $\chi, \delta, \gamma, \mu$, and φ, as well as the maximum value of $|\varphi'(t)|$ on the interval $[0, 1]$. The proof of the theorem uses a method similar to those employed by various authors, including Hussien et al. [54] and Altınkaya and Yalçın ([55,56]).

Theorem 1. *Let Φ be given by (1). For $\chi \geq 1$, $0 \leq \alpha < 1$, $(\mu, \beta, \gamma, \delta \geq 0)$, $\gamma > \delta, \beta > \mu$ and $k \in \mathbb{N}_0$. If $\Phi \in D^{\ell}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi)$ and $\varrho_m = 0$; $m = 2, \ldots, \Bbbk - 1$, then*

$$|\varrho_{\Bbbk}| \leq \frac{2(1-q)}{\left|1 + (q - q^{\Bbbk})\chi\right|\left|\Omega^{\ell}_{\Bbbk}\right|}; \qquad (\Bbbk = 4, 5, 6, \ldots). \tag{13}$$

Proof. Since $\Phi \in D^{\ell}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi)$ of form (1), we have:

$$(1-\chi)\frac{\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta)}{\eta} + \chi D_q \Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta) = 1 + \sum_{\Bbbk=1}^{\infty}\Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega^{\ell}_{\Bbbk}\varrho_{\Bbbk}\eta^{\Bbbk-1} \tag{14}$$

and for $h = \Phi^{-1}$, we have

$$\begin{aligned}(1-\chi)\frac{\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}h(\rho)}{\rho} + \chi D_q \Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}h(\rho) &= 1 + \sum_{\Bbbk=1}^{\infty}\Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega^{\ell}_{\Bbbk} b_{\Bbbk}\rho^{\Bbbk-1} \\ = 1 + \sum_{\Bbbk=1}^{\infty}\Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega^{\ell}_{\Bbbk}\left(\frac{1}{\Bbbk}K^{-\Bbbk}_{\Bbbk-1}(\varrho_2, \varrho_3, \ldots, \varrho_{\Bbbk})\right)\rho^{\Bbbk-1}, &\end{aligned} \tag{15}$$

where $\Omega^{\ell}_{\Bbbk}$ and $K^{-\Bbbk}_{\Bbbk-1}$ are given by (4) and (9), respectively.

Since $\Phi, \Phi^{-1} \in D^{\ell}_{\Sigma_q}(\chi, \delta, \gamma, \mu; \varphi)$. Then, by using the definition of subordination, two Schwartz functions exist,

$$u(\eta) = \sum_{\Bbbk=1}^{\infty}\gimel_{\Bbbk}\eta^{\Bbbk} \quad \text{and} \quad v(\rho) = \sum_{\Bbbk=1}^{\infty}\daleth_{\Bbbk}\rho^{\Bbbk},$$

which are analytic in ∇, such that

$$\varphi(u(\eta)) = (1-\chi)\frac{\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta)}{\eta} + \chi D_q \Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}\Phi(\eta), \quad (\eta \in \nabla) \tag{16}$$

$$\varphi(v(\rho)) = (1-\chi)\frac{\Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}h(\rho)}{\rho} + \chi D_q \Upsilon^{(\ell)}_{q,\mu,\beta,\gamma,\delta}h(\rho), \quad (\rho \in \nabla), \tag{17}$$

where

$$\varphi(u(\eta)) = 1 + \sum_{\Bbbk=1}^{\infty}\sum_{\ell=1}^{\Bbbk}\varphi_k E^{\ell}_{\Bbbk}(\gimel_1, \gimel_2, \ldots, \gimel_{\Bbbk})\eta^{\Bbbk}. \tag{18}$$

and

$$\varphi(v(\rho)) = 1 + \sum_{\Bbbk=1}^{\infty}\sum_{\ell=1}^{\Bbbk}\varphi_k E^{\ell}_{\Bbbk}(\daleth_1, \daleth_2, \ldots, \daleth_{\Bbbk})\rho^{\Bbbk}. \tag{19}$$

From (14), (16), and (18), we have

$$\Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega^{\ell}_{\Bbbk}\varrho_{\Bbbk} = \sum_{\ell=1}^{\Bbbk-1}\varphi_k E^{\ell}_{n-1}(\gimel_1, \gimel_2, \ldots, \gimel_{n-1}) \qquad (n \geq 2), \tag{20}$$

Similarly, from (15), (17), and (19), we have

$$\Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega^{\ell}_{\Bbbk} b_{\Bbbk} = \sum_{\ell=1}^{\Bbbk-1}\varphi_k E^{\ell}_{\Bbbk-1}(\daleth_1, \daleth_2, \ldots, \daleth_{\Bbbk-1}) \quad (\Bbbk \geq 2), \tag{21}$$

by the given assumption

$$\varrho_m = 0, \quad (2 \leq m \leq \Bbbk - 1),$$

which is equivalent to

$$\gimel_m = \daleth_m = 0; \quad (1 \leq m \leq \Bbbk - 2),$$

and from Equations (20) and (21), we have $b_{\Bbbk} = -\varrho_{\Bbbk}$ and so

$$\begin{cases} \Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega_{\Bbbk}^{\ell}\varrho_{\Bbbk} = \varphi_1 \daleth_{\Bbbk-1}, \\ \Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega_{\Bbbk}^{\ell}\varrho_{\Bbbk} = -\varphi_1 \urcorner_{\Bbbk-1}. \end{cases} \tag{22}$$

Taking the absolute value for Equation (22), we obtain

$$\begin{aligned} |\varrho_{\Bbbk}| &\le \frac{|\varphi_1||\daleth_{\Bbbk-1}|}{\Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega_{\Bbbk}^{\ell}} \\ &= \frac{|\varphi_1||\urcorner_{\Bbbk-1}|}{\Big(1-\chi(1-[\Bbbk]_q)\Big)\Omega_{\Bbbk}^{\ell}}, \qquad (n \ge 4). \end{aligned} \tag{23}$$

Using Caratheodory's Lemma, we obtain

$$|\varrho_{\Bbbk}| \le \frac{2(1-q)}{\Big|1-q+(q-q^{\Bbbk})\chi\Big|\Big|\Omega_{\Bbbk}^{\ell}\Big|}.$$

This completes the proof of the theorem. □

In the next theorem, we estimate the initial coefficients of the functions from the indicated class $D_{\Sigma_q}^{\ell}(\chi,\delta,\gamma,\mu;\varphi)$.

Theorem 2. *For $\chi \ge 1$, $0 \le \alpha < 1$, $(\mu,\beta,\gamma,\delta \ge 0)$, $\gamma > \delta, \beta > \mu$ and $k \in \Bbbk_0$, if $\Phi \in D_{\Sigma_q}^{\ell}(\chi,\delta,\gamma,\mu;\varphi)$ where $\Phi(\eta)$ is given by (1), then we have the following consequence*

$$|\varrho_2| \le min\left\{\frac{2}{(1+\chi q)\Omega_2^{\ell}}, \frac{2}{\sqrt{(1+\chi(q^2+q))\Omega_3^{\ell}}}\right\},$$

$$|\varrho_3| \le min\left\{\frac{4}{\big(1+\chi(2+q)\Omega_2^{\ell}\big)^2} + \frac{2}{(1+\chi(q^2+q))\Omega_3^{\ell}}, \frac{6}{(1+\chi(q^2+q))\Omega_3^{\ell}}\right\},$$

and

$$|2\varrho_2^2 - \varrho_3| \le \frac{4}{|(1+\chi([3]_q-1))\Omega_3^{\ell}|}.$$

Proof. Replacing $\Bbbk$ by 2 and 3 in (20) and (21), respectively, we obtain:

$$(1-\chi(1-[2]_q))\Omega_2^{\ell}\varrho_2 = \varphi_1\daleth_1, \tag{24}$$

$$(1-\chi(1-[3]_q))\Omega_3^{\ell}\varrho_3 = \varphi_1\daleth_2 + \varphi_2 c_1^2, \tag{25}$$

$$(1-\chi(1-[2]_q))\Omega_2^{\ell}\varrho_2 = -\varphi_1\urcorner_1, \tag{26}$$

and

$$\Big(1-\chi(1-[3]_q)\Big)\Omega_3^{\ell}(2\varrho_2^2-\varrho_3) = \varphi_1\urcorner_2 + \varphi_2 d_1^2. \tag{27}$$

From (24) and (26), we have $\daleth_1 = -\gimel_1$ and

$$|\varrho_2| = \frac{|\varphi_1 \gimel_1|}{\Big|1-\chi(1-[2]_q)\Big|\Omega_2^\ell} = \frac{|\varphi_1 \daleth_1|}{\Big|1-\chi(1-[2]_q)\Big|\Omega_2^\ell} \leq \frac{2}{1+\chi(1+[2]_q)\Omega_2^\ell}. \tag{28}$$

Now, by adding (25) and (27)

$$2\Big(1-\chi(1-[3]_q)\Big)\Omega_3^\ell \varrho_2^2 = \varphi_1(\gimel_2 + \daleth_1) + \varphi_2(c_1^2 + d_1^2),$$

or, equivalently,

$$|\varrho_2| \leq \frac{2}{\sqrt{\Big(1+\chi(q^2+q)\Big)\Omega_3^\ell}}. \tag{29}$$

Next, in order to find the bounds of $|\varrho_3|$, subtract (25) from (27), we have

$$2\Big(1+\chi([3]_q-1)\Big)\Omega_3^\ell(\varrho_3 - \varrho_2^2) = \varphi_1(\gimel_2 - \daleth_2) + \varphi_2(c_1^2 - d_1^2), \tag{30}$$

or

$$2\Big(1+\chi([3]_q-1)\Big)\Omega_3^\ell(\varrho_3 - \varrho_2^2) \leq \varphi_2(\gimel_2 - \daleth_2),$$

$$|\varrho_3| \leq \varrho_2^2 + \frac{|\varphi_2(\gimel_2 - \daleth_2)|}{2\Big|\Big(1+\chi([3]_q-1)\Big)\Omega_3^\ell\Big|}. \tag{31}$$

Equivalent to

$$|\varrho_3| \leq \varrho_2^2 + \frac{|\varphi_2(\gimel_2 - \daleth_2)|}{2\Big|\Big(1+\chi(q^2+q)\Big)\Omega_3^\ell\Big|},$$

Substituting the value ϱ_2 from (29) and (30) into (31), one obtains

$$|\varrho_3| \leq \frac{4}{(1+\chi(2+q)\Omega_2^\ell)^2} + \frac{2}{\Big(1+\chi(q^2+q)\Big)\Omega_3^\ell},$$

and

$$|\varrho_3| \leq \frac{6}{\Big(1+\chi(q^2+q)\Big)\Omega_3^\ell}.$$

Finally, from (30), by applying the Caratheodory Lemma, we obtain

$$|2\varrho_2^2 - \varrho_3| = \frac{|\varphi_1 \daleth_2 + \varphi_2 d_1^2|}{\Big|(1-\chi(1-[3]_q))\Omega_3^\ell\Big|} \leq \frac{4}{\Big|(1+\chi([3]_q-1))\Omega_3^\ell\Big|}. \tag{32}$$

This completes the proof of Theorem 2. □

5. Corollaries

The following corollaries, which roughly match Examples 1 and 2, are produced by Theorems 1 and 2.

By putting $\ell = 0$ in Theorem 1, we obtain the following corollary.

Corollary 1 ([37]). *Let $\chi \geq 1$. A bi-univalent function Φ given by (1) belongs to the class $D_\Sigma(q,\chi;\varphi)$ $(\chi \geq 1)$. If $\varrho_m = 0$; $m = 2,\ldots,\Bbbk - 1$. Then*

$$|\varrho_\Bbbk| \leq \frac{2(1-q)}{1-q+(q-q^\Bbbk)\chi} \qquad (n \geq 4).$$

Applying the limit $q \to 1^-$ in Theorem 1 and considering the case when $\ell = 0$, we obtain the following corollary.

Corollary 2 ([37]). *Let $\chi \geq 1$. A bi-univalent function Φ given by (1) belongs to the class $R_\sigma(\chi, \varphi)(\chi \geq 1)$. If $\varrho_m = 0$; $m = 2, \ldots, \Bbbk - 1$. Then*

$$|\varrho_\Bbbk| \leq \frac{2}{1 + \chi(\Bbbk - 1)} \qquad (n \geq 4).$$

For $k = 0$ in Theorem 2, we obtain the following corollary.

Corollary 3 ([37]). *Let $\chi \geq 1$. A bi-univalent function Φ given by (1) belongs to the class $D_\Sigma(q; \chi, \varphi)$. Then*

(1) $|\varrho_2| \leq \frac{2}{1+q\chi}$,

(2) $|\varrho_3| \leq \frac{4}{(1+q\chi)^2} + \frac{2}{1+(q^2+q)\chi}$,

(3) $|2\varrho_2^2 - \varrho_3| \leq \frac{4}{1+(q^2+q)\chi}$.

For $k = 0$ and $q \to 1^-$ in Theorem 2, we obtain the following corollary.

Corollary 4. *A bi-univalent function Φ given by (1) belongs to the class $R_\sigma(\chi, \varphi)(\chi \geq 1)$. Then*

(1) $|\varrho_2| \leq \frac{2}{1+\chi}$,

(2) $|\varrho_3| \leq \frac{4}{(1+3\chi)^2} + \frac{2}{1+2\chi}$.

6. Conclusions

This article investigated a novel subclass of bi-univalent functions, $D^\ell \Sigma q(\chi, \delta, \gamma, \mu; \varphi)$, on the symmetry disk ∇. For functions belonging to each of these three classes of bi-univalent functions, we calculated estimates for the upper bound of the Taylor–Maclaurin coefficients of these functions in the aforementioned subset. By concentrating on the variables employed in our primary findings, several additional novel findings were made.

The study of bi-univalent functions is an important and active area of research in complex analysis and its applications. The investigation of this subclass provides deeper insights into the theory and applications of bi-univalent functions. The results obtained in this article can be generalized in the future using post-quantum calculus and other q-analogs of the fractional derivative operator. Additionally, further analysis can be conducted to explore additional subclasses and their characteristics.

Overall, this article contributes to the ongoing research in the field of complex analysis by providing a detailed study of three important subclasses of bi-univalent functions. Further research can be conducted to investigate more subclasses and their properties to enhance our understanding of the theory and applications of bi-univalent functions.

Author Contributions: Conceptualization, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah); methodology, A.A. (Ala Amourah); validation, A.A. (Ala Amourah), A.A. (Abdullah Alsoboh), O.O., G.M.G. and N.Z.; formal analysis, A.A. (Ala Amourah); investigation, F.M.S., A.A. (Abdullah Alsoboh); writing—original draft preparation, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah); writing—review and editing, A.A. (Ala Amourah) and O.O.; supervision, A.A. (Abdullah Alsoboh) and F.M.S. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work under grant code: (23UQU4320576DSR003).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare that there are no conflict of interest.

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