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A Nonconstant Gradient Constrained Problem for Nonlinear Monotone Operators

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Abstract: The purpose of the research is the study of a nonconstant gradient constrained problem for nonlinear monotone operators. In particular, we study a stationary variational inequality, defined by a strongly monotone operator, in a convex set of gradient-type constraints. We investigate the relationship between the nonconstant gradient constrained problem and a suitable double obstacle problem, where the obstacles are the viscosity solutions to a Hamilton–Jacobi equation, and we show the equivalence between the two variational problems. To obtain the equivalence, we prove that a suitable constraint qualification condition, Assumption S, is fulfilled at the solution of the double obstacle problem. It allows us to apply a strong duality theory, holding under Assumption S. Then, we also provide the proof of existence of Lagrange multipliers. The elements in question can be not only functions in L^2, but also measures.

Keywords: variational inequalities; non-constant gradient constraints; obstacle problem; nonlinear monotone operators; Lagrange multipliers

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1. Introduction

A very interesting problem, which has attracted much interest for many decades because of its simple formulation in terms of differential equations, is the elastic–plastic torsion problem, namely, the problem of minimizing the functional

$$\frac{1}{2}\int_{\Omega}(|Dv|^2 - hv)dx \tag{1}$$

on the class of functions $\{v \in H_0^1(\Omega) \,:\, |Dv| \leq 1\}$.

The elastic–plastic torsion problem arises when a long elastic bar with cross section Ω is twisted by an angle. In particular, the formulation due to R. von Mises (see [1]) of the elastic–plastic torsion problem of a cylindrical bar is the following one:

"Find a function $u(x)$, which vanishes on $\partial\Omega$ and is continuous, together with its first derivatives on Ω; on Ω the gradient of u, Du, must have an absolute value less than or equal to a given positive constant t; whenever, in Ω, $|Du| < t$, the function u must satisfy the differential equation $\Delta u = -2\nu\alpha$, where the positive constants ν and α denote the shearing modulus and the angle of twist per unit length respectively".

The plastic region, P, refers to the range of deformation in which the material exhibits significant plastic or irreversible behavior. It is the region beyond the elastic limit where the material undergoes permanent changes in shape, and the deformation is not recoverable. When a material is loaded within its elastic limit, it deforms elastically, meaning that it can return to its original shape once the load is removed. However, beyond the elastic limit, the material enters the plastic region, and plastic deformation occurs. In particular, the set

$$E = \{x \in \Omega : |Du(x)| < t\}$$

is the set of points where the cross section still remains elastic, namely the elastic set, and the set

$$P = \{x \in \Omega : |Du(x)| = t\},$$

is the set of points where the material has become plastic due to the torsion, namely the plastic set.

The ridge R of Ω is, by definition, the set of points in Ω where $dist(x, \partial\Omega)$ is not $C^{1,1}$, whereas the part of ∂E, which is contained in Ω, is called the free boundary (see [2]).

For the derivation of the variational inequality from the physical problem see [3].

Ting [4] investigated problem (1) for $n = 2$, whereas the existence of a Lagrange multiplier formulation for (1) (and hence of a corresponding system of partial differential equations) was proved for constant h in [5] by Brézis.

Glowinski et al. [6] studied the numerical aspects; for results on the elastic and plastic sets E and P and on the free boundary we refer to Caffarelli and Friedman [2]. In [7] Brezis and Sibony proved that the elastic–plastic torsion problem is equivalent to an obstacle-type problem, in which the distance function represents the obstacle. Moreover, they proposed two numerical methods for the obstacle problem.

In [8] Chiadò Piat and Percivale proved the existence of measure-type Lagrange multipliers under more general assumptions on the operator and on h.

Daniele et al. [9] obtained similar results, solving a problem unsolved for a long time by using a new infinite dimensional duality theory. They show, for a class of problems including Problem (1), the existence of an L^∞ Lagrange multiplier, if the problem admits solution and a constraint qualification condition is fulfilled at this solution (see Section 3). The Lagrange multiplier is the solution to a dual problem (see also [10–12] for other results related to linear and nonlinear monotone operators).

Many other studies in the past years are related to the problem when the gradient constraints are no longer constant, since it models many interesting physical and biological phenomena (see [13] for an overview of constrained and unconstrained free boundary problems).

Studying variational problems with gradient constraints involves techniques from the calculus of variations and constrained optimization. Some common methods include: Lagrange multipliers, penalty methods, augmented Lagrangian methods, and projection methods. As is well known, Lagrange multipliers introduce additional unknowns and allow the constraints to be incorporated into the objective function through a modified Lagrangian. The resulting problem can then be solved using variational methods or numerical optimization techniques.

Relevant issues related to the problem with gradient constraints are existence and regularity of the solution, existence of Lagrange multipliers, connection with double obstacle problem, and numerical aspects.

Regarding the existence and regularity of the solution, we refer to L. Evans in [14], who studied general linear elliptic equations with a non-constant gradient constraint $g(x) \in C^2(\overline{\Omega})$, and proved that there exists a unique solution in the space $W^{2,p}_{loc}(\Omega) \cap W^{1,\infty}_0(\Omega)$, with $1 < p < \infty$ (see also [15–18] for other regularity results).

The conditions required for the existence of a Lagrange multiplier are typically related to the regularity of the problem, such as the smoothness of the objective function and constraints.

One of the important conditions for the existence of a Lagrange multiplier is the constraint qualification. There are different types of constraint qualifications, such as the linear independence constraint qualification (LICQ), the Mangasarian–Fromovitz constraint qualification (MFCQ), and Slater's condition (see Theorem 4).

If the qualification condition is satisfied, then according to the Lagrange multiplier theorem, there exists a Lagrange multiplier associated with the optimal solution. The Lagrange multiplier, generally, provides information about the sensitivity of the objective function to changes in the constraints.

However, let us stress that the existence of a Lagrange multiplier does not guarantee a unique solution to the optimization problem. It only indicates the existence of a necessary condition for an optimum.

A very interesting property of variational problems with gradient constraints is the relationship with double obstacle problems. Let us remark that the equivalence is not always true, as observed in [14] (see also [19,20]). Equivalence results between the two problems associated to the Laplacian or to a linear operator are contained in [19–21] (see also [22]).

The equivalence also holds for the problem associated to a nonlinear strongly monotone operator $a(Du)$ with nonconstant gradient constraint of type $G(Du) \leq M$, where G is a strictly convex function (see [23]).

Let us note that monotone operators play a fundamental role in various branches of mathematics, including optimization theory, to analyze several mathematical problems involving nonlinear operators. Monotone operators are extensively used in the study of variational inequalities too (see [24]).

The paper adds to the literature on nonconstant gradient constrained problem further results related to the relationship with double obstacle problem and the existence of Lagrange multipliers. Here we investigate the problem associated to a nonlinear strongly monotone operator as in [23], but we consider the nonconstant gradient constraint of type $|Du| \leq g(x)$, with $g(x) \in C^2(\overline{\Omega})$, $g(x) > 0$. We also prove the existence of L^2 Lagrange multipliers and, under less restrictive assumptions, an existence result of measure-type Lagrange multipliers. Let us note that the existence of Lagrange multipliers as measures is not proved for gradient contraints of type $G(Du) \leq M$.

In particular, the problem under consideration is

Find $u \in K_g = \left\{ v \in H_0^{1,2}(\Omega) : |Dv|^2 = \sum_{i=1}^{n} (D_i v)^2 \leq g(x), \text{ a.e. in } \Omega \right\}$ *such that*:

$$\int_\Omega \sum_{i=1}^{n} a_i(Du)(D_i v - D_i u)\, dx \geq \int_\Omega f(v-u) dx, \quad \forall v \in K_g, \tag{2}$$

where $a(p) : \mathbb{R}^n \to \mathbb{R}^n$ is a strongly monotone operator of class C^2 (see (6)).

Let us note that it follows from classical results in the literature that there exists a unique solution to problem (2) (see [25]).

In the first result of the paper (Theorem 1) we show that, under a condition on the gradient constraint g, problem (2) is equivalent to the following double obstacle problem

Find $u \in K$ *such that*:

$$\int_\Omega \sum_{i=1}^{n} a_i(Du)(D_i v - D_i u)\, dx \geq \int_\Omega f(v-u)\, dx, \quad \forall v \in K, \tag{3}$$

where $K = \left\{ v \in H_0^{1,2}(\Omega) : w_1(x) \leq v(x) \leq w_2(x) \text{ a.e. in } \Omega \right\}$, and

$$w_1 = \inf_{v \in K} v(x), \quad w_2 = \sup_{v \in K} v(x).$$

From Theorem 5.1 in [26] (see also [20]), $w_2 \in H^{1,\infty}(\Omega)$ is the viscosity solution to the Hamilton–Jacobi equation

$$\begin{cases} |Du| = \sqrt{g(x)} & a.e.\ in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{4}$$

and

$$w_2(x) = inf_{x_0 \in \partial\Omega} L(x, x_0) \tag{5}$$

where

$$L(x,x_0) =$$
$$inf\left\{\int_0^{T_0} \sqrt{g(\xi(s))}ds : \xi : [0,T_0] \to \overline{\Omega},\ \xi(0) = x,\ \xi(T_0) = x_0,\ |\xi'(s)| \le 1\ a.e.\ in\ [0,T_0]\right\}$$

w_1 can be calculated analagously.

It is important to note that the regularity of these obstacles follows from the theory of the viscosity solutions to Hamilton–Jacobi equations (see [17], p. 31), even if the solutions to the Hamilton–Jacobi equations are, in general, not smooth.

As already recalled, the problems are, generally, not equivalent, but a condition on the sign of the second derivatives of g is required.

Before proving the equivalence, in Section 3 we achieve a regularity result for solutions to (3) (Theorem 8), that we need in the sequel.

Then, thanks to the equivalence, it is possible to prove that Lagrange multipliers exist in L^2 (Theorem 2).

Finally, an existence result of Lagrange multipliers as Radon measures holds, under less restrictive assumptions (Theorem 3).

The results are obtained following variational arguments and the strong duality theory.

Let us remark, that, during the past several decades, the variational methods have played a key role in solving many problems arising in nonlinear analysis and optimization theory such as differential hemivariational inequalities systems (see [27]), monotone bilevel equilibrium problems, generalized global fractional-order composite dynamical systems, generalized time-dependent hemivariational inequalities systems, optimal control of feedback control system, and so on.

Moreover, let us emphasize that real-life applications have been investigated on the basis of the theory of variational inequalities with operators of monotone type (see [28–30] for mathematical models describing flows of Bingham-type fluids and flows of an Oldroyd type by means of a variational inequality approach).

Finally, let us stress that the problem under consideration is strictly connected to the Monge–Kantorovich mass transfer problem. In particular, in [31] the authors study the integrability of the Lagrange multiplier, assuming that f belongs to $L^p(\Omega)$ in the case of constant gradient constraint (see also [32] for variable constraint g). The Monge–Kantorovich mass transfer problem has applications in diverse fields such as economics, image processing, computer vision, transportation planning, and statistical physics. It provides a mathematical framework for studying the optimal flow of mass, resources, or information between different distributions or regions.

The paper is organized as follows: in Section 2 we state our main results of equivalence between the variational problems and existence of Lagrange multipliers, in Section 3 we provide a preliminary regularity result and some results of the theory of strong duality are recalled. In Section 4 we prove Theorem 1 and Section 5 is devoted to the proofs of Theorems 2 and 3. Finally, in Section 6 we provide our conclusions and suggest new problems that may be of interest for future research.

2. Results

The main results of the paper are presented in this section.

In what follows we assume that Ω is an open bounded convex subset of $\mathbb{R}^n$ and the boundary $\partial\Omega$ is of class C^2.

Moreover, the operator a is of class C^2, with $a(0) = 0$.

In the first two results, we assume that a is a strongly monotone operator, that is, there exists $\lambda > 0$, such that

$$(a(p) - a(q), p - q) \ge \lambda \|p - q\|^2 \quad \forall p, q \in \mathbb{R}^n, p \ne q. \tag{6}$$

Theorem 1. *Assuming that a satisfies assumption (6), $f \equiv constant > 0$, and the following condition is fulfilled*

$$- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial a_i}{\partial p_j} \frac{\partial g}{\partial x_j} \right) \geq 0 \quad in\ \Omega, \tag{7}$$

then, the solution u to problem (2) is also the solution to problem (3).

Moreover, the following coincidence of sets holds:

$$P = \{x \in \Omega : |Du|^2 = g(x)\} = I = \{x \in \Omega : u(x) = w_1(x) \text{ or } u(x) = w_2(x)\}.$$

Regarding the Lagrange multipliers, we prove the existence in two different cases. In the first one, the Lagrange multipliers are L^2 functions, whereas, in the second one, under less restrictive assumptions, they are measures.

Let us stress that the second result (Theorem 3) holds under assumption of strictly monotonicity on the operator a, namely

$$(a(P) - a(Q), P - Q) > 0 \quad \forall P, Q \in \mathbb{R}^n, P \neq Q. \tag{8}$$

Theorem 2. *Under the same assumptions as in Theorem 1, if $u \in K_g \cap W^{2,p}(\Omega)$ solves problem (2), then, there exists a Lagrange multiplier $\nu \in L^2(\Omega)$, $\nu \geq 0$ a.e. in Ω, that is*

$$\begin{cases} \nu \left(\displaystyle\sum_{i=1}^{n} (D_i u)^2 - g(x) \right) = 0 \ \text{a.e. in } \Omega \\ \displaystyle\sum_{i=1}^{n} \frac{\partial a_i(Du)}{\partial x_i} + f = \nu \ \text{a.e. in } \Omega. \end{cases} \tag{9}$$

Theorem 3. *Assume that a satisfies assumption (8) and $f \in L^p(\Omega)$, $p > 1$. If $u \in K_g$ solves problem (2), then there exists a Lagrange multiplier $\mu^* \in (L^\infty(\Omega))^*$, that is*

$$\begin{cases} \langle \mu^*, y \rangle \geq 0 \quad \forall y \in L^\infty(\Omega),\ y \geq 0 \quad \text{a.e. in } \Omega; \\ \langle \mu^*, \displaystyle\sum_{i=1}^{n} \Big((D_i u)^2 - g(x) \Big) \rangle = 0; \\ \displaystyle\int_\Omega \left\{ \sum_{i=1}^{n} a_i(Du) \frac{\partial \varphi}{\partial x_i} - f\varphi \right\} dx = \langle \mu^*, -2 \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \rangle \quad \forall \varphi \in H_0^{1,\infty}(\Omega). \end{cases} \tag{10}$$

3. Preliminary Results

This section is devoted to some preliminary results that we need to prove our theorems.

In particular, first we recall the strong duality theory and, then, we prove a regularity result for the solution to the double obstacle problem (3) that we need to apply in Section 4 a maximum principle.

For the sake of clarity, here we provide the main results of classical strong duality theory and a new strong duality theory, obtained using new separation theorems based on the notion of quasi-relative interior.

For the classical results of strong duality theory we refer to ([33], Theorems 6.7 and 6.11).

It is important to note that strong duality has important implications in optimization theory. It allows us to obtain lower bounds on the optimal value of the primal problem by solving the dual problem. It also provides a way to assess optimality and obtain dual solutions that can provide additional information about the primal problem, such as shadow prices or sensitivity analysis.

The framework, in which the classical theory works, is the following one: X is a real linear space and $S \subset X$ is a nonempty subset; $(Y, \|\cdot\|)$ is a partially ordered real normed space with ordering cone C, and $C^* = \{\lambda \in Y^* : \langle \lambda, y \rangle \geq 0\ \forall y \in C\}$ is the dual cone of C, whereas Y^* is the topological dual of Y. Moreover, $F : S \to \mathbb{R}$ is a given objective functional, $G : S \to Y$ is a given constraint mapping and the constraint set is given as $\mathbb{K} := \{v \in S : G(v) \in -C\}$.

We consider the primal problem

$$\min_{\substack{G(v)\in -C\\ v\in S}} F(v) \tag{11}$$

and the dual problem

$$\max_{\lambda\in C^*} \inf_{v\in S}[F(v) + \lambda(G(v))], \tag{12}$$

where λ is the Lagrange multiplier associated with the sign constraints.

As is well known (see [33]), the weak duality always holds, namely,

$$\max_{\lambda\in C^*} \inf_{v\in S}[F(v) + \lambda(G(v))] \leq \min_{\substack{G(v)\in -C\\ v\in S}} F(v) \tag{13}$$

Moerover, if problem (11) is solvable and in (13) the equality holds, the strong duality between the primal problem (11) and the dual problem (12) holds.

Theorem 4 (classical strong duality property [33])**.** *Assume that the composite mapping $(F, G) : S \to \mathbb{R} \times Y$ is convex-like with respect to product cone $\mathbb{R}_+ \times C$ in $\mathbb{R} \times Y$, $\mathbb{K}$ is nonempty and the ordering cone C has a nonempty interior $int(C)$. If the primal problem (11) is solvable and the generalized Slater condition is satisfied, namely there is a vector $\bar{v} \in S$ with $G(\bar{v}) \in -int(C)$, then the dual problem (12) is also solvable and the extremal values of the two problems are equal. Moreover, if u is the optimal solution to problem (11) and $\overline{\nu} \in C^*$ is a solution to problem (12), it follows that*

$$\overline{\nu}(G(u)) = 0. \tag{14}$$

Moreover, if

$$\mathcal{L}(v, \nu) = F(v) + \nu(G(v)),$$

is the Lagrange functional, then the following relationship with the saddle points of $\mathcal{L}(v, \nu)$ holds.

Theorem 5 (see [33])**.** *Under the same assumptions as in Theorem 4, if the ordering cone C is closed, then a point $(u, \overline{\nu}) \in S \times C^*$ is a saddle point of the Lagrange functional $\mathcal{L}(v, \nu)$, namely*

$$\mathcal{L}(u, \nu) \leq \mathcal{L}(u, \overline{\nu}) \leq \mathcal{L}(v, \overline{\nu}),\ \forall v \in S, \forall \nu \in C^*,$$

if and only if u is a solution to the primal problem (11), $\overline{\nu}$ is a solution to the dual problem (12) and the extremal values of the two problems are equal.

Let us stress that we apply classical strong duality theory to prove Theorem 3, whereas we need a new theory (see [9]) to obtain the other results. Indeed, in our framework, as in many applications in infinite dimensional settings, the classical theory does not work, since the assumption of nonemptiness of the ordering cone is not fulfilled.

Here, we recall the new strong duality theory in its complete version, namely in the case of inequality and equality constraints.

The assumptions read as follows:

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be real normed spaces with Y^*, Z^* topological dual of Y and Z, respectively; Y is partially ordered by a convex cone C, $C^* = \{\mu \in Y^* : \langle \mu, y\rangle \geq 0\ \forall y \in C\}$ is the dual cone of C. S is a nonempty subset of X, and $F : S \to \mathbb{R}$, $G : S \to Y$, $H : S \to Z$ are three functions.

Moreover, we define the feasible set

$$\mathbb{K} = \{v \in S :\ G(v) \in -C,\ \ H(v) = \theta_Z\}.$$

We recall the definition of tangent cone to $S^* \subset X$ at a point $v \in X$:

$$T_{S^*}(v) := \left\{ l \in X : l = \lim_n \mu_n(v_n - v), \mu_n > 0,\, v_n \in S^* \ \forall n \in N,\ \lim_n v_n = v \right\}$$

We introduce the following constraint qualification assumption: we say that Assumption S is satisfied at a point $v_0 \in \mathbb{K}$ if

$$T_{\widetilde{N}}(0, \theta_Y, \theta_Z) \cap \Big( ]-\infty, 0[\times \{\theta_Y\} \times \{\theta_Z\} \Big) = \emptyset, \tag{15}$$

where

$$\widetilde{N} = \{(F(v) - F(v_0) + \alpha, G(v) + w, H(v)) :\ v \in S \setminus \mathbb{K},\ \alpha \geq 0,\ w \in C\}.$$

Under Assumption S the following strong duality property holds (see [9]).

Theorem 6. *Let us assume that F and G are convex functions, H is an affine-linear mapping and $v_0 \in \mathbb{K}$ is a solution to the primal problem*

$$\min_{v \in \mathbb{K}} F(v). \tag{16}$$

Then, if Assumption S is fulfilled at v_0, the dual problem

$$\max_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{v \in S} \{F(v) + \langle \lambda, G(v) \rangle + \langle \mu, H(v) \rangle\} \tag{17}$$

is also solvable and the extreme values of the primal problem and of the dual problem coincide. Moreover, if $(v_0, \lambda^*, \mu^*) \in \mathbb{K} \times C^* \times Z^*$ solves problem (17), then $\langle \lambda^*, G(v_0) \rangle = 0$.

Moreover, if

$$\mathcal{L}(v, \lambda, \mu) = F(v) + \langle \lambda, G(v) \rangle + \langle \mu, H(v) \rangle$$

is the Lagrange functional, then the following result on the saddle points of the Lagrange functional holds.

Theorem 7 ([9])**.** *Under the same assumptions as in Theorem 6, $v_0 \in \mathbb{K}$ solves problem (16) if and only if there exist $\lambda^* \in C^*$ and $\mu^* \in Z^*$ such that (x_0, λ^*, μ^*) is a saddle point of the Lagrange functional, namely*

$$\mathcal{L}(v_0, \lambda, \mu) \leq \mathcal{L}(v_0, \lambda^*, \mu^*) \leq \mathcal{L}(v, \lambda^*, \mu^*), \quad \forall v \in S,\ \lambda \in C^*,\ \mu \in Z^*.$$

Now, we prove the following regularity result, that we will use in Section 4.

Theorem 8. *Let the assumptions of Theorem 1 be satisfied and u be the solution to problem (3). Then, $u \in W^{2,p}(\Omega)$. In particular, if $p > n$, $Du \in C^{0,\alpha}(\Omega)$.*

Proof. The first goal is an estimate for

$$|u|_1 = \sup\left\{ \frac{|u(x) - u(y)|}{|x - y|} :\ x, y, \in \overline{\Omega},\ x \neq y \right\},$$

obtained using similar arguments as in [34].

Let u be the solution to (3), we set $\tilde{u}$ the extension by zero of u to $\mathbb{R}^n$ and

$$u^h(x) = \max\{\tilde{u}(x+h) - \tilde{u}(x) - M|h|, 0\} \quad \forall x, h \in R^n, \tag{18}$$

where $M = \max\{|w_1|_1, |w_2|_1\}$.

Defining

$$u_1(x) = \max\{\tilde{u}(x), \tilde{u}(x+h) - M\,|h|\} = \tilde{u}(x) + u^h(x),$$

$$u_2(x) = \min\{\tilde{u}(x), \tilde{u}(x-h) + M\,|h|\} = \tilde{u}(x) - u^h(x-h).$$

as in [19,23], we have $u_{1/\Omega}, u_{2/\Omega} \in K$ and

$$\tilde{w}_1(x) \leq u_2(x) \leq u_1(x) \leq \tilde{w}_2(x) \ a.e.\ in\ \Omega.$$

Following the same arguments as in [12], we get

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} (a_i(D\tilde{u}(x+h)) - a_i(D\tilde{u}(x)))D_i u^h(x)dx \leq 0. \tag{19}$$

Setting $X_h^+ = \{x \in R^n \ : \tilde{u}(x+h) - \tilde{u}(x) - M\,|h| \geq 0\}$, from (18) and (19) it follows that

$$\int_{X_h^+} \sum_{i=1}^{n} (a_i(D\tilde{u}(x+h)) - a_i(D\tilde{u}(x)))(D_i\,\tilde{u}(x+h) - D_i\tilde{u}(x))dx \leq 0. \tag{20}$$

Thanks to strong monotonicity assumption (6) and to inequality (20), we may conclude that $u^h = 0$ in X_h^+ and then

$$\tilde{u}(x+h) - \tilde{u}(x) - M\,|h| \leq 0 \quad \forall x, h \in \mathbb{R}^n,$$

namely,

$$|u|_1 \leq M$$

and

$$|Du| \leq M \quad a.e.\ in\ \Omega. \tag{21}$$

To conclude, we consider the following elastic–plastic torsion problem
Find $w \in K_M = K \cap \{v \in H_0^1 :\ |Dv| \leq M \text{ a.e. in } \Omega\}$ such that:

$$\int_\Omega \sum_{i=1}^{n} a_i(Dw)(D_i v - D_i w)\,dx \geq \int_\Omega f(v-w)dx, \quad \forall v \in K_M. \tag{22}$$

Since the feasible set K_M is a bounded, closed, and convex set, from classical results (see [35]), the unique solution $u \in K_M$ to the variational inequality (22) belongs to $W^{2,p}(\Omega)$. Then, the thesis is achieved. □

4. The Equivalence of the Two Variational Problems

Now, we may prove Theorem 1.
Obviously,

$$K_g \subseteq K. \tag{23}$$

Then, to prove the equivalence of the two problems, we have to show that if $u \in K$ is the solution to (3), then u belongs to K_g.
To this aim, setting

$$F(v) = \int_\Omega \left\{ \sum_{i=1}^{n} a_i(Du)(D_i v - D_i u) - f(v-u) \right\} dx$$

we note that problem (3) may be rewritten as the optimization problem

$$\min_{v \in K} F(v), \tag{24}$$

which satisfies Assumption S.

Indeed, if we set

$$X = S = L^2(\Omega),\ Y = L^2(\Omega) \times L^2(\Omega),$$
$$C = C^* = \{(a(x), b(x)) \in L^2(\Omega) \times L^2(\Omega) : a(x), b(x) \geq 0 \text{ a.e. in } \Omega\},$$
$$G(v) = (G_1(v), G_2(v)) = (w_1 - v, v - w_2),$$

we have

$$\tilde{N} = \{(F(v) + \alpha, w_1 - v + a, v - w_2 + b), v \in L^2 \setminus K, \alpha \geq 0, y = (a, b) \in C\}.$$

Following similar arguments as in [23,36] we may show that, if

$$\left(l, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)}\right) = \lim_n [\mu_n(F(v_n) + \alpha_n, w_1 - v_n + a_n, v_n - w_2 + b_n)],$$

with $\mu_n > 0$, $\lim_n (F(v_n) + \alpha_n) = 0$, $\alpha_n \geq 0$, $v_n \in L^2(\Omega) \setminus K$, $\lim_n \mu_n(w_1 - v_n + a_n) = \theta_{L^2(\Omega)}$, $\lim_n \mu_n(v_n - w_2 + b_n) = \theta_{L^2(\Omega)}$, $y_n = (a_n, b_n) \in C$, then

$$l \geq 0,$$

namely, Assumption S is fulfilled at the solution to problem (24).

Then, if we consider the Lagrange functional

$$\mathcal{L}(v, \lambda, \mu) = \tag{25}$$
$$= \int_\Omega (-\sum_{i=1}^n \frac{\partial a_i(Du)}{\partial x_i} - f)(v - u)\, dx + \int_\Omega \lambda(w_1(x) - v(x))\, dx + \int_\Omega \mu(v(x) - w_2(x))\, dx,$$

thanks to Theorem 7, there exists a saddle point $(\lambda^*, \mu^*) \in C$, namely,

$$\mathcal{L}(u, \lambda, \mu) \leq \mathcal{L}(u, \lambda^*, \mu^*) \leq \mathcal{L}(v, \lambda^*, \mu^*) \quad \forall v \in L^2(\Omega),\ \forall(\lambda, \mu) \in C, \tag{26}$$

and

$$\int_\Omega \lambda^*(w_1(x) - u(x))\, dx = 0, \quad \int_\Omega \mu^*(u(x) - w_2(x))\, dx = 0, \tag{27}$$

that is,

$$\lambda^*(w_1(x) - u(x)) = 0, \quad \mu^*(u(x) - w_2(x)) = 0, \quad a.e.\ in\ \Omega. \tag{28}$$

Using variational arguments (see [12]), it follows

$$-\sum_{i=1}^n \frac{\partial a_i(Du)}{\partial x_i} - f - \lambda^* + \mu^* = 0 \quad \text{a.e. in } \Omega. \tag{29}$$

Now, we consider the coincidence set $I = \{x \in \Omega : \ u(x) = w_1(x)\ or\ u(x) = w_2(x)\}$ and the non-coincidence set $N = \{x \in \Omega : \ w_1(x) < u(x) < w_2(x)\}$.

From [26], Theorem 5.1, we have that $|Dw_1(x)| = |Dw_2(x)| = \sqrt{g(x)}$ a.e. in Ω, then

$$|Du| = \sqrt{g(x)}\ in\ I. \tag{30}$$

Moreover, from (28) and (29) it follows that $\lambda^* = \mu^* = 0$ a.e. in N and

$$-\sum_{i=1}^n \frac{\partial a_i(Du)}{\partial x_i} = f \quad a.e.\ in\ N. \tag{31}$$

Thanks to the regularity of u, stated in Theorem 8, and since f is a constant function, we follow the same steps used in [35], Lemma III.10. We differentiate (31) with respect to x_k, multiply it by $\frac{\partial u}{\partial x_k}$ and sum it with respect to k. Then, it follows that

$$\sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_j \partial x_i}\frac{\partial u}{\partial x_k} + \sum_{i,j,k=1}^{n} \frac{\partial a_i(Du)}{\partial p_j}\frac{\partial^3 u}{\partial x_j \partial x_i \partial x_k}\frac{\partial u}{\partial x_k} = 0. \tag{32}$$

It follows that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial}{\partial x_j}\left(|Du|^2 - g(x)\right)\right] &= \sum_{i,j,k=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_j} \\ &+ \sum_{i,j,k=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial^2 u}{\partial x_k \partial x_j}\frac{\partial^2 u}{\partial x_k \partial x_i} + \sum_{i,j,k=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial u}{\partial x_k}\frac{\partial^3 u}{\partial x_j \partial x_i \partial x_k} \\ &- \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial g(x)}{\partial x_j}\right).\end{aligned} \tag{33}$$

From assumptions (6) and (7), we have

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial}{\partial x_j}\left(|Du|^2 - g(x)\right)\right] \\ &\geq \sum_{i,j,k=1}^{n}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_k \partial x_j} - \frac{\partial}{\partial x_k}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_j \partial x_i}\right]\frac{\partial u}{\partial x_k} \\ &= \sum_{i,j,k=1}^{n}\left[\sum_{l=1}^{n}\frac{\partial^2 a_i(Du)}{\partial p_j \partial p_l}\frac{\partial^2 u}{\partial x_l \partial x_i}\frac{\partial^2 u}{\partial x_k \partial x_j} - \sum_{l=1}^{n}\frac{\partial^2 a_i(Du)}{\partial p_j \partial p_l}\frac{\partial^2 u}{\partial x_k \partial x_l}\frac{\partial^2 u}{\partial x_j \partial x_i}\right]\frac{\partial u}{\partial x_k} = 0.\end{aligned} \tag{34}$$

Finally, since the coefficients are bounded, N is an open set, applying the maximum principle to the operator

$$-\mathcal{A}(\varphi) = -\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial a_i}{\partial p_j}\frac{\partial \varphi}{\partial x_j}\right),$$

acting on $|Du|^2 - g(x)$ on N, we have

$$|Du(x)| < \sqrt{g(x)} \quad a.e. \in N. \tag{35}$$

From (30) and (35) it follows that, if $u \in K$ is a solution to (3), then

$$|Du(x)| \leq \sqrt{g(x)} \quad a.e. \in \Omega. \tag{36}$$

Taking into account the uniqueness of the solution, we may conclude that the solution to (3) is also the solution to (2) and Theorem 1 is proved.

Finally, the following interesting coincidence of sets follows from (30) and (35)

$$E = \{x \in \Omega : |Du| < \sqrt{g(x)}\} = N.$$

5. Lagrange Multipliers

In this section we provide the proofs of the existence of Lagrange multipliers.

A first result, the existence of L^2 Lagrange multipliers, holds under the assumption $f \equiv constant > 0$ and a strongly monotone operator. It follows from (28) and (29) as in the proof of Theorem 1.

The second result holds assuming that $f \in L^p(\Omega)$, $p > 1$, and the operator a is strictly monotone. In this case the Lagrange multipliers exist in the dual of L^∞.

Indeed, we set

$$X = S = W_0^{1,\infty}(\Omega); \quad C = \{v \in L^\infty(\Omega) \,:\, v(x) \geq 0 \; a.e. \; in \; \Omega\}.$$

In this case C has a nonempty interior, then we may apply the classical strong duality theory (see [33]).

We may rewrite problem (2) as

$$\textit{Find } u \in K_1 = \left\{ v \in H_0^{1,\infty}(\Omega) : \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \leq g(x), \text{ a.e. on } \Omega \right\} \textit{ such that:}$$

$$\int_\Omega \left\{ \sum_{i=1}^{n} a_i(Du)(D_i v - D_i u) - f(v - u) \right\} dx \geq 0, \quad \forall v \in K_1. \tag{37}$$

Following the same steps as in [19], we may prove that C is closed and the generalized Slater condition is verified. Moreover, since F and G are convex, then the composite mapping (F, G) is convex-like, namely all the assumptions of Theorems 6.7 and 6.11 in [33] are fulfilled.

Then, it follows that there exists $\mu^* \in C^*$ solution to the dual problem

$$\max_{\mu \in C^*} \inf_{v \in S} [F(v) + \langle \mu, G(v) \rangle], \tag{38}$$

with

$$F(v) = \int_\Omega \left\{ \sum_{i=1}^{n} a_i(Du)(D_i v - D_i u) - f(v - u) \right\} dx \tag{39}$$

and

$$G(v) = |Dv|^2 - g(x).$$

Moreover, (u, μ^*) is a saddle point of the Lagrange functional

$$\mathcal{L}(v, \mu) = F(v) + \langle \mu, G(v) \rangle, \, \forall v \in H_0^{1,\infty}(\Omega), \forall \mu \in C^*,$$

that is

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \mu^*) \leq \mathcal{L}(v, \mu^*), \; \forall v \in H_0^{1,\infty}(\Omega), \forall \mu \in C^*. \tag{40}$$

Using variational arguments as in [19], we obtain that $\mu^* \in (L^\infty(\Omega))^*$ satisfies conditions (10).

6. Discussions

The paper adds to the already existing literature on nonconstant gradient constrained problem further results related to the relationship with double obstacle problem and the existence of Lagrange multipliers.

In particular, in the paper we focused on the nonconstant gradient constraint $|Du| \leq g(x)$ associated with a nonlinear monotone operator $a(Du)$.

The existence of Lagrange multipliers as Lebesgue functions is guaranteed in the case $f \equiv constant > 0$ and strong monotonicity assumption on the operator, whereas the Lagrange multipliers exist as Radon measure in the case $f \in L^p$, $p > 1$, and strict monotonicity assumption is required.

In the future, several studies could be carried out in several directions in this framework. For example it will be interesting to consider a regular, nonconstant, free term f, or studying the problem associated with different nonlinear operators. Moreover, the properties of the Lagrange multiplier may be investigated. Finally, one could analyze the natural parabolic counterpart.

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