

## Article

# A Fractional Analysis of Zakharov–Kuznetsov Equations with the Liouville–Caputo Operator

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**Abstract:** In this study, we used two unique approaches, namely the Yang transform decomposition method (YTDM) and the homotopy perturbation transform method (HPTM), to derive approximate analytical solutions for nonlinear time-fractional Zakharov–Kuznetsov equations (ZKEs). This framework demonstrated the behavior of weakly nonlinear ion-acoustic waves in plasma containing cold ions and hot isothermal electrons in the presence of a uniform magnetic flux. The density fraction and obliqueness of two compressive and rarefactive potentials are depicted. In the Liouville–Caputo sense, the fractional derivative is described. In these procedures, we first used the Yang transform to simplify the problems and then applied the decomposition and perturbation methods to obtain comprehensive results for the problems. The results of these methods also made clear the connections between the precise solutions to the issues under study. Illustrations of the reliability of the proposed techniques are provided. The results are clarified through graphs and tables. The reliability of the proposed procedures is demonstrated by illustrative examples. The proposed approaches are attractive, though these easy approaches may be time-consuming for solving diverse nonlinear fractional-order partial differential equations.

**Keywords:** Yang transform; fractional Zakharov–Kuznetsov equations; Liouville–Caputo operator; Adomian decomposition method (ADM); homotopy perturbation method (HPM)

**MSC:** 34A25; 34A08; 26A33; 35A20



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## 1. Introduction

The generalization of integer-order to arbitrary-order calculus is known as fractional calculus (FC), and it was first developed around the end of the seventeenth century. The main advantage of fractional calculus is that it has been shown to be a highly useful tool for understanding the memory and hereditary characteristics of numerous phenomena. Additionally, ordinary calculus represents a small subset of fractional calculus. The fractional-order derivative groundwork was laid through the combined effort of pioneers such as Riemann [1], Liouville [2], Caputo [3], Podlubny [4], and Miller and Ross [5], among many others [6,7]. The early theory of fractional derivatives has been rapidly advanced throughout the past few decades. Authors such as Srivastava [8], Kilbas et al. [9], Legnani et al. [10], and Hilfer [11] have gone into additional detail and developed the area. The main focus of their studies was the systematic comprehension of FC, including uniqueness and existence. The theory of fractional-order calculus has been linked to real-world projects and used in a variety of fields, including electrodynamics [12], chaos theory [13], optics [14], signal processing [15], and other areas [16–22].

The aforementioned works played a crucial role in our understanding of the nature and behavior of nonlinear problems that arise in daily life, as well as the analytical and numeri-

cal solutions for differential equations of an arbitrary order. Through spatial and temporal fractional-order derivatives, fractional-order models have expanded our understanding of differentiability and added nonlocal and system memory effects. These characteristics enable us to model phenomena at various temporal and spatial scales without dividing the problem into even more subsets. The assumption that fractional derivatives can highlight or capture key characteristics of complicated events is the foundation for using fractional-order models to span various scales. Furthermore, because classical derivatives are local in nature, we can only use them to describe changes in a point's immediate surroundings; however, fractional derivatives allow us to express changes in an interval. Because of this characteristic, arbitrary-order derivatives can be used to model a wider range of physical processes related to quantum physics, signal analysis, diffusion, elasticity, and seismic vibrations. It has been discovered that fractional differential equations may explain many systems in transdisciplinary fields more effectively and conveniently. Fractional derivatives are now frequently employed to examine a variety of difficult issues. For instance, fractional calculus is used for the mathematical modeling of viscoelastic materials [23].

It is becoming noticeable that fractional partial differential equations (FPDEs) are a useful modeling tool for complicated multiscale events, particularly those combining overlapping microscopic and macroscopic dimensions. In contrast to integer-order PDEs, the fractional order of the derivatives in FPDEs can be a function of both space and time or even distribution. This has created outstanding prospects for simulating and modeling multi-physics phenomena, such as the smooth transition from wave propagation to diffusion or from local to non-local dynamics. Numerous well-known scholars have made contributions to this area due to the importance of analytically solving FPDEs in engineering and science [24–29]. With the aid of the Elzaki transform decomposition approach, the approximate analytical solution for time-fractional Swift–Hohenberg equations with conformable derivatives was studied in [30]. The authors of [31] used the natural decomposition approach to obtain the solutions of the fractional modified Boussinesq and approximate long wave equations. In [32], a new fractional sub-equation method was used to study an exact solution for fractional partial differential equations. A new analytical solution for fractional nonlinear systems of third-order Korteweg–de Vries (KdV) equations and systems of coupled Burgers equations in one and two dimensions were investigated by the variational iteration method in [33] using a conformable fractional derivative. Fractional calculus has been used by many scholars as a tool to ascertain the nature of complex problems [34–41].

The Korteweg–de Vries equations are essential for scientific applications. One of the well-known variations are the ZKEs, which analyze electrostatic-acoustic pulses in magnetized ions. They were developed in an ocean-based study of coastal waves [42]. To show nonlinear phenomena such as isotope waves in high-magnetization lossless plasma, the ZKEs were initially developed in two dimensions [43]. In this study, we investigated the time-fractional Zakharov–Kuznetsov (FZK) equation  $(\alpha_1, \alpha_2, \alpha_3)$  with a fractional time-derivative of the order  $0 < \lambda \leq 1$ , possessing the following form:

$$D_{\vartheta}^{\lambda} \mathbb{K} + a_1 (\mathcal{J}^{\alpha_1})_{\zeta} + b_1 (\mathcal{J}^{\alpha_2})_{\zeta\zeta\zeta} + b_1 (\mathcal{J}^{\alpha_3})_{\zeta\psi\psi} = 0, \quad (1)$$

where  $\mathcal{J} = \mathcal{J}(\zeta, \psi, \vartheta)$ ,  $D_{\vartheta}^{\lambda}$  denotes the Liouville–Caputo fractional derivative of the order  $\lambda$ ;  $a_1$  and  $b_1$  are arbitrary constants;  $\alpha_i, i = 1, 2, 3$  are integers; and the nature of nonlinear events such as ion-acoustic waves in the context of a symmetrical magnetic field in plasma containing hot isothermal electrons and cold ions is depicted by the expression  $\alpha_i \neq 0$  ( $i = 1, 2, 3$ ) [44,45]. The ZKEs were suggested to examine a shallow nonlinear isotope ripple in a plasma with significant magnetization impairment in three dimensions by the authors of [43]. The variation iteration method [46] and the HPM [47] have been used, respectively, to examine the approximative analytical solutions of fractional ZKEs. Several of the above-mentioned strategies suffered from the drawback that they were invariably hierarchical and involved a great deal of computational complexity. The innovations of this research are the YTDM and HPTM, which combine the Yang transform (YT), ADM, and HPM to

solve the time-fractional ZKE. The Yang transform was introduced by Xiao-Jun Yang and can be used to handle various kinds of differential equations with constant coefficients [48]. The proposed method made it simpler to estimate the series terms as compared to the traditional Adomian process [49,50], since it does not need to compute the fractional derivative or fractional integrals in the recursive mechanism. Since the YTDM does not require prescribed assumptions, linearization, discretization, or perturbation, round-off errors are avoided. In the literature, the YTDM has been used to solve a wide range of differential equations, including Lax’s time-fractional Korteweg–de Vries equation [51] and the phi-four equations [52]. He’s polynomials, the Yang transform, and the homotopy perturbation method have been combined to generate the HPTM [53–55]. He’s polynomials can be applied to simply manage nonlinear terms. The proposed method’s analytical results demonstrated how easily implemented and highly desirable this method is computationally. It appears that these new methods could be used to reduce the time and cost of computing. The abovementioned methods produce convergent series.

The remainder of this article is structured as follows. We introduce some definitions and YT features in Section 2. We describe the suggested methods for resolving fractional partial differential equations (FPDEs) in Sections 3 and 4. After that, in Section 5, we use the methods discussed to find approximations of solutions to the fractional space-time ZK equation. In Section 6, we discuss the numerical simulations for present methods. The paper’s conclusions are stated in Section 7.

**2. Preliminaries**

For our analysis, a number of definitions and axiom results from the literature were necessary.

**Definition 1.** The fractional Liouville–Caputo derivative is as follows [56]:

$$D_{\vartheta}^{\lambda} \mathcal{J}(\zeta, \vartheta) = \frac{1}{\Gamma(k - \lambda)} \int_0^{\vartheta} (\vartheta - \gamma)^{k-\lambda-1} \mathcal{J}^{(k)}(\zeta, \gamma) d\gamma, \quad k - 1 < \lambda \leq k, \quad k \in \mathbf{N}. \quad (2)$$

**Definition 2.** The YT of the stated function is as follows [48]:

$$Y\{\mathcal{J}(\vartheta)\} = M(u) = \int_0^{\infty} e^{-\frac{\vartheta}{u}} \mathcal{J}(\vartheta) d\vartheta, \quad \vartheta > 0, \quad (3)$$

where  $u$  is the transform variable.

Some important functions for the YT are stated as follows:

$$\begin{aligned} Y[1] &= u, \\ Y[\vartheta] &= u^2, \\ Y[\vartheta^q] &= \Gamma(q + 1)u^{q+1}, \end{aligned} \quad (4)$$

representing the inverse YT as

$$Y^{-1}\{M(u)\} = \mathcal{J}(\vartheta). \quad (5)$$

**Definition 3.** The YT of the stated function with a derivative of  $n$ th order is given as follows [48]:

$$Y\{\mathcal{J}^n(\vartheta)\} = \frac{M(u)}{u^n} - \sum_{k=0}^{n-1} \frac{\mathcal{J}^k(0)}{u^{n-k-1}}, \quad \forall n = 1, 2, 3, \dots \quad (6)$$

**Definition 4.** The YT of the stated function with derivative of order fractional is given as [48]

$$Y\{\mathcal{J}^{\lambda}(\vartheta)\} = \frac{M(u)}{u^{\lambda}} - \sum_{k=0}^{n-1} \frac{\mathcal{J}^k(0)}{u^{\lambda-(k+1)}}, \quad n - 1 < \lambda \leq n. \quad (7)$$

### 3. Configuration for HPTM

In this subsection, we explain the basic concept behind the HPTM for solving the FPDE:

$$D_{\vartheta}^{\lambda} \mathcal{J}(\zeta, \vartheta) = \mathcal{P}_1[\zeta] \mathcal{J}(\zeta, \vartheta) + \mathcal{Q}_1[\zeta] \mathcal{J}(\zeta, \vartheta), \quad 0 < \lambda \leq 1, \tag{8}$$

with the initial guesses

$$\mathcal{J}(\zeta, 0) = \xi(\zeta).$$

Here,  $D_{\vartheta}^{\lambda} = \frac{\partial^{\lambda}}{\partial \vartheta^{\lambda}}$  denotes the Liouville–Caputo fractional derivative, and  $\mathcal{P}_1[\zeta]$  and  $\mathcal{Q}_1[\zeta]$  are linear and nonlinear operators, respectively.

Upon implementing the YT, we have

$$Y[D_{\vartheta}^{\lambda} \mathcal{J}(\zeta, \vartheta)] = Y[\mathcal{P}_1[\zeta] \mathcal{J}(\zeta, \vartheta) + \mathcal{Q}_1[\zeta] \mathcal{J}(\zeta, \vartheta)], \tag{9}$$

$$\frac{1}{u^{\lambda}} \{M(u) - u\mathcal{J}(0)\} = Y[\mathcal{P}_1[\zeta] \mathcal{J}(\zeta, \vartheta) + \mathcal{Q}_1[\zeta] \mathcal{J}(\zeta, \vartheta)], \tag{10}$$

which yields

$$M(u) = u\mathcal{J}(0) + u^{\lambda} Y[\mathcal{P}_1[\zeta] \mathcal{J}(\zeta, \vartheta) + \mathcal{Q}_1[\zeta] \mathcal{J}(\zeta, \vartheta)]. \tag{11}$$

By utilizing the inverse YT, we obtain

$$\mathcal{J}(\zeta, \vartheta) = \mathcal{J}(0) + Y^{-1}[u^{\lambda} Y[\mathcal{P}_1[\zeta] \mathcal{J}(\zeta, \vartheta) + \mathcal{Q}_1[\zeta] \mathcal{J}(\zeta, \vartheta)]]. \tag{12}$$

Utilizing the HPM, we obtain

$$\mathcal{J}(\zeta, \vartheta) = \sum_{k=0}^{\infty} \epsilon^k \mathcal{J}_k(\zeta, \vartheta), \tag{13}$$

with the parameter  $\epsilon \in [0, 1]$ .

The nonlinear term is taken as

$$\mathcal{Q}_1[\zeta] \mathcal{J}(\zeta, \vartheta) = \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}). \tag{14}$$

In addition,  $H_k(\mathcal{J})$  describes He’s polynomials and is written as follows:

$$H_n(\mathcal{J}_0, \mathcal{J}_1, \dots, \mathcal{J}_n) = \frac{1}{\Gamma(n+1)} D_{\epsilon}^k \left[ \mathcal{Q}_1 \left( \sum_{k=0}^{\infty} \epsilon^i \mathcal{J}_i \right) \right]_{\epsilon=0}, \tag{15}$$

where  $D_{\epsilon}^k = \frac{\partial^k}{\partial \epsilon^k}$ .

Substituting (12) and (13) into (11), we obtain

$$\sum_{k=0}^{\infty} \epsilon^k \mathcal{J}_k(\zeta, \vartheta) = \mathcal{J}(0) + \epsilon \times \left( Y^{-1} \left[ u^{\lambda} Y \left\{ \mathcal{P}_1 \sum_{k=0}^{\infty} \epsilon^k \mathcal{J}_k(\zeta, \vartheta) + \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) \right\} \right] \right). \tag{16}$$

On comparing the  $\epsilon$  coefficients on both sides, we obtain

$$\begin{aligned} \epsilon^0 : \mathcal{J}_0(\zeta, \vartheta) &= \mathcal{J}(0), \\ \epsilon^1 : \mathcal{J}_1(\zeta, \vartheta) &= Y^{-1} \left[ u^{\lambda} Y (\mathcal{P}_1[\zeta] \mathcal{J}_0(\zeta, \vartheta) + H_0(\mathcal{J})) \right], \\ \epsilon^2 : \mathcal{J}_2(\zeta, \vartheta) &= Y^{-1} \left[ u^{\lambda} Y (\mathcal{P}_1[\zeta] \mathcal{J}_1(\zeta, \vartheta) + H_1(\mathcal{J})) \right], \\ &\vdots \\ \epsilon^k : \mathcal{J}_k(\zeta, \vartheta) &= Y^{-1} \left[ u^{\lambda} Y (\mathcal{P}_1[\zeta] \mathcal{J}_{k-1}(\zeta, \vartheta) + H_{k-1}(\mathcal{J})) \right], \end{aligned} \tag{17}$$

$$k > 0, k \in \mathbf{N}.$$

Lastly, the approximate analytical solution  $\mathcal{J}_k(\zeta, \vartheta)$  is stated as

$$\mathcal{J}(\zeta, \vartheta) = \lim_{M \rightarrow \infty} \sum_{k=1}^M \mathcal{J}_k(\zeta, \vartheta). \tag{18}$$

#### 4. Configuration for YTDM

In this subsection, we illustrate the basic concept behind the YTDM for solving the FPDE:

$$D_{\vartheta}^{\lambda} \mathcal{J}(\zeta, \vartheta) = \mathcal{P}_1(\zeta, \vartheta) + \mathcal{Q}_1(\zeta, \vartheta), \quad 0 < \lambda \leq 1, \tag{19}$$

with the initial guesses

$$\mathcal{J}(\zeta, 0) = \zeta(\zeta).$$

Here,  $D_{\vartheta}^{\lambda} = \frac{\partial^{\lambda}}{\partial \vartheta^{\lambda}}$  denotes the Liouville–Caputo fractional derivative, and  $\mathcal{P}_1$  and  $\mathcal{Q}_1$  are linear and nonlinear operators, respectively.

Upon implementing the YT, we have

$$\begin{aligned} Y[D_{\vartheta}^{\lambda} \mathcal{J}(\zeta, \vartheta)] &= Y[\mathcal{P}_1(\zeta, \vartheta) + \mathcal{Q}_1(\zeta, \vartheta)], \\ \frac{1}{u^{\lambda}} \{M(u) - u\mathcal{J}(0)\} &= Y[\mathcal{P}_1(\zeta, \vartheta) + \mathcal{Q}_1(\zeta, \vartheta)], \end{aligned} \tag{20}$$

which yields

$$M(\mathcal{J}) = u\mathcal{J}(0) + u^{\lambda} Y[\mathcal{P}_1(\zeta, \vartheta) + \mathcal{Q}_1(\zeta, \vartheta)]. \tag{21}$$

By utilizing the inverse YT, we obtain

$$\mathcal{J}(\zeta, \vartheta) = \mathcal{J}(0) + Y^{-1}[u^{\lambda} Y[\mathcal{P}_1(\zeta, \vartheta) + \mathcal{Q}_1(\zeta, \vartheta)]]. \tag{22}$$

Utilizing the YTDM, we have

$$\mathcal{J}(\zeta, \vartheta) = \sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \vartheta). \tag{23}$$

The nonlinear term is taken as

$$\mathcal{Q}_1(\zeta, \vartheta) = \sum_{m=0}^{\infty} \mathcal{A}_m, \tag{24}$$

with

$$\mathcal{A}_m = \frac{1}{m!} \left[ \frac{\partial^m}{\partial \ell^m} \left\{ \mathcal{Q}_1 \left( \sum_{k=0}^{\infty} \ell^k \zeta_k, \sum_{k=0}^{\infty} \ell^k \vartheta_k \right) \right\} \right]_{\ell=0}. \tag{25}$$

Substituting (22) and (23) into (21), we obtain

$$\sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \vartheta) = \mathcal{J}(0) + Y^{-1} \left[ u^{\lambda} Y \left\{ \mathcal{P}_1 \left( \sum_{m=0}^{\infty} \zeta_m, \sum_{m=0}^{\infty} \vartheta_m \right) + \sum_{m=0}^{\infty} \mathcal{A}_m \right\} \right]. \tag{26}$$

Thus, we obtain the following approximation:

$$\mathcal{J}_0(\zeta, \vartheta) = \mathcal{J}(0), \tag{27}$$

$$\mathcal{J}_1(\zeta, \vartheta) = Y^{-1} \left[ u^{\lambda} Y \{ \mathcal{P}_1(\zeta_0, \vartheta_0) + \mathcal{A}_0 \} \right].$$

In general, for  $m \geq 1$ , we can write

$$\mathcal{J}_{m+1}(\zeta, \vartheta) = Y^{-1} \left[ u^{\lambda} Y \{ \mathcal{P}_1(\zeta_m, \vartheta_m) + \mathcal{A}_m \} \right].$$

### 5. Numerical Applications

**Example 1.** Let us assume a nonlinear FZK equation of the form

$$D_{\vartheta}^{\lambda} \mathcal{J}(\zeta, \psi, \vartheta) + \mathcal{J}_{\zeta}^2(\zeta, \psi, \vartheta) + \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) + \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta) = 0, \quad 0 < \lambda \leq 1, \quad (28)$$

with the initial guess

$$\mathcal{J}(\zeta, \psi, 0) = \frac{4}{3} \mu \sinh^2(\zeta + \psi).$$

Upon implementing the YT, we have

$$Y\left(\frac{\partial^{\lambda} \mathcal{J}}{\partial \vartheta^{\lambda}}\right) = Y\left(-\mathcal{J}_{\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta)\right), \quad (29)$$

which yields

$$\frac{1}{u^{\lambda}} \{M(u) - u\mathcal{J}(0)\} = Y\left(-\mathcal{J}_{\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta)\right), \quad (30)$$

$$M(u) = u\mathcal{J}(0) + u^{\lambda} \left(-\mathcal{J}_{\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta)\right). \quad (31)$$

By utilizing the inverse YT, we obtain

$$\mathcal{J}(\zeta, \psi, \vartheta) = \mathcal{J}(0) + Y^{-1} \left[ u^{\lambda} \left\{ Y\left(-\mathcal{J}_{\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta)\right) \right\} \right], \quad (32)$$

$$\mathcal{J}(\zeta, \psi, \vartheta) = \frac{4}{3} \mu \sinh^2(\zeta + \psi) + Y^{-1} \left[ u^{\lambda} \left\{ Y\left(-\mathcal{J}_{\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta)\right) \right\} \right].$$

Utilizing the HPM, we obtain

$$\sum_{k=0}^{\infty} \epsilon^k \mathcal{J}_k(\zeta, \psi, \vartheta) = \frac{4}{3} \mu \sinh^2(\zeta + \psi) + \epsilon \left( Y^{-1} \left[ u^{\lambda} Y \left[ - \left( \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) \right)_{\zeta} - \frac{1}{8} \left( \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) \right)_{\zeta\zeta\zeta} - \frac{1}{8} \left( \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) \right)_{\psi\psi\zeta} \right] \right] \right). \quad (33)$$

In addition, the nonlinear terms obtained by means of He’s polynomial  $H_k(\mathcal{J})$  are as follows:

$$\sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) = \mathcal{J}^2(\zeta, \psi, \vartheta). \quad (34)$$

Some nonlinear terms are presented as below:

$$\begin{aligned} H_0(\mathcal{J}) &= \mathcal{J}_0^2, \\ H_1(\mathcal{J}) &= 2\mathcal{J}_0\mathcal{J}_1, \\ H_2(\mathcal{J}) &= 2\mathcal{J}_0\mathcal{J}_2 + (\mathcal{J}_1)^2. \end{aligned}$$

On comparing the  $\epsilon$  coefficients on both sides, we obtain

$$\begin{aligned}
 \epsilon^0 : \mathcal{J}_0(\zeta, \psi, \vartheta) &= \frac{4}{3}\mu \sinh^2(\zeta + \psi), \\
 \epsilon^1 : \mathcal{J}_1(\zeta, \psi, \vartheta) &= Y^{-1} \left( u^\lambda Y \left[ -H_0(\mathcal{J}) - \frac{1}{8}H_0(\mathcal{J}) - \frac{1}{8}H_0(\mathcal{J}) \right] \right) \\
 &= - \left[ \frac{224}{9}\mu^2 \sinh^2(\zeta + \psi) \cosh(\zeta + \psi) + \frac{32}{3}\mu^2 \sinh(\zeta + \psi) \cosh^3(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)}, \\
 \epsilon^2 : \mathcal{J}_2(\zeta, \psi, \vartheta) &= Y^{-1} \left( u^\lambda Y \left[ -H_1(\mathcal{J}) - \frac{1}{8}H_1(\mathcal{J}) - \frac{1}{8}H_1(\mathcal{J}) \right] \right) \\
 &= \frac{128}{27}\mu^3 \left[ 1200 \cosh^6(\zeta + \psi) - 2080 \cosh^4(\zeta + \psi) + 968 \cosh^2(\zeta + \psi) - 79 \right] \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)}, \\
 \epsilon^3 : \mathcal{J}_3(\zeta, \psi, \vartheta) &= Y^{-1} \left( u^\lambda Y \left[ -H_2(\mathcal{J}) - \frac{1}{8}H_2(\mathcal{J}) - \frac{1}{8}H_2(\mathcal{J}) \right] \right) = -\frac{2048}{81} \sinh(\zeta + \psi) \cosh(\zeta + \psi) \\
 &\left[ 884,000 \cosh^6(\zeta + \psi) - 160,200 \cosh^4(\zeta + \psi) + 85,170 \cosh^2(\zeta + \psi) - 11,903 \right] \mu^4 \frac{\vartheta^{3\lambda}}{\Gamma(3\lambda + 1)}, \\
 &\vdots
 \end{aligned}$$

The approximate analytical solution obtained by means of the HPTM is as follows:

$$\begin{aligned}
 \mathcal{J}(\zeta, \psi, \vartheta) &= \mathcal{J}_0(\zeta, \psi, \vartheta) + \mathcal{J}_1(\zeta, \psi, \vartheta) + \mathcal{J}_2(\zeta, \psi, \vartheta) + \mathcal{J}_3(\zeta, \psi, \vartheta) + \dots \\
 &= \frac{4}{3}\mu \sinh^2(\zeta + \psi) - \left[ \frac{224}{9}\mu^2 \sinh^2(\zeta + \psi) \cosh(\zeta + \psi) + \frac{32}{3}\mu^2 \sinh(\zeta + \psi) \cosh^3(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} \\
 &+ \frac{128}{27}\mu^3 \left[ 1200 \cosh^6(\zeta + \psi) - 2080 \cosh^4(\zeta + \psi) + 968 \cosh^2(\zeta + \psi) - 79 \right] \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)} \\
 &- \frac{2048}{81} \sinh(\zeta + \psi) \cosh(\zeta + \psi) \left[ 884,000 \cosh^6(\zeta + \psi) - 160,200 \cosh^4(\zeta + \psi) + 85,170 \cosh^2(\zeta + \psi) \right. \\
 &\left. - 11,903 \right] \mu^4 \frac{\vartheta^{3\lambda}}{\Gamma(3\lambda + 1)} + \dots
 \end{aligned}$$

**Implementation of YTDM**

Upon implementing the YT, we have

$$Y \left\{ \frac{\partial^\lambda \mathcal{J}}{\partial \vartheta^\lambda} \right\} = Y \left( -\mathcal{J}_\zeta^2(\zeta, \psi, \vartheta) - \frac{1}{8}\mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8}\mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta) \right), \tag{35}$$

which yields

$$\frac{1}{u^\lambda} \{M(u) - u\mathcal{J}(0)\} = Y \left( -\mathcal{J}_\zeta^2(\zeta, \psi, \vartheta) - \frac{1}{8}\mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8}\mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta) \right), \tag{36}$$

$$M(u) = u\mathcal{J}(0) + u^\lambda Y \left( -\mathcal{J}_\zeta^2(\zeta, \psi, \vartheta) - \frac{1}{8}\mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8}\mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta) \right). \tag{37}$$

By utilizing the inverse YT, we obtain

$$\begin{aligned} \mathcal{J}(\zeta, \psi, \vartheta) &= \mathcal{J}(0) + Y^{-1} \left[ u^\lambda \left\{ Y \left( -\mathcal{J}_\zeta^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta) \right) \right\} \right] \\ &= \frac{4}{3} \mu \sinh^2(\zeta + \psi) + Y^{-1} \left[ u^\lambda \left\{ Y \left( -\mathcal{J}_\zeta^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\zeta\zeta\zeta}^2(\zeta, \psi, \vartheta) - \frac{1}{8} \mathcal{J}_{\psi\psi\zeta}^2(\zeta, \psi, \vartheta) \right) \right\} \right]. \end{aligned} \tag{38}$$

Utilizing the YTDM, we have

$$\mathcal{J}(\zeta, \vartheta) = \sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \psi, \vartheta). \tag{39}$$

The nonlinear term obtained by means of the Adomian polynomial is taken as  $\mathcal{J}^2(\zeta, \psi, \vartheta) = \sum_{m=0}^{\infty} \mathcal{A}_m$ . Thus, we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \psi, \vartheta) &= \mathcal{J}(\zeta, \psi, 0) + Y^{-1} \left[ u^\lambda Y \left[ - \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_\zeta - \frac{1}{8} \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\zeta\zeta\zeta} - \frac{1}{8} \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\psi\psi\zeta} \right] \right] \\ &= \frac{4}{3} \mu \sinh^2(\zeta + \psi) + Y^{-1} \left[ u^\lambda Y \left[ - \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_\zeta - \frac{1}{8} \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\zeta\zeta\zeta} - \frac{1}{8} \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\psi\psi\zeta} \right] \right]. \end{aligned} \tag{40}$$

Some nonlinear terms are presented below:

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{J}_0^2, \\ \mathcal{A}_1 &= 2\mathcal{J}_0\mathcal{J}_1, \\ \mathcal{A}_2 &= 2\mathcal{J}_0\mathcal{J}_2 + \mathcal{J}_1^2. \end{aligned}$$

Thus, we obtain the following approximation:

$$\mathcal{J}_0(\zeta, \psi, \vartheta) = \frac{4}{3} \mu \sinh^2(\zeta + \psi).$$

For  $m = 0$

$$\mathcal{J}_1(\zeta, \psi, \vartheta) = - \left[ \frac{224}{9} \mu^2 \sinh^2(\zeta + \psi) \cosh(\zeta + \psi) + \frac{32}{3} \mu^2 \sinh(\zeta + \psi) \cosh^3(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)}.$$

For  $m = 1$

$$\mathcal{J}_2(\zeta, \psi, \vartheta) = \frac{128}{27} \mu^3 \left[ 1200 \cosh^6(\zeta + \psi) - 2080 \cosh^4(\zeta + \psi) + 968 \cosh^2(\zeta + \psi) - 79 \right] \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)}.$$

For  $m = 2$

$$\begin{aligned} \mathcal{J}_3(\zeta, \psi, \vartheta) &= - \frac{2048}{81} \sinh(\zeta + \psi) \cosh(\zeta + \psi) \left[ 884,000 \cosh^6(\zeta + \psi) - 160,200 \cosh^4(\zeta + \psi) \right. \\ &\quad \left. + 85,170 \cosh^2(\zeta + \psi) - 11,903 \right] \mu^4 \frac{\vartheta^{3\lambda}}{\Gamma(3\lambda + 1)}. \end{aligned}$$

Consequently, we determine the series solutions by continuing the same process in order to calculate the components for  $(m \geq 3)$  as follows:



$$\begin{aligned}
 \mathcal{J}(\zeta, \psi, \vartheta) &= \sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \psi, \vartheta) = \mathcal{J}_0(\zeta, \psi, \vartheta) + \mathcal{J}_1(\zeta, \psi, \vartheta) + \mathcal{J}_2(\zeta, \psi, \vartheta) + \mathcal{J}_3(\zeta, \psi, \vartheta) + \dots \\
 &= \frac{4}{3}\mu \sinh^2(\zeta + \psi) - \left[ \frac{224}{9}\mu^2 \sinh^2(\zeta + \psi) \cosh(\zeta + \psi) + \frac{32}{3}\mu^2 \sinh(\zeta + \psi) \cosh^3(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} \\
 &\quad + \frac{128}{27}\mu^3 \left[ 1200 \cosh^6(\zeta + \psi) - 2080 \cosh^4(\zeta + \psi) + 968 \cosh^2(\zeta + \psi) - 79 \right] \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)} \\
 &\quad - \frac{2048}{81} \sinh(\zeta + \psi) \cosh(\zeta + \psi) \left[ 884,000 \cosh^6(\zeta + \psi) - 160,200 \cosh^4(\zeta + \psi) + 85,170 \cosh^2(\zeta + \psi) \right. \\
 &\quad \left. - 11,903 \right] \mu^4 \frac{\vartheta^{3\lambda}}{\Gamma(3\lambda + 1)} + \dots
 \end{aligned}$$

By setting  $\lambda = 1$ , we obtain

$$\mathcal{J}(\zeta, \psi, \vartheta) = \frac{4}{3}\mu \sinh^2(\zeta + \psi - \mu\vartheta). \tag{41}$$

**Example 2.** Let us assume a nonlinear FZK equation of the form

$$D_\vartheta^\lambda \mathcal{J}(\zeta, \psi, \vartheta) + \mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) + 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) + 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) = 0, \quad 0 < \lambda \leq 1, \tag{42}$$

with the initial guess

$$\mathcal{J}(\zeta, \psi, 0) = \frac{3}{2}\mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right].$$

Upon implementing the YT, we have

$$Y \left( \frac{\partial^\lambda \mathcal{J}}{\partial \vartheta^\lambda} \right) = Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right), \tag{43}$$

which yields

$$\frac{1}{u^\lambda} \{M(u) - u\mathcal{J}(0)\} = Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right), \tag{44}$$

$$M(u) = u\mathcal{J}(0) + u^\lambda \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right). \tag{45}$$

By utilizing the inverse YT, we obtain

$$\begin{aligned}
 \mathcal{J}(\zeta, \psi, \vartheta) &= \mathcal{J}(0) + Y^{-1} \left[ u^\lambda \left\{ Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right) \right\} \right] \\
 &= \frac{3}{2}\mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] + Y^{-1} \left[ u^\lambda \left\{ Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right) \right\} \right].
 \end{aligned} \tag{46}$$

Utilizing the HPM, we obtain

$$\sum_{k=0}^{\infty} \epsilon^k \mathcal{J}_k(\zeta, \psi, \vartheta) = \frac{3}{2} \mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] + \epsilon \left( Y^{-1} \left[ u^\lambda Y \left[ - \left( \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) \right)_{\zeta} - 2 \left( \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) \right)_{\zeta\zeta\zeta} - 2 \left( \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) \right)_{\psi\psi\zeta} \right] \right] \right). \tag{47}$$

In addition, the nonlinear terms obtained by means of He’s polynomial  $H_k(\mathcal{J})$  are as follows:

$$\sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{J}) = \mathcal{J}^3(\zeta, \psi, \vartheta). \tag{48}$$

Some nonlinear terms are presented below:

$$\begin{aligned} H_0(\mathcal{J}) &= \mathcal{J}_0^3, \\ H_1(\mathcal{J}) &= 3\mathcal{J}_0^2 \mathcal{J}_1, \\ H_2(\mathcal{J}) &= 3\mathcal{J}_0^2 \mathcal{J}_2 + 3\mathcal{J}_0 \mathcal{J}_1^2. \end{aligned}$$

On comparing the  $\epsilon$  coefficients on both sides, we obtain

$$\begin{aligned} \epsilon^0 : \mathcal{J}_0(\zeta, \psi, \vartheta) &= \frac{3}{2} \mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right], \\ \epsilon^1 : \mathcal{J}_1(\zeta, \psi, \vartheta) &= Y^{-1} \left( u^\lambda Y \left[ - H_0(\mathcal{J}) - 2H_0(\mathcal{J}) - 2H_0(\mathcal{J}) \right] \right) \\ &= -3\mu^3 \sinh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] \cosh \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} - \frac{3}{8} \mu^3 \cosh^3 \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)}, \\ \epsilon^2 : \mathcal{J}_2(\zeta, \psi, \vartheta) &= Y^{-1} \left( u^\lambda Y \left[ - H_1(\mathcal{J}) - 2H_1(\mathcal{J}) - 2H_1(\mathcal{J}) \right] \right) \\ &= \frac{3}{32} \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)} \mu^5 \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] \left[ 765 \cosh^4 \left[ \frac{1}{6}(\zeta + \psi) \right] - 729 \cosh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] + 91 \right], \\ &\vdots \end{aligned}$$

The approximate analytical solution obtained by means of the HPTM is as follows:

$$\begin{aligned} \mathcal{J}(\zeta, \psi, \vartheta) &= \mathcal{J}_0(\zeta, \psi, \vartheta) + \mathcal{J}_1(\zeta, \psi, \vartheta) + \mathcal{J}_2(\zeta, \psi, \vartheta) + \dots \\ &= \frac{3}{2} \mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] - 3\mu^3 \sinh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] \cosh \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} \\ &\quad - \frac{3}{8} \mu^3 \cosh^3 \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} + \frac{3}{32} \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)} \mu^5 \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] \left[ 765 \cosh^4 \left[ \frac{1}{6}(\zeta + \psi) \right] \right. \\ &\quad \left. - 729 \cosh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] + 91 \right] + \dots \end{aligned}$$

**Implementation of YTDM**

Upon implementing the YT, we have

$$Y \left\{ \frac{\partial^\lambda \mathcal{J}}{\partial \vartheta^\lambda} \right\} = Y \left( - \mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right), \tag{49}$$

which yields

$$\frac{1}{u^\lambda} \{M(u) - u\mathcal{J}(0)\} = Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right), \tag{50}$$

$$M(u) = u\mathcal{J}(0) + u^\lambda Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right). \tag{51}$$

By utilizing the inverse YT, we obtain

$$\begin{aligned} \mathcal{J}(\zeta, \psi, \vartheta) &= \mathcal{J}(0) + Y^{-1} \left[ u^\lambda \left\{ Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right) \right\} \right] \\ &= \frac{3}{2} \mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] + Y^{-1} \left[ u^\lambda \left\{ Y \left( -\mathcal{J}_\zeta^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\zeta\zeta\zeta}^3(\zeta, \psi, \vartheta) - 2\mathcal{J}_{\psi\psi\zeta}^3(\zeta, \psi, \vartheta) \right) \right\} \right]. \end{aligned} \tag{52}$$

Utilizing the YTDM, we have

$$\mathcal{J}(\zeta, \vartheta) = \sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \psi, \vartheta). \tag{53}$$

The nonlinear term obtained by means of the Adomian polynomial is taken as  $\mathcal{J}^3(\zeta, \psi, \vartheta) = \sum_{m=0}^{\infty} \mathcal{A}_m$ . Thus, we have

$$\begin{aligned} \sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \psi, \vartheta) &= \mathcal{J}(\zeta, \psi, 0) + Y^{-1} \left[ u^\lambda Y \left[ - \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_\zeta - 2 \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\zeta\zeta\zeta} - 2 \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\psi\psi\zeta} \right] \right] \\ &= \frac{3}{2} \mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] + Y^{-1} \left[ u^\lambda Y \left[ - \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_\zeta - 2 \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\zeta\zeta\zeta} \right. \right. \\ &\quad \left. \left. - 2 \left( \sum_{m=0}^{\infty} \mathcal{A}_m \right)_{\psi\psi\zeta} \right] \right]. \end{aligned} \tag{54}$$

Some nonlinear terms are presented below:

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{J}_0^3, \\ \mathcal{A}_1 &= 3\mathcal{J}_0^2 \mathcal{J}_1, \\ \mathcal{A}_2 &= 3\mathcal{J}_0^2 \mathcal{J}_2 + 3\mathcal{J}_0 \mathcal{J}_1^2. \end{aligned}$$

Thus, we obtain the following approximation:

$$\mathcal{J}_0(\zeta, \psi, \vartheta) = \frac{3}{2} \mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right].$$

For  $m = 0$

$$\mathcal{J}_1(\zeta, \psi, \vartheta) = -3\mu^3 \sinh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] \cosh \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} - \frac{3}{8} \mu^3 \cosh^3 \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)}.$$

For  $m = 1$

$$\mathcal{J}_2(\zeta, \psi, \vartheta) = \frac{3}{32} \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)} \mu^5 \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] \left[ 765 \cosh^4 \left[ \frac{1}{6}(\zeta + \psi) \right] - 729 \cosh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] + 91 \right].$$

Consequently, we determine the series solutions by continuing the same process in order to calculate the components for  $(m \geq 3)$  as follows:

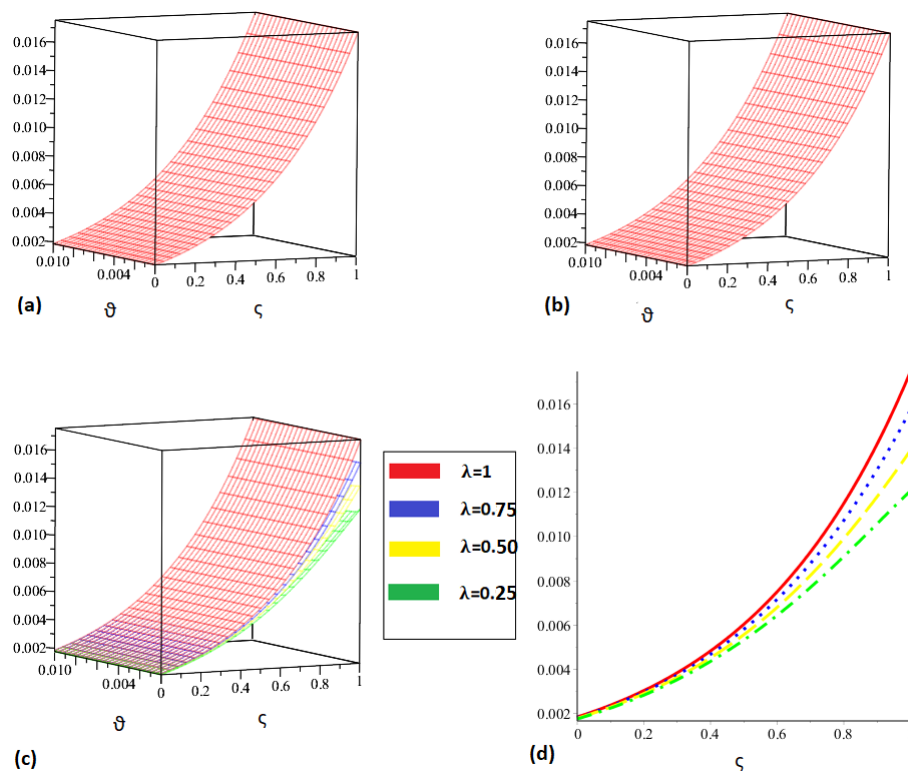
$$\begin{aligned} \mathcal{J}(\zeta, \psi, \vartheta) &= \sum_{m=0}^{\infty} \mathcal{J}_m(\zeta, \psi, \vartheta) = \mathcal{J}_0(\zeta, \psi, \vartheta) + \mathcal{J}_1(\zeta, \psi, \vartheta) + \mathcal{J}_2(\zeta, \psi, \vartheta) + \dots \\ &= \frac{3}{2}\mu \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] - 3\mu^3 \sinh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] \cosh \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} \\ &\quad - \frac{3}{8}\mu^3 \cosh^3 \left[ \frac{1}{6}(\zeta + \psi) \right] \frac{\vartheta^\lambda}{\Gamma(\lambda + 1)} + \frac{3}{32} \frac{\vartheta^{2\lambda}}{\Gamma(2\lambda + 1)} \mu^5 \sinh \left[ \frac{1}{6}(\zeta + \psi) \right] \left[ 765 \cosh^4 \left[ \frac{1}{6}(\zeta + \psi) \right] \right. \\ &\quad \left. - 729 \cosh^2 \left[ \frac{1}{6}(\zeta + \psi) \right] + 91 \right] + \dots \end{aligned}$$

By setting  $\lambda = 1$ , we obtain

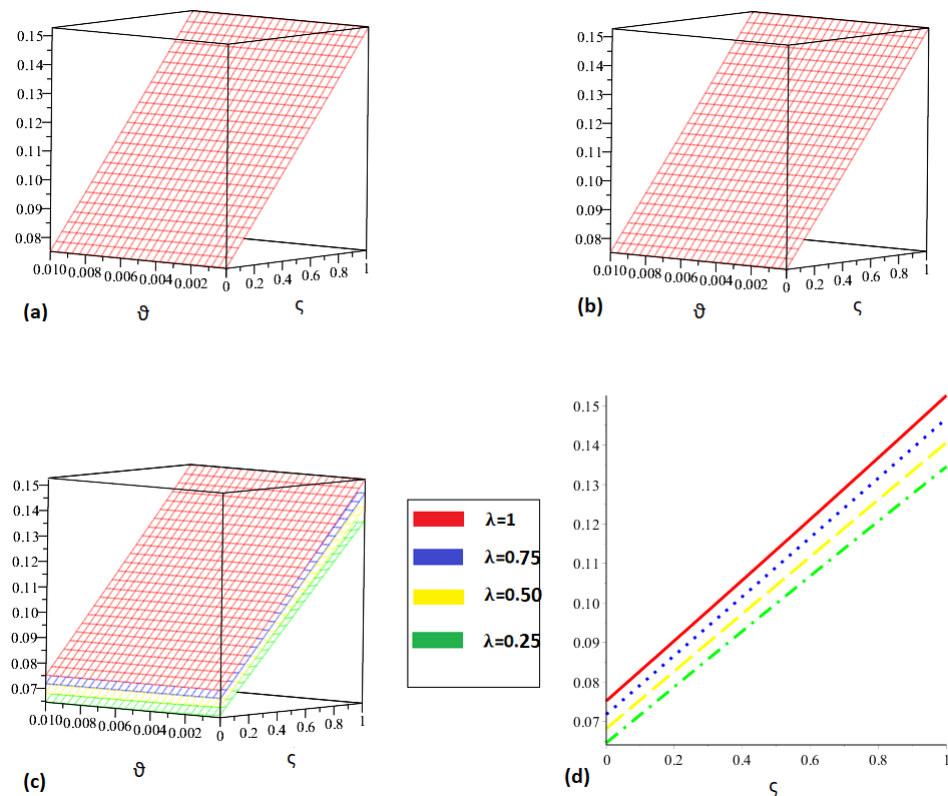
$$\mathcal{J}(\zeta, \psi, \vartheta) = \frac{3}{2}\mu \sinh^2 \left[ \frac{1}{6}(\zeta + \psi - \mu\vartheta) \right]. \tag{55}$$

### 6. Numerical Simulation Studies

In this section, we present a numerical analysis to verify the precision of the numerical solution obtained by the two effective techniques. The third-order series solution was taken into consideration to assess the corresponding behavior in the proposed approaches. Figure 1a shows the behavior of the exact solution, while Figure 1b shows the graphical behavior for the approximative solution derived using the proposed approaches with  $\lambda = 1$ . A surface plot of approximations for various fractional orders with  $\lambda = 0.25, 0.50, 0.75$ , and 1 is shown in Figure 1c,d for  $\vartheta = 0.01$ . We illustrate the behavior of the exact solution in Figure 2a and the graphical behavior of the approximative solution derived by the proposed approaches in Figure 2b for  $\lambda = 1$ . Similarly, the surface plot of approximations for various fractional orders with  $\lambda = 0.25, 0.50, 0.75$ , and 1 is shown in Figure 2c,d for  $\vartheta = 0.01$ . The domains for all the figures are  $\zeta \in [0, 1]$ ,  $\vartheta \in [0, 0.01]$ , with  $\psi = 1$  and  $\mu = 0.001$ . A comparison of the exact and approximative solutions for various values of  $\lambda$  is shown in Table 1. In Table 2, we compare our solution with the solutions derived by the perturbation–iteration algorithm (PIA) and residual power series method (RPSM) in terms of absolute error at  $\mu = 0.001$  and  $\lambda = 1$ , as an example. Table 3 represents the comparison between the exact and the approximate solution for various values of  $\lambda$ , whereas in Table 4 we compare our solution with the solution derived by the variational iteration method (VIM) in terms of absolute error at  $\mu = 0.001$  and  $\lambda = 1$ , as an example. The numerical simulation is presented to show the precision and demonstrate how the resulting solution converged to the exact solution as the fractional order transitioned into the classical order. Finally, we can conclude that the analysis under consideration could help researchers better understand the nature of various nonlinear and complex problems describing a variety of events. Both the couple and the system of equations describing real-world situations could be solved using the proposed techniques and fractional operator.



**Figure 1.** Graphical representation of the accuracy of the solutions of Example 1 obtained using the proposed techniques and various fractional orders.



**Figure 2.** Graphical representation of the accuracy of the solutions of Example 2 obtained using the proposed techniques and various fractional orders.

**Table 1.** Numerical simulation of the accuracy of the solutions of Example 1 obtained using the proposed techniques for different orders of  $\lambda$ .

$\vartheta$	$\zeta$	$\lambda = 0.85$	$\lambda = 0.90$	$\lambda = 0.95$	$\lambda = 1$ (Approx)	$\lambda = 1$ (Exact)
0.01	0.2	0.002999	0.003012	0.003025	0.003037	0.003037
	0.4	0.004747	0.004776	0.004805	0.004835	0.004835
	0.6	0.007327	0.007392	0.007457	0.007524	0.007524
	0.8	0.011097	0.011245	0.011392	0.011541	0.011541
	1	0.016543	0.016873	0.017203	0.017538	0.017538
0.02	0.2	0.002999	0.003012	0.003024	0.003037	0.003037
	0.4	0.004746	0.004775	0.004805	0.004835	0.004835
	0.6	0.007324	0.007390	0.007456	0.007524	0.007524
	0.8	0.011091	0.011240	0.011388	0.011541	0.011541
	1	0.016529	0.016863	0.017195	0.017538	0.017538
0.03	0.2	0.002998	0.003011	0.003024	0.003037	0.003037
	0.4	0.004745	0.004775	0.004804	0.004835	0.004835
	0.6	0.007322	0.007388	0.007454	0.007524	0.007524
	0.8	0.011086	0.011236	0.011384	0.011541	0.011541
	1	0.016517	0.016853	0.017187	0.017538	0.017538
0.04	0.2	0.002998	0.003011	0.003024	0.003037	0.003037
	0.4	0.004744	0.004774	0.004803	0.004835	0.004835
	0.6	0.007319	0.007386	0.007453	0.007524	0.007524
	0.8	0.011081	0.011231	0.011381	0.011541	0.011541
	1	0.016505	0.016843	0.017179	0.017538	0.017538
0.05	0.2	0.002997	0.003010	0.003023	0.003037	0.003037
	0.4	0.004743	0.004773	0.004802	0.004835	0.004835
	0.6	0.004743	0.007384	0.007451	0.007524	0.007524
	0.8	0.011076	0.011227	0.011377	0.011541	0.011541
	1	0.016494	0.016834	0.017171	0.017538	0.017538

**Table 2.** Comparison between our solution and the solutions derived by the perturbation–iteration algorithm (PIA) and residual power series method (RPSM) at  $\mu = 0.001$  and  $\lambda = 1$  for example 1 in terms of absolute error.

$\vartheta$	$\zeta$	$\psi$	PIA Error	RPSM Error	Our Method Error
0.2	0.1	0.1	$3.85217 \times 10^{-7}$	$3.85217 \times 10^{-7}$	$3.8519486000 \times 10^{-7}$
0.3	0.1	0.1	$5.75911 \times 10^{-7}$	$5.75912 \times 10^{-7}$	$5.7583616000 \times 10^{-7}$
0.4	0.1	0.1	$7.65359 \times 10^{-7}$	$7.65352 \times 10^{-7}$	$7.6517330000 \times 10^{-7}$
0.2	0.6	0.6	$4.66337 \times 10^{-5}$	$4.66389 \times 10^{-5}$	$4.6473505000 \times 10^{-5}$
0.3	0.6	0.6	$6.86056 \times 10^{-5}$	$6.86314 \times 10^{-5}$	$6.8073230000 \times 10^{-5}$
0.4	0.6	0.6	$8.98263 \times 10^{-5}$	$8.99046 \times 10^{-5}$	$8.8581596000 \times 10^{-5}$
0.2	0.9	0.9	$5.12131 \times 10^{-4}$	$5.14241 \times 10^{-4}$	$4.9248718000 \times 10^{-4}$
0.3	0.9	0.9	$7.38186 \times 10^{-4}$	$7.48450 \times 10^{-4}$	$6.7502883000 \times 10^{-4}$
0.4	0.9	0.9	$9.57942 \times 10^{-4}$	$9.89139 \times 10^{-4}$	$8.1510249000 \times 10^{-4}$

**Table 3.** Numerical simulation of the accuracy of the solutions of Example 2 obtained using the proposed techniques for different orders of  $\lambda$ .

$\vartheta$	$\zeta$	$\lambda = 0.80$	$\lambda = 0.90$	$\lambda = 0.95$	$\lambda = 1$ (Approx)	$\lambda = 1$ (Exact)
0.01	0.2	0.030120	0.030147	0.030173	0.030197	0.030197
	0.4	0.035230	0.035259	0.035288	0.035315	0.035315
	0.6	0.040377	0.040410	0.040442	0.040473	0.040473
	0.8	0.045568	0.045604	0.045641	0.045675	0.045675
	1	0.050807	0.050848	0.050889	0.050928	0.050928

**Table 3.** Cont.

$\vartheta$	$\zeta$	$\lambda = 0.80$	$\lambda = 0.90$	$\lambda = 0.95$	$\lambda = 1$ (Approx)	$\lambda = 1$ (Exact)
0.02	0.2	0.030119	0.030146	0.030172	0.030195	0.030195
	0.4	0.035229	0.035258	0.035288	0.035313	0.035313
	0.6	0.040376	0.040409	0.040441	0.040470	0.040470
	0.8	0.045566	0.045603	0.045640	0.045672	0.045672
	1	0.050806	0.050847	0.050888	0.050925	0.050925
0.03	0.2	0.030118	0.030145	0.030172	0.030192	0.030192
	0.4	0.035228	0.035257	0.035287	0.035310	0.035310
	0.6	0.040375	0.040408	0.040441	0.040467	0.040467
	0.8	0.045565	0.045602	0.045639	0.045670	0.045670
	1	0.050804	0.050846	0.050887	0.050923	0.050923
0.04	0.2	0.030117	0.030144	0.030171	0.030190	0.030190
	0.4	0.035227	0.035256	0.035286	0.035308	0.035308
	0.6	0.040374	0.040407	0.040440	0.040465	0.040465
	0.8	0.045564	0.045601	0.045638	0.045667	0.045667
	1	0.050803	0.050845	0.050886	0.050920	0.050920
0.05	0.2	0.030116	0.030144	0.030170	0.030187	0.030187
	0.4	0.035226	0.035256	0.035285	0.035305	0.035305
	0.6	0.040372	0.040406	0.040439	0.040462	0.040462
	0.8	0.045563	0.045600	0.045637	0.045664	0.045664
	1	0.050802	0.050843	0.050885	0.050917	0.050917

**Table 4.** Comparison between our solution and the solution derived by the variational iteration method (VIM) at  $\mu = 0.001$  and  $\lambda = 1$  for example 2 in terms of absolute error.

$\vartheta$	$\zeta$	$\psi$	VIM Error	Our Method Error
0.2	0.1	0.1	$5.00091 \times 10^{-5}$	$4.9951950000 \times 10^{-8}$
0.3	0.1	0.1	$5.00091 \times 10^{-5}$	$7.4927920000 \times 10^{-8}$
0.4	0.1	0.1	$5.00091 \times 10^{-5}$	$9.9903860000 \times 10^{-8}$
0.2	0.6	0.6	$3.02003 \times 10^{-4}$	$5.0898600000 \times 10^{-8}$
0.3	0.6	0.6	$3.02003 \times 10^{-4}$	$7.6348000000 \times 10^{-8}$
0.4	0.6	0.6	$3.02003 \times 10^{-4}$	$1.0179720000 \times 10^{-7}$
0.2	0.9	0.9	$4.56780 \times 10^{-4}$	$5.2122800000 \times 10^{-8}$
0.3	0.9	0.9	$4.56780 \times 10^{-4}$	$7.8184000000 \times 10^{-8}$
0.4	0.9	0.9	$4.56780 \times 10^{-4}$	$1.0424500000 \times 10^{-7}$

### 7. Conclusions

The time-fractional ZK equation, which governs the nonlinear evolution of ion-acoustic waves in a magnetized plasma with hot and cold electrons, was investigated in this study using the proposed YTD and HPT methods. Both negative (rarefactive) and positive (compressive) potential structures that were symmetric about the origin were produced according to the different physical properties. Due to the limited number of estimations used in the proposed procedures, they were more effective than alternative analytical approaches. We gained a clear understanding of the technique, because it entailed directly applying the YT to the anticipated problem before modifying the ADM and HPM. The approximate solution to the considered problem was then derived using the inverse Yang transform. We presented 2D and 3D plots to illustrate the compatibility of the generated model and the precise solutions to the problems, respectively. The results obtained by existing studies were very well in line with the solutions presented in examples 1 and 2 in this paper. The simulations demonstrated that the proposed techniques attained remarkable agreement, suggesting that the proposed methods are quite effective and simple to use

for obtaining approximative analytical solutions to a variety of fractional physical and biological models.

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