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Some Stability Results and Existence of Solutions for a Backward Differential Equation with Time Advance via ζ—Caputo Fractional Derivative

Abdellatif Ben Makhlouf [1], **Lassaad Mchiri [2,*]** and **Mohamed Rhaima [3]**

[1] Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Sfax 1171, Tunisia
[2] ENSIIE, University of Evry-Val-d'Essonne, 1 square de la Résistance, CEDEX, 91025 Evry-Courcouronnes, France
[3] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[*] Correspondence: lassaad.mchiri@ensiie.fr

Abstract: In this paper, using a fixed point method, we proved the existence and uniqueness of solutions for a backward differential equation with time advance via $\zeta-$Caputo fractional derivative. Furthermore, the Ulam–Hyers–Rassias and the Ulam–Hyers stabilities of the backward differential equation with time advance via $\zeta-$Caputo fractional derivative are investigated. Finally, some experiments are given to illustrate the theoretical results.

1. Introduction

The study of ordinary differential equations and partial differential equations of fractional order has interested several researchers for a long time [1–4]. Problems of the stability of these equations have been studied in a very large way. For example, the authors in [5,6] studied the finite-time stability, and in [7–10], the authors studied the stability in the sense of Ulam–Hyers–Rassias (UHR) and Ulam–Hyers (UH). Additionally, some problems of controllability and optimal control of these equations have been completed by several researchers [11–15].

In the literature, several types of fractional derivatives have been introduced by some researchers such as [16–18]. These different types of fractional derivatives have greatly contributed to the development and enrichment of many basic sciences such as mathematics, physics, medicine, engineering, stochastics, etc. For example, the authors in [19] studied a fractional model for COVID-19, in [20] the authors studied a fractional order eco-epidemiological system with infected prey, and a fractional differential system in hepatitis B has been investigated in [21].

The existence of a solution and the stability in the sense of Ulam have been studied abundantly by several researchers [7–10,22]. In [2,23–25], and the stability with delay was investigated for different types of fractional derivatives. To the best of our knowledge, the study of stability with time advance for a backward differential equation with time advance via $\zeta-$Caputo fractional derivative has never been investigated. In this paper, we considered a backward differential equation with time advance via $\zeta-$Caputo fractional derivative. In the first time, we proved the existence and uniqueness of solutions by using a fixed point method. Next, we gave two stability results, in the sense of Ulam, for the backward differential equation with time advance. Finally, some numerical experiments

have been presented at the end of the paper to illustrate the theoretical results. Then, we can summarize the novelties of this work as follows:

◇ The presentation a new problem defined by a backward differential equation with time advance via $\zeta-$Caputo fractional derivative.

◇ The study of the existence and uniqueness of solutions for the backward differential equation with time advance via $\zeta-$Caputo fractional derivative by using Banach fixed-point Theorem.

◇ Study of the UHR and UH stabilities for the backward differential equation with time advance via $\zeta-$Caputo fractional derivative.

◇ Numerical implementations.

In Section 2, the main results of the paper are given. Section 4 is devoted to the numerical implementations and discussion of the numerical experiments.

2. Preliminaries and Definitions

In this paper, we consider following notations:

- The space $\mathcal{AC}([a,b];\mathbb{R})$ defined by:

$$\mathcal{AC}([a,b];\mathbb{R}) = \left\{ f : [a,b] \longrightarrow \mathbb{R}, \quad f \text{ is absolutely continuous} \right\}.$$

- The Banach space $\mathcal{C}([a,b];\mathbb{R})$ of continuous functions defined from $[a,b]$ into $\mathbb{R}$.
- The Banach space $\mathcal{D}_r([-r,0];\mathbb{R})$ of continuous functions defined from $[-r,0]$ into $\mathbb{R}$, where $r > 0$.

Let a, b, r be some positive real numbers such that $a < b$ and $r > 0$. We define a metric space $(\mathfrak{E}, \mathbf{d})$, where the space $\mathfrak{E} = \mathcal{C}([a,b+r];\mathbb{R})$ and the metric $\mathbf{d}$ is given by

$$\mathbf{d}(u,v) = \sup_{x \in [a,b+r]} \left\{ \frac{|u(x) - v(x)|}{\sigma(x)\delta(x)} \right\},$$

where the function σ and δ are defined by

$$\sigma(x) = \begin{cases} 1, & \forall \quad x \in [b, b+r], \\ e^{\lambda(\zeta(b) - \zeta(x))}, & \forall \quad x \in [a, b]. \end{cases}$$

$$\delta(x) = \begin{cases} \gamma(b), & \forall \quad x \in [b, b+r], \\ \gamma(x), & \forall \quad x \in [a, b], \end{cases}$$

where γ is a non-increasing continuous positive function and ζ is an increasing continuous function. It is clear that the two functions σ and δ are non-increasing functions on $[a, b]$.

Definition 1 ([26]). *Let $\alpha \in (0,1)$ and $\zeta \in \mathcal{C}^1([a,b])$ be functions such that ζ is increasing and $\zeta'(t) \neq 0$, for all $t \in [a,b]$. The $\zeta-$Caputo fractional derivative of a function $v(t)$ is defined by:*

$$^{C}D_b^{\alpha,\zeta}v(t) = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{1}{\zeta'(t)} \frac{d}{dt} \right) \int_t^b \zeta'(s)(\zeta(s) - \zeta(t))^{-\alpha} v(s)\, ds.$$

Lemma 1. *If $v \in \mathcal{AC}([a,b];\mathbb{R})$, then the $\zeta-$Caputo fractional derivative of the function $v(t)$ is given by*

$$^{C}D_b^{\alpha,\zeta}v(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\zeta(s) - \zeta(t))^{-\alpha} v'(s)\, ds,$$

where $\alpha \in (0,1)$ and $\zeta \in \mathcal{C}^1([a,b])$ is a functions such that ζ is increasing and $\zeta'(t) \neq 0$, for all $t \in [a,b]$.

Theorem 1 (Theorem 2.1 in [9]). *Suppose* (F, d) *is a complete metric space and* $L : F \longrightarrow F$ *is a contraction (with* $\tau \in [0, 1)$*). Suppose that* $v \in F$, $\lambda > 0$ *and* $d(v, L(v)) \leq \lambda$. *So, there exists a unique* $\beta \in F$ *that satisfies* $\beta = L(\beta)$. *Moreover,*

$$d(v, \beta) \leq \frac{\lambda}{1 - \tau}.$$

3. Main Results

Consider the backward differential equation with time advance:

$$^{C}D_{T}^{\alpha,\zeta} \mathbf{u}(t) = f(t, \mathbf{u}^t), \quad \forall t \in [t_0, T], \tag{1}$$

$$\mathbf{u}(t) = \varphi(T - t), \quad \forall t \in [T, T + r], \tag{2}$$

where $^{C}D_{T}^{\alpha,\zeta}(\cdot)$ is the well-known $\zeta-$Caputo fractional derivative with order $\alpha \in (0, 1)$, see [26]. The variable $\mathbf{u}^t$ is defined by

$$\mathbf{u}^t(x) = \mathbf{u}(t - x), \quad \text{for all} \quad x \in [-r, 0], \quad r > 0. \tag{3}$$

Let $\iota = T - t$, for $t \in [T, T + r]$, then $\iota \in [-r, 0]$. Consequently, the initial condition defined by Equation (2) can be rewritten as follows:

$$\mathbf{u}^T(\iota) = \varphi(\iota), \quad \forall \iota \in [-r, 0],$$

where φ is a continuous function belonging to the space $\mathcal{D}_r = \mathcal{C}([-r, 0]; \mathbb{R})$. The second member f in Equation (1) is a continuous function and is defined by

$$f : [t_0, T] \times \mathcal{D}_r \longrightarrow \mathbb{R}.$$

The correspondent integral equation of (1) is given by [26]

$$\mathbf{u}(t) \quad = \quad \mathbf{u}(T) + \frac{1}{\Gamma(\alpha)} \int_t^T \zeta'(s) \Big(\zeta(s) - \zeta(t) \Big)^{\alpha - 1} f(s, \mathbf{u}^s) ds.$$

Let us now consider the following assumption:

$$(\mathcal{H}): \qquad |f(t, \mathbf{v}_1) - f(t, \mathbf{v}_2)| \leq \mathcal{L} \|\mathbf{v}_1 - \mathbf{v}_2\|, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{D}_r \quad \text{and} \quad \forall t \in [t_0, T],$$

where $\mathcal{L}$ is a positive constant.

Definition 2. *The problem* (1)–(2) *is UH stable if there exists a real number* $\lambda_f > 0$ *such that for every* $\varepsilon > 0$ *and for each solution* $\tilde{\mathbf{u}} \in \mathcal{AC}([t_0, T + r]; \mathbb{R})$ *of the following inequality:*

$$\left| {}^{C}D_{T}^{\alpha,\zeta} \tilde{\mathbf{u}}(t) - f(t, \tilde{\mathbf{u}}^t(t)) \right| \leq \varepsilon,$$

there exists a solution $\mathbf{u}$ *of Equation* (1) *with*

$$\mathbf{u}(t) = \tilde{\mathbf{u}}(t), \quad \forall t \in [T, T + r],$$

such that

$$|\tilde{\mathbf{u}}(t) - \mathbf{u}(t)| \leq \lambda_f \varepsilon, \quad t \in [t_0, T].$$

Definition 3. *The problem* (1)–(2) *is UHR stable with respect to* $\gamma \in \mathcal{C}([t_0, T]; \mathbb{R})$, *if there exists a real number* $C_{f,\gamma} > 0$ *such that for every* $\varepsilon > 0$ *and for each solution* $\tilde{\mathbf{u}} \in \mathcal{AC}([t_0, T + r]; \mathbb{R})$ *of the following inequality:*

$$\left| {}^{C}D_{T}^{\alpha,\zeta} \tilde{\mathbf{u}}(t) - f(t, \tilde{\mathbf{u}}^t(t)) \right| \leq \varepsilon \gamma(t), \quad t \in [t_0, T],$$

there exists a solution **u** *of Equation (1) with*

$$\mathbf{u}(t) = \tilde{\mathbf{u}}(t), \quad \forall t \in [T, T+r],$$

$$|\tilde{\mathbf{u}}(t) - \mathbf{u}(t)| \leq C_{f,\gamma} \varepsilon \gamma(t), \quad t \in [t_0, T].$$

Let $M_T^{\alpha,\zeta}$ be the constant defined by

$$M_T^{\alpha,\zeta} = \frac{\lambda^\alpha (\zeta(T) - \zeta(t_0))^\alpha}{(\lambda^\alpha - \mathcal{L})\Gamma(\alpha+1)} e^{\lambda(\zeta(T) - \zeta(t_0))}, \tag{4}$$

where λ is a positive constant such that $\mathcal{L} < \lambda^\alpha$.

We present in the following the first main result, Theorem 2, which expresses the UHR stability for the system (1)–(2).

Theorem 2. *Suppose that the assumption* $(\mathcal{H})$ *holds. If* $y \in \mathcal{AC}([t_0, T+r]; \mathbb{R})$ *satisfies the inequality:*

$$|^C D_T^{\alpha,\zeta} y(t) - f(t, y^t)| \leq \varepsilon \gamma(t), \quad \text{for all} \quad t \in [t_0, T], \tag{5}$$

where $\varepsilon > 0$ *and* γ *is a non-increasing continuous positive function, then there exists a unique solution* $\mathbf{u}^*$ *of (1)–(2) with*

$$\mathbf{u}^*(t) = y(t), \quad \forall t \in [T, T+r],$$

such that

$$|\mathbf{u}^*(t) - y(t)| \leq \varepsilon M_T^{\alpha,\zeta} \gamma(t), \quad \forall t \in [t_0, T],$$

where the constant $M_T^{\alpha,\zeta}$ *is given by the relation (4).*

Remark 1. *Let* $y \in \mathcal{C}([t_0, T]; \mathbb{R})$. *Then, the function* y *is a solution to the inequality (5) if and only if there exists a function* $\theta \in \mathcal{C}([t_0, T]; \mathbb{R})$ *and a function* $\chi \in \mathcal{C}([t_0, T]; \mathbb{R})$:

$$^C D_T^{\alpha,\zeta} y(t) = f(t, y^t) + \theta(t) \quad \text{and} \quad |\theta(t)| \leq \varepsilon \chi(t), \quad \forall t \in [t_0, T].$$

Let $\mathfrak{E} = \mathcal{C}([t_0, T+r]; \mathbb{R})$. Now, we define the operator $\mathfrak{A} : \mathfrak{E} \longrightarrow \mathfrak{E}$ as follows:

$$(\mathfrak{A}\mathbf{u})(t) = \begin{cases} y(t), & \forall t \in [T, T+r], \\[2mm] y(T) + \frac{1}{\Gamma(\alpha)} \int_t^T \zeta'(s) \left(\zeta(s) - \zeta(t) \right)^{\alpha-1} f(s, \mathbf{u}^s) ds, & \forall t \in [t_0, T]. \end{cases}$$

Immediately, we have the following result.

Proposition 1. *The operator* $\mathfrak{A} : \mathfrak{E} \longrightarrow \mathfrak{E}$ *is contractive.*

Proof. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathfrak{E}$. Then, we have

$$(\mathfrak{A}\mathbf{u}_1)(t) - (\mathfrak{A}\mathbf{u}_2)(t) = 0, \quad \forall t \in [T, T+r].$$

For $t \in [t_0, T]$, we obtain

$$\begin{aligned} \left| (\mathfrak{A}\mathbf{u}_1)(t) - (\mathfrak{A}\mathbf{u}_2)(t) \right| &= \left| \frac{1}{\Gamma(\alpha)} \int_t^T \zeta'(s) \left(\zeta(s) - \zeta(t) \right)^{\alpha-1} \left[f(s, \mathbf{u}_1^s) - f(s, \mathbf{u}_2^s) \right] ds \right|, \\ &\leq \frac{\mathcal{L}}{\Gamma(\alpha)} \int_t^T \zeta'(s) \left(\zeta(s) - \zeta(t) \right)^{\alpha-1} \|\mathbf{u}_1^s - \mathbf{u}_2^s\| ds, \end{aligned}$$

where:

$$\|\mathbf{u}_1^s - \mathbf{u}_2^s\| = \sup_{\iota \in [-r,0]} \left(|\mathbf{u}_1(s-\iota) - \mathbf{u}_2(s-\iota)| \right).$$

For $s \in [t, T]$, there is $\iota \in [-r, 0]$ such that

$$
\begin{aligned}
\|\mathbf{u}_1^s - \mathbf{u}_2^s\| &= |\mathbf{u}_1(s-\iota) - \mathbf{u}_2(s-\iota)|, \\
&= \frac{|\mathbf{u}_1(s-\iota) - \mathbf{u}_2(s-\iota)|}{\sigma(s-\iota)\delta(s-\iota)}\sigma(s-\iota)\delta(s-\iota), \\
&\leq \mathbf{d}(\mathbf{u}_1, \mathbf{u}_2)\sigma(s)\delta(s).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left| (\mathfrak{A}\mathbf{u}_1)(t) - (\mathfrak{A}\mathbf{u}_2)(t) \right| &\leq \frac{\mathcal{L}\mathbf{d}(\mathbf{u}_1, \mathbf{u}_2)}{\Gamma(\alpha)} \int_t^T \zeta'(s)\left(\zeta(s) - \zeta(t)\right)^{\alpha-1} \sigma(s)\delta(s)ds, \\
&\leq \frac{\mathcal{L}\mathbf{d}(\mathbf{u}_1, \mathbf{u}_2)}{\Gamma(\alpha)}\delta(t) \int_t^T \zeta'(s)\left(\zeta(s) - \zeta(t)\right)^{\alpha-1} e^{\lambda(\zeta(T)-\zeta(s))}ds.
\end{aligned}
\tag{6}
$$

Let $\rho = \zeta(s)$. Then, $d\rho = \zeta'(s)ds$. Consequently, we obtain

$$
\begin{aligned}
\int_t^T \zeta'(s)\left(\zeta(s) - \zeta(t)\right)^{\alpha-1} e^{\lambda(\zeta(T)-\zeta(s))}ds &= \int_{\zeta(t)}^{\zeta(T)} \left(\rho - \zeta(t)\right)^{\alpha-1} e^{\lambda(\zeta(T)-\rho)}d\rho, \\
&= \int_{\zeta(t)}^{\zeta(T)} \left(\rho - \zeta(t)\right)^{\alpha-1} e^{\lambda(\zeta(T)-\zeta(t))} e^{-\lambda(\rho-\zeta(t))}d\rho, \\
&= e^{\lambda(\zeta(T)-\zeta(t))} \int_{\zeta(t)}^{\zeta(T)} \left(\rho - \zeta(t)\right)^{\alpha-1} e^{-\lambda(\rho-\zeta(t))}d\rho.
\end{aligned}
$$

Let $s = \lambda(\rho - \zeta(t))$. Then, $ds = \lambda d\rho$. Therefore,

$$
\begin{aligned}
\int_t^T \zeta'(s)\left(\zeta(s) - \zeta(t)\right)^{\alpha-1} e^{\lambda(\zeta(T)-\zeta(s))}ds &= e^{\lambda(\zeta(T)-\zeta(t))} \int_0^{\lambda(\zeta(T)-\zeta(t)} \frac{s^{\alpha-1}}{\lambda^\alpha}e^{-s}ds, \\
&\leq \frac{e^{\lambda(\zeta(T)-\zeta(t))}}{\lambda^\alpha}\Gamma(\alpha).
\end{aligned}
\tag{7}
$$

Thus, from relations (6) and (7), we deduce that

$$\left| (\mathfrak{A}\mathbf{u}_1)(t) - (\mathfrak{A}\mathbf{u}_2)(t) \right| \leq \frac{\mathcal{L}}{\lambda^\alpha}\mathbf{d}(\mathbf{u}_1, \mathbf{u}_2)\delta(t)e^{\lambda(\zeta(T)-\zeta(t))} = \frac{\mathcal{L}}{\lambda^\alpha}\mathbf{d}(\mathbf{u}_1, \mathbf{u}_2)\sigma(t)\delta(t), \quad \forall t \in [t_0, T].$$

Therefore, we obtain

$$\mathbf{d}(\mathfrak{A}\mathbf{u}_1, \mathfrak{A}\mathbf{u}_2) \leq \frac{\mathcal{L}}{\lambda^\alpha}\mathbf{d}(\mathbf{u}_1, \mathbf{u}_2).$$

Recall that $\mathcal{L} < \lambda^\alpha$. Thus, the operator $\mathfrak{A}$ is contractive. □

We can now establish the proof of Theorem 2.

Proof of Theorem 2. We have

$$(\mathfrak{A}y)(t) - y(t) = 0, \quad \text{for all} \quad t \in [T, T+r].$$

It follows from (5) that

$$|y(t) - (\mathfrak{A}y)(t)| \leq \frac{\varepsilon}{\Gamma(\alpha)} \int_t^T \gamma(s)\zeta'(s)\left(\zeta(s) - \zeta(t)\right)^{\alpha-1}ds,$$

$$\leq \frac{\varepsilon \gamma(t)}{\Gamma(\alpha)} \int_t^T \zeta'(s) \Big(\zeta(s) - \zeta(t)\Big)^{\alpha-1} ds,$$

$$\leq \frac{\varepsilon \gamma(t)}{\Gamma(\alpha+1)} \Big(\zeta(T) - \zeta(t)\Big)^{\alpha}.$$

Therefore,

$$\mathbf{d}(y, \mathfrak{A}y) \leq \frac{\varepsilon}{\Gamma(\alpha+1)} \Big(\zeta(T) - \zeta(t_0)\Big)^{\alpha}.$$

Using Theorem 2.1 in [9], there exists a unique solution $\mathbf{u}^*$ of (1)–(2), with $\mathbf{u}^*(t) = y(t)$ for all $t \in [T, T+r]$, such that

$$\mathbf{d}(y, \mathbf{u}^*) \leq \frac{\varepsilon}{\Gamma(\alpha+1)} \Big(\zeta(T) - \zeta(t_0)\Big)^{\alpha} \frac{1}{1 - \frac{\mathcal{L}}{\lambda^{\alpha}}},$$

$$\leq \frac{\lambda^{\alpha} \Big(\zeta(T) - \zeta(t_0)\Big)^{\alpha}}{(\lambda^{\alpha} - \mathcal{L})\Gamma(\alpha+1)} \varepsilon.$$

Therefore,

$$|y(t) - \mathbf{u}^*(t)| \leq \frac{\lambda^{\alpha} \Big(\zeta(T) - \zeta(t_0)\Big)^{\alpha}}{(\lambda^{\alpha} - \mathcal{L})\Gamma(\alpha+1)} e^{\lambda(\zeta(T) - \zeta(t_0))} \gamma(t)\varepsilon = \varepsilon M_T^{\alpha,\zeta} \gamma(t), \quad \forall t \in [t_0, T].$$

$\square$

The second main result of this paper is given by the following Corollary 1, which expresses the UH stability for the system (1)–(2).

Corollary 1. *Suppose that the assumption $(\mathcal{H})$ holds. If $y \in \mathcal{AC}([t_0, T+r]; \mathbb{R})$ satisfies the identity*

$$|{}^C D_T^{\alpha,\zeta} y(t) - f(t, y^t)| \leq \varepsilon, \quad \text{for all} \quad t \in [t_0, T], \tag{8}$$

where $\varepsilon > 0$, then there exists a unique solution $\mathbf{u}^*$ of (1)–(2) with

$$\mathbf{u}^*(t) = y(t), \quad \forall t \in [T, T+r],$$

such that

$$|\mathbf{u}^*(t) - y(t)| \leq M_T^{\alpha,\zeta} \varepsilon, \quad \forall t \in [t_0, T], \tag{9}$$

where the constant $M_T^{\alpha,\zeta}$ is given by the relation (4).

Proof. The proof of Corollary 1 can be deduced from that of Theorem 2, where the considered metric function $\mathbf{d}$, in this case, is defined by

$$\mathbf{d}(u, v) = \sup_{x \in [t_0, T+r]} \left\{ \frac{|u(x) - v(x)|}{\beta(x)} \right\},$$

where the positive function β is given by

$$\beta(x) = \begin{cases} 1 & , \quad \forall \quad x \in [T, T+r], \\ e^{\lambda(\zeta(T) - \zeta(x))}, & \forall \quad x \in [t_0, T]. \end{cases}$$

$\square$

4. Numerical Illustration

In this section, we consider the case when $f(s, \mathbf{u}^s)$ is written in the form $g(s, \mathbf{u}(s + r))$. Consider the following integral equation:

$$\mathbf{u}(t) \;=\; \mathbf{u}(T) + \frac{1}{\Gamma(\alpha)} \int_t^T \zeta'(s) \Big(\zeta(s) - \zeta(t) \Big)^{\alpha-1} g(s, \mathbf{u}(s + r)) ds. \tag{10}$$

We divide the interval $[t_0, T]$ into N sub-intervals $[t_i, t_{i+1}]$, for $i = 0, \cdots, N - 1$, of equal amplitude h, where

$$t_i = t_0 + ih, \quad i = 0, \cdots, N \quad \text{and} \quad h = \frac{T - t_0}{N}.$$

Then, it is clear that

$$
\begin{aligned}
T &= t_N, \\
h &= t_{i+1} - t_i, \quad 0, \cdots, N - 1
\end{aligned}
$$

At the grid point t_i, for $i = N - 1, \cdots, 0$, Equation (10) takes the form

$$
\begin{aligned}
\mathbf{u}(t_i) &= \mathbf{u}(t_N) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^T \zeta'(s) \Big(\zeta(s) - \zeta(t_i) \Big)^{\alpha-1} g(s, \mathbf{u}(s + r)) ds, \\
&= \mathbf{u}(t_N) + \frac{1}{\Gamma(\alpha)} \sum_{k=i}^{N-1} \int_{t_k}^{t_{k+1}} \zeta'(s) \Big(\zeta(s) - \zeta(t_i) \Big)^{\alpha-1} g(s, \mathbf{u}(s + r)) ds, \\
&\approx \mathbf{u}(t_N) + \frac{1}{\Gamma(\alpha)} \sum_{k=i}^{N-1} g(t_k, \mathbf{u}(t_k + r)) \int_{t_k}^{t_{k+1}} \zeta'(s) \Big(\zeta(s) - \zeta(t_i) \Big)^{\alpha-1} ds.
\end{aligned}
\tag{11}
$$

By integrating the integral in the right side of Equation (11), we obtain

$$\mathbf{u}(t_i) = \mathbf{u}(t_N) + \frac{1}{\Gamma(1 + \alpha)} \sum_{k=i}^{N-1} g(t_k, \mathbf{u}(t_k + r)) \omega_k^i, \tag{12}$$

where the coefficients ω_k^{i+1} are given by

$$\omega_k^i = (\zeta(t_{k+1}) - \zeta(t_i))^\alpha - (\zeta(t_k) - \zeta(t_i))^\alpha, \quad k = i, \cdots, N - 1, \quad i = N - 1, \cdots, 0.$$

Now, let us consider the following notation:

$$\mathbf{u}_i = \mathbf{u}(t_i), \quad i = 0, \cdots, N.$$

Assume that there exists a positive integer $p > 0$ such that $r = ph$. The coefficient $\mathbf{u}(t_k + r)$ can be rewritten as

$$\mathbf{u}(t_k + r) = \mathbf{u}(t_0 + kh + ph) = \mathbf{u}(t_0 + (k + p)h) = \mathbf{u}(t_{k+p}) \approx \mathbf{u}_{k+p}.$$

Therefore, Equation (12) can be rewritten as

$$\mathbf{u}_i = \mathbf{u}_N + \frac{1}{\Gamma(1 + \alpha)} \sum_{k=i}^{N-1} g(t_k, \mathbf{u}_{k+p}) \omega_k^i, \quad i = N - 1, \cdots, 0. \tag{13}$$

Example 1. *Let $[t_0, T] = [0, 1]$, $r = 0$. Consider the system:*

$$
\begin{aligned}
{}^C D_T^{\alpha, \zeta} \mathbf{u}(t) &= 5\mathbf{u}(t) + F(t), \quad \forall t \in [t_0, T], \tag{14} \\
\mathbf{u}(T) &= \mathbf{u}_T. \tag{15}
\end{aligned}
$$

whose exact solution is $\mathbf{u}(t) = 1 + t^2$. The initial condition $\mathbf{u}(T) = \mathbf{u}(1) = 2$ and $\alpha = 0.75$, $\zeta(t) = t$. The source term F is given by

$$F(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 5(1+t^2).$$

The system (14)–(15) is solved using the scheme (13) by the software Matlab 7.5.0 (R2007b). Figure 1 show the convergence of the numerical solution to the exact solution.
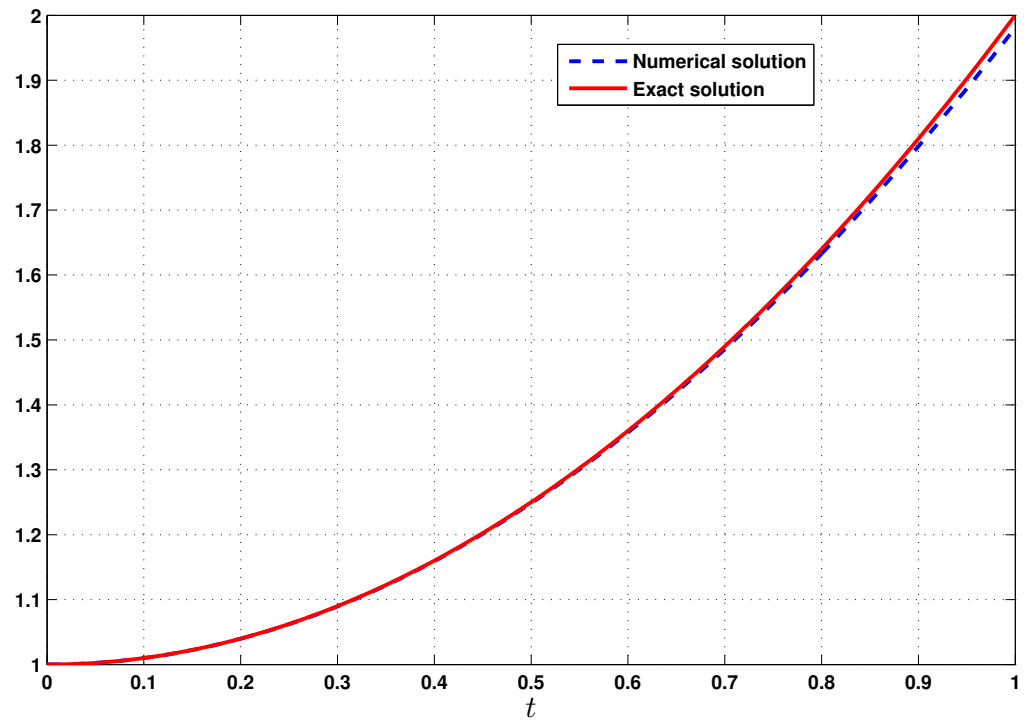

Figure 1. The exact solution and the numerical solution for a time step $h = 10^{-3}$.

Example 2. *This example is devoted to the UHR stability.*
The data used in this experiment are chosen as follows:

$$[t_0, T] = [1, e], \quad [-r, 0] = [-\frac{1}{10}(e-1), 0], \quad \lambda = 0.2, \quad \alpha = 0.75.$$

Let $\zeta(t) = \ln(t)$. In this case, we obtain ${}^C D_T^{\alpha,\zeta}\mathbf{u}(t) = {}^{CH} D_e^{0.75}\mathbf{u}(t)$, where ${}^{CH} D_e^{0.75}\mathbf{u}(t)$ is the Caputo–Hadamard fractional derivative of $\mathbf{u}(t)$ (see [26]).
Consider the Caputo–Hadamard fractional problem:

$$^{CH}D_e^{0.75}\mathbf{u}(t) = f(t, \mathbf{u}^t), \quad \forall t \in [1, e], \tag{16}$$

where $f(t, \mathbf{u}^t) = \frac{1}{10}\cos(t)\mathbf{u}(t+r)$. Note that the function f satisfies the assumption $(\mathcal{H})$:

$$|f(t, \mathbf{u}) - f(t, \mathbf{v})| \leq \frac{1}{10}\|\mathbf{u} - \mathbf{v}\|, \quad \forall t \in [1, e].$$

Now, let us define the fractional system:

$$\begin{aligned}
^{CH}D_e^{0.75}y(t) &= f(t, y^t) + \theta(t), \quad \forall t \in [1, e], \\
y(t) &= 0.6\cos(2.5\pi(e-t))e^{-3(e-t)}, \quad \forall t \in [e, e+r],
\end{aligned}$$

where $\theta(t) = \frac{1}{30t^2+1}$. Since, we have

$$|\theta(t)| = \frac{1}{30t^2+1} \le \frac{1}{30}\frac{1}{t^2}, \quad \forall t \in [1,e],$$

then, from Remark 1, $y \in \mathcal{C}([1,e];\mathbb{R})$ is a solution to the inequality (5), with $\varepsilon = \frac{1}{30}$ and $\gamma(t) = \frac{1}{t^2}$:

$$|^{CH}D_e^{0.75}y(t) - f(t,y^t)| \le \frac{1}{30}\frac{1}{t^2}, \quad \forall t \in [1,e].$$

Therefore, from Theorem 2, we deduce that Equation (16) has a unique solution $\mathbf{u}^*$ such that

$$\mathbf{u}^*(t) = y(t), \quad \forall t \in [e,e+r],$$

and

$$|\mathbf{u}^*(t) - y(t)| \le \frac{1}{30}\frac{1}{t^2}M_e^{0.75,\zeta}, \quad \forall t \in [1,e],$$

where the constant $M_e^{0.75,\zeta}$ is given by

$$
\begin{aligned}
M_e^{0.75,\zeta} &= \frac{\lambda^\alpha(\zeta(T)-\zeta(t_0))^\alpha}{(\lambda^\alpha - \mathcal{L})\Gamma(\alpha+1)}e^{\lambda(\zeta(T)-\zeta(t_0))}, \\
&= \frac{0.2^{0.75}(\ln(e)-\ln(1))^{0.75}}{(0.2^{0.75}-0.1)\Gamma(0.75+1)}e^{0.2(\ln(e)-\ln(1))} \approx 1.9966.
\end{aligned}
$$

So, we obtain

$$|\mathbf{u}^*(t) - y(t)| \le 0.06655\frac{1}{t^2}, \quad \forall t \in [1,e].$$

In Figure 2, we plotted the solution $\mathbf{u}^*$ of Equation (16) for $t \in [1,e]$ and the corresponding initial condition $\mathbf{u}(t) = y(t) = \varphi(e-t)$, for $t \in [e,e+r]$. In Figure 3, we plotted the solution $\mathbf{u}^*$ of Equation (16) and the solution $y(t)$ of the inequality (5), for $t \in [1,e]$, with the same initial condition $\mathbf{u}(t) = y(t) = \varphi(e-t)$, for $t \in [e,e+r]$. The difference $|\mathbf{u}^*(t) - y(t)|$ and the curve of the function $\varepsilon\gamma(t)M_e^{0.75,\zeta}$, for $t \in [1,e]$, are plotted in Figure 4. In fact, it is clear that there is consistency between the UHR stability result and the numerical experiment given in Figure 4.
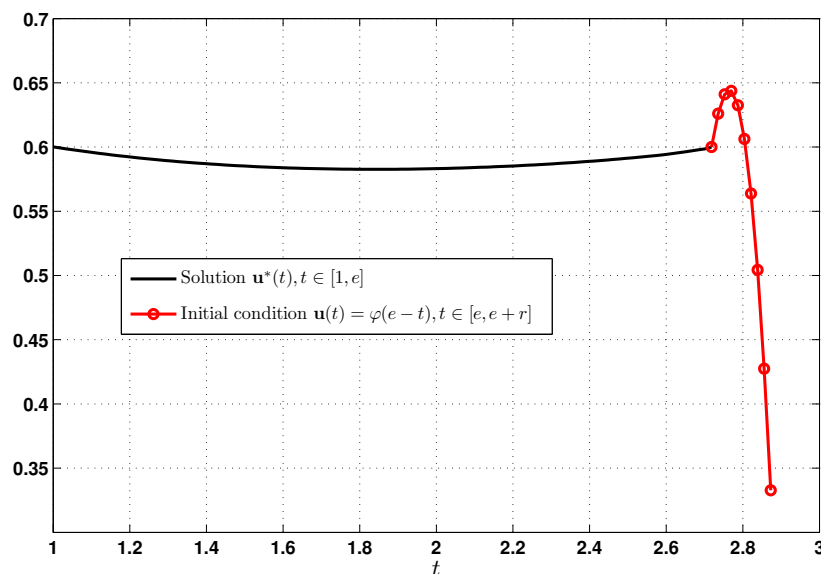

Figure 2. The numerical solution $\mathbf{u}^*(t)$ for $t \in [1,e]$, with the initial condition $\varphi(e-t)$, for $t \in [e,e+r]$.
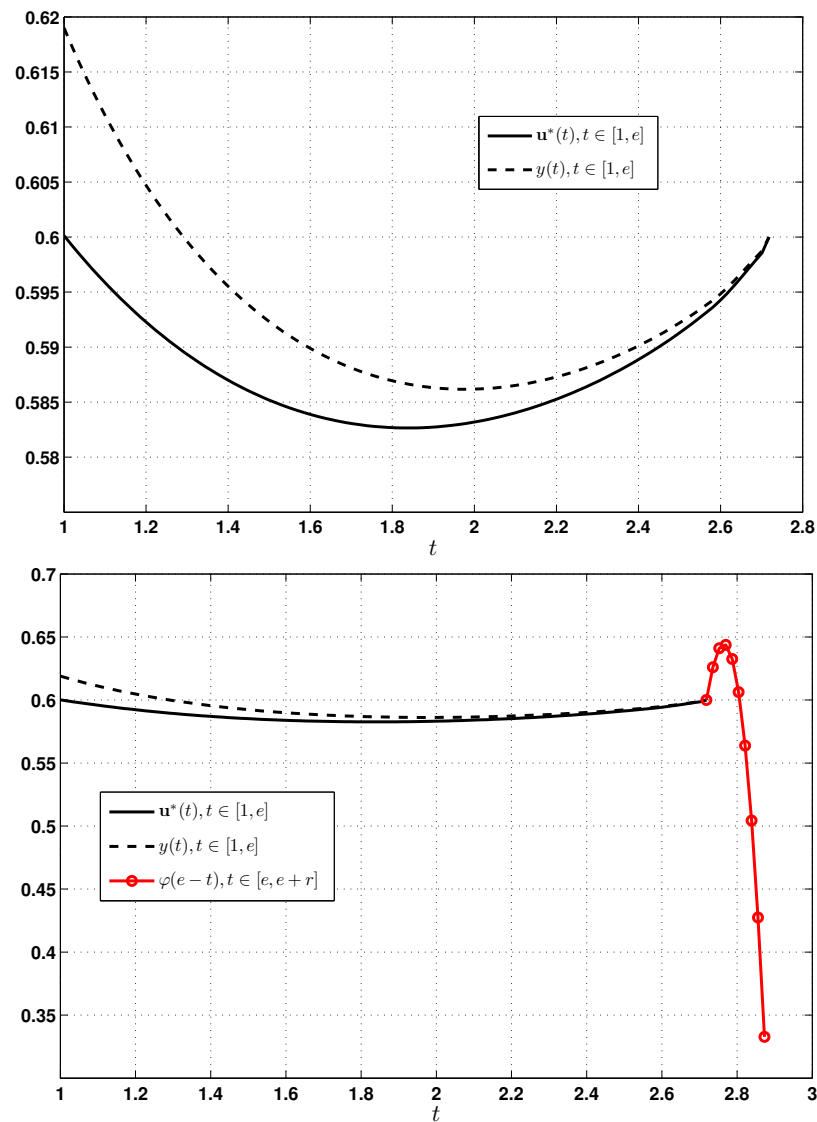
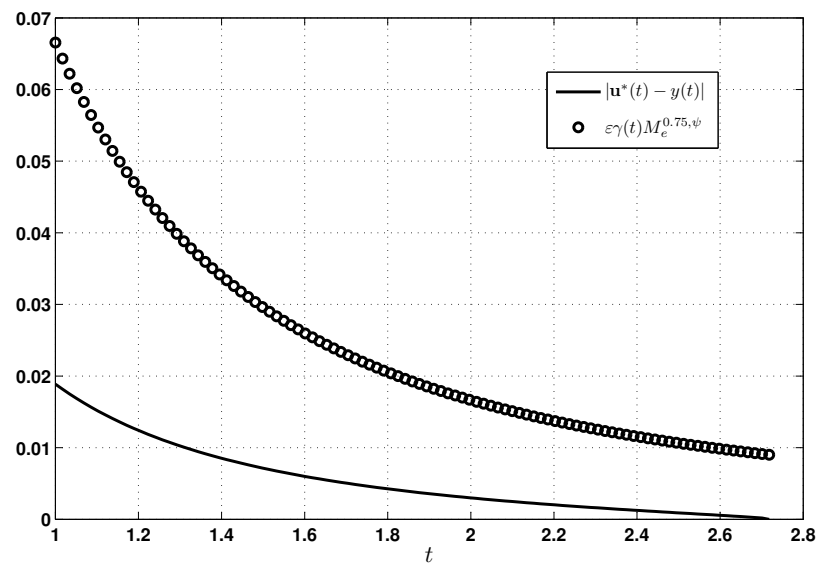**Figure 3.** The numerical solution $\mathbf{u}^*(t)$ and the numerical solution $y(t)$, for $t \in [1, e]$.

Figure 4. The difference $|\mathbf{u}^*(t) - y(t)|$ and the curve of the function $\varepsilon\gamma(t)M_e^{0.75,\zeta}$, for $t \in [1, e]$.

Example 3. *This example is devoted to the UH stability.*
The data used in this experiment are chosen as follows:

$$[t_0, T] = [1, 2e], \ [-r, 0] = [-\frac{1}{5}(2e-1), 0], \ \lambda = 0.2, \ \alpha = 0.7.$$

Let $\zeta(t) = \sqrt{t^2 + 1}$. Consider the system defined by $\zeta-$Caputo fractional derivative:

$$^C D_{2e}^{0.7,\zeta} \mathbf{u}(t) \ = \ f(t, \mathbf{u}^t), \quad \forall t \in [1, 2e], \tag{17}$$

where $f(t, \mathbf{u}^t) = 0.3 \sin(5t) \mathbf{u}(\mathbf{t} + \mathbf{r})$. Note that the function f satisfies the assumption $(\mathcal{H})$:

$$|f(t, \mathbf{u}) - f(t, \mathbf{v})| \le 0.3 \|\mathbf{u} - \mathbf{v}\|, \quad \forall t \in [1, 2e].$$

Now, let us define the fractional system:

$$\begin{aligned} ^C D_{2e}^{0.7,\zeta} y(t) \ &= \ f(t, y^t) + \varepsilon, \quad \forall t \in [1, 2e], \\ y(t) \ &= \ 6 \cos(12\pi(2e - t)) \sin(13\pi(2e - t)), \quad \forall t \in [2e, 2e + r], \end{aligned}$$

where $\varepsilon = 0.01$. If $y \in \mathcal{C}([1, 2e]; \mathbb{R})$ is a solution to the inequality

$$|^C D_{2e}^{0.7,\zeta} y(t) - f(t, y^t)| \le \varepsilon, \quad \forall t \in [1, 2e],$$

then, from Corollary 1, we deduce that Equation (17) has a unique solution $\mathbf{u}^*$ such that

$$\mathbf{u}^*(t) = y(t), \quad \forall t \in [2e, 2e + r],$$

and

$$|\mathbf{u}^*(t) - y(t)| \le \varepsilon M_{2e}^{0.7,\zeta}, \quad \forall t \in [1, 2e],$$

where the constant $M_{2e}^{0.7,\zeta}$ is given by:

$$\begin{aligned} M_{2e}^{0.7,\zeta} \ &= \ \frac{\lambda^\alpha (\zeta(T) - \zeta(t_0))^\alpha}{(\lambda^\alpha - \mathcal{L})\Gamma(\alpha + 1)} e^{\lambda(\zeta(T) - \zeta(t_0))}, \\ &= \ \frac{0.2^{0.7} (\zeta(2e) - \zeta(1))^{0.7}}{(0.2^{0.7} - 0.3)\Gamma(0.7 + 1)} e^{0.2(\zeta(2e) - \zeta(1))} \approx 90.5732. \end{aligned}$$

So, we obtain

$$|\mathbf{u}^*(t) - y(t)| \le 0.905732, \quad \forall t \in [1, 2e].$$

In Figure 5, the solution $\mathbf{u}^*$ of Equation (17) for $t \in [1, 2e]$ and the corresponding initial condition $\mathbf{u}(t) = y(t) = \varphi(2e - t)$, for $t \in [2e, 2e + r]$ are plotted. In Figure 6, we plotted the solution $\mathbf{u}^*$ of Equation (17) and the solution $y(t)$ of the inequality (8), for $t \in [1, 2e]$, with the same initial condition $\mathbf{u}(t) = y(t) = \varphi(2e - t)$, for $t \in [2e, 2e + r]$. The difference $|\mathbf{u}^*(t) - y(t)|$ and the horizontal line $\xi(t) = \varepsilon M_{2e}^{0.7,\zeta}$, for $t \in [1, 2e]$, are plotted in Figure 7. Again, it is clear that there is consistency between the UH stability result and the numerical experiment given in Figure 7.
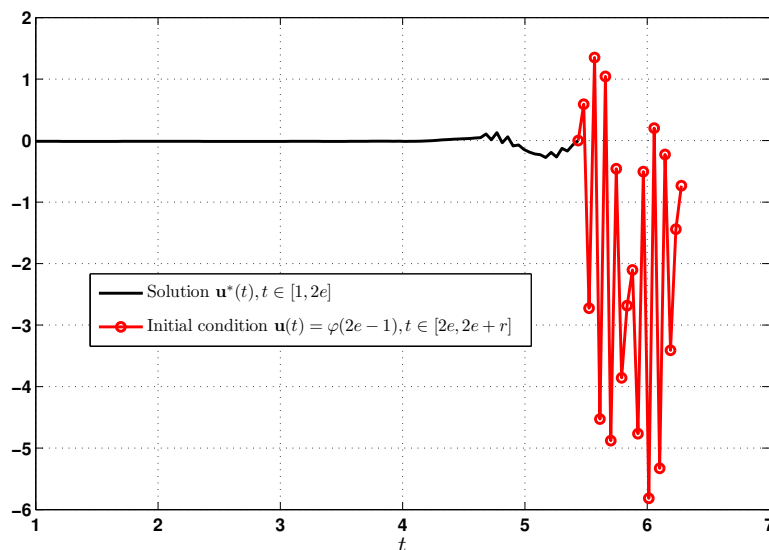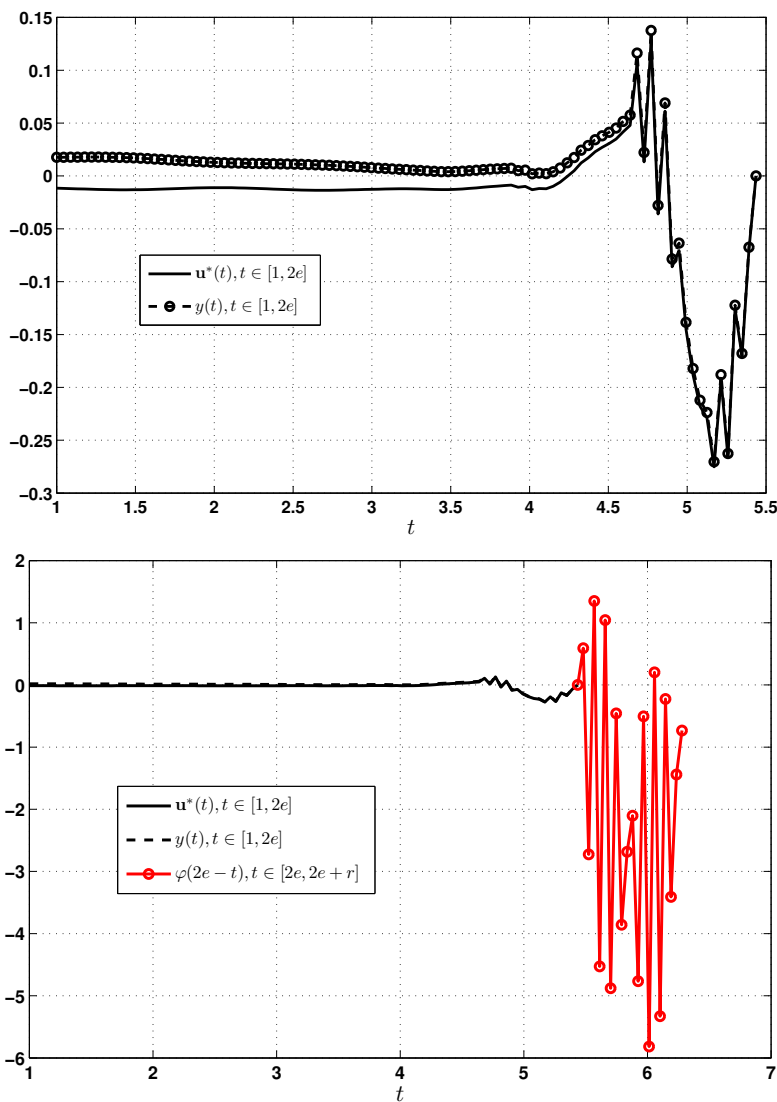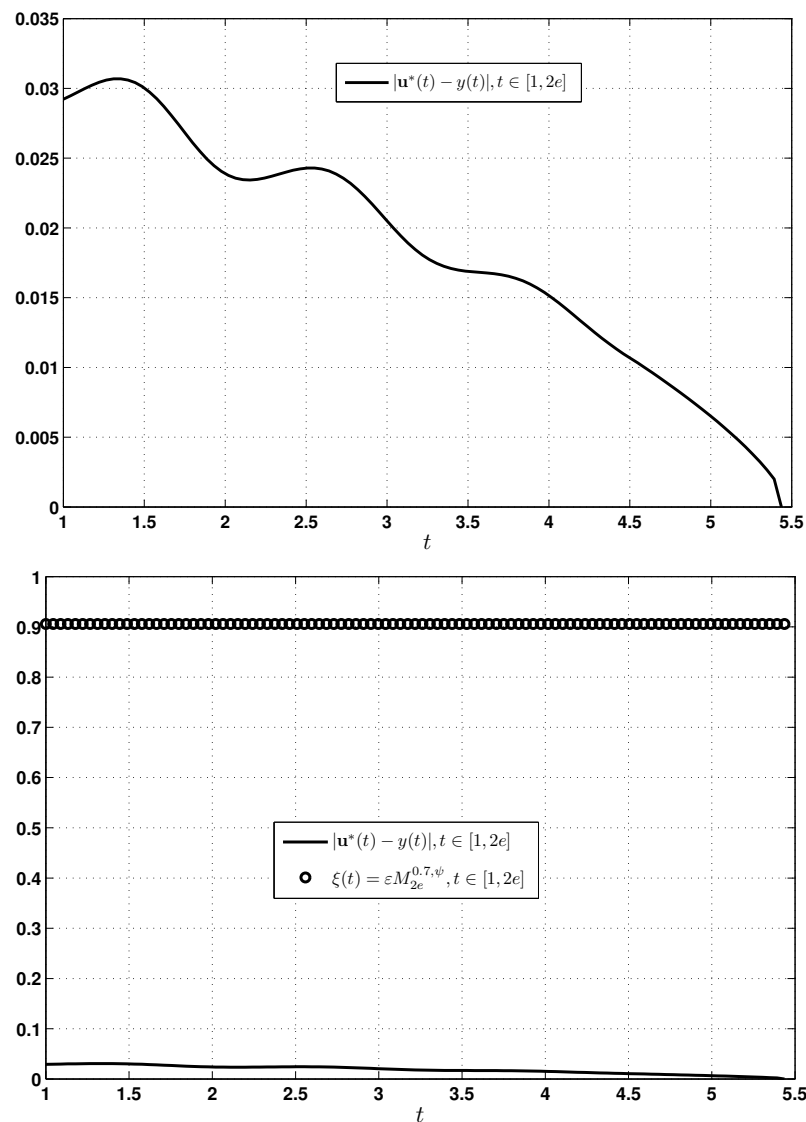
Figure 5. We plotted the solution $\mathbf{u}^*$ of Equation (17) for $t \in [1, 2e]$, and the corresponding initial condition $\mathbf{u}(t) = y(t) = \varphi(2e - t)$, for $t \in [2e, 2e + r]$.

Figure 6. We plotted the solution $\mathbf{u}^*$ of Equation (17) and the solution $y(t)$ of the inequality (5), for $t \in [1, 2e]$, with the same initial condition $\mathbf{u}(t) = y(t) = \varphi(2e - t)$, for $t \in [2e, 2e + r]$.

Figure 7. We plotted both the difference $|\mathbf{u}^*(t) - y(t)|$ and the horizontal line $\xi(t) = \varepsilon M_{2e}^{0.7,\zeta}$, for $t \in [1, 2e]$.

5. Conclusions

In this work, several goals are achieved. We have proved the existence and the uniqueness of solutions for the problem defined by backward differential equations with time advance via $\zeta-$Caputo fractional derivative. Moreover, two stability results, in the sense of UHR and UH, have been established. Finally, we presented numerical results to confirm the theoretical results obtained.

In future work, it would be interesting to study the finite-time stability for this type of backward differential equation with time advance with other types of fractional derivatives.

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