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Applications of Fuzzy Semiprimary Ideals under Group Action

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Abstract: Group actions are a valuable tool for investigating the symmetry and automorphism features of rings. The concept of fuzzy ideals in rings has been expanded with the introduction of fuzzy primary, weak primary, and semiprimary ideals. This paper explores the existence of fuzzy ideals that are semiprimary but neither weak primary nor primary. Furthermore, it defines a group action on a fuzzy ideal and examines the properties of fuzzy ideals and their level cuts under this group action. In fact, it aims to investigate the relationship between fuzzy semiprimary ideals and the radical of fuzzy ideals under group action. Additionally, it includes the results related to the radical of fuzzy ideals and fuzzy $\mathfrak{G}$-semiprimary ideals. Moreover, the preservation of the image and inverse image of a fuzzy $\mathfrak{G}$-semiprimary ideal of a ring $\mathscr{R}$ under certain conditions is also studied. It delves into the algebraic nature of fuzzy ideals and the radical under $\mathfrak{G}$-homomorphism of fuzzy ideals.

Keywords: fuzzy primary ideals; fuzzy $\mathfrak{G}$-primary ideals; radical of fuzzy ideals; fuzzy ideals weak primary; fuzzy semiprimary ideals

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1. Introduction

Since the pioneering work of L.A. Zadeh [1] on fuzzy sets, there has been a growing interest in this field due to its wide-ranging applications in engineering and computer science. Initially, the focus was on fuzzy set theory and fuzzy logic. However, over the past two decades, there has been increasing interest in the development of fuzzy algebra, which generalizes the well-established properties of algebraic structures. A significant portion of mathematical research has been dedicated to the study of fuzzy ideals in rings. Unlike classical subrings (ideals), fuzzy subrings (fuzzy ideals) are characterized by the inability to precisely determine which elements of a ring $\mathscr{R}$ belong to a fuzzy subring (fuzzy ideal). The concept of fuzzy subgroups in groups was introduced and investigated by Rosenfeld [2] in 1971, and since then, numerous researchers [3] have explored the properties of fuzzy subgroups. In 1991, D.S. Malik and J.N. Mordeson [4] introduced and studied the maximality of fuzzy ideals, fuzzy primary ideals, and the fuzzy radical of fuzzy ideals in rings. Subsequently, the concepts of fuzzy nil radicals [5], fuzzy primary ideals [6], and fuzzy primary ideals [7] were introduced for a ring. In 1992, R. Kumar [8] redefined fuzzy semiprimary, fuzzy primary ideals of rings. Furthermore, he characterized fuzzy ideals by their level cuts. In 1993, H.V. Kumbhojkar and M.S. Bapat [9] examined the advantages and disadvantages of various formulations of fuzzy primary ideals. Kalita et al. [10] introduced and investigated singular fuzzy ideals of commutative rings. H.S. Kim et al. [11] explored the radical structure of fuzzy polynomial ideals. Additionally, many researchers [12,13] studied fuzzy ideals in various other algebraic structures. In [14], P. Yiarayong discussed weakly fuzzy primary and weakly fuzzy quasi-primary ideals in LA-semigroups as well as fuzzy primary, fuzzy quasi-primary, and fuzzy completely primary concepts.

In this paper, we assume that $\mathscr{R}$ is a commutative ring with unity and consider the definition of fuzzy primary ideals of a ring $\mathscr{R}$, introduced in [9], and study $\mathfrak{G}$-invariant fuzzy primary and weak primary ideals of the ring $\mathscr{R}$ and their properties. We also studied the properties of primary and fuzzy weak primary ideals of $\mathscr{R}$ under $\mathfrak{G}$-homomorphism.

This paper is categorized into the following sections. With some of the fundamental concepts and outcomes, we start Section 2. Section 3 studies the relationship between primary and radical fuzzy ideals under the group action. In Section 4, we generalize the concept of a fuzzy primary ideal as a weak primary (w-primary) fuzzy ideal and study properties of a fuzzy w-primary ideal under the group action. In Section 5, we broaden the concept of the fuzzy w-primary ideal to the semiprimary fuzzy ideals and explore the characteristics of the semiprimary fuzzy ideals under the group action.

2. Preliminaries

Definition 1. *A map $\phi : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$, with $\phi(a, x)$ written $a * x$, is an action of group $\mathfrak{G}$ on set $\mathfrak{X}$. If $\forall a_1, a_2 \in \mathfrak{G}, z \in \mathfrak{X}$,*
(i) $a_1 * (a_2 * z) = (a_1 a_2) * z$,
(ii) $e * z = z$, where e is the identity element of group $\mathfrak{G}$.

Definition 2. *A homomorphism $\phi : \mathscr{R} \to \mathscr{S}$ from a ring $\mathscr{R}$ to a commutative ring $\mathscr{S}$ with unity is called $\mathfrak{G}$-homomorphism, if for all $\mathfrak{g} \in \mathfrak{G}, r \in \mathscr{R}$, $\phi(\mathfrak{g} * x) = \mathfrak{g} * \phi(x)$, where group $\mathfrak{G}$ acts on both rings.*

A fuzzy set ζ of the ring $\mathscr{R}$ is a map from $\mathscr{R}$ to $[0, 1]$. If this map ζ satisfies the conditions (i) $\zeta(x - y) \geq \min\{\zeta(x), \zeta(y)\}$ and (ii) $\zeta(xy) \geq \max\{\zeta(x), \zeta(y)\}$, then ζ is said to be a fuzzy ideal of $\mathscr{R}$. For rings $\mathscr{R}$ and $\mathscr{S}$, the sets $F(\mathscr{R})$ and $F(\mathscr{S})$ are the collection of all fuzzy ideals of $\mathscr{R}$ and $\mathscr{S}$, respectively, i.e., $F(\mathscr{R}) = \{\zeta : \mathscr{R} \to [0, 1] \mid \zeta \text{ is a fuzzy ideal of } \mathscr{R}\}$ and $F(\mathscr{S}) = \{\zeta : \mathscr{S} \to [0, 1] \mid \zeta \text{ is a fuzzy ideal of } \mathscr{S}\}$.

Definition 3. *Let $\zeta \in F(\mathscr{R}), \zeta' \in F(\mathscr{S})$ and $\phi : \mathscr{R} \to \mathscr{S}$ be a homomorphism. Then, we define the image of ζ under ϕ as follows:*

$$\phi(\zeta)(\mathfrak{z}) = \begin{cases} \bigvee\{\zeta(\mathfrak{y}) | \mathfrak{y} \in \phi^{-1}(\mathfrak{z})\} & \text{if } \phi^{-1}(\mathfrak{z}) \text{ is nonempty} \\ 0 & \text{otherwise} \end{cases}$$

and inverse image of ζ' is a fuzzy subset $\phi^{-1}(\zeta')$ defined by $\phi^{-1}(\zeta')(\mathfrak{z}) = \zeta'(\phi(\mathfrak{z}))$.

Definition 4. *For any fuzzy subset ζ of a set $\mathscr{X}$, set $\zeta_t = \{\mathfrak{z} \in \mathscr{X} | \zeta(\mathfrak{z}) \geq t\}$, where $t \in [0, 1]$, is said to be a level subset of ζ.*

Definition 5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be any sets, $\phi : \mathscr{X} \to \mathscr{Y}$ be any function. A fuzzy subset ζ of $\mathscr{X}$ is called ϕ-invariant if $\phi(p) = \phi(q)$ implies that $\zeta(p) = \zeta(q)$, where $p, q \in \mathscr{X}$.*

Consider $\mathscr{R}$ to be a ring and $\mathfrak{G}$ a finite group acting on $\mathscr{R}$. Now, we define an action of $\mathfrak{G}$ on a fuzzy set ζ of $\mathscr{R}$ as follows:

Definition 6. *An action of $\mathfrak{G}$ on fuzzy set ζ of $\mathscr{R}$ is given by*

$$\zeta^{\mathfrak{g}} = \{\zeta(x^{g}) | \ x \in \mathscr{R}, \mathfrak{g} \in \mathfrak{G}\}$$

where $x^{\mathfrak{g}}$ means $\mathfrak{g}$ acts on x.

Proposition 1. *Let $\phi : \mathscr{R} \to \mathscr{S}$ be a ring homomorphism and $\zeta \in F(\mathscr{R})$ which is a constant on $ker\phi$. Then, $\phi(\zeta)(\phi(\mathfrak{r})) = \zeta(\mathfrak{r})$, for all $\mathfrak{r} \in \mathscr{R}$.*

Proof. Assume that ζ is constant on $ker\phi$ and $\mathfrak{r} \in \mathscr{R}$. Then, for $\mathfrak{s} \in \mathscr{S}$, $\phi(\mathfrak{r}) = \mathfrak{s}$. Moreover, for any $\mathfrak{t} \in \phi^{-1}(\mathfrak{s})$, we have $\phi(\mathfrak{t}) = \mathfrak{s} = \phi(\mathfrak{r})$, and consequently, $\zeta(\mathfrak{t}) = \zeta(\mathfrak{r})$. Now,

$$\begin{aligned}\phi(\zeta)(\phi(\mathfrak{r})) &= \phi(\zeta)(s)\\ &= \sup\{\zeta(\mathfrak{t})|\mathfrak{t} \in \phi^{-1}(\mathfrak{s})\}\\ &= \sup \zeta(\mathfrak{r}) = \zeta(\mathfrak{r}).\end{aligned}$$

□

Definition 7. *Let $\mathfrak{G}$ be a group acting on a ring $\mathscr{R}$. Then, action of the group $\mathfrak{G}$ on $\delta \in F(\mathscr{R})$ is defined as $\delta^{\mathfrak{g}}(r) = \delta(r^{\mathfrak{g}})$.*

3. Fuzzy $\mathfrak{G}$-Primary Ideals

This section deals with the study of fuzzy primary ideals and the properties of fuzzy primary ideals, such as the preservation of images, inverse images under group action, as well as the relationship between the primary and radical of fuzzy ideals under the group action.

Definition 8 ([9]). *A nonconstant fuzzy ideal ζ is said to be primary if for any $\mathfrak{y}, \mathfrak{z} \in \mathscr{R}$, $n \in \mathbb{Z}_+$, $\zeta(\mathfrak{y}\mathfrak{z}) = \zeta(\mathfrak{y})$ or $\zeta(\mathfrak{z}^n)$.*

If ζ is $\mathfrak{G}$-invariant, then ζ is a fuzzy $\mathfrak{G}$-primary ideal of $\mathscr{R}$.

Example 1. *Suppose that ζ is a fuzzy ideal of the ring of integers $\mathbb{Z}$, which is defined as*

$$\zeta(\mathfrak{z}) = \begin{cases} 0.8 & \mathfrak{z} = 0 \\ 0.1 & \mathfrak{z} \neq 0. \end{cases}$$

Then, ζ is a primary.

Definition 9. *Let $\zeta \in F(\mathscr{R})$. Then, the fuzzy set $\sqrt{\zeta}$, defined as $\sqrt{\zeta}(\mathfrak{r}) = \vee\{\zeta(\mathfrak{r}^{\mathfrak{m}})|\mathfrak{m} > 0\}$, is called the radical of fuzzy ideal ζ.*

Example 2. *Consider the ring $\mathbb{Z}_8$ of integers modulo 8. Then, a fuzzy ideal ζ of $\mathbb{Z}_8$ is given by*

z	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\zeta(z)$	1	0	0.5	0	0.6	0	0.5	0

and the radical of fuzzy ideal ζ is given by

z	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\sqrt{\zeta}(z)$	1	0	1	0	1	0	1	0

Proposition 2. *Let $\zeta \in F(\mathscr{R})$ be primary. Then, ζ^g is also a fuzzy primary ideal of $\mathscr{R}$.*

Proof. Let $\zeta \in F(\mathscr{R})$ be primary. Then, for $r, s \in \mathscr{R}$,

$$\begin{aligned}\zeta^{\mathfrak{g}}(rs) &= \zeta(rs)^{\mathfrak{g}} = \zeta(r^{\mathfrak{g}}s^{\mathfrak{g}})\\ &= \zeta(r^{\mathfrak{g}}) \text{ or } \zeta(s^{\mathfrak{g}})^n\\ &= \zeta^{\mathfrak{g}}(r) \text{ or } \zeta(s^n)^{\mathfrak{g}}\\ &= \zeta^{\mathfrak{g}}(r) \text{ or } \zeta^{\mathfrak{g}}(s^n).\end{aligned}$$

This implies that $\zeta^{\mathfrak{g}}$ is a fuzzy primary ideal of $\mathscr{R}$. □

Lemma 1. *Let $\zeta \in F(\mathscr{R})$ be primary. Then, $(\sqrt{\zeta})^{\mathfrak{g}} \in F(\mathscr{R})$ is also primary.*

Proof. By (Proposition 7.2, [9]), which states, "If ζ is a fuzzy ideal of a ring $\mathscr{R}$, then $\sqrt{\zeta}$ is a fuzzy ideal of $\mathscr{R}$", and Proposition 2 , $(\sqrt{\zeta})^{\mathfrak{g}}$ is primary. $\square$

Lemma 2. *Let $\mathfrak{h} : \mathscr{R} \to \mathscr{S}$ be a $\mathfrak{G}$-epimorphism. Then,*

(i) *Image of any fuzzy $\mathfrak{G}$-primary ideal ζ which is constant on ker$\mathfrak{h}$ of $\mathscr{R}$ is a fuzzy $\mathfrak{G}$-primary ideal of $\mathscr{S}$.*

(ii) *Inverse image of any fuzzy $\mathfrak{G}$-primary ideal η of $\mathscr{S}$ is a fuzzy $\mathfrak{G}$-primary ideal of $\mathscr{R}$.*

Proof. (i) Let $\zeta \in F(\mathscr{R})$ be $\mathfrak{G}$-primary. Then, for $r, s \in \mathscr{S}$, $\mathfrak{h}(\zeta)(rs) = \sup_{\mathfrak{h}(t)=rs} \zeta(t)$. Because $\mathfrak{h}$ is onto, there exists $t \in \mathscr{R}$ such that $\mathfrak{h}(t) = rs$ also there exist $r_1, s_1 \in \mathscr{R}$ such that $\mathfrak{h}(r_1) = r$ and $\mathfrak{h}(s_1) = s$. Thus, $\mathfrak{h}(\zeta)(rs) = \sup_{\mathfrak{h}(t)=rs} \zeta(t)$. Because ζ is a $\mathfrak{G}$-primary ideal and ζ is constant on *ker*$\mathfrak{h}$, then by Proposition 1,

$$\begin{aligned}\mathfrak{h}(\zeta)(rs) &= \underset{\mathfrak{h}(r_1)=r}{\zeta(r_1)} \text{ or } \underset{\mathfrak{h}(s_1)=s}{\zeta(s_1^n)} \\ &= \mathfrak{h}(\zeta)(r) \text{ or } \mathfrak{h}(\zeta)(s^n).\end{aligned}$$

Thus, $\mathfrak{h}(\zeta) \in F(\mathscr{R})$ is primary.

Now, we will show that $\mathfrak{h}(\zeta)$ is $\mathfrak{G}$-invariant. Suppose that $r \in \mathscr{R}$. Then,

$$\begin{aligned}\mathfrak{h}(\zeta)(r^{\mathfrak{g}}) &= \sup_{\mathfrak{h}(z)=r^{\mathfrak{g}}} \zeta(z) = \sup_{(\mathfrak{h}(z))^{1/\mathfrak{g}}=r} \zeta(z) \\ &= \sup_{\mathfrak{h}(z^{1/\mathfrak{g}})=r} \zeta(z^{1/\mathfrak{g}}) \\ &= \mathfrak{h}(\zeta)(r).\end{aligned}$$

This shows that $\mathfrak{h}(\zeta) \in F(\mathscr{S})$ is $\mathfrak{G}$-primary.

(ii) Let $\eta \in F(\mathscr{S})$ be $\mathfrak{G}$-primary. Then, for $s, t \in \mathscr{S}$,

$$\begin{aligned}\mathfrak{h}^{-1}(\eta)(st) &= \eta(\mathfrak{h}(st)) = \eta(\mathfrak{h}(s)\mathfrak{h}(t)) \\ &= \eta(\mathfrak{h}(s)) \text{ or } \eta(\mathfrak{h}(t)^n) \\ &= \mathfrak{h}^{-1}(\eta)(s) \text{ or } \mathfrak{h}^{-1}(\eta)(t^n).\end{aligned}$$

This implies that $\mathfrak{h}^{-1}(\eta) \in F(\mathscr{R})$ is primary. Because η is $\mathfrak{G}$-invariant, for $s \in \mathscr{S}$,

$$\begin{aligned}\mathfrak{h}^{-1}(\eta)(s^g) &= \eta(\mathfrak{h}(s^g)) = \eta((\mathfrak{h}(s))^g) \\ &= \eta(\mathfrak{h}(s)) \\ &= \mathfrak{h}^{-1}(\eta)(s).\end{aligned}$$

This implies that $\mathfrak{h}^{-1}(\eta) \in F(\mathscr{R})$ is $\mathfrak{G}$-primary. $\square$

Proposition 3. *$\zeta \in F(\mathscr{R})$ is primary ($\mathfrak{G}$-primary) iff each of its level cuts is primary ($\mathfrak{G}$-primary).*

Proof. Let $\zeta \in F(\mathscr{R})$ be $\mathfrak{G}$-primary. Then, for any $\mathfrak{y}, \mathfrak{z} \in \mathscr{R}$, $\mathfrak{y}\mathfrak{z} \in \zeta_t$, i.e., $\zeta(\mathfrak{y}\mathfrak{z}) \geq t$. Because ζ is primary ideal, we have

$$\zeta(\mathfrak{y}\mathfrak{z}) = \zeta(\mathfrak{y}) \geq t \text{ or } \zeta(\mathfrak{y}\mathfrak{z}) = \zeta(\mathfrak{z}^n) \geq t.$$

This implies that ζ_t is a primary ideal.

Now, we will show that ζ_t is $\mathfrak{G}$-invariant, i.e., $\zeta_t^{\mathfrak{G}} = \zeta_t$.

$$\begin{aligned}\zeta_t^{\mathfrak{g}} &= \{x^{\mathfrak{g}} \in \mathscr{R} | \zeta(x^{\mathfrak{g}}) \geq t\} \\ &= \{x^{\mathfrak{g}} \in \mathscr{R} | \zeta^{\mathfrak{g}}(x) \geq t\}.\end{aligned}$$

Because ζ is $\mathfrak{G}$-invariant, then $\zeta_t^{\mathfrak{g}} = \zeta_t$. The converse holds directly by definition of the semiprimary ideal of $\mathscr{R}$. $\square$

Theorem 1. *If $\mathfrak{h}$ is a homomorphism from a ring $\mathscr{R}$ onto ring a $\mathscr{S}$, ζ_1, ζ_2 and η_1, η_2 are fuzzy ideals of $\mathscr{R}$ and $\mathscr{S}$, respectively, and then the following hold:*

(i) $(\mathfrak{h}^{-1}(\eta_1)\mathfrak{h}^{-1}(\eta_2))^{\mathfrak{G}} \subseteq (\mathfrak{h}^{-1}(\eta_1\eta_2))^{\mathfrak{G}}$.

(ii) $(\mathfrak{h}(\zeta_1)\mathfrak{h}(\zeta_2))^{\mathfrak{G}} \subseteq (\mathfrak{h}(\zeta_1\zeta_2))^{\mathfrak{G}}$.

(iii) *If $\mathfrak{h}$ is $\mathfrak{G}$-homomorphism and $\eta_1 \subseteq \eta_2$, then $(\mathfrak{h}^{-1}(\eta_1))^{\mathfrak{G}} \subseteq \mathfrak{h}^{-1}(\eta_2)^{\mathfrak{G}}$.*

Proof. (i) Let $s \in \mathscr{S}$. Then,

$$\begin{aligned}
(\mathfrak{h}^{-1}(\eta_1)\mathfrak{h}^{-1}(\eta_2))^{\mathfrak{G}}(s) &= \bigcap_{g\in\mathfrak{G}} (\mathfrak{h}^{-1}(\eta_1)\mathfrak{h}^{-1}(\eta_2))^{g}(s) \\
&= \bigcap_{g\in\mathfrak{G}} [\sup_{s^g=ab} \{\min(\mathfrak{h}^{-1}(\eta_1)(a), \mathfrak{h}^{-1}(\eta_2)(b))\}] \\
&\leq \bigcap_{g\in\mathfrak{G}} \{\min(\eta_1(\mathfrak{h}(a)), \eta_2(\mathfrak{h}(b)))\} = \bigcap_{g\in\mathfrak{G}} \{\min(\eta_1(\mathfrak{h}(a)), \eta_2(\mathfrak{h}(b)))\} \\
&= \bigcap_{g\in\mathfrak{G}} \eta_1\eta_2(\mathfrak{h}(ab)) = \bigcap_{g\in\mathfrak{G}} \mathfrak{h}^{-1}(\eta_1\eta_2)(s^g) \\
&= \bigcap_{g\in\mathfrak{G}} (\mathfrak{h}^{-1}(\eta_1\eta_2))^{g}(s) \\
&= (\mathfrak{h}^{-1}(\eta_1\eta_2))^{\mathfrak{G}}(s).
\end{aligned}$$

Thus, $(\mathfrak{h}^{-1}(\eta_1)\mathfrak{h}^{-1}(\eta_2))^{\mathfrak{G}} \subseteq (\mathfrak{h}^{-1}(\eta_1\eta_2))^{\mathfrak{G}}$. Equality holds, if $\mathfrak{h}$ is injective. For $s \in \mathscr{R}$,

$$\begin{aligned}
(\mathfrak{h}^{-1}(\eta_1\eta_2))^{\mathfrak{G}}(s) &= \bigcap_{g\in\mathfrak{G}} (\mathfrak{h}^{-1}(\eta_1\eta_2))^{g}(s) = \bigcap_{g\in\mathfrak{G}} \mathfrak{h}^{-1}(\eta_1\eta_2)(s^g) \\
&= \bigcap_{g\in\mathfrak{G}} [\sup_{\mathfrak{h}(s^g)=ab} \{\min((\eta_1)(a), (\eta_2)(b))\}].
\end{aligned}$$

Because $\mathfrak{h}$ is onto and $a, b \in \mathscr{S}$, there exist $a_1, b_1 \in \mathscr{R}$ such that $\mathfrak{h}(a_1) = a$ and $\mathfrak{h}(b_1) = b$. Moreover, given that $\mathfrak{h}$ is injective, $\mathfrak{h}(s^g) = \mathfrak{h}(ab)$ hence implies that $s^g = ab$. Thus,

$$\begin{aligned}
\bigcap_{g\in\mathfrak{G}} [\sup_{\mathfrak{h}(s^g)=ab} \{\min((\eta_1)(a), (\eta_2)(b))\}] &= \bigcap_{g\in\mathfrak{G}} [\sup_{\mathfrak{h}(s^g)=\mathfrak{h}(a_1)\mathfrak{h}(b_1)} \{\min((\eta_1)(\mathfrak{h}(a_1)), (\eta_2)(\mathfrak{h}(b_1)))\}] \\
&= \bigcap_{g\in\mathfrak{G}} [\sup_{s^g=a_1b_1} \{\min(\mathfrak{h}^{-1}(\eta_1)(a_1), \mathfrak{h}^{-1}(\eta_2)(b_1))\}] \\
&= \bigcap_{g\in\mathfrak{G}} (\mathfrak{h}^{-1}(\eta_1)\mathfrak{h}^{-1}(\eta_2))^{g}(s) \\
&= (\mathfrak{h}^{-1}(\eta_1)\mathfrak{h}^{-1}(\eta_2))^{\mathfrak{G}}(s).
\end{aligned}$$

Therefore, $(\mathfrak{h}^{-1}(\eta_1)\mathfrak{h}^{-1}(\eta_2))^{\mathfrak{G}} = (\mathfrak{h}^{-1}(\eta_1\eta_2))^{\mathfrak{G}}$.

(ii) Let $\mathfrak{r} \in \mathscr{R}$. Then,

$$\begin{aligned}
(\mathfrak{h}(\zeta_1)\mathfrak{h}(\zeta_2))^{\mathfrak{G}}(\mathfrak{r}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} (\mathfrak{h}(\zeta_1)\mathfrak{h}(\zeta_2))(\mathfrak{r}^{\mathfrak{g}}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}} [\sup_{\mathfrak{r}^{\mathfrak{g}}=\mathfrak{a}\mathfrak{b}} \{\min(\mathfrak{h}(\zeta_1)(\mathfrak{a}), \mathfrak{h}(\zeta_2)(\mathfrak{b}))\}] \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}} [\sup_{\mathfrak{r}^{\mathfrak{g}}=\mathfrak{a}\mathfrak{b}} \{\min(\sup_{\mathfrak{a}_1\in\mathfrak{h}^{-1}(\mathfrak{a})} \zeta_1(\mathfrak{a}_1), \sup_{\mathfrak{b}_1\in\mathfrak{h}^{-1}(\mathfrak{b})} \zeta_2(\mathfrak{b}_1))\}].
\end{aligned}$$

Now, for some $\mathfrak{a}_2 \in \mathfrak{h}(\mathfrak{a}), \mathfrak{b}_2 \in \mathfrak{h}(\mathfrak{b}), \mathfrak{a}_2\mathfrak{b}_2 = \mathfrak{c}$, and $\mathfrak{r}^{\mathfrak{g}} = \mathfrak{a}\mathfrak{b}$,

$$\begin{aligned}
\bigcap_{\mathfrak{g}\in\mathfrak{G}}[\sup_{\mathfrak{r}^{\mathfrak{g}}=\mathfrak{a}\mathfrak{b}}\{\min(\sup_{\mathfrak{a}_1\in\mathfrak{h}^{-1}(\mathfrak{a})}\zeta_1(\mathfrak{a}_1), \sup_{\mathfrak{b}_1\in\mathfrak{h}^{-1}(\mathfrak{b})}\zeta_2(\mathfrak{b}_1))\}] &\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{\min(\zeta_1(\mathfrak{a}_2),\zeta_2(\mathfrak{b}_2))\} \\
&\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}}(\zeta_1\zeta_2)(\mathfrak{a}_2\mathfrak{b}_2) \\
&\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}}\sup_{\mathfrak{c}\in\mathfrak{h}^{-1}(\mathfrak{r}^{\mathfrak{g}})}(\zeta_1\zeta_2)(\mathfrak{c}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\mathfrak{h}(\zeta_1\zeta_2)(\mathfrak{r}^{\mathfrak{g}}) \\
&= \mathfrak{h}(\zeta_1\zeta_2)^{\mathfrak{G}}(\mathfrak{r}).
\end{aligned}$$

This shows that $(\mathfrak{h}(\zeta_1)\mathfrak{h}(\zeta_2))^{\mathfrak{G}} \subseteq \mathfrak{h}(\zeta_1\zeta_2)^{\mathfrak{G}}$.

(iii) Let $\mathfrak{s} \in \mathscr{S}$. Then,

$$\begin{aligned}
(\mathfrak{h}^{-1}(\eta_1))^{\mathfrak{G}}(\mathfrak{s}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(\mathfrak{h}^{-1}(\eta_1))^{\mathfrak{g}}(\mathfrak{s}) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}(\mathfrak{h}^{-1}(\eta_1))(\mathfrak{s}^{\mathfrak{g}}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\eta_1(\mathfrak{h}(\mathfrak{s}^{\mathfrak{g}})) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}\eta_1((\mathfrak{h}(\mathfrak{s}))^{\mathfrak{g}}) \\
&\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}}\eta_2((\mathfrak{h}(\mathfrak{s}))^{\mathfrak{g}}) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}\eta_2^{\mathfrak{g}}((\mathfrak{h}(\mathfrak{s}))) \\
&= \mathfrak{h}^{-1}(\eta_2^{\mathfrak{G}})(\mathfrak{s}).
\end{aligned}$$

This implies that $(\mathfrak{h}^{-1}(\eta_1))^{\mathfrak{G}} \subseteq \mathfrak{h}^{-1}(\eta_2^{\mathfrak{G}})$. □

4. Weak Primary and Fuzzy $\mathfrak{G}$-Weak Primary Ideals

In this section, we study fuzzy weak primary ideals of the ring $\mathscr{R}$, a concept more general than that of fuzzy primary ideals.

Definition 10 ([9]). *A fuzzy ideal ζ of a ring $\mathscr{R}$ is said to be weak primary (w-primary), if $\zeta(uv) = \zeta(u)$ or $\zeta(uv) \leq \zeta(v^n)$, for some $n \in \mathbb{Z}^+$.*

Proposition 4 ([9]). *Every fuzzy primary ideal is w-primary. However, the converse is not true in general.*

Proof. Straightforward. □

Example 3 ([9]). *Assume that ζ is a fuzzy ideal of the ring of integers $\mathbb{Z}$, defined as follows:*

$$\zeta(\mathfrak{z}) = \begin{cases} 0 & \text{for } \mathfrak{z} \notin \langle p\rangle, \\ \frac{n}{n+1} & \text{for } \mathfrak{z} \in \langle p^n\rangle \sim \langle p^{n+1}\rangle, n = 1,2,\ldots,5, \\ 1 & \ \mathfrak{z} \in \langle p^6\rangle \end{cases}$$

For $\zeta(p^2p^3) = \zeta(p^5) = \frac{5}{6} < 1 = \zeta((p^3)^2)$, $\zeta(p^2) = \frac{2}{3} \neq \frac{5}{6}$ and $\zeta(p^3) = \frac{3}{4} > \zeta((p^3)^2 = 1$. This shows that ζ is not primary but a fuzzy weak primary ideal.

Example 4. *Let $\zeta \in F(\mathbb{Z}_4)$, defined as follows:*

$$\zeta(\mathfrak{z}) = \begin{cases} 0.8 & \mathfrak{z} = 0 \\ 0.1 & \mathfrak{z} \neq 0. \end{cases}$$

Because $\zeta(2 \cdot 1) = \zeta(2) = 0.1 < \zeta(2^2) = \zeta(4) = 0.8$, then ζ is weak primary but not primary.

Proposition 5. *If $\zeta \in F(\mathscr{R})$ is the weak primary, then $\zeta^g \in F(\mathscr{R})$ is weak primary.*

Proof. Let $\zeta \in F(\mathscr{R})$ be weak primary. Then, for any $r, s \in \mathscr{R}$, $\zeta^g(rs) = \zeta(rs)^g = \zeta(r^g s^g)$. Because ζ is weak primary and $r^g, s^g \in \mathscr{R}$, then $\zeta(r^g s^g) = \zeta(r^g)$ or $\zeta(r^g s^g) \leq \zeta(s^g)^n$, for some $n \in \mathbb{Z}^+$. Hence, ζ^g is w-primary. □

Proposition 6. *A fuzzy ideal ζ is $\mathfrak{G}$-weak primary iff each of the fuzzy ideal ζ's level cuts is $\mathfrak{G}$-primary.*

Proof. Let $\zeta \in F(\mathscr{R})$ be $\mathfrak{G}$-weak primary. Then, we have to show that each level cut ζ_t is $\mathfrak{G}$-weak primary.
For any $a, b \in \mathscr{R}$, $ab \in \zeta_t$, i.e., $\zeta(ab) \geq t$. Because ζ is primary ideal, we have

$$\zeta(ab) = \zeta(a) \geq t \text{ or } t \leq \zeta(ab) \leq \zeta(b^n).$$

This implies that ζ_t is a weak primary ideal.

Now, we will show that ζ_t is $\mathfrak{G}$-invariant, i.e., $\zeta_t^{\mathfrak{G}} = \zeta_t$.

$$\begin{aligned} \zeta_t^g &= \{x^g \in \mathscr{R} | \zeta(x^g) \geq t\} \\ &= \{x^g \in \mathscr{R} | \zeta^g(x) \geq t\}. \end{aligned}$$

Because ζ is $\mathfrak{G}$-invariant, then $\zeta_t^g = \zeta_t$. □

Theorem 2. *Let $\mathfrak{h} : \mathscr{R} \to \mathscr{S}$ be a $\mathfrak{G}$-homomorphism of rings.*

(i) *If $\zeta_1 \in F(\mathscr{S})$ is $\mathfrak{G}$-weak primary, then so $\mathfrak{h}^{-1}(\zeta_1) \in F(\mathscr{R})$ is $\mathfrak{G}$-weak primary. Converse is true if $\mathfrak{h}$ is an epimorphism.*

(ii) *Let $\mathfrak{h}$ be an epimorphism. Then, $\zeta \in F(\mathscr{R})$ is $\mathfrak{G}$-weak primary iff $\mathfrak{h}(\zeta) \in F(\mathscr{S})$ is $\mathfrak{G}$-weak primary.*

Proof. (i) Let $\zeta_1 \in F(\mathscr{S})$ be $\mathfrak{G}$-weak primary. Then, for $r_1, r_2 \in \mathscr{R}$,

$$\begin{aligned} \mathfrak{h}^{-1}(\zeta_1)(r_1 r_2) &= \zeta_1(\mathfrak{h}(r_1 r_2)) = \zeta_1(\mathfrak{h}(r_1)\mathfrak{h}(r_2)) \\ &= \zeta_1(\mathfrak{h}(r_1)) \text{ or } \leq \zeta_1(\mathfrak{h}(r_2)^n) \\ &= \mathfrak{h}^{-1}(\zeta_1)(r_1) \text{ or } \leq \mathfrak{h}^{-1}(\zeta_1)(r_2^n). \end{aligned}$$

This implies that $\mathfrak{h}^{-1}(\zeta_1)$ is weak primary. Because ζ_1 is $\mathfrak{G}$-invariant, then for $r \in \mathscr{R}$,

$$\begin{aligned} (\mathfrak{h}^{-1}(\zeta_1))^{\mathfrak{g}}(r) &= \zeta_1(\mathfrak{h}(r^g)) = \zeta_1^{\mathfrak{g}}(\mathfrak{h}(r)) \\ &= \zeta_1^{\mathfrak{G}}(\mathfrak{h}(r)) = \zeta_1(\mathfrak{h}(r)) \\ &= \mathfrak{h}^{-1}(\zeta_1)(r). \end{aligned}$$

This implies that $\mathfrak{h}^{-1}(\zeta_1) \in F(\mathscr{R})$ is $\mathfrak{G}$-weak primary.

Conversely, suppose that $\zeta_1 \in F(\mathscr{S})$, $\mathfrak{h}^{-1}(\zeta_1) \in F(\mathscr{R})$ is $\mathfrak{G}$-weak primary and $\mathfrak{h}$ is an epimorphism. Then, for any $s_1, s_2 \in \mathscr{S}$ there exist $r_1, r_2 \in \mathscr{R}$ such that $\mathfrak{h}(r_1) = s_1$ and $\mathfrak{h}(r_2) = s_2$. Thus, we have

$$\begin{aligned} \zeta_1(s_1 s_2) &= \zeta_1(\mathfrak{h}(r_1)\mathfrak{h}(r_2)) = \zeta_1(\mathfrak{h}(r_1 r_2)) \\ &= \mathfrak{h}^{-1}(\zeta_1)(r_1 r_2). \end{aligned}$$

Because $\mathfrak{h}^{-1}(\zeta_1)$ is $\mathfrak{G}$-weak primary, we have

$$\begin{aligned} \mathfrak{h}^{-1}(\zeta_1)(r_1 r_2) &= \mathfrak{h}^{-1}(\zeta_1)(r_1) \text{ or } \mathfrak{h}^{-1}(\zeta_1)(r_1 r_2) \leq \mathfrak{h}^{-1}(\zeta_1)(r_2^n), \text{ for some } n \in \mathbb{Z}^+ \\ &= \zeta_1(\mathfrak{h}(r_1)) \text{ or } \mathfrak{h}^{-1}(\zeta_1)(r_1 r_2) \leq \zeta_1(\mathfrak{h}(r_2))^n \\ &= \zeta_1(s_1) \text{ or } \mathfrak{h}^{-1}(\zeta_1)(r_1 r_2) \leq \zeta_1(s_2)^n. \end{aligned}$$

Therefore, $\zeta_1 \in F(\mathscr{S})$ is weak primary. For $s \in \mathscr{S}$,

$$\begin{aligned}\zeta_1(s^g) &= \zeta_1(\mathfrak{h}(r^g)) = \mathfrak{h}^{-1}(\zeta_1)(r^g) \\ &= \mathfrak{h}^{-1}(\zeta_1)(r) = \zeta_1(\mathfrak{h}(r)) \\ &= \zeta_1(s).\end{aligned}$$

This shows that $\zeta_1 \in F(\mathscr{S})$ is $\mathfrak{G}$-weak primary.

(ii) Assume that $\zeta \in F(\mathscr{R})$ be $\mathfrak{G}$-weak primary. Then, for $s_1, s_2 \in \mathscr{S}$, $\mathfrak{h}(\zeta)(s_1 s_2) = \sup_{\mathfrak{h}(z)=s_1s_2} \zeta(z)$. Because $\mathfrak{h}$ is onto, there exists $z \in \mathscr{R}$ such that $\mathfrak{h}(z) = s_1 s_2$ also there exist $t_1, t_2 \in \mathscr{R}$ such that $\mathfrak{h}(t_1) = s_1$ and $\mathfrak{h}(t_2) = s_2$. Thus, $\mathfrak{h}(\zeta)(s_1 s_2) = \sup_{\mathfrak{h}(z)=s_1s_2} \zeta(z)$. Because ζ is $\mathfrak{G}$-weak primary, we obtain

$$\begin{aligned}\mathfrak{h}(\zeta)(s_1 s_2) &= \underset{\mathfrak{h}(t_1)=s_1}{\zeta(t_1)} \ \text{ or } \ \leq \underset{\mathfrak{h}(t_2)=s_2}{\zeta(t_2^n)} \\ &= \mathfrak{h}(\zeta)(s_1) \ \text{ or } \ \leq \mathfrak{h}(\zeta)(s_2^n).\end{aligned}$$

This shows that $\mathfrak{h}(\zeta) \in F(\mathscr{S})$ is weak primary.

Now, we will show that $\mathfrak{h}(\zeta)$ is $\mathfrak{G}$-invariant. Suppose that $s \in \mathscr{S}$. Then,

$$\begin{aligned}\mathfrak{h}(\zeta)(s^g) &= \sup_{\mathfrak{h}(z)=s^g} \zeta(z) = \sup_{(\mathfrak{h}(z))^{g^{-1}}=s} \zeta(z) \\ &= \sup_{\mathfrak{h}(z^{g^{-1}})=s} \zeta(z^{g^{-1}}) \\ &= \mathfrak{h}(\zeta)(s).\end{aligned}$$

Hence, $\mathfrak{h}(\zeta) \in F(\mathscr{S})$ is $\mathfrak{G}$-weak primary. $\square$

5. Fuzzy Semiprimary Ideals and Their Applications

This section is devoted to studying some fundamental characteristics of fuzzy semiprimary ideals and their images as well as studying the relationship between fuzzy semiprimary, fuzzy w-primary, primary, prime, and maximal ideals.

Definition 11 ([8])**.** *A fuzzy ideal η of a ring $\mathscr{R}$ is called a fuzzy semiprimary if either $\eta(xy) \leq \eta(x^m)$ or $\eta(xy) \leq \eta(y^n)$, for all $x, y \in \mathscr{R}$ and for some $m, n \in \mathbb{Z}_+$.*

Definition 12. *A $\mathfrak{G}$-invariant fuzzy ideal of a ring $\mathscr{R}$ which is semiprimary is called a fuzzy $\mathfrak{G}$-semiprimary ideal of $\mathscr{R}$.*

The following lemma is straightforward, from the Definitions 8 and 10.

Lemma 3. *Every primary (or w-primary) fuzzy ideal is a fuzzy semiprimary ideal.*

Remark 1. *Converse of Lemma 3 need not be true in general.*

Example 5. *Let $\mathbb{Z}$ be a ring of integers and $\zeta \in F(\mathbb{Z})$ is defined as*

$$\zeta(z) = \begin{cases} 1 & \text{if } z \in 4\mathbb{Z} \\ \frac{1}{2} & \text{if } z \in 2\mathbb{Z} \setminus 4\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Then, ζ is not w-primary because $\zeta(2 \cdot 3) = \zeta(6) = \frac{1}{2} < 1 = \zeta(2^2)$ and $\zeta(2 \cdot 3) = \zeta(6) = \frac{1}{2} > 0 = \zeta(3^m)$ for any $m \in \mathbb{Z}_+$.

Lemma 4. *Let $\mathfrak{h} : \mathscr{R} \to \mathscr{S}$ be a $\mathfrak{G}$-epimorphism. Then,*

(i) *Image of any fuzzy $\mathfrak{G}$-semiprimary ideal ζ which is constant on ker$\mathfrak{h}$ of $\mathscr{R}$ is a fuzzy $\mathfrak{G}$-semiprimary ideal of $\mathscr{S}$.*

(ii) *Inverse image of any fuzzy $\mathfrak{G}$-semiprimary ideal η of $\mathscr{S}$ is a fuzzy $\mathfrak{G}$-semiprimary ideal of $\mathscr{R}$.*

Proof. Follows from Lemma 2. □

The following proposition is easy to prove.

Proposition 7. *ζ is a fuzzy $\mathfrak{G}$-semiprimary ideal of $\mathscr{R}$ iff each of its level cuts is a $\mathfrak{G}$-semiprimary ideal of $\mathscr{R}$.*

Theorem 3. *Let $\mathscr{R}$ be a ring and $\zeta, \delta \in F(\mathscr{R})$. Then, ζ and δ adhere to the following properties:*

(i) *If ζ is semiprimary, then $\sqrt{\zeta}$ is w-primary.*

(ii) $(\sqrt{\zeta \cap \delta})^{\mathfrak{G}} = (\sqrt{\zeta})^{\mathfrak{G}} \cap (\sqrt{\delta})^{\mathfrak{G}}$.

Proof. (i) Suppose on the contrary that $\sqrt{\zeta}$ is not w-primary, i.e., neither $\sqrt{\zeta}(xy) = \sqrt{\zeta}(x)$ nor $\sqrt{\zeta}(xy) \leq \sqrt{\zeta}(y^n)$ for $x, y \in \mathscr{R}$ for $n \in \mathbb{Z}_+$. Then, for all $x, y \in \mathscr{R}$,

$$
\begin{aligned}
\sqrt{\zeta}(xy) &= \sup\{\zeta(x^n y^n) \mid n > 0\} \\
&\leq \sup\{\zeta(x^n)^l \mid n > 0 \text{ and for some } l \in \mathbb{Z}_+\} \\
&\quad \text{or } \leq \sup\{\zeta(y^n)^m \mid n > 0 \text{ and for some } m \in \mathbb{Z}_+\} \\
&= \sup\{\zeta(x^n)^l \mid n > 0 \text{ and for some } l \in \mathbb{Z}_+\} \\
&\quad \text{or } = \sup\{\zeta(y^n)^m \mid n > 0 \text{ and for some } m \in \mathbb{Z}_+\}.
\end{aligned}
$$

This implies that $\sqrt{\zeta}(xy) \leq \sqrt{\zeta}(x)$ or $\sqrt{\zeta}(xy) \leq \sqrt{\zeta}(y)$. However, $\sqrt{\zeta}(xy) \neq \sqrt{\zeta}(x)$ and $\sqrt{\zeta}(xy) > \sqrt{\zeta}(y^n)$, a contradiction. Hence, $\sqrt{\zeta}$ is w-primary.

(ii) Because $\zeta \subseteq \sqrt{\zeta}$ and $\delta \subseteq \sqrt{\delta}$, then $\zeta^{\mathfrak{G}} \subseteq \sqrt{\zeta}^{\mathfrak{G}}$ and $\delta^{\mathfrak{G}} \subseteq \sqrt{\delta}^{\mathfrak{G}}$. This implies that $(\sqrt{\zeta})^{\mathfrak{G}} \cap (\sqrt{\delta})^{\mathfrak{G}} \subseteq \sqrt{\zeta} \cap \sqrt{\delta} \subseteq \sqrt{\zeta \cap \delta}^{\mathfrak{G}}$.

For any $x \in \mathscr{R}$,

$$
\begin{aligned}
(\sqrt{\zeta \cap \delta})^{\mathfrak{G}}(x) &= \bigcap_{g \in \mathfrak{G}} (\sqrt{\zeta \cap \delta})(x^g) = \bigcap_{g \in \mathfrak{G}} [\sup\{(\zeta \cap \delta)(x^g)^{\mathfrak{l}} \mid \mathfrak{l} > 0\}] \\
&= \bigcap_{g \in \mathfrak{G}} \big[\sup\{\inf\big(\zeta(x^g)^{\mathfrak{l}}, \delta(x^g)^{\mathfrak{l}}\big) \mid \mathfrak{l} > 0\}\big] \\
&\leq \bigcap_{g \in \mathfrak{G}} \big[\inf\{\sup\big(\zeta(x^g)^{\mathfrak{l}} \mid \mathfrak{l} > 0\big), \sup\big(\delta(x^g)^{\mathfrak{l}} \mid \mathfrak{l} > 0\big)\}\big] \\
&= \inf\big[\bigcap_{g \in \mathfrak{G}} \{\sqrt{\zeta}(x^g), \sqrt{\delta}(x^g)\}\big] = \inf\{\bigcap_{g \in \mathfrak{G}} \sqrt{\zeta}(x^g), \bigcap_{g \in \mathfrak{G}} \sqrt{\delta}(x^g)\} \\
&= ((\sqrt{\zeta})^{\mathfrak{G}} \cap (\sqrt{\delta})^{\mathfrak{G}})(x).
\end{aligned} \tag{1}
$$

Thus, $(\sqrt{\zeta \cap \delta})^{\mathfrak{G}} = (\sqrt{\zeta})^{\mathfrak{G}} \cap (\sqrt{\delta})^{\mathfrak{G}}$.

Theorem 4. *Let $\zeta \in F(\mathscr{R})$ be $\mathfrak{G}$-semiprimary. Then, $\zeta = (\sqrt{\zeta})^{\mathfrak{G}}$.*

Proof. Suppose that $\zeta \in F(\mathscr{R})$ is $\mathfrak{G}$-semiprimary. Then, for $x \in \mathscr{R}$,

$$
\begin{aligned}
(\sqrt{\zeta})^{\mathfrak{G}}(x) &= \bigcap_{g \in \mathfrak{G}} [\sqrt{\zeta}(x^g)] \\
&= \bigcap_{g \in \mathfrak{G}} [\sup\{\zeta(x^m)^g \mid m > 0\}] \\
&\leq \sup\{\zeta(x^m) \mid m > 0\}.
\end{aligned} \tag{2}
$$

Because ζ is semiprimary, we obtain $\zeta(x^m) \leq \zeta(x)$. Equation (2) implies that $(\sqrt{\zeta})^{\mathfrak{G}} \subseteq \zeta$. For other inclusion,

$$\zeta(x) = \zeta^{\mathfrak{G}}(x) = \underset{g\in\mathfrak{G}}{\cap} \zeta^g(x) = \underset{g\in\mathfrak{G}}{\cap} \zeta(x^g). \tag{3}$$

Due to the fact that ζ is a fuzzy ideal, $\zeta(x) \leq \zeta(x^n)$, for some $n \in \mathbb{Z}_+$. This implies that

$$\underset{g\in\mathfrak{G}}{\cap} \zeta(x^g) \leq \underset{g\in\mathfrak{G}}{\cap} \zeta(x^g)^m. \tag{4}$$

By Equations (3) and (4), we obtain

$$\begin{aligned}\zeta(x) &\leq \underset{g\in\mathfrak{G}}{\cap} \zeta(x^g)^m \leq \underset{g\in\mathfrak{G}}{\cap} [\sup\{\zeta^g(x^m) \mid m > 0\}] \\ &= \underset{g\in\mathfrak{G}}{\cap} (\sqrt{\zeta})^g(x) \\ &= (\sqrt{\zeta})^{\mathfrak{G}}(x).\end{aligned}$$

This shows that $\zeta = (\sqrt{\zeta})^{\mathfrak{G}}$.

We know that, in general, a fuzzy prime ideal need not be maximal and a fuzzy semiprimary ideal need not be w-primary. Next, the theorem shows under what condition a fuzzy semiprimary ideal is w-primary. □

Theorem 5. *Let $\mathscr{R}$ be a ring. If every fuzzy prime ideal in $\mathscr{R}$ is maximal, then every fuzzy semiprimary ideal of $\mathscr{R}$ is w-primary.*

Proof. Suppose that $\zeta \in F(\mathscr{R})$ is semiprimary. Then, by (Theorem 5.2, [8]), which states, "A fuzzy ideal ζ of ring $\mathscr{R}$ is fuzzy semiprimary iff $\sqrt{\zeta}$ is a fuzzy prime ideal of $\mathscr{R}$", $\sqrt{\zeta} \in F(\mathscr{R})$ is prime. By assumption, $\sqrt{\zeta}$ is maximal. Hence, from (Theorem 5.4, [8]), which states, "Let ζ be any fuzzy ideal of ring $\mathscr{R}$. If $\sqrt{\zeta}$ is fuzzy maximal, then $\sqrt{\zeta}$ is fuzzy w-primary", ζ is w-primary. □

Theorem 6. *If $\zeta \in F(\mathscr{R})$ is semiprimary, then $\zeta^{\mathfrak{G}} \in F(\mathscr{R})$ is $\mathfrak{G}$-semiprimary. Conversely, if $\eta \in F(\mathscr{R})$ is $\mathfrak{G}$-semiprimary, then there exists a $\zeta \in F(\mathscr{R})$ which is semiprimary such that $\zeta^{\mathfrak{G}} = \eta$.*

Proof. Let $\zeta \in F(\mathscr{R})$ be semiprimary. Then, $\zeta^{\mathfrak{G}}(xy) = \underset{g\in\mathfrak{G}}{\cap} \{\zeta(x^g y^g)\} \leq \underset{g\in\mathfrak{G}}{\cap} \zeta(x^g)^m$ or $\leq \underset{g\in\mathfrak{G}}{\cap} \zeta(y^g)^n$. Thus, $\zeta^{\mathfrak{G}}(xy) \leq \zeta^{\mathfrak{G}}(x^m)$ or $\leq \zeta^{\mathfrak{G}}(y^n)$, for some $m, n \in \mathbb{Z}_+$. For the converse part, it is obvious that $\eta^{\mathfrak{G}} = \eta$. By Zorn's lemma, we can obtain a maximal ideal, say $\zeta \in F(\mathscr{R})$. Our claim is that ζ is semiprimary. Because $\zeta^{\mathfrak{G}} \subseteq \zeta \subseteq \eta$, for any $x, y \in \mathscr{R}$, $\zeta(xy) \leq \eta(xy)$. Due to the maximality of ζ and the semiprimariness of η, ζ is a fuzzy semiprimary ideal such that $\zeta^{\mathfrak{G}} = \eta$. □

6. Conclusions

In this manuscript, we have investigated the conditions under which fuzzy $\mathfrak{G}$-semiprimary ideals and the radical of fuzzy ideals are related to each other. Thus, our understanding of fuzzy algebra and its applications might be much improved with the help of the idea of fuzzy semiprimary ideals. In the future, we may extend this work to more general structures, such as near rings and semirings.

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