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Faber Polynomial Coefficient Estimates for Bi-Close-to-Convex Functions Defined by the q-Fractional Derivative

Hari Mohan Srivastava [1,2,3,4,5], Isra Al-Shbeil [6,*], Qin Xin [7], Fairouz Tchier [8], Shahid Khan [9] and Sarfraz Nawaz Malik [10]

1 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
2 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
3 Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
4 Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan
5 Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
6 Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
7 Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100 Torshavn, Faroe Islands, Denmark; qinx@setur.fo
8 Mathematics Department, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia; ftchier@ksu.edu.sa
9 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22500, Pakistan; shahidmath761@gmail.com
10 Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt 47040, Pakistan; snmalik110@ciitwah.edu.pk or snmalik110@yahoo.com
* Correspondence: i.shbeil@ju.edu.jo

Abstract: By utilizing the concept of the q-fractional derivative operator and bi-close-to-convex functions, we define a new subclass of $\mathcal{A}$, where the class $\mathcal{A}$ contains normalized analytic functions in the open unit disk $\mathbb{E}$ and is invariant or symmetric under rotation. First, using the Faber polynomial expansion (FPE) technique, we determine the lth coefficient bound for the functions contained within this class. We provide a further explanation for the first few coefficients of bi-close-to-convex functions defined by the q-fractional derivative. We also emphasize upon a few well-known outcomes of the major findings in this article.

Keywords: quantum (or q-) calculus; analytic functions; q-derivative operator; bi-univalent functions; Faber polynomial expansions

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1. Introduction, Definitions and Motivation

Alexander [1] established the first integral operator in 1915, which he successfully applied in the investigation of analytical functions. This area of study of analytic functions, encompassing derivative and fractional derivative operators, has been a focus of ongoing research in geometric function theory of complex analysis. Several combinations of such operators are continually being developed [2,3]. Recent publications such as [4] provide an example of how important differential and integral fractional operators are in research. Recent research on differential and integral operators from a variety of angles, including quantum (or q-) calculus, has produced intriguing findings that have applications in other branches of physics and mathematics. Further investigation may reveal that such operators play a role in providing solutions to partial differential equations, since they have a role in the investigation of differential equations using functional analysis and operator theory. In his survey-cum-expository review study, Srivastava [5] highlights the intriguing operator applications that are emerging from such a methodology.

Many applications of the q-calculus can be found in both the field of mathematics and in other scientific disciplines such as numerical analysis, fractional calculus, special polynomials, analytic number theory and quantum group theory. Mathematicians and physicists are becoming interested in the large field of fractional calculus. The theory of analytical functions has been integrated with the theory of fractional calculus. The fractional differential equations are used in numerous mathematical models. In fact, nonlinear differential equations are considered to be a rival to fractional differential equations as a model (see, for example, Refs. [6–9]).

Researchers, who have created and examined a significant number of new analytic function subclasses in the field of geometric function theory (GFT), have extensively used the q-calculus. In the year 1909, Jackson [10,11] is to be acknowledged for the formal beginning of q-calculus because he provided the first definitions of the q-integrals and the q-derivatives. He proposed the q-calculus operator and the q-difference operator (D_q), which are extensions of the derivative and integral operators. Several mathematical and scientific disciplines, including mechanics, the theory of relativity, control theory, basic hypergeometric functions, combinatorics, number theory, and statistics, use the q-calculus. Ismail et al. [12] established the generalized version of the starlike functions, which was one of the very first contributions of the use of q-calculus in GFT. They gave this newly created class the name "class of q-starlike functions" because they defined it by using q-derivatives. It took a while for this area of research to advance, but the recent works of Anastassiu and Gal [13,14] based upon their complex operators research with their separate q-generalizations happen to provide a fine addition. Those were termed as q-Gauss–Weierstrass and q-Picard singular integral operators, respectively (see also the work of Mason [15] on the solution of linear q-difference equations with entire-function coefficients). By utilizing fundamental q-hypergeometric functions, Srivastava [5] built a solid foundation for the use of the q-calculus in GFT. Aral and Gupta [16–18] provided a further set of contributions by using q-beta functions. Aldweby et al. [19,20] established the q-analogue of certain operators by utilizing the convolution techniques for analytic functions. Additionally, they explored the composition of q-operators in the context of analytic functions that involve the q-version of hypergeometric functions. The subject of q-calculus has drawn the interest of several researchers in recent years, and the papers [21–23] contain a variety of new observations. Further current details on convex and starlike functions with regard to their symmetric points can be found in [24,25] and the references therein. As a consequence of ongoing research on differential and integral operators, we in this study present a novel fractional differential operator. With the aid of this operator, we intend to introduce a new family of analytic functions which are geometrically close-to-convex.

The class $\mathcal{A}$ contains all functions h which are analytic in $\mathbb{E}$ and which also satisfy the normalization condition given by

$$h(0) = 0 \quad \text{and} \quad h'(0) = 1,$$

where

$$\mathbb{E} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

$\mathbb{C}$ being the set of complex numbers. Thus, clearly, each function $h \in \mathcal{A}$ can be expressed as follows:

$$h(z) = \sum_{l=1}^{\infty} a_l z^l \qquad (z \in \mathbb{E};\ a_1 := 1). \tag{1}$$

Let the class $\mathcal{S} \subset \mathcal{A}$ consist of univalent functions in $\mathbb{E}$. The commonly known subclasses of $\mathcal{S}$ are the classes of convex, starlike and close-to-convex functions, which are denoted by and defined, respectively, as follows:

$$\mathcal{C} := \left\{ h : h \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{\left(zh'(z)\right)'}{h'(z)} \right) > 0 \right\} \qquad (z \in \mathbb{E}),$$

$$\mathcal{S}^* := \left\{ h : h \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zh'(z)}{h(z)}\right) > 0 \right\} \qquad (z \in \mathbb{E})$$

and

$$\mathcal{K} := \left\{ h : h \in \mathcal{S},\ g \in \mathcal{S}^* \quad \text{and} \quad \Re\left(\frac{zh'(z)}{g(z)}\right) > 0 \right\} \qquad (z \in \mathbb{E})$$

or, equivalently,

$$\mathcal{K} := \left\{ h : h \in \mathcal{A},\ g \in \mathcal{C} \quad \text{and} \quad \Re\left(\frac{h'(z)}{g'(z)}\right) > 0 \right\} \qquad (z \in \mathbb{E})$$

For $h_1, h_2 \in \mathcal{A}$, h_1 is said to be subordinate to h_2 in $\mathbb{E}$, denoted by

$$h_1(z) \prec h_2(z) \qquad (z \in \mathbb{E}),$$

if we have a Schwarz function ℓ in $\mathbb{E}$ such that $\ell \in \mathcal{A}$, $|\ell(z)| < 1$ and $\ell(0) = 0$, and

$$h_1(z) = h_2\big(\ell(z)\big) \qquad (z \in \mathbb{E}).$$

The image of $\mathbb{E}$ under every $h \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$ and each function $h \in \mathcal{S}$ has an inverse $h^{-1} = \gamma$ given by

$$\gamma\big(h(z)\big) = z \qquad (z \in \mathbb{E})$$

and

$$h\big(\gamma(\vartheta)\big) = \vartheta \qquad \big(|\vartheta| < r_0(h)\big),$$

where $r_0(h)$ is the radius of the disk with $r_0(h) \geqq \frac{1}{4}$. The inverse function $\gamma(\vartheta)$ has the following series expansion:

$$\gamma(\vartheta) = \vartheta - a_2\vartheta^2 + (2a_2^2 - a_3)\vartheta^3 - (5a_2^3 - 5a_2a_3 + a_4)\vartheta^4 + \cdots. \tag{2}$$

If both h and h^{-1} are in the univalent function class $\mathcal{S}$, then the function h is called bi-univalent in $\mathbb{E}$. The set of bi-univalent functions in $\mathbb{E}$ is denoted by Σ. In GFT, the issue of finding bounds on the coefficients has always been important. Many characteristics of analytic functions, such as univalency, rate of growth and distortion, can be affected by the size of their coefficients. The pioneering work, which actually revived the study of analytic and bi-univalent functions, was presented in the year 2010 by Srivastava et al. [26]. In 1914, for $0 \leqq \alpha < 1$, Hamidi and Jahangiri [27] defined the class of bi-close-to-convex functions and investigated some useful results by using the Faber polynomial expansion technique. To overcome some the aforementioned problems, several researchers employed various other techniques. Finding coefficient estimates of functions belonging to Σ had already attracted a lot of interest, just like for univalent functions. For $h \in \Sigma$, Levin [28] demonstrated that $|a_2| < 1.51$ and after that, Branan and Clunie [29] contributed the improvement of $|a_2|$ and demonstrated that $|a_2| \leqq \sqrt{2}$. Furthermore, for $h \in \Sigma$, Netanyahu [30] proved that (see, for details, Refs. [31,32])

$$\max|a_2| = \frac{4}{3}.$$

In many of these efforts, only non-sharp estimates of the initial coefficients were derived. In [33], Alharbi et al. investigated two new subclasses of Σ by using the Sălăgean-Erdélyi-Kober operator and problems related to coefficients, such as the Fekete-Szegä problem, were also investigated. Recently, Oros et al. [34] defined some new subfamilies of bi-univalent functions and found the coefficient estimates for these subfamilies.

Our current work is primarily driven by the discovery of numerous intriguing and productive applications of special polynomials in GFT. One of these is the well-known Faber polynomial that has recently gained immense importance in the study of mathematics and other scientific disciplines. Faber [35] introduced Faber polynomials and these polynomials have important uses in many areas of mathematics, especially in GFT of Complex Analysis. Schiffer [36] discussed the applications of the Faber polynomials in 1948 (see also [37]). Following that, Pommerenke [38–40] significantly added to the facts that were already known about the structure of the Faber polynomial expansion (FPE). By using the FPE technique and defining subclasses of the bi-univalent function class Σ, Hamidi and Jahangiri [27,41] found some new coefficient bounds. Furthermore, many authors (see, for example, Refs. [42–51]) applied the technique of Faber polynomials and determined some interesting results for bi-univalent functions (see, for details, Ref. [44]).

For understanding the concepts of this article, it is now necessary to review certain fundamental definitions and notions relevant to the q-calculus.

Definition 1. *The q-shifted factorial $(\varkappa;q)_l$ is presented as*

$$(\varkappa;q)_l = \prod_{j=0}^{l-1}\left(1-\varkappa q^j\right) \qquad (l \in \mathbb{N};\ \varkappa, q \in \mathbb{C}), \tag{3}$$

where, as usual, $\mathbb{C}$ is the set of complex numbers. If $\varkappa \neq q^{-m}$ $(m \in \mathbb{N}_0 := \{0,1,2,3,\cdots\})$, then

$$(\varkappa;q)_\infty = \prod_{j=0}^{\infty}\left(1-\varkappa q^j\right) \qquad (\varkappa \in \mathbb{C};\ |q|<1). \tag{4}$$

In the case when $\varkappa \neq 0$ and $q \geqq 1$, $(\varkappa;q)_\infty$ diverges. Therefore, when we take $(\varkappa;q)_\infty$, then we will assume that $|q|<1$.

Remark 1. *For $q \to 1-$ in $(\varkappa;q)_l$, we have*

$$(\varkappa;q)_l = (\varkappa)_l = \prod_{j=0}^{l-1}(\varkappa+j) \qquad (l \in \mathbb{N}).$$

The q-factorial $[l]_q!$ is defined by

$$[n]_q! = \prod_{l=1}^{n}[l]_q \qquad (l \in \mathbb{N}), \tag{5}$$

where the q-number $[l]_q$ is given below:

$$[l]_q = \frac{1-q^l}{1-q} \qquad (l \in \mathbb{N}).$$

If $l=0$, then $[l]_q! = 1$.

Definition 2. *The $(\varkappa;q)_l$ in (3) can be given more precisely in the form of the q-gamma function as follows:*

$$\Gamma_q(\varkappa) = \frac{(1-q)^{1-\alpha}(q;q)_\infty}{(q^a;q)_\infty} \qquad (0<q<1),$$

or

$$(q^\varkappa;q)_l = \frac{\left(1-q^l\right)\Gamma_q(\varkappa+l)}{\Gamma_q(\varkappa)} \qquad (l \in \mathbb{N}).$$

Definition 3 (Jackson [10])**.** *For $h \in \mathcal{A}$, the q-difference operator is defined by*

$$D_q h(z) = \frac{h(z)-h(qz)}{z(1-q)} \qquad (z \in \mathbb{E}).$$

We recall that for $l \in \mathbb{N}$ and $z \in \mathbb{E}$, we have

$$D_q(z^l) = [l]_q z^{l-1} \qquad \text{and} \qquad D_q\left(\sum_{l=1}^{\infty} a_l z^l\right) = \sum_{l=1}^{\infty} [l]_q a_l z^{l-1},$$

where the q-number $[l]_q$ is already given along with (5).

The q-generalized Pochhammer symbol is expressed as follows:

$$[\varkappa]_{q,l} = \frac{\Gamma_q(\varkappa + l)}{\Gamma_q(\varkappa)} \qquad (l \in \mathbb{N};\ \varkappa \in \mathbb{C}).$$

Remark 2. *If $q \to 1-$, then*

$$[\varkappa]_{q,l} = (\varkappa)_l = \frac{\Gamma(\varkappa + l)}{\Gamma(\varkappa)}.$$

Definition 4 (see [52])**.** *For $\varrho > 0$, the fractional q-integral operator is defined by*

$$I_{q,z}^{\varrho} h(z) = \frac{1}{\Gamma_q(\varrho)} \int_0^z (z - tq)_{\varrho-1}\, h(t)\, d_q(t), \tag{6}$$

where $(z - tq)_{\varrho-1}$ is given by

$$(z - tq)_{\varrho-1} = z^{\varrho-1}{}_1\Phi_0\left(q^{-\varrho+1}; —; q, \frac{tq^{\varrho}}{z}\right).$$

The representation of the q-binomial series ${}_1\Phi_0$ is given by

$${}_1\Phi_0(a; —; q, z) = 1 + \sum_{l=1}^{\infty} \frac{(a,q)_l}{(q,q)_l} z^l \qquad (|q| < 1;\ |z| < 1).$$

Definition 5 (see, for example, [53,54])**.** *For an analytic function h, the fractional q-derivative operator $\mathfrak{D}_q$ of order ϱ is described by*

$$\begin{aligned} \mathfrak{D}_q h(z) &= D_q I_{q,z}^{1-\varrho} h(z) \\ &= \frac{1}{\Gamma_q(1-\varrho)} D_q \int_0^z (z - tq)_{-\varrho} h(t) d_q(t) \qquad (0 \leqq \varrho < 1). \end{aligned}$$

In Geometric Function Theory, linear operators (both derivative and integral operators) are extensively utilized. The most important aspect of this study is that we are simultaneously examining the characteristics of many classes of analytic functions under a certain linear operator. Taking the aforementioned importance of linear operators into consideration, we now define the operator below.

Definition 6. *The extended fractional q-derivative $\mathfrak{D}_q^{\varrho}$ of order ϱ is specified as follows:*

$$\mathfrak{D}_q^{\varrho} h(z) = D_q^m I_{q,z}^{m-\varrho} h(z), \tag{7}$$

where m is assumed to be the smallest integer. We find from (7) that

$$\mathfrak{D}_q^{\varrho} z^l = \frac{\Gamma_q(l+1)}{\Gamma_q(l+1-\varrho)} z^{l-\varrho} \qquad (0 \leqq \varrho;\ l > -1).$$

Remark 3. *For $-\infty < \varrho < 0$, $\mathfrak{D}_q^{\varrho}$ denotes a fractional q-integral of h of order ϱ. Additionally, for $0 \leqq \varrho < 2$, $\mathfrak{D}_q^{\varrho}$ denotes a q-derivative of h of order ϱ.*

Definition 7. *Following the work of Selvakumaran et al. [55], we introduce the (ϱ, q)-differintegral operator $\mathbf{\Omega}_q^{\varrho} : \mathcal{A} \to \mathcal{A}$, which they defined as follows:*

$$\mathbf{\Omega}_q^{\varrho} h(z) = \frac{\Gamma_q(2-\varrho)}{\Gamma_q(2)} z^{\varrho} \mathfrak{D}_q^{\varrho} h(z)$$
$$= z + \sum_{l=2}^{\infty} \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} a_l z^l \qquad (z \in \mathbb{E}), \tag{8}$$

where $0 \leqq \varrho < 2$ *and* $0 < q < 1$.

Each of the following properties of the (ϱ, q)-differintegral operator $\mathbf{\Omega}_q^{\varrho} h$ are worthy of note.

Property 1.
$$\lim_{\varrho \to 1} \mathbf{\Omega}_q^{\varrho} h(z) = \mathbf{\Omega}_q^{\varrho} h(z) = z D_q h(z).$$

Property 2.
$$\mathbf{\Omega}_q^{\varrho}\left(\Omega_q^{\delta} h(z)\right) = \Omega_q^{\delta}\left(\mathbf{\Omega}_q^{\varrho} h(z)\right) = z + \sum_{l=2}^{\infty} \frac{\Gamma_q(2-\varrho)\Gamma_q(2-\delta)\left(\Gamma_q(l+1)\right)^2}{\Gamma_q(2)\Gamma_q(l+1-\varrho)\Gamma_q(l+1-\delta)} a_l z^l.$$

Property 3.
$$\frac{D_q\left(\mathbf{\Omega}_q^{\varrho} h(z)\right)}{\mathbf{\Omega}_q^{\varrho} h(z)} = \begin{cases} \dfrac{z D_q h(z)}{h(z)} & (\varrho = 0) \\[2ex] 1 + z \frac{D_q\left(D_q h(z)\right)}{D_q h(z)} & (\varrho = 1). \end{cases}$$

Considering the operator $\mathbf{\Omega}_q^{\varrho}$ defined in Definition 7 and inspired by the work given in [27], a new subclass of the class Σ is introduced by means of this operator. The next section will provide proofs of the original findings by using the Faber polynomial method and one lemma.

Definition 8. *Let the function h be of the form (1). Then, h is referred to as ϱ-fractional bi-close-to-convex function in $\mathbb{E}$ if a suitable function $g \in \mathcal{S}^*$ exists such that*

$$\Re\left(\frac{D_q\left(\mathbf{\Omega}_q^{\varrho} h(z)\right)}{g(z)}\right) > \alpha$$

and

$$\Re\left(\frac{D_q\left(\mathbf{\Omega}_q^{\varrho} \alpha(\vartheta)\right)}{\delta(\vartheta)}\right) > \alpha,$$

where $0 \leqq \alpha < 1,\ 0 \leqq \varrho < 2$ *and* $z, \vartheta \in \mathbb{E}$*. All such functions are symbolized by* $\mathcal{K}_{\Sigma}(q, \alpha, \varrho)$.

Remark 4. *If we let $q \to 1-$ and $\varrho = 0$, then $\mathcal{K}_{\Sigma}(q, \alpha, \varrho)$ reduces to the class introduced by Hamidi and Jahangiri in [27].*

Remark 5. *If $q \to 1-$ and $\alpha = 0$, then $\mathcal{K}_{\Sigma}(q, \alpha, \varrho)$ reduces to the class introduced by Sakar and Güney in [56].*

2. The Faber Polynomial Expansion Method and Its Applications

The coefficients of the inverse mapping $\gamma = h^{-1}$ can be expressed by using the Faber polynomial method for analytic functions h and as follows (see [43,57]):

$$\gamma(\vartheta) = h^{-1}(\vartheta) = \vartheta + \sum_{l=2}^{\infty} \frac{1}{l} \mathfrak{q}_{l-1}^{l}(a_2, a_3, \cdots, a_l)\vartheta^l,$$

where

$$\mathfrak{q}_{l-1}^{-l} = \frac{(-l)!}{(-2l+1)!(l-1)!}a_2^{l-1} + \frac{(-l)!}{[2(-l+1)]!(l-3)!}a_2^{l-3}a_3$$

$$+ \frac{(-l)!}{(-2l+3)!(l-4)!}a_2^{l-4}a_4$$

$$+ \frac{(-l)!}{[2(-l+2)]!(l-5)!}a_2^{l-5}\left[a_5 + (-l+2)a_3^2\right]$$

$$+ \frac{(-l)!}{(-2l+5)!(l-6)!}a_2^{l-6}[a_6 + (-2l+5)a_3a_4]$$

$$+ \sum_{\mathfrak{i} \geqq 7} a_2^{l-\mathfrak{i}} S_{\mathfrak{i}}$$

and a homogeneous polynomial in $a_2, a_3, \cdots, a_l$ is denoted by $S_{\mathfrak{i}}$ for $7 \leqq \mathfrak{i} \leqq l$. Especially, the first three terms of $\mathfrak{q}_{l-1}^{-l}$ are given below:

$$\frac{1}{2}\mathfrak{q}_1^{-2} = -a_2,$$

$$\frac{1}{3}\mathfrak{q}_2^{-3} = 2a_2^2 - a_3$$

and

$$\frac{1}{4}\mathfrak{q}_3^{-4} + -(5a_2^3 - 5a_2a_3 + a_4).$$

Generally, an extension of $\mathfrak{q}_l^r$ of the following type is used for $r \in \mathbb{Z}$ $(\mathbb{Z} := 0, \pm1, \pm2, \cdots)$ and $l \geqq 2$:

$$\mathfrak{q}_l^r = ra_l + \frac{r(r-1)}{2}\mathcal{V}_l^2 + \frac{r!}{(r-3)!3!}\mathcal{V}_l^3 + \cdots + \frac{r!}{(r-l)!\,(l)!}\mathcal{V}_l^l,$$

where

$$\mathcal{V}_l^r = \mathcal{V}_l^r(a_2, a_3, \cdots)$$

and, by using [57], we have

$$\mathcal{V}_l^v(a_2, \cdots, a_l) = \sum_{l=1}^{\infty} \frac{v!(a_2)^{\mu_1} \cdots (a_l)^{\mu_l}}{\mu_1!, \cdots, \mu_l!} \qquad (a_1 = 1;\ v \leqq l).$$

Clearly, upon adding all non-negative integers $\mu_1, \cdots, \mu_l$, which satisfy

$$\mu_1 + \mu_2 + \cdots + \mu_l = v \quad \text{and} \quad \mu_1 + 2\mu_2 + \cdots + l\mu_l = l,$$

we find that

$$\mathcal{V}_l^l(a_1, \cdots, a_l) = \mathcal{V}_1^l$$

and that the first and last polynomials are given by

$$\mathcal{V}_l^l = a_1^l \quad \text{and} \quad \mathcal{V}_l^1 = a_l.$$

Lemma 1 (see [58]). *If p is a function with a positive real part and*

$$p(z) = 1 + \sum_{l=1}^{\infty} c_l z^l,$$

then

$$|c_l| \leqq 2.$$

The problem of finding bounds for the coefficients has always been a key concern in geometric function theory. The size of their coefficients can determine a number of properties of analytic functions, including univalency, rate of growth and distortion. Many scholars have used a variety of methods to overcome the aforementioned issues. Similar to univalent functions, bi-univalent function coefficient estimation has received a lot of interest lately. As a result of the significance of studying the coefficient problems described above, in this section, we utilize the $(varrho, q)$-fractional derivative operator and the Fabor polynomial technique to obtain coefficient estimates for $|a_l|$ and discuss the unpredictable behavior of the initial coefficient bounds for $|a_2|$ and $|a_3|$. We also investigate the Fekete–Szegö problem and give some examples. We also demonstrate how some of the previously published results would be improved and generalized as a result of our primary findings as well as their corollaries and consequences.

3. Main Results

Our first main result is asserted by Theorem 1 below.

Theorem 1. *If h has the series representation stated in (1) and belongs to the class $\mathcal{K}_\Sigma(q, \alpha, \varrho)$, and if $a_\mathfrak{i} = 0$ and $2 \leqq \mathfrak{i} \leqq l - 1$, then*

$$|a_l| \leqq \frac{\Gamma_q(2)\Gamma_q(l+1-\varrho)(2(1-\alpha)+l)}{[l]_q\Gamma_q(2-\varrho)\Gamma_q(l+1)} \qquad (l \geqq 3).$$

Proof. For $h \in \mathcal{K}_\Sigma(q, \alpha, \varrho)$, there exists a function g. The FPE for $\frac{D_q\left(\mathbf{\Omega}_q^\varrho h(z)\right)}{g(z)}$ is given by

$$\frac{D_q\left(\mathbf{\Omega}_q^\varrho h(z)\right)}{g(z)} = 1 + \sum_{l=2}^{\infty} \left[\begin{array}{c} \left([l]_q \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} a_l - b_l\right) \sum\limits_{l=1}^{l-2} \mathfrak{q}_l^{-1}(b_2, b_3, \cdots, b_{l+1}) \\ \cdot\left(\left([l]_q - l\right) \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} a_{l-l} - b_{l-l}\right) \end{array} \right] z^{l-1}. \quad (9)$$

Additionally, regarding the inverse maps $\gamma = h^{-1}$ and $\delta = g^{-1}$, we obtain

$$\frac{D_q\left(\mathbf{\Omega}_q^\varrho \alpha(\vartheta)\right)}{\delta(\vartheta)} = 1 + \sum_{l=2}^{\infty} \left[\begin{array}{c} \left([l]_q \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} A_l - B_l\right) \sum\limits_{l=1}^{l-2} \mathfrak{q}_l^{-1}(B_2, B_3, \cdots, B_{l+1}) \\ \cdot\left(\left([l]_q - l\right) \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} A_{l-l} - B_{l-l}\right) \end{array} \right] \vartheta^{l-1}. \quad (10)$$

As opposed to that, since

$$\Re\left(\frac{D_q\left(\mathbf{\Omega}_q^\varrho h(z)\right)}{g(z)}\right) > \alpha \qquad (z \in \mathbb{E}),$$

there must exist a function $p(z)$ given by

$$p(z) = 1 + \sum_{l=1}^{\infty} c_l z^l$$

such that

$$\begin{aligned} \frac{D_q\left(\mathbf{\Omega}_q^{\varrho} h(z)\right)}{g(z)} &= 1 + (1-\alpha)p(z) \\ &= 1 + (1-\alpha)\sum_{l=1}^{\infty} c_l z^l. \end{aligned} \tag{11}$$

Similarly, since

$$\Re\left(\frac{D_q\left(\mathbf{\Omega}_q^{\varrho}\gamma(\vartheta)\right)}{\delta(\vartheta)}\right) > \alpha \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{E}),$$

there must exist a function $\mathfrak{r}$ given by

$$\mathfrak{r}(\vartheta) = 1 + \sum_{l=1}^{\infty} d_l \vartheta^l$$

such that

$$\begin{aligned} \frac{D_q\left(\mathbf{\Omega}_q^{\varrho}\gamma(\vartheta)\right)}{\delta(\vartheta)} &= 1 + (1-\alpha)q(\vartheta) \\ &= 1 + (1-\alpha)\sum_{l=1}^{\infty} d_l \vartheta^l. \end{aligned} \tag{12}$$

For each $l \geqq 2$, evaluating the coefficients of the Equations (9) and (11), we obtain

$$\left\{ \begin{array}{c} \left([l]_q \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} a_l - b_l\right) \sum\limits_{l=1}^{l-2} \mathfrak{q}_l^{-1}(b_2, b_3, \cdots, b_{l+1}) \\ \cdot\left(([l]_q - l)\frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} a_{l-l} - b_{l-l}\right) \end{array} \right\} = (1-\alpha)c_{l-1}. \tag{13}$$

Additionally, by evaluating the coefficients of the Equations (10) and (12), for any $l \geqq 2$, we have

$$\left\{ \begin{array}{c} \left([l]_q \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} A_l - B_l\right) \sum\limits_{l=1}^{l-2} \mathfrak{q}_l^{-1}(B_2, B_3, \cdots, B_{l+1}) \\ \\ \cdot\left(([l]_q - l)\frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)} A_{l-l} - B_{l-l}\right) \end{array} \right\} = (1-\alpha)d_{l-1}. \tag{14}$$

Using the Equations (13) and (14), we derive the following for the particular case when $l = 2$:

$$\frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)} a_2 - b_2 = (1-\alpha)c_1$$
$$\frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)} A_2 - B_2 = (1-\alpha)d_1$$

and

$$a_2 = \frac{\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}((1-\alpha)c_1 + b_2)$$
$$A_2 = \frac{\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}((1-\alpha)d_1 + B_2).$$

We now solve for a_l and apply Lemma 1 and the moduli, so that

$$|a_2| \leqq \frac{2[\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}(2-\alpha).$$

However, assuming that $2 \leqq k \leqq l-1$ and $a_k = 0$ are true, the following results are obtained.

$$A_l = -a_l$$

and

$$\frac{[l]_q\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)}a_l - b_l = (1-\alpha)c_{l-1},$$
$$-\frac{[l]_q\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)}a_l - B_l = (1-\alpha)d_{l-1}$$

and

$$a_l = \frac{\Gamma_q(2)\Gamma_q(l+1-\varrho)}{[l]_q\Gamma_q(2-\varrho)\Gamma_q(l+1)}((1-\alpha)c_{l-1} + b_l)$$
$$-a_l = \frac{\Gamma_q(2)\Gamma_q(l+1-\varrho)}{[l]_q\Gamma_q(2-\varrho)\Gamma_q(l+1)}((1-\alpha)d_{l-1} + B_l)$$

By solving for a_l and using Lemma 1 and the moduli, we can derive

$$|a_l| \leqq \frac{\Gamma_q(2)\Gamma_q(l+1-\varrho)(2(1-\alpha)+l)}{[l]_q\Gamma_q(2-\varrho)\Gamma_q(l+1)},$$

upon noticing that

$$|b_l| \leqq l \text{ and } |B_l| \leqq l.$$

This completes the proof of Theorem 1. □

The following corollaries can be obtained by putting different values of the parameters involved.

Corollary 1. *If the function h has the series representation stated in* (1) *and belongs to the class* $\mathcal{K}_\Sigma(q,0,1)$, *and if*

$$a_i = 0 \quad (2 \leqq i \leqq l-1),$$

then

$$|a_l| \leqq \frac{\Gamma_q(2)\Gamma_q(l)(2+l)}{[l]_q\Gamma_q(l+1)} \qquad (l \geqq 3).$$

Corollary 2. *If the function h has the series representation stated in* (1) *and belongs to* $\mathcal{K}_\Sigma(q,\alpha,1)$, *and if* $a_i = 0 \quad (2 \leqq i \leqq l-1)$, *then*

$$|a_l| \leqq \frac{\Gamma_q(2)\Gamma_q(l)(2(1-\alpha)+l)}{[l]_q\Gamma_q(l+1)} \qquad (l \geqq 3).$$

Corollary 3. *If the function h has the series representation stated in* (1) *and belongs to the class* $\mathcal{K}_\Sigma(q \to 1-, \alpha, \varrho)$, *and if* $a_i = 0 \;\; (2 \leqq i \leqq l-1)$, *then*

$$|a_l| \leqq \frac{\Gamma(2)\Gamma(l+1-\varrho)(2(1-\alpha)+l)}{l\Gamma(2-\varrho)\Gamma(l+1)} \qquad (l \geqq 3).$$

Corollary 4. *If the h has the series representation stated in* (1) *and belongs to the class* $\mathcal{K}_\Sigma(q \to 1-, \alpha, 1)$, *and if* $a_i = 0 \;\; (2 \leqq i \leqq l-1)$, *then*

$$|a_l| \leqq \frac{\Gamma(2)\Gamma(l)(2(1-\alpha)+l)}{l\Gamma(l+1)} \qquad (l \geqq 3).$$

The following known consequence of Theorem 1 for $\varrho = 0$ and $q \to 1-$ was demonstrated in [27].

Corollary 5 (see [27]). *Let* $h \in \mathcal{K}_\Sigma(\alpha)$. *If* $a_{i+1} = 0 \;\; (1 \leqq i \leqq l)$, *then*

$$|a_l| \leqq 1 + \frac{2(1-\alpha)}{l} \qquad (l \geqq 3).$$

Corollary 6 (see [56]). *If the function h has the series representation stated in* (1) *and belongs to the class* $\mathcal{K}_\Sigma(q \to 1-, 0, \varrho)$, *and if* $a_i = 0 \;\; (2 \leqq i \leqq l-1)$, *then*

$$|a_l| \leqq \frac{(2+l)\Gamma(l+1-\varrho)}{l\Gamma(2-\varrho)\Gamma(l+1)} \qquad (l \geqq 3).$$

As a special form of Theorem 1, our next result (Theorem 2 below) provides estimates for the initial coefficients $|a_2|$ and $|a_3|$, and also for the Fekete–Szegö-type functional involved in $|a_3 - a_2^2|$ for functions in the class $\mathcal{K}_\Sigma(m, \alpha, q)$.

Theorem 2. *Let the function* $h \in \mathcal{K}_\Sigma(q, \alpha, \varrho)$ *be given by* (1). *Then,*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2\Gamma_q(2)\Gamma_q(3-\varrho)\Gamma_q(4-\varrho)(1-\alpha)}{\Gamma_q(2-\varrho)\Big([3]_q\Gamma_q(4)\Gamma_q(3-\varrho) - [2]_q\Gamma_q(3)\Gamma_q(4-\varrho)\Big)}} \\ (0 \leqq \alpha < 1 - \phi(q,\varrho)) \\ \\ \dfrac{2\Gamma_q(2)\Gamma_q(3-\varrho)(1-\alpha)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3) - \Gamma_q(2)\Gamma_q(3-\varrho)} \\ (1 - \phi(q,\varrho) \leqq \alpha < 1) \end{cases}$$

and

$$|a_3| \leqq \frac{2\Gamma_q(2)\Gamma_q(4-\varrho)(1-\alpha)}{[3]_q\Gamma_q(4)\Gamma_q(2-\varrho) - \Gamma_q(2)\Gamma_q(4-\varrho)} \cdot \frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3) - \Gamma_q(2)\Gamma_q(3-\varrho) + 2(1-\alpha)\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3) - \Gamma_q(2)\Gamma_q(3-\varrho)},$$

where

$$\phi(q,\varrho) := \frac{\Gamma_q(2)\Gamma_q(4-\varrho)\{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3) - \Gamma_q(2)\Gamma_q(3-\varrho)\}^2}{2\Gamma_q(3-\varrho)\Gamma_q(2)([3]_q\Gamma_q(4)\Gamma_q(2-\varrho)\Gamma_q(3-\varrho) - [2]_q\Gamma_q(3)\Gamma_q(2-\varrho)\Gamma_q(4-\varrho))}.$$

Furthermore, it is asserted that

$$\left|a_3 - a_2^2\right| \leqq \frac{2\Gamma_q(2)\Gamma_q(4-\varrho)(1-\alpha)}{[3]_q\Gamma_q(2-\varrho)\Gamma_q(4) - \Gamma_q(2)\Gamma_q(4-\varrho)}.$$

Proof. Taking a function $g(z) = \mathbf{\Omega}_q^{\varrho} h(z)$ in the proof of Theorem 1, we obtain $a_l = -b_l$. For $l = 2$, the Equations (13) and (14), respectively, yield

$$a_2\left(\frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)} - 1\right) = (1-\alpha)c_1,$$
$$a_2\left(\frac{-[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)} + 1\right) = (1-\alpha)d_1;$$

and

$$a_2 = \frac{\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3) - \Gamma_q(2)\Gamma_q(3-\varrho)}(1-\alpha)c_1,$$
$$-a_2 = \frac{\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3) - \Gamma_q(2)\Gamma_q(3-\varrho)}(1-\alpha)d_1.$$

If we use moduli of either of these two equations, we obtain

$$|a_2| \leqq \frac{2\Gamma_q(2)\Gamma_q(3-\varrho)(1-\alpha)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3) - \Gamma_q(2)\Gamma_q(3-\varrho)}.$$

For $l = 3$, the Equations (13) and (14), respectively, yield

$$\left(\frac{[3]_q\Gamma_q(2-\varrho)\Gamma_q(4)}{\Gamma_q(2)\Gamma_q(4-\varrho)} - 1\right)a_3 - \left(\frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)} - 1\right)a_2^2 = (1-\alpha)c_2 \tag{15}$$

and

$$\left(2a_2^2 - a_3\right)\left(\frac{[3]_q\Gamma_q(2-\varrho)\Gamma_q(4)}{\Gamma_q(2)\Gamma_q(4-\varrho)} - 1\right) - \left(\frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)} - 1\right)a_2^2 = (1-\alpha)d_2. \tag{16}$$

By combining the two equations mentioned above, we obtain

$$2a_2^2\left(\frac{[3]_q\Gamma_q(2-\varrho)\Gamma_q(4)}{\Gamma_q(2)\Gamma_q(4-\varrho)} - 1\right) - 2\left(\frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)} - 1\right)a_2^2 = (1-\alpha)(c_2+d_2),$$
$$2a_2^2\left(\frac{[3]_q\Gamma_q(2-\varrho)\Gamma_q(4)}{\Gamma_q(2)\Gamma_q(4-\varrho)} - \frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)}\right) = (1-\alpha)(c_2+d_2)$$

or

$$2a_2^2 = \frac{[3]_q\Gamma_q(3-\varrho)\Gamma_q(2-\varrho)\Gamma_q(4) - [2]_q\Gamma_q(4-\varrho)\Gamma_q(2-\varrho)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\varrho)\Gamma_q(4-\varrho)} = (1-\alpha)(c_2+d_2)$$

Now, by finding $|a_2|$, we arrive at

$$\left|a_2^2\right| = \frac{\Gamma_q(2)\Gamma_q(3-\varrho)\Gamma_q(4-\varrho)(1-\alpha)|d_2+c_2|}{2\Gamma_q(2-\varrho)\left\{[3]_q\Gamma_q(4)\Gamma_q(3-\varrho) - [2]_q\Gamma_q(3)\Gamma_q(4-\varrho)\right\}}.$$

Additionally, by applying Lemma 1, we obtain

$$|a_2| \leqq \sqrt{\frac{2\Gamma_q(2)\Gamma_q(3-\varrho)\Gamma_q(4-\varrho)(1-\alpha)}{\Gamma_q(2-\varrho)\left([3]_q\Gamma_q(4)\Gamma_q(3-\varrho) - [2]_q\Gamma_q(3)\Gamma_q(4-\varrho)\right)}}.$$

As a result, we obtain the following estimate:

$$\sqrt{\frac{2\Gamma_q(2)\Gamma_q(3-\varrho)\Gamma_q(4-\varrho)(1-\alpha)}{\Gamma_q(2-\varrho)\Big([3]_q\Gamma_q(4)\Gamma_q(3-\varrho)-[2]_q\Gamma_q(3)\Gamma_q(4-\varrho)\Big)}} < \frac{2\Gamma_q(2)\Gamma_q(3-\varrho)\Gamma_q(4-\varrho)(1-\alpha)}{\Gamma_q(2-\varrho)\Big([3]_q\Gamma_q(4)\Gamma_q(3-\varrho)-[2]_q\Gamma_q(3)\Gamma_q(4-\varrho)\Big)}.$$

Upon substituting

$$a_2=\frac{c_1(1-\alpha)\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)-\Gamma_q(2)\Gamma_q(3-\varrho)}$$

into (15), we have

$$a_3=\frac{\Gamma_q(2)\Gamma_q(4-\varrho)(1-\alpha)}{[3]_q\Gamma_q(4)\Gamma_q(2-\varrho)-\Gamma_q(2)\Gamma_q(4-\varrho)} \cdot\left(c_2+\frac{(1-\alpha)\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)-\Gamma_q(2)\Gamma_q(3-\varrho)}c_1^2\right).$$

Taking the moduli on both sides, we find that

$$|a_3|\leqq\frac{\Gamma_q(2)\Gamma_q(4-\varrho)(1-\alpha)}{[3]_q\Gamma_q(4)\Gamma_q(2-\varrho)-\Gamma_q(2)\Gamma_q(4-\varrho)} \cdot\left(|c_2|+\frac{(1-\alpha)\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)-\Gamma_q(2)\Gamma_q(3-\varrho)}\left|c_1^2\right|\right).$$

Applying Lemma 1, we obtain

$$|a_3|\leqq\frac{\Gamma_q(2)\Gamma_q(4-\varrho)(1-\alpha)}{[3]_q\Gamma_q(4)\Gamma_q(2-\varrho)-\Gamma_q(2)\Gamma_q(4-\varrho)} \cdot\left(2+\frac{4(1-\alpha)\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)-\Gamma_q(2)\Gamma_q(3-\varrho)}\right),$$

that is,

$$|a_3|\leqq\frac{2\Gamma_q(2)\Gamma_q(4-\varrho)(1-\alpha)}{[3]_q\Gamma_q(4)\Gamma_q(2-\varrho)-\Gamma_q(2)\Gamma_q(4-\varrho)} \cdot\left(\frac{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)-\Gamma_q(2)\Gamma_q(3-\varrho)+2(1-\alpha)\Gamma_q(2)\Gamma_q(3-\varrho)}{[2]_q\Gamma_q(2-\varrho)\Gamma_q(3)-\Gamma_q(2)\Gamma_q(3-\varrho)}\right).$$

Lastly, upon subtracting Equation (15) from Equation (16), we have

$$\left|a_3-a_2^2\right|\leqq\frac{2\Gamma_q(2)\Gamma_q(4-\varrho)(1-\alpha)}{[3]_q\Gamma_q(2-\varrho)\Gamma_q(4)-\Gamma_q(2)\Gamma_q(4-\varrho)}.$$

Our proof of Theorem 2 is thus completed. □

Several corollaries and consequences of Theorem 2 are presented below.

Corollary 7. *Let the function $h \in \mathcal{K}_\Sigma(q, \alpha, 1)$ be given by (1). Then,*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2\Gamma_q(2)\Gamma_q(2)\Gamma_q(3)(1-\alpha)}{\left([3]_q\Gamma_q(4)\Gamma_q(2) - [2]_q\Gamma_q(3)\Gamma_q(3)\right)}} \\ \left(0 \leqq \alpha < 1 - \varphi_1(q)\right) \\ \\ \dfrac{2\Gamma_q(2)\Gamma_q(2)(1-\alpha)}{[2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2)} \\ \left(1 - \varphi_1(q) \leqq \alpha < 1\right), \end{cases}$$

$$|a_3| \leqq \frac{2\Gamma_q(2)\Gamma_q(3)(1-\alpha)}{[3]_q\Gamma_q(4) - \Gamma_q(2)\Gamma_q(3)} \cdot \left(\frac{[2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2) + 2(1-\alpha)\Gamma_q(2)\Gamma_q(2)}{[2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2)}\right)$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{2\Gamma_q(2)\Gamma_q(3)(1-\alpha)}{[3]_q\Gamma_q(4) - \Gamma_q(2)\Gamma_q(3)},$$

where

$$\varphi_1(q) = \frac{\Gamma_q(2)\Gamma_q(3)\left([2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2)\right)^2}{2\Gamma_q(2)\Gamma_q(2)\left([3]_q\Gamma_q(4)\Gamma_q(2) - [2]_q\Gamma_q(3)\Gamma_q(3)\right)}.$$

Corollary 8. *Let $h \in \mathcal{K}_\Sigma(q, 0, 1)$ be given by (1). Then,*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2\Gamma_q(2)\Gamma_q(2)\Gamma_q(3)}{[3]_q\Gamma_q(4)\Gamma_q(2) - [2]_q\Gamma_q(3)\Gamma_q(3)}} \\ \left(0 \leqq \alpha < 1 - \varphi_2(q)\right) \\ \\ \dfrac{2\Gamma_q(2)\Gamma_q(2)}{[2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2)} \left(1 - \varphi_2(q) \leqq \alpha < 1\right), \end{cases}$$

$$|a_3| \leqq \frac{2\Gamma_q(2)\Gamma_q(3)}{[3]_q\Gamma_q(4) - \Gamma_q(2)\Gamma_q(3)} \cdot \frac{[2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2) + \Gamma_q(2)\Gamma_q(2)}{[2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2)}$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{2\Gamma_q(2)\Gamma_q(3)}{[3]_q\Gamma_q(4) - \Gamma_q(2)\Gamma_q(3)},$$

where

$$\varphi_2(q) = \frac{\Gamma_q(2)\Gamma_q(3)\left\{[2]_q\Gamma_q(3) - \Gamma_q(2)\Gamma_q(2)\right\}^2}{2\Gamma_q(2)\Gamma_q(2)\left\{[3]_q\Gamma_q(4)\Gamma_q(2) - [2]_q\Gamma_q(3)\Gamma_q(3)\right\}}.$$

Corollary 9. *Let $h \in \mathcal{K}_\Sigma(q \to 1-, \alpha, \varrho)$ be given by* (1). *Then,*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2\Gamma(2)\Gamma(3-\varrho)\Gamma(4-\varrho)(1-\alpha)}{\Gamma(2-\varrho)\{3\Gamma(4)\Gamma(3-\varrho)-2\Gamma(3)\Gamma(4-\varrho)\}}} \\ (0 \leqq \alpha < 1-\varphi_3(\varrho)) \\ \dfrac{2\Gamma(2)\Gamma(3-\varrho)(1-\alpha)}{2\Gamma(2-\varrho)\Gamma(3)-\Gamma(2)\Gamma(3-\varrho)} \\ (1-\varphi_3(\varrho) \leqq \alpha < 1), \end{cases}$$

$$|a_3| \leqq \frac{2\Gamma(2)\Gamma(4-\varrho)(1-\alpha)}{3\Gamma(4)\Gamma(2-\varrho)-\Gamma(2)\Gamma(4-\varrho)} \cdot \frac{2\Gamma(2-\varrho)\Gamma(3)-\Gamma(2)\Gamma(3-\varrho)+2(1-\alpha)\Gamma(2)\Upsilon(3-\varrho)}{2\Gamma(2-\varrho)\Gamma(3)-\Gamma(2)\Gamma(3-\varrho)}$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{2\Gamma(2)\Gamma(4-\varrho)(1-\alpha)}{3\Gamma(2-\varrho)\Gamma(4)-\Gamma(2)\Gamma(4-\varrho)},$$

where

$$\varphi_3(\varrho) = \frac{\Gamma(2)\Gamma(4-\varrho)\{2\Upsilon(2-\varrho)\Gamma(3)-\Gamma(2)\Gamma(3-\varrho)\}^2}{2\Gamma(3-\varrho)\Gamma(2)\{3\Gamma(4)\Gamma(2-\varrho)\Gamma(3-\varrho)-2\Gamma(3)\Gamma(2-\varrho)\Gamma(4-\varrho)\}}.$$

As another application of Theorem 2 for $\varrho = 4$ and $q \to 1-$, we obtain the result given in [27].

Corollary 10 (see [27]). *Let $h \in \mathcal{K}_\Sigma(q \to 1-, \alpha, 0)$. Then,*

$$|a_2| \leqq \begin{cases} \sqrt{2(1-\alpha)} & \left(0 \leqq \alpha < \frac{1}{2}\right) \\ 2(1-\alpha) & \left(\frac{1}{2} \leqq \alpha < 1\right) \end{cases}$$

and

$$|a_3| \leqq \begin{cases} 2(1-\alpha) & \left(0 \leqq \alpha < \frac{1}{2}\right) \\ (1-\alpha)(3-2\alpha) & \left(\frac{1}{2} \leqq \alpha < 1\right). \end{cases}$$

Corollary 11 (see [56]). *Let $h \in \mathcal{K}_\Sigma(q \to 1-, 0, \varrho)$ be given by* (1). *Then,*

$$|a_2| \leqq \min\left(\sqrt{\frac{2\Gamma(3-\varrho)\Gamma(4-\varrho)}{\Gamma(2-\varrho)\{3\Gamma(4)\Gamma(3-\varrho)-2\Gamma(3)\Gamma(4-\varrho)\}}}, \quad \frac{2\Gamma(2)\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\Gamma(3)-\Gamma(2)\Gamma(3-\varrho)} \right),$$

$$|a_3| \leqq \frac{2\Gamma_q(4-\varrho)}{3\Gamma(4)\Gamma(2-\varrho)-\Gamma(4-\varrho)} \cdot \left(\frac{2\Gamma(2-\varrho)\Gamma(3)-\Gamma(3-\varrho)+2\Gamma(3-\varrho)}{2\Gamma(2-\varrho)\Gamma(3)-\Gamma(3-\varrho)}\right)$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{2\Gamma(2)\Gamma(4-\varrho)}{3\Gamma(2-\varrho)\Gamma(4) - \Gamma(4-\varrho)}.$$

Example 1. *For $l \geqq 3$, we will demonstrate that $h(z)$ given by*

$$h(z) = z + \frac{1-\alpha}{l-1} z^l$$

is a bi-close-to-convex function of order α, where $\alpha \in [0,1)$ in $\mathbb{E}$. Indeed, since the function

$$g(z) = z - \frac{1-\alpha}{l-\alpha} z^l$$

is starlike in $\mathbb{E}$, we have

$$\begin{aligned}
\frac{D_q \mathbf{\Omega}_q^{\varrho} h(z)}{g(z)} &= \frac{1 + \left(\frac{\Psi_l(q,\varrho)[l]_q(1-\alpha)}{l-1}\right) z^{l-1}}{1 - \left(\frac{1-\alpha}{l-\alpha}\right) z^{l-1}} \\
&= 1 + \sum_{j=1}^{\infty} \left(\frac{(1-\alpha)^j}{(l-\alpha)^j} + \frac{\Psi_l(q,\varrho)[l]_q(1-\alpha)j}{(l-1)(l-\alpha)^{j-1}} \right) z^{(l-1)k},
\end{aligned}$$

where

$$\Psi_l(q,\varrho) = \frac{\Gamma_q(2-\varrho)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\varrho)}.$$

Therefore, we obtain

$$\begin{aligned}
\frac{\frac{D_q \mathbf{\Omega}_q^{\varrho} h(z)}{g(z)} - \alpha}{1-\alpha} = 1 + \sum_{j=1}^{\infty} &\left(\frac{l\left(\Psi_l(q,\varrho)[l]_q + 1\right) - \Psi_l(q,\varrho)[l]_q \alpha - 1}{(l-1)(l-\alpha)} \right) \\
&\cdot \left(\frac{1-\alpha}{l-\alpha} \right)^{j-1} z^{(l-1)j}.
\end{aligned}$$

Obviously, we also have

$$\Re\left(\frac{D_q \mathbf{\Omega}_q^{\varrho} h(z)}{g(z)} \right) - \alpha > 0 \qquad (z \in \mathbb{E}).$$

For $\gamma = h^{-1}$ and $\delta = g^{-1}$, it is easily seen that

$$\gamma(\vartheta) = \vartheta - \frac{1-\alpha}{l-1}\, \vartheta^l,$$

and if we set

$$\delta(\vartheta) = \vartheta + \frac{1-\alpha}{l-\alpha}\, \vartheta^l$$

which is starlike in $\mathbb{E}$. As a result, we have

$$\begin{aligned}
\frac{\frac{D_q \mathbf{\Omega}_q^{\varrho} \gamma(\vartheta)}{\delta(z)} - \alpha}{1-\alpha} = 1 + \sum_{j=1}^{\infty} (-1)^j &\left(\frac{l\left(\Psi_l(q,\varrho)[l]_q + 1\right) - \Psi_l(q,\varrho)[l]_q \alpha - 1}{(l-1)(l-\alpha)} \right) \\
&\cdot \left(\frac{1-\alpha}{l-\alpha} \right)^{j-1} \vartheta^{(l-1)j}.
\end{aligned}$$

Thus, clearly, we find that

$$\Re\left(\frac{D_q\boldsymbol{\Omega}_q^{\varrho}\gamma(\vartheta)}{\delta(\vartheta)}-\alpha\right)>0 \qquad (z\in\mathbb{E}).$$

4. Conclusions

In this article, we have used the notions of the q-fractional derivative, bi-univalent functions and FPE to define some new subfamilies of Σ. We investigated lth coefficient bounds and the Fekete–Szegö functional for these newly defined classes. Our study has also demonstrated how the results are enhanced and expanded by appropriate specialization of the parameters, including some recently released findings.

This article is composed of three sections. We briefly reviewed some fundamental geometric function theory ideas in Section 1 because they were important to deriving our main findings. All of these components are well-known, and we have correctly cited them. In Section 2, we provide the Faber polynomial approach and its applications and some initial lemmas. In Section 3, we present our key findings.

For future studies, researchers can use other extended q-operators instead of the $(\varrho;q)$-differintegral operator and define a number of new subclasses of the bi-univalent function class Σ. Furthermore, by using the Faber polynomial technique, the interested researchers can discuss the behavior of coefficient estimates for different types of newly defined subclasses of bi-univalent functions. Researchers may also investigate a variety of methods, depending on how inspired they are by the knowledge gained in this subject. Fractional derivative operators have made it possible to study differential equations from the perspectives of functional analysis and operator theory. Using the operator method for resolving differential equations, various properties fractional derivative operator are used extensively.

It is a clearly presented fact that the transition from our q-results to the corresponding $(\mathfrak{p},q)$-results is a rather trivial exercise because the additional forced-in parameter $\mathfrak{p}$ is obviously redundant (see, for details, ([5], p. 340) and ([54], Section 5, pp. 1511–1512); see also [59–62]).

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