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Cut-Free Gentzen Sequent Calculi for Tense Logics

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Abstract: The cut-free single-succedent Gentzen sequent calculus GK_t for the minimal tense logic K_t is introduced. This sequent calculus satisfies the displaying property. The proof proceeds in terms of a Kolmogorov translation and three intermediate sequent systems. Finally, we show that tense logics axiomatized by strictly positive implication have cut-free Gentzen sequent calculi uniformly.

Keywords: proof theory; tense logic; sequent calculus; cut elimination

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1. Introduction

Basic tense logic is the extension of classical propositional logic with tense operators (cf. e.g., [1]). The minimal tense logic K_t can be formulated as the minimal modal logic K with a past modality. Viewed as a kind of bimodal logic, tense logic is investigated mainly by using tools and techniques from modal logic (cf. e.g., [2,3]). As far as the proof theory of modal logic is concerned, some approaches have taken to consider cut-free sequent calculi for modal logics. Since Gentzen sequent calculus for classical logic was developed, it has been extended with rules for modal operators, and such sequent systems are explored in the literature (cf. e.g., [4]). In particular, some aspects of Gentzen sequent calculi for modal logics are explored by Poggiolesi [5]. Belnap's display logic [6] provides an alternative to Gentzen-type sequent calculi for various non-classical logics (cf. e.g., [7,8]). Labelled sequent calculi are developed in terms of the relational semantics for modal logics (cf. e.g., [9–11]). Deep inference (cf. e.g., [12]) calculi are developed where nested sequents are used.

Proof theory for the basic tense logic K_t and its extensions has also been investigated in the literature (cf. [13,14]). However, sequent calculi for tense logics in the sense of Gentzen have not been well-developed. A fundamental problem lies in the elimination of the cut rule. Inspired by the proof-theoretic study of Lambek calculus (cf. [15]), a cut-free Gentzen sequent calculus for intuitionistic modal logic has been provided in [16], and such a sequent calculus is also developed for intuitionistic tense logic in [17]. In these works, structural operators for $\wedge$ (conjunction), $\Diamond$ (future possibility) and $\blacklozenge$ (past possibility) are introduced such that formula structures instead of multisets of formulas are provided for defining sequents. Eventually cut-free Gentzen sequent calculi for intuitionistic modal and tense logics are developed. It is this road we take to provide cut-free sequent calculi for classical tense logics.

Gentzen's sequent calculus for classical propositional logic is multi-succedent. One way of constructing a single-succedent sequent calculus is that the rule of excluded middle is added to the G3-style sequent calculus for intuitionistic propositional logic (cf. [18], p. 114). In the present paper, we take an alternative approach to this problem. The central point is that we can eliminate cut by considering appropriate rules for negation. Double negation rules and two particular rules for negation are provided. Eventually, the proof of cut elimination for GK_t is not a direct induction on the cut height or the complexity of the cut formula but an indirect proof in terms of two additional auxiliary sequent systems. It is this machinery that allows uniform cut elimination for Gentzen sequent calculi for tense logics axiomatized by strictly positive axioms.

The structure of this paper is as follows. Section 2 provides some preliminaries on tense logics. Section 3 introduces a single-succedent sequent calculus GK_t for the minimal tense logic K_t. Section 4 provides a proof of the cut elimination for GK_t. Section 5 proves that all tense logics axiomatized by strictly positive implications have a single-succedent sequent calculus obtained from GK_t by adding structural rules, which are transformed from the strictly positive axioms. Section 6 offers some concluding remarks.

2. Preliminaries on Tense Logics

Let $\mathrm{Var} = \{p_i : i < \omega\}$ be a denumerable set of propositional variables. Primitive connectives are $\bot$ (falsum), $\neg$ (negation) and $\wedge$ (conjunction). We use four unary operators, which are $\Diamond$ (future possibility), $\blacklozenge$ (past possibility), $\Box$ (future necessity) and $\blacksquare$ (past necessity). The formula algebra $\mathcal{F}$ is defined inductively as follows:

$$\mathcal{F} \ni \alpha ::= p \mid \bot \mid \neg\alpha \mid (\alpha_1 \wedge \alpha_2) \mid \Diamond\alpha \mid \Box\alpha \mid \blacklozenge\alpha \mid \blacksquare\alpha$$

where $p \in \mathrm{Var}$. We use abbreviations $\top := \neg\bot$ (true), $\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$ (disjunction), $\alpha \to \beta := \neg(\alpha \wedge \neg\beta)$ (implication) and $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$, which are defined as usual. The complexity $c(\beta)$ of a formula β is defined inductively as follows:

$$\begin{aligned} c(\beta) &= 0, \text{ if } \beta \in \mathrm{Var} \cup \{\bot\}. \\ c(\beta \wedge \gamma) &= \max\{c(\beta), c(\gamma)\} + 1. \\ c(\odot\beta) &= c(\beta) + 1, \text{ if } \odot \in \{\neg, \Diamond, \Box, \blacklozenge, \blacksquare\}. \end{aligned}$$

A substitution is a homomorphism $s : \mathcal{F} \to \mathcal{F}$. Let α^s be the substitution of α under s. For every formula α, let $var(\alpha)$ be the set of all variables appearing in α.

Let $W \neq \varnothing$ and $Q \subseteq W \times W$. For every $w \in W$, let $Q(w) = \{u \in W : wQu\}$. The inverse of Q is defined as $Q^{-1} = \{\langle w, u\rangle : uRw\}$. For every $X \subseteq W$, let $Q(X) = \bigcup_{w \in X} Q(w)$, $[Q]X = \{w \in W : Q(w) \subseteq X\}$ and $\langle Q\rangle X = Q^{-1}(X)$. We use the Boolean operations $\cap, \cup$ and $\overline{(.)}$ (complementation) on the power set $\mathcal{P}(W)$.

Definition 1. *A frame is a pair $\mathfrak{F} = (W, R)$ where $W \neq \varnothing$ (the set of possible worlds) and $R \subseteq W \times W$ (the accessibility relation). A valuation in $\mathfrak{F} = (W, R)$ is a function $V : \mathrm{Var} \to \mathcal{P}(W)$. For every formula $\alpha \in \mathcal{F}$, the truth set $V(\alpha)$ of α under a valuation V in $\mathfrak{F}$ is defined inductively as follows:*

$$\begin{aligned} V(\bot) &= \varnothing & V(\neg\alpha) &= \overline{V(\alpha)} \\ V(\alpha \wedge \beta) &= V(\alpha) \cap V(\beta) & V(\Diamond\alpha) &= \langle R\rangle V(\alpha) \\ V(\Box\alpha) &= [R]V(\alpha) & V(\blacklozenge\alpha) &= \langle R^{-1}\rangle V(\alpha) \\ V(\blacksquare\alpha) &= [R^{-1}]V(\alpha) \end{aligned}$$

Note that $V(\top) = W$, $V(\alpha \vee \beta) = V(\alpha) \cup V(\beta)$ and $V(\alpha \to \beta) = \overline{V(\alpha)} \cup V(\beta)$. A formula α is valid in a frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \alpha$) if $V(\alpha) = W$ for all valuations V in $\mathfrak{F}$. A formula α is valid *in a class of frames $\mathcal{K}$ (notation: $\mathcal{K} \models \alpha$) if $\mathfrak{F} \models \alpha$ for all $\mathfrak{F} \in \mathcal{K}$. The* logic *of $\mathcal{K}$ is defined as the set of formulas $\mathsf{Th}(\mathcal{K}) = \{\alpha \in \mathcal{F} : \mathcal{K} \models \alpha\}$. For a set of formulas Σ, we write $\mathfrak{F} \models \Sigma$ if $\mathfrak{F} \models \alpha$ for all $\alpha \in \Sigma$. The class of all frames for Σ is defined as $\mathsf{Fr}(\Sigma) = \{\mathfrak{F} : \mathfrak{F} \models \Sigma\}$.*

Definition 2. *A tense logic is a set of formulas L such that the following conditions hold:*

(1) *L contains the following formulas:*

$(A1)$ $p \to (q \to p)$
$(A2)$ $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
$(A3)$ $(\neg q \to \neg p) \to (p \to q)$
$(A4)$ $\bot \to p$
$(A5)$ $\Diamond p \leftrightarrow \neg \Box \neg p$
$(A6)$ $\blacklozenge p \leftrightarrow \neg \blacksquare \neg p$

(2) *L is closed under the following rules:*

$$\frac{\alpha \to \beta \quad \alpha}{\beta}\,(MP) \qquad \frac{\Diamond\alpha \to \beta}{\alpha \to \blacksquare\beta}\,(\mathrm{Adj}_{\Diamond\blacksquare}) \qquad \frac{\blacklozenge\alpha \to \beta}{\alpha \to \Box\beta}\,(\mathrm{Adj}_{\blacklozenge\Box})$$

(3) *L is closed under substitution, i.e., if $\beta \in L$, then $\beta^s \in L$ for every substitution s.*

We write $\vdash_L \beta$ (β is a theorem *of L) if $\beta \in L$.*

Fact 1. *For every tense logic L, the following hold:*

(1) *if $\vdash_L \alpha \to \beta$, then $\vdash_L \odot\alpha \to \odot\beta$ for every $\odot \in \{\Diamond, \Box, \blacklozenge, \blacksquare\}$.*
(2) $\vdash_L \Diamond\bot \leftrightarrow \bot$; $\vdash_L \blacklozenge\bot \leftrightarrow \bot$; $\vdash_L \Box\top \leftrightarrow \top$ *and* $\vdash_L \blacksquare\top \leftrightarrow \top$.
(3) *for every $\odot \in \{\Diamond, \blacklozenge\}$, $\vdash_L \odot(\alpha \vee \beta) \leftrightarrow \odot\alpha \vee \odot\beta$.*
(4) *for every $\odot \in \{\Box, \blacksquare\}$, $\vdash_L \odot(\alpha \wedge \beta) \leftrightarrow \odot\alpha \wedge \odot\beta$ and $\odot(\alpha \to \beta) \to (\odot\alpha \to \odot\beta)$.*
(5) $\vdash_L \alpha \to \Box\blacklozenge\alpha$ *and* $\vdash_L \alpha \to \blacksquare\Diamond\alpha$.

Let $\{L_i : i \in I\}$ be a collection of tense logics. It is obvious that $\bigcap_{i \in I} L_i$ is a tense logic. The minimal tense logic is denoted by $\mathsf{K}_t = \bigcap\{L \subseteq \mathcal{F} : L \text{ is a tense logic}\}$. Let L be a tense logic and Σ a set of formulas. The tense logic generated by Σ over L is defined as $L \oplus \Sigma = \bigcap\{L' : L \cup \Sigma \subseteq L' \text{ and } L' \text{ is a tense logic}\}$. If $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$, we write $L \oplus \alpha_1 \oplus \ldots \oplus \alpha_n$ for $L \oplus \Sigma$. Let $\mathsf{Ext}(L)$ be the set of all tense logics containing L. Clearly, $\langle \mathsf{Ext}(L), \cap, \oplus, L, \mathcal{F}\rangle$ forms a bounded distributive lattice. The set of all tense logics is $\mathsf{Ext}(\mathsf{K}_t)$.

A tense logic L is consistent if $\bot \notin L$. The only inconsistent tense logic is $\mathcal{F}$. A tense logic L is Kripke-complete if $L = \mathsf{Th}(\mathsf{Fr}(L))$. Using the standard method of canonical model, we obtain many results on the Kripke completeness (cf. e.g., [2]).

Theorem 1. *K_t is Kripke-complete, i.e., $\vdash_{\mathsf{K}_t} \alpha$ iff $\mathsf{Fr}(\mathsf{K}_t) \models \alpha$.*

Proof. See, e.g., ([2], Corollary 4.36). □

3. A Gentzen Sequent Calculus for K_t

We introduce a single-succedent Gentzen sequent calculus GK_t for the minimal tense logic K_t. We follow the formulation of a sequent calculus for intuitionistic tense logic IK_t given in [17], and use formula structures to define a sequent.

Definition 3. *Let the comma, $\circ$ and $\bullet$ be structural operators for $\wedge$, $\Diamond$ and $\blacklozenge$, respectively. The set of all formula structures $\mathcal{FS}$ is defined inductively as follows:*

$$\mathcal{FS} \ni \Gamma ::= \alpha \mid (\Gamma_1, \Gamma_2) \mid \circ\Gamma \mid \bullet\Gamma, \textit{ where } \alpha \in \mathcal{F}.$$

Let $\mathcal{FS}^* = \mathcal{FS} \cup \{\epsilon\}$, where ϵ stands for the empty. For each $\Delta \in \mathcal{FS}$, the formula $\Delta^\flat$ is defined inductively as follows:

$$\beta^\flat = \beta \qquad (\Delta_1, \Delta_2)^\flat = \Delta_1^\flat \wedge \Delta_2^\flat$$
$$(\circ\Delta)^\flat = \Diamond\Delta^\flat \qquad (\bullet\Delta)^\flat = \blacklozenge\Delta^\flat$$

In particular, let $\varepsilon^\flat = \top$. By $d(\Delta)$, we denote the degree of a formula structure Γ, i.e., the number of occurrences of structural operators in Δ. A sequent is an expression $\Delta \Rightarrow \beta$, where

$\Delta \in \mathcal{FS}^*$ and $\beta \in \mathcal{F}$. We write $\Rightarrow \beta$ instead of $\varepsilon \Rightarrow \beta$. We use s, t, etc., with or without subscripts to denote sequents. A sequent rule is a figure

$$\frac{s_1 \ldots s_n}{s_0}(R)$$

where $s_1, \ldots, s_n$ are premises and s_0 is the conclusion of (R).

Definition 4. *Let $-$ be the given symbol called the position. The set of all contexts $\mathcal{C}$ is defined recursively as follows:*

$$\mathcal{C} \ni \Gamma[-] ::= - \mid (\Gamma_1[-], \Gamma_2) \mid (\Gamma_1, \Gamma_2[-]) \mid \circ\Gamma[-] \mid \bullet\Gamma[-]$$

where $\Gamma_1, \Gamma_2 \in \mathcal{FS}$. The set of all formula contexts $\mathcal{FC}$ is defined inductively as follows:

$$\mathcal{FC} \ni \beta[-] ::= - \mid \neg\beta[-] \mid (\beta_1[-] \wedge \beta_2) \mid (\beta_1 \wedge \beta_2[-]) \mid \Diamond\beta[-] \mid \Box\beta[-] \mid \blacklozenge\beta[-] \mid \blacksquare\beta[-]$$

where $\beta_1, \beta_2 \in \mathcal{F}$. For every context $\Gamma[-]$, we define $\Gamma[-]^\sharp$ inductively as follows:

$$-^\sharp = - \qquad (\Gamma_1[-], \Gamma_2)^\sharp = \Gamma_1[-]^\sharp \wedge \Gamma_2^\flat$$
$$(\Gamma_1, \Gamma_2[-])^\sharp = \Gamma_1^\flat \wedge \Gamma_2[-]^\sharp \qquad (\circ\Gamma)^\sharp = \Diamond\Gamma^\sharp$$
$$(\bullet\Gamma)^\sharp = \blacklozenge\Gamma^\sharp$$

The degree of a context $\Gamma[-]$, denoted by $d(\Gamma[-])$, is defined as the number of occurrences of structural operators appearing in $\Gamma[-]$. For every context $\Gamma[-]$ and formula structure $\Delta \in \mathcal{FS}$, let $\Gamma[\Delta]$ be the formula structure obtained from $\Gamma[-]$ by replacing $-$ with Δ.

The degree of a formula context $\alpha[-]$, denoted by $d(\alpha[-])$, is defined as the number of occurrences of logical operators appearing in $\alpha[-]$. For each $\alpha[-]$ and $\beta \in \mathcal{F}$, let $\alpha[\beta]$ be the formula obtained from $\alpha[-]$ by replacing $-$ with β.

Obviously, $\Gamma[-]$ can be considered as a formula structure with a position, and $\Gamma[-]^\sharp$ as a formula with a position. For each Δ, we have $\Gamma[\Delta]^\flat = \Gamma[-]^\sharp(\Delta^\flat/-)$ arising from $\Gamma[-]^\sharp$ by replacing $-$ with $\Delta^\flat$.

Example 1. *Let us provide examples of context and formula context. The expression $\Gamma[-] = \circ(\bullet(-, \neg q), p \wedge q)$ is a context. If we replace the formula structure $\Delta = \bullet(p, (q, \circ q))$ for the position $-$ in $\Gamma[-]$, we obtain the formula structure $\Gamma[\Delta] = \circ(\bullet(\bullet(p, (q, \circ q)), \neg q), p \wedge q)$. The expression $\alpha[-] = \neg(p \wedge -) \to r$ is a formula context. If we replace the formula $\beta = p \wedge q$ for the position $-$ in $\alpha[-]$, we obtain the formula $\alpha[\beta] = \neg(p \wedge (p \wedge q)) \to r$.*

Definition 5. *The sequent calculus GK_t is defined by the following axiom and inference rules:*

(1) *Axiom:*

$$(\mathrm{Id})\ \varphi \Rightarrow \varphi$$

(2) *Logical rules:*

$$\frac{\Theta \Rightarrow \bot}{\Pi[\Theta] \Rightarrow \varphi}(\bot)$$

$$\frac{\Pi[\varphi_1, \varphi_2] \Rightarrow \psi}{\Pi[\varphi_1 \wedge \varphi_2] \Rightarrow \psi}(\wedge\mathrm{L}) \quad \frac{\Pi_1 \Rightarrow \varphi_1 \quad \Pi_2 \Rightarrow \varphi_2}{\Pi_1, \Pi_2 \Rightarrow \varphi_1 \wedge \varphi_2}(\wedge\mathrm{R})$$

$$\frac{\Pi \Rightarrow \varphi}{\Pi, \neg\varphi \Rightarrow \bot}(\neg\mathrm{L}) \quad \frac{\varphi, \Pi \Rightarrow \bot}{\Pi \Rightarrow \neg\varphi}(\neg\mathrm{R})$$

$$\frac{\Pi[\varphi] \Rightarrow \psi}{\Pi[\neg\neg\varphi] \Rightarrow \psi}(\neg\neg\mathrm{L}) \quad \frac{\Pi \Rightarrow \varphi}{\Pi \Rightarrow \neg\neg\varphi}(\neg\neg\mathrm{R})$$

$$\frac{\Pi[\circ\varphi] \Rightarrow \psi}{\Pi[\Diamond\varphi] \Rightarrow \psi}(\Diamond\mathrm{L}) \quad \frac{\Pi \Rightarrow \varphi}{\circ\Pi \Rightarrow \Diamond\varphi}(\Diamond\mathrm{R}) \quad \frac{\Pi[\bullet\varphi] \Rightarrow \psi}{\Pi[\blacklozenge\varphi] \Rightarrow \psi}(\blacklozenge\mathrm{L}) \quad \frac{\Pi \Rightarrow \varphi}{\bullet\Pi \Rightarrow \blacklozenge\varphi}(\blacklozenge\mathrm{R})$$

$$\frac{\Pi[\varphi] \Rightarrow \psi}{\Pi[\circ\blacksquare\varphi] \Rightarrow \psi}(\blacksquare\mathrm{L}) \quad \frac{\circ\Pi \Rightarrow \varphi}{\Pi \Rightarrow \blacksquare\varphi}(\blacksquare\mathrm{R}) \quad \frac{\Pi[\varphi] \Rightarrow \psi}{\Pi[\bullet\Box\varphi] \Rightarrow \psi}(\Box\mathrm{L}) \quad \frac{\bullet\Pi \Rightarrow \varphi}{\Pi \Rightarrow \Box\varphi}(\Box\mathrm{R})$$

The derived formula in the below sequent of a logical rule is called principal.

(3) *Structural rules:*

$$\frac{\Pi[\Theta,\Theta] \Rightarrow \psi}{\Pi[\Theta] \Rightarrow \psi}(\mathrm{Ctr}) \quad \frac{\Pi[\Theta_i] \Rightarrow \psi}{\Pi[\Theta_1,\Theta_2] \Rightarrow \psi}(\mathrm{Wek})(i=1,2)$$

$$\frac{\circ\Pi_1,\Pi_2 \Rightarrow \bot}{\Pi_1,\bullet\Pi_2 \Rightarrow \bot}(\mathrm{Dual}_{\circ\bullet}) \quad \frac{\bullet\Pi_1,\Pi_2 \Rightarrow \bot}{\Pi_1,\circ\Pi_2 \Rightarrow \bot}(\mathrm{Dual}_{\bullet\circ})$$

(4) *Cut rule:*

$$\frac{\Theta \Rightarrow \varphi \quad \Pi[\varphi] \Rightarrow \psi}{\Pi[\Theta] \Rightarrow \varphi}(\mathrm{Cut})$$

A derivation in GK_t *is a finite tree of sequents in which each node is either an axiom or derived from child node(s) by a sequent rule. Derivations are denoted by* $\mathcal{D}, \mathcal{E}$*, etc., with or without subscripts. The height of a derivation* $\mathcal{D}$*, denoted by* $|\mathcal{D}|$*, is defined as the maximal length of branches in* $\mathcal{D}$*. A single node derivation has height* 0*. A sequent* s *is derivable in* GK_t*, notation* $\mathsf{GK}_t \vdash s$*, if there exists a derivation in* GK_t *with root node* s*. For every* $k \geq 0$*, we write* $\mathsf{GK}_t \vdash_k s$ *if there exists a derivation of* s *in* GK_t *with height at most* k*. A sequent rule with premises* $s_1, \ldots, s_n$ *and conclusion* s_0 *is admissible in* GK_t *if* $\mathsf{GK}_t \vdash s_0$ *whenever* $\mathsf{GK}_t \vdash s_i$ *for all* $1 \leq i \leq n$*. The prefix* GK_t *is omitted if no confusion can arise.*

Lemma 1. *The following sequent rules are admissible in* GK_t*:*

$$\frac{\Pi[\Theta_1,\Theta_2] \Rightarrow \psi}{\Pi[\Theta_2,\Theta_1] \Rightarrow \psi}(\mathrm{Ex}) \quad \frac{\Pi[\Theta_1,(\Theta_2,\Theta_3)] \Rightarrow \psi}{\Pi[(\Theta_1,\Theta_2),\Theta_3] \Rightarrow \psi}(\mathrm{As}_1) \quad \frac{\Pi[(\Theta_1,\Theta_2),\Theta_3] \Rightarrow \psi}{\Pi[\Theta_1,(\Theta_2,\Theta_3)] \Rightarrow \psi}(\mathrm{As}_2)$$

Proof. We have the following derivations:

$$\dfrac{\dfrac{\dfrac{\Pi[\Theta_1,\Theta_2] \Rightarrow \psi}{\Pi[(\Theta_2,\Theta_1),\Theta_2] \Rightarrow \psi}(\mathrm{Wek})}{\Pi[(\Theta_2,\Theta_1),(\Theta_2,\Theta_1)] \Rightarrow \psi}(\mathrm{Wek})}{\Pi[\Theta_2,\Theta_1] \Rightarrow \psi}(\mathrm{Ctr}) \quad \dfrac{\dfrac{\dfrac{\dfrac{\Pi[\Theta_1,(\Theta_2,\Theta_3)] \Rightarrow \psi}{\Pi[\Theta_1,((\Theta_1,\Theta_2),\Theta_3)] \Rightarrow \psi}(\mathrm{Wek})}{\Pi[(\Theta_1,\Theta_2),((\Theta_1,\Theta_2),\Theta_3)] \Rightarrow \psi}(\mathrm{Wek})}{\Pi[((\Theta_1,\Theta_2),\Theta_3),((\Theta_1,\Theta_2),\Theta_3)] \Rightarrow \psi}(\mathrm{Wek})}{\Pi[(\Theta_1,\Theta_2),\Theta_3] \Rightarrow \psi}(\mathrm{Ctr})$$

The admissibility of (As_2) can be shown similarly. □

Note that, in Lemma 1, (Ex) is the the rule of exchange, and (As_1) and (As_2) are rules of associativity with respect to the structural operator of comma. Henceforth, applications of these rules are so obvious that we skip them in derivations. A formula α is GK_t-equivalent to β (notation: $\mathsf{GK}_t \vdash \alpha \Leftrightarrow \beta$) if $\mathsf{GK}_t \vdash \alpha \Rightarrow \beta$ and $\mathsf{GK}_t \vdash \beta \Rightarrow \alpha$.

Lemma 2. *The following hold in* GK_t*:*

(1) $\vdash \beta, \Gamma \Rightarrow \beta$.
(2) $\vdash \beta \Leftrightarrow \neg\neg\beta$ *and* $\vdash\ \Rightarrow \beta \vee \neg\beta$.
(3) $\vdash \beta \Rightarrow \alpha$ *iff* $\vdash \neg\alpha \Rightarrow \neg\beta$.
(4) $\vdash \neg\beta \Rightarrow \alpha$ *iff* $\vdash \neg\alpha \Rightarrow \beta$.
(5) $\vdash \beta \Rightarrow \neg\alpha$ *iff* $\vdash \alpha \Rightarrow \neg\beta$.
(6) $\vdash \Diamond\beta \Leftrightarrow \neg\Box\neg\beta$ *and* $\vdash \blacklozenge\beta \Leftrightarrow \neg\blacksquare\neg\beta$.
(7) $\vdash \Box\beta \Leftrightarrow \neg\Diamond\neg\beta$ *and* $\vdash \blacksquare\beta \Leftrightarrow \neg\blacklozenge\neg\beta$.
(8) $\vdash \Diamond\blacksquare\beta \Rightarrow \beta$ *and* $\vdash \blacklozenge\Box\beta \Rightarrow \beta$.

(9) $\vdash \beta \Rightarrow \Box\blacklozenge\beta$ *and* $\vdash \beta \Rightarrow \blacksquare\Diamond\beta$.
(10) *if* $\vdash \beta \Rightarrow \alpha$, *then* $\vdash \odot\beta \Rightarrow \odot\alpha$ *for every* $\odot \in \{\Diamond, \Box, \blacklozenge, \blacksquare\}$.

Proof. For (1), it is obtained by (Wek). For (2), from (Id) by $(\neg\neg\mathrm{L})$ and $(\neg\neg\mathrm{R})$, we obtain $\vdash \beta \Leftrightarrow \neg\neg\beta$. We also have the following derivation:

$$\dfrac{\dfrac{\dfrac{\neg\beta \Rightarrow \neg\beta}{\neg\beta, \neg\neg\beta \Rightarrow \bot}\,(\neg\mathrm{L})}{\neg\beta \wedge \neg\neg\beta \Rightarrow \bot}\,(\wedge\mathrm{L})}{\Rightarrow \neg(\neg\beta \wedge \neg\neg\beta)}\,(\neg\mathrm{R})$$

For (3), we have the following derivations:

$$\dfrac{\dfrac{\beta \Rightarrow \alpha}{\neg\alpha, \beta \Rightarrow \bot}\,(\neg\mathrm{L})}{\neg\alpha \Rightarrow \neg\beta}\,(\neg\mathrm{R}) \qquad \dfrac{\dfrac{\beta \Rightarrow \neg\neg\beta \quad \dfrac{\dfrac{\neg\alpha \Rightarrow \neg\beta}{\neg\alpha, \neg\neg\beta \Rightarrow \bot}\,(\neg\mathrm{L})}{\neg\neg\beta \Rightarrow \neg\neg\alpha}\,(\neg\mathrm{R})}{\beta \Rightarrow \neg\neg\alpha}\,(\mathrm{Cut}) \quad \neg\neg\alpha \Rightarrow \alpha}{\beta \Rightarrow \alpha}\,(\mathrm{Cut})$$

By (3), (2) and (Cut), we obtain (4) and (5). For (6), we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta}{\neg\beta, \beta \Rightarrow \bot}\,(\neg\mathrm{L})}{\bullet\Box\neg\beta, \beta \Rightarrow \bot}\,(\Box\mathrm{L})}{\Box\neg\beta, \circ\beta \Rightarrow \bot}\,(\mathrm{Dual}_{\bullet\circ})}{\Box\neg\beta, \Diamond\beta \Rightarrow \bot}\,(\Diamond\mathrm{L})}{\Diamond\beta \Rightarrow \neg\Box\neg\beta}\,(\neg\mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta}{\circ\beta \Rightarrow \Diamond\beta}\,(\Diamond\mathrm{R})}{\circ\beta, \neg\Diamond\beta \Rightarrow \bot}\,(\neg\mathrm{L})}{\beta, \bullet\neg\Diamond\beta \Rightarrow \bot}\,(\mathrm{Dual}_{\circ\bullet})}{\bullet\neg\Diamond\beta \Rightarrow \neg\beta}\,(\neg\mathrm{R})}{\neg\Diamond\beta \Rightarrow \Box\neg\beta}\,(\Box\mathrm{R})}{\neg\Box\neg\beta \Rightarrow \Diamond\beta}\,(4)$$

The proof of (7) is similar. For (8) and (9), we have the following derivations:

$$\dfrac{\dfrac{\beta \Rightarrow \beta}{\circ\blacksquare\beta \Rightarrow \beta}\,(\blacksquare\mathrm{L})}{\Diamond\blacksquare\beta \Rightarrow \beta}\,(\Diamond\mathrm{L}) \quad \dfrac{\dfrac{\beta \Rightarrow \beta}{\bullet\Box\beta \Rightarrow \beta}\,(\Box\mathrm{L})}{\blacklozenge\Box\beta \Rightarrow \beta}\,(\blacklozenge\mathrm{L}) \quad \dfrac{\dfrac{\beta \Rightarrow \beta}{\bullet\beta \Rightarrow \blacklozenge\beta}\,(\blacklozenge\mathrm{R})}{\beta \Rightarrow \Box\blacklozenge\beta}\,(\Box\mathrm{R}) \quad \dfrac{\dfrac{\beta \Rightarrow \beta}{\circ\beta \Rightarrow \Diamond\beta}\,(\Diamond\mathrm{R})}{\beta \Rightarrow \blacksquare\Diamond\beta}\,(\blacksquare\mathrm{R})$$

For (10), assume $\vdash \beta \Rightarrow \alpha$. By $(\Diamond\mathrm{R})$, $\vdash \circ\beta \Rightarrow \Diamond\alpha$. By $(\Diamond\mathrm{L})$, $\vdash \Diamond\beta \Rightarrow \Diamond\alpha$. By $(\Box\mathrm{L})$, $\vdash \bullet\Box\beta \Rightarrow \alpha$. By $(\Box\mathrm{R})$, $\vdash \Box\beta \Rightarrow \Box\alpha$. The cases of $\blacklozenge$ and $\blacksquare$ are similar. □

Lemma 3. *If* $\mathsf{GK}_t \vdash \alpha \Leftrightarrow \beta$, *then* $\mathsf{GK}_t \vdash \chi \Leftrightarrow \chi(\alpha/\beta)$, *where* $\chi(\alpha/\beta)$ *is obtained from* χ *by replacing one or more occurrences of* α *in* χ *by* β.

Proof. The proof proceeds by induction on the complexity of χ. The case $\chi \in \mathrm{Var} \cup \{\bot\}$ or $\chi = \alpha$ is trivial. Let $\chi \neq \alpha$. Suppose $\chi = \chi_1 \wedge \chi_2$. By induction hypothesis, $\vdash \chi_1 \Leftrightarrow \chi_1(\alpha/\beta)$ and $\vdash \chi_2 \Leftrightarrow \chi_2(\alpha/\beta)$. It is easy to obtain $\vdash \chi \Leftrightarrow \chi(\alpha/\beta)$. Suppose $\chi = \neg\xi$. By induction hypothesis, $\vdash \xi \Leftrightarrow \xi(\alpha/\beta)$. By Lemma 2 (3), $\vdash \neg\xi \Leftrightarrow \neg\xi(\alpha/\beta)$. Suppose $\chi = \Diamond\xi$. By induction hypothesis, $\vdash \xi \Leftrightarrow \xi(\alpha/\beta)$. By Lemma 2 (10), $\vdash \Diamond\xi \Leftrightarrow \Diamond\xi(\alpha/\beta)$. The remaining cases for $\chi = \odot\xi$ with $\odot \in \{\Box, \blacklozenge, \blacksquare\}$ can be shown similarly. □

Lemma 4. *The following hold in* GK_t:

(1) *if* $\vdash \alpha \Rightarrow \gamma$ *and* $\vdash \beta \Rightarrow \gamma$, *then* $\vdash \alpha \vee \beta \Rightarrow \gamma$.
(2) *if* $\vdash \alpha \Rightarrow \beta_1$ *or* $\vdash \alpha \Rightarrow \beta_2$, *then* $\vdash \alpha \Rightarrow \beta_1 \vee \beta_2$.
(3) $\vdash \alpha, \neg\alpha \vee \beta \Rightarrow \beta$.
(4) $\vdash \alpha \wedge \beta \Rightarrow \gamma$ *iff* $\vdash \alpha \Rightarrow \neg\beta \vee \gamma$.
(5) $\vdash \alpha \wedge \neg\beta \Rightarrow \gamma$ *iff* $\vdash \alpha \Rightarrow \beta \vee \gamma$.

Proof. For (1), we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \gamma}{\neg\gamma \Rightarrow \neg\alpha}\text{ Lemma 2 (3)} \qquad \dfrac{\beta \Rightarrow \gamma}{\neg\gamma \Rightarrow \neg\beta}\text{ Lemma 2 (3)}}{\neg\gamma, \neg\gamma \Rightarrow \neg\alpha \wedge \neg\beta}\,(\wedge\mathrm{R})}{\neg\gamma \Rightarrow \neg\alpha \wedge \neg\beta}\,(\mathrm{Ctr})}{\neg(\neg\alpha \wedge \neg\beta) \Rightarrow \gamma}\text{ Lemma 2 (4)}$$

For (2), by $\vdash \neg\beta_1 \wedge \neg\beta_2 \Rightarrow \neg\beta_i$, where $i = 1, 2$ and Lemma 2 (5), $\vdash \beta_i \Rightarrow \neg(\neg\beta_1 \wedge \neg\beta_2)$. Assume $\vdash \alpha \Rightarrow \beta_i$. By (Cut), $\vdash \alpha \Rightarrow \neg(\neg\beta_1 \wedge \neg\beta_2)$. For (3), we have the derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha, \neg\beta \Rightarrow \alpha}{\alpha, \neg\beta \Rightarrow \neg\neg\alpha}\,(\neg\neg\mathrm{R}) \qquad \alpha, \neg\beta \Rightarrow \neg\beta}{\alpha, \neg\beta, \alpha, \neg\beta \Rightarrow \neg\neg\alpha \wedge \neg\beta}\,(\wedge\mathrm{R})}{\alpha, \neg\beta \Rightarrow \neg\neg\alpha \wedge \neg\beta}\,(\mathrm{Ctr})}{\alpha, \neg\beta, \neg(\neg\neg\alpha \wedge \neg\beta) \Rightarrow \bot}\,(\neg\mathrm{L})}{\alpha, \neg(\neg\neg\alpha \wedge \neg\beta) \Rightarrow \neg\neg\beta}\,(\neg\mathrm{R}) \qquad \neg\neg\beta \Rightarrow \beta}{\alpha, \neg(\neg\neg\alpha \wedge \neg\beta) \Rightarrow \beta}\,(\mathrm{Cut})$$

For (4), we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \alpha \wedge \beta}\,(\wedge\mathrm{R}) \qquad \alpha \wedge \beta \Rightarrow \gamma}{\alpha, \beta \Rightarrow \gamma}\,(\mathrm{Cut})}{\alpha, \beta, \neg\gamma \Rightarrow \bot}\,(\neg\mathrm{L})}{\alpha, \neg\neg\beta, \neg\gamma \Rightarrow \bot}\,(\neg\neg\mathrm{L})}{\alpha, \neg\neg\beta \wedge \neg\gamma \Rightarrow \bot}\,(\wedge\mathrm{L})}{\alpha \Rightarrow \neg(\neg\neg\beta \wedge \neg\gamma)}\,(\neg\mathrm{R}) \qquad \dfrac{\dfrac{\alpha \Rightarrow \neg\beta \vee \gamma \quad \beta, \neg\beta \vee \gamma \Rightarrow \gamma}{\alpha, \beta \Rightarrow \gamma}\,(\mathrm{Cut})}{\alpha \wedge \beta \Rightarrow \gamma}\,(\wedge\mathrm{L})$$

Note that $\vdash \beta, \neg\beta \vee \gamma \Rightarrow \gamma$ by (3). For (5), by (4), $\vdash \alpha \wedge \neg\beta \Rightarrow \gamma$ iff $\vdash \alpha \Rightarrow \neg\neg\beta \vee \gamma$. By Lemma 3, $\vdash \neg\neg\beta \vee \gamma \Leftrightarrow \beta \vee \gamma$. Hence, $\vdash \alpha \wedge \neg\beta \Rightarrow \gamma$ iff $\vdash \alpha \Rightarrow \beta \vee \gamma$. $\square$

Remark 1. *Every formula structure Θ can be replaced by a corresponding formula $\Theta^\flat$. Clearly,* $\mathsf{GK}_t \vdash \Pi[\Theta] \Rightarrow \alpha$ *iff* $\mathsf{GK_t} \vdash \Pi[\Theta^\flat] \Rightarrow \alpha$. *Then, by Lemma 4 (4), one obtains* $\mathsf{GK}_t \vdash \alpha, \Theta \Rightarrow \beta$ *iff* $\mathsf{GK}_t \vdash \Theta \Rightarrow \neg\alpha \vee \beta$.

Definition 6. *We define the displaying formula $FD(\Delta[-] : \alpha)$ with respect to a formula context $\Delta[-]$ and a formula α recursively as follows:*

$$\begin{aligned} FD(- : \alpha) &= \alpha \\ FD(\Delta_1[-], \Delta_2 : \alpha) &= FD(\Delta_1[-] : \neg\Delta_2^\flat \vee \alpha) \\ FD(\Delta_1, \Delta_2[-] : \alpha) &= FD(\Delta_2[-] : \neg\Delta_1^\flat \vee \alpha) \\ FD(\circ\Delta[-] : \alpha) &= FD(\Delta[-] : \blacksquare\alpha) \\ FD(\bullet\Delta[-] : \alpha) &= FD(\Delta[-] : \Box\alpha) \end{aligned}$$

Theorem 2 (Displaying)**.** $\mathsf{GK}_t \vdash \Gamma[\Delta] \Rightarrow \alpha$ *iff* $\mathsf{GK}_t \vdash \Delta \Rightarrow FD(\Gamma[-] : \alpha)$.

Proof. We proceed by the induction on $d(\Gamma[-])$. Basic cases for $d(\Gamma[-]) = 0$ are trivial. We consider the following cases for the induction:

(1) $\Gamma[-] = (\Gamma_1[-], \Gamma_2)$ or $(\Gamma_1, \Gamma_2[-])$. We prove only the former one and the other can be treated similarly. Clearly, $FD(\Gamma[-] : \beta) = FD(\Gamma_1[-] : \neg\Gamma_2^\flat \vee \beta)$. By induction hypothesis, $\vdash \Gamma_1[\Delta] \Rightarrow \neg\Gamma_2^\flat \vee \beta$ iff $\vdash \Delta \Rightarrow FD(\Gamma_1[-] : \neg\Gamma_2^\flat \vee \beta)$. It needs only to show that

$\vdash \Gamma_1[\Delta], \Gamma_2 \Rightarrow \beta$ iff $\vdash \Gamma_1[\Delta] \Rightarrow \neg\Gamma_2^\flat \vee \beta$. Let $\vdash \Gamma_1[\Delta], \Gamma_2 \Rightarrow \beta$. Hence, $\vdash \Gamma_1[\Delta], \Gamma_2^\flat \Rightarrow \beta$. Then, $\vdash \Gamma_1[\Delta] \Rightarrow \neg\Gamma_2^\flat \vee \beta$. Suppose $\vdash \Gamma_1[\Delta] \Rightarrow \neg\Gamma_2^\flat \vee \beta$. Clearly, $\vdash \Gamma_1[\Delta], \Gamma_2^\flat \Rightarrow \beta$. Consequently, $\vdash \Gamma_1[\Delta], \Gamma_2 \Rightarrow \beta$.

(2) $\Gamma[-] = \bullet\Sigma[-]$ or $\circ\Sigma[-]$. We prove only the former one and the other can be treated similarly. Clearly, $FD(\Gamma[-] : \beta) = D(\Sigma[-] : \Box\beta)$. According to induction hypothesis, $\vdash \Sigma[\Delta] \Rightarrow \Box\beta$ iff $\vdash \Delta \Rightarrow FD(\Sigma[-] : \Box\beta)$. It needs only to show that $\vdash \bullet\Sigma[\Delta] \Rightarrow \beta$ iff $\vdash \Sigma[\Delta] \Rightarrow \Box\beta$. Let $\vdash \circ\Sigma[\Delta] \Rightarrow \beta$. By ($\Box$R), $\vdash \Sigma[\Delta] \Rightarrow \Box\beta$. Let $\vdash \Sigma[\Delta] \Rightarrow \Box\beta$. By ($\blacklozenge$R), $\vdash \circ\Sigma[\Delta] \Rightarrow \blacklozenge\Box\beta$. By Lemma 2 (8), $\vdash \blacklozenge\Box\beta \Rightarrow \beta$. By (Cut), $\vdash \bullet\Sigma[\Delta] \Rightarrow \beta$. □

A sequent $\Gamma \Rightarrow \beta$ is valid in a frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \Gamma \Rightarrow \beta$) if $\mathfrak{F} \models \neg\Gamma^\flat \vee \beta$. A sequent s is valid in a class of frames $\mathcal{K}$ (notation: $\mathcal{K} \models s$) if $\mathfrak{F} \models s$ for all $\mathfrak{F} \in \mathcal{K}$. An inference rule (R) preserves validity if the validity of premise(s) implies that of the conclusion. Now, we shall prove the soundness and completeness of GK_t.

Lemma 5. *For every frame $\mathfrak{F}$, $\mathfrak{F} \models \Gamma[\Delta] \Rightarrow \beta$ iff $\mathfrak{F} \models \Delta \Rightarrow FD(\Gamma[-] : \beta)$.*

Proof. We proceed by induction on $d(\Gamma[-])$. The basic cases are trivial. Assume $\Gamma[-] = (\Gamma_1[-], \Gamma_2)$ or $(\Gamma_1, \Gamma_2[-])$. We prove only the former case. Obviously, $FD(\Gamma[-] : \beta) = FD(\Gamma_1[-] : \neg\Gamma_2^\flat \vee \beta)$. By induction hypothesis, $\mathfrak{F} \models \Gamma_1[\Delta] \Rightarrow \neg\Gamma_2^\flat \vee \beta$ iff $\mathfrak{F} \models \Delta \Rightarrow FD(\Gamma_1[-] : \neg\Gamma_2^\flat \vee \beta)$. Clearly, $\mathfrak{F} \models \Gamma_1[\Delta], \Gamma_2 \Rightarrow \beta$ iff $\mathfrak{F} \models \Gamma_1[\Delta] \Rightarrow \neg\Gamma_2^\flat \vee \beta$. Let $\Gamma[-] = \circ\Sigma[-]$ or $\bullet\Sigma[-]$. We prove only the former case. Obviously, $FD(\Gamma[-] : \beta) = FD(\Sigma[-] : \blacksquare\beta)$. By induction hypothesis, $\mathfrak{F} \models \Sigma[\Delta] \Rightarrow \blacksquare\beta$ iff $\mathfrak{F} \models \Delta \Rightarrow FD(\Sigma[-] : \blacksquare\beta)$. Clearly, $\mathfrak{F} \models \circ\Sigma[\Delta] \Rightarrow \beta$ iff $\mathfrak{F} \models \Sigma[\Delta] \Rightarrow \blacksquare\beta$. □

Theorem 3 (Soundness)**.** *If $\mathsf{GK}_t \vdash \Gamma \Rightarrow \beta$, then $\mathsf{Fr}(\mathsf{K}_t) \models \Gamma \Rightarrow \beta$.*

Proof. Assume $\vdash_k \Gamma \Rightarrow \beta$. The case $k = 0$ holds obviously. Suppose $k > 0$ and $\Gamma \Rightarrow \beta$ is derived by a rule (R). Right rules for $\wedge, \neg, \Diamond, \blacklozenge, \Box$ and $\blacksquare$ preserve validity obviously. Other logical rules, (Ctr), (Wek) and (Cut) preserve validity by Lemma 5. For ($\mathrm{Dual}_{\circ\bullet}$), assume $\mathfrak{F} \models \circ\Delta_1, \Delta_2 \Rightarrow \bot$. Then, $\mathfrak{F} \models \Diamond\Delta_1^\flat \wedge \Delta_2^\flat \Rightarrow \bot$. Let V be any valuation in $\mathfrak{F}$. Then, $V(\Diamond\Delta_1^\flat \wedge \Delta_2^\flat) = \varnothing$ and so $V(\Diamond\Delta_1^\flat) = \varnothing = V(\Delta_2^\flat)$. Then, $V(\Delta_1^\flat) = \varnothing = V(\blacklozenge\Delta_2^\flat)$. Then, $\mathfrak{F} \models \Delta_1, \blacklozenge\Delta_2 \Rightarrow \bot$. The case for ($\mathrm{Dual}_{\bullet\circ}$) is similar. □

Lemma 6. *If $\vdash_{\mathsf{K}_t} \alpha$, then $\mathsf{GK}_t \vdash \Rightarrow \alpha$.*

Proof. Assume $\vdash_{\mathsf{K}_t} \alpha$. By Lemma 2, Lemma 4 and Remark 1, for every instance χ of axioms (A1)–(A6), we have $\mathsf{GK}_t \vdash \Rightarrow \chi$. Assume $\mathsf{GK}_t \vdash \Rightarrow \beta$ and $\mathsf{GK}_t \vdash \Rightarrow \neg\beta \vee \gamma$. By Lemma 4 (3), $\mathsf{GK}_t \vdash \beta, \neg\beta \vee \gamma \Rightarrow \gamma$. By (Cut), $\mathsf{GK}_t \vdash \Rightarrow \gamma$. Assume $\mathsf{GK}_t \vdash \Rightarrow \neg\Diamond\alpha \vee \beta$. By Remark 1, $\mathsf{GK}_t \vdash \Diamond\alpha \Rightarrow \beta$. By Lemma 2 (9) and (10), using (Cut), $\mathsf{GK}_t \vdash \alpha \Rightarrow \blacksquare\beta$. By Remark 1, $\mathsf{GK}_t \vdash \Rightarrow \neg\alpha \vee \blacksquare\beta$. The case for ($\mathrm{Adj}_{\blacklozenge\Box}$) is shown similarly. □

Theorem 4 (Completeness)**.** *If $\mathsf{Fr}(\mathsf{K}_t) \models \Gamma \Rightarrow \beta$, then $\mathsf{GK}_t \vdash \Gamma \Rightarrow \beta$.*

Proof. Assume $\mathsf{Fr}(\mathsf{K}_t) \models \Gamma \Rightarrow \beta$. Then, $\mathsf{Fr}(\mathsf{K}_t) \models \neg\Gamma^\flat \vee \beta$. By Theorem 1, $\vdash_{\mathsf{K}_t} \neg\Gamma^\flat \vee \beta$. By Lemma 6, $\mathsf{GK}_t \vdash \Rightarrow \neg\Gamma^\flat \vee \beta$. Then, $\mathsf{GK}_t \vdash \Gamma^\flat \Rightarrow \beta$ and so $\mathsf{GK}_t \vdash \Gamma \Rightarrow \beta$. □

4. Cut Elimination

In this section, we show the cut elimination of GK_t. Let GK_t^* be the sequent calculus obtained from GK_t by removing the rule (Cut) and replacing the rules ($\neg$L) and ($\neg$R) by the following four rules:

$$\frac{\Gamma \Rightarrow \neg\alpha}{\alpha, \Gamma \Rightarrow \bot}(\neg\mathrm{L}_1) \quad \frac{\Gamma \Rightarrow \delta}{\neg\delta, \Gamma \Rightarrow \bot}(\neg\mathrm{L}_2) \text{ where } \delta \neq \neg\delta' \text{ for any formula } \delta'$$

$$\frac{\neg\alpha, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \alpha}(\neg\mathrm{R}_1) \quad \frac{\delta, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \neg\delta}(\neg\mathrm{R}_2) \text{ where } \delta \neq \neg\delta' \text{ for any formula } \delta'$$

The cut elimination of GK_t is obtained by showing that GK_t is equivalent to GK_t^*.

Lemma 7. *If* $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \beta$, *then* $\mathsf{GK}_t \vdash \Gamma \Rightarrow \beta$.

Proof. Note that $(\neg\mathrm{L}_2)$ and $(\neg\mathrm{R}_2)$ are special cases of $(\neg\mathrm{L})$ and $(\neg\mathrm{R})$, respectively. Assume $\mathsf{GK}_t \vdash \Gamma \Rightarrow \neg\alpha$. Clearly, $\mathsf{GK}_t \vdash \neg\alpha, \alpha \Rightarrow \bot$. By (Cut), $\mathsf{GK}_t \vdash \alpha, \Gamma \Rightarrow \bot$. Hence, $(\neg\mathrm{L}_1)$ is admissible in GK_t. Assume $\mathsf{GK}_t \vdash \neg\alpha, \Gamma \Rightarrow \bot$. By $(\neg\mathrm{R})$, $\mathsf{GK}_t \vdash \Gamma \Rightarrow \neg\neg\alpha$. By $\mathsf{GK}_t \vdash \neg\neg\alpha \Rightarrow \alpha$ and (Cut), $\mathsf{GK}_t \vdash \Gamma \Rightarrow \alpha$. Hence, $(\neg\mathrm{R}_1)$ is admissible in GK_t. □

The converse of Lemma 7 shall be shown in three steps. First, $\mathsf{GK}_t^\dagger$ is the sequent calculus obtained from GK_t by removing the rules $(\neg\neg\mathrm{L})$ and $(\neg\neg\mathrm{R})$. We shall define a translation ko such that, for every sequent s, $\mathsf{GK}_t \models s$ iff $\mathsf{GK}_t^\dagger \vdash ko(s)$. Second, we show the cut elimination of $\mathsf{GK}_t^\dagger$ and so obtain a sequent calculus $\mathsf{GK}_t^\ddagger$, which is obtained from $\mathsf{GK}_t^\dagger$ by removing the rule (Cut). Third, we show that $\mathsf{GK}_t^\ddagger \vdash ko(s)$ implies $\mathsf{GK}_t^* \vdash s$.

$$\begin{tikzcd}
\mathsf{GK}_t \arrow[r, dotted, "ko", "\text{Theorem 5}"'] & \mathsf{GK}_t^\dagger \arrow[d, dotted, leftrightarrow, "\text{Cut elimination (Theorem 6)}"] \\
\mathsf{GK}_t^* \arrow[u, dotted, "\text{Lemma 7}"] & \mathsf{GK}_t^\ddagger \arrow[l, dotted, "\text{Theorem 7}"']
\end{tikzcd}$$

Lemma 8. *If* $\mathsf{GK}_t^\dagger \vdash \Gamma_1[\alpha], \Gamma_2 \Rightarrow \bot$, *then* $\mathsf{GK}_t^\dagger \vdash \Gamma_1[\neg\neg\alpha], \Gamma_2 \Rightarrow \bot$.

Proof. Let $\mathsf{GK}_t^\dagger \vdash \Gamma_1[\alpha], \Gamma_2 \Rightarrow \bot$. We prove $\mathsf{GK}_t^\dagger \vdash \Gamma_1[\neg\neg\alpha], \Gamma_2 \Rightarrow \bot$ by induction on $d(\Gamma_1[-])$. Suppose $\Gamma_1[\alpha] = \alpha$. Then, $\mathsf{GK}_t^\dagger \vdash \alpha, \Gamma_2 \Rightarrow \bot$. By $(\neg\mathrm{R})$ and $(\neg\mathrm{L})$, $\mathsf{GK}_t^\dagger \vdash \neg\neg\alpha, \Gamma_2 \Rightarrow \bot$. Suppose $\Gamma_1[-] = (\Sigma_1[-], \Sigma_2)$ or $(\Sigma_2, \Sigma_1[-])$. We prove only the former case. Clearly, $\mathsf{GK}_t^\dagger \vdash \Sigma_1[\alpha], \Sigma_2, \Gamma_2 \Rightarrow \bot$. By induction hypothesis, $\mathsf{GK}_t^\dagger \vdash \Sigma_1[\neg\neg\alpha], \Sigma_2, \Gamma_2 \Rightarrow \bot$. Suppose $\Gamma_1[-] = \circ\Sigma[-]$ or $\bullet\Sigma[-]$. We prove only the former cases. Clearly, $\mathsf{GK}_t^\dagger \vdash \circ\Sigma[\alpha], \Gamma_2 \Rightarrow \bot$. By $(\mathrm{Dual}_{\circ\bullet})$, $\mathsf{GK}_t^\dagger \vdash \Sigma[\alpha], \bullet\Gamma_2 \Rightarrow \bot$. By induction hypothesis, $\mathsf{GK}_t^\dagger \vdash \Sigma[\neg\neg\alpha], \bullet\Gamma_2 \Rightarrow \bot$. By $(\mathrm{Dual}_{\bullet\circ})$, $\mathsf{GK}_t^\dagger \vdash \circ\Sigma[\neg\neg\alpha], \Gamma_2 \Rightarrow \bot$. □

Lemma 9. *For every formula* α, $\mathsf{GK}_t^\dagger \vdash \alpha \Rightarrow \neg\neg\alpha$ *and* $\mathsf{GK}_t^\dagger \vdash \neg\neg\neg\alpha \Leftrightarrow \neg\alpha$.

Proof. Apply $(\neg\mathrm{L})$ and $(\neg\mathrm{R})$ to $\alpha \Rightarrow \alpha$. □

Lemma 10. *If* $\mathsf{GK}_t^\dagger \vdash \Gamma[\alpha] \Rightarrow \neg\beta$, *then* $\mathsf{GK}_t^\dagger \vdash \Gamma[\neg\neg\alpha] \Rightarrow \neg\beta$.

Proof. Assume $\mathsf{GK}_t^\dagger \vdash \Gamma[\alpha] \Rightarrow \neg\beta$. By $(\neg\mathrm{L})$ and Lemma 8, $\mathsf{GK}_t^\dagger \vdash \Gamma[\neg\neg\alpha], \neg\neg\beta \Rightarrow \bot$. By $(\neg\mathrm{R})$, $\mathsf{GK}_t^\dagger \vdash \Gamma[\neg\neg\alpha] \Rightarrow \neg\neg\neg\beta$. By Lemma 9, $\mathsf{GK}_t^\dagger \vdash \neg\neg\neg\beta \Rightarrow \neg\beta$. By (Cut), $\mathsf{GK}_t^\dagger \vdash \Gamma[\neg\neg\alpha] \Rightarrow \neg\beta$. □

Definition 7. *The Kolmogorov translation* $kol : \mathcal{F} \to \mathcal{F}$ *is recursively defined as follows:*

$$\begin{aligned}
kol(p) &= \neg\neg p \textit{ where } p \in \mathsf{Var}. \\
kol(\bot) &= \neg\neg\bot \\
kol(\alpha \wedge \beta) &= \neg\neg(kol(\alpha) \wedge kol(\beta)) \\
kol(\neg\alpha) &= \neg ko(\alpha) \\
kol(\odot\alpha) &= \neg\neg \odot kol(\alpha) \textit{ where } \odot \in \{\Diamond, \blacklozenge, \Box, \blacksquare\}.
\end{aligned}$$

For every formula structure Γ, *by* $kol(\Gamma)$, *we denote the formula structure obtained from* Γ *by replacing each formula* β *in* Γ *by* $kol(\beta)$. *For every context* $\Gamma[-]$, $kol(\Gamma[-])$ *is defined natrually.*

Lemma 11. *For every formula* α, $\mathsf{GK}_t^\dagger \vdash kol(\neg\neg\alpha) \Rightarrow kol(\alpha)$.

Proof. Clearly, $kol(\neg\neg\alpha) = \neg\neg kol(\alpha)$ and $kol(\alpha) = \neg\delta$ for some δ. By Lemma 9, $\mathsf{GK}_t^\dagger \vdash \neg\neg\neg\delta \Rightarrow \neg\delta$. Then, $\mathsf{GK}_t^\dagger \vdash kol(\neg\neg\alpha) \Rightarrow kol(\alpha)$. $\square$

Theorem 5. $\mathsf{GK}_t \vdash \Gamma \Rightarrow \beta$ *iff* $\mathsf{GK}_t^\dagger \vdash kol(\Gamma) \Rightarrow kol(\beta)$.

Proof. Let $\mathsf{GK}_t^\dagger \vdash kol(\Gamma) \Rightarrow kol(\beta)$. Then, $\mathsf{GK}_t \vdash kol(\Gamma) \Rightarrow kol(\beta)$. Obviously, $\mathsf{GK}_t \vdash \alpha \Leftrightarrow kol(\alpha)$ for any α. Due to Lemma 3 and (Cut), $\mathsf{GK}_t \vdash \Gamma \Rightarrow \beta$. The opposite direction is proved by induction on the derivation of $\Gamma \Rightarrow \beta$. Let $\Gamma \Rightarrow \beta$ be obtained by rule (R). We provide only details of proof for the case $(R) = (\mathrm{Dual}_{\circ\bullet})$ or $(\mathrm{Dual}_{\bullet\circ})$. Other cases are analogous to the proof of [17], Lemma 5.5. We prove only the former case. Let the premise and the conclusion of (R) be $\mathsf{GK}_t \vdash_{k-1} \circ\Delta_1, \Delta_2 \Rightarrow \bot$ and $\mathsf{GK}_t \vdash_k \Delta_1, \bullet\Delta_2 \Rightarrow \bot$, respectively. By induction hypothesis, $\mathsf{GK}_t^\dagger \vdash \circ kol(\Delta_1), kol(\Delta_2) \Rightarrow \neg\neg\bot$. By $\mathsf{GK}_t^\dagger \vdash \neg\neg\bot \Rightarrow \bot$ and (Cut), $\mathsf{GK}_t^\dagger \vdash \circ kol(\Delta_1), kol(\Delta_2) \Rightarrow \bot$. By $(\mathrm{Dual}_{\circ\bullet})$, $\mathsf{GK}_t^\dagger \vdash kol(\Delta_1), \bullet kol(\Delta_2) \Rightarrow \bot$. $\square$

Now, we prove the cut elimination holds for $\mathsf{GK}_t^\dagger$. For every $n \geq 0$, let $\Gamma[\Delta]^n$ be the formula structure in which Δ appears at n places. In particular, if $n = 0$, then $\Gamma[\Delta]^n$ denotes a formula structure in which Δ does not appear. We introduce the following mix rule:

$$\frac{\Delta \Rightarrow \alpha \quad \Gamma[\alpha]^n \Rightarrow \beta}{\Gamma[\Delta]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

Clearly, (Mix) is admissible in $\mathsf{GK}_t^\dagger$, and (Cut) is a special case of (Mix). Thus, the cut elimination is equivalent to the mix elimination of $\mathsf{GK}_t^\dagger$.

Theorem 6. *If* $\mathsf{GK}_t^\dagger \vdash \Gamma \Rightarrow \chi$, *then* $\mathsf{GK}_t^\dagger \vdash \Gamma \Rightarrow \chi$ *without any application of* (Mix).

Proof. Let $\mathcal{D}$ be a derivation of $\Gamma \Rightarrow \chi$ in $\mathsf{GK}_t^\dagger$. It suffices to show that (Mix) can be eliminated from $\mathcal{D}$. Let an application of (Mix) in $\mathcal{D}$ be as follows:

$$\frac{\vdash_k \Delta \Rightarrow \alpha \quad \vdash_m \Sigma[\alpha]^n \Rightarrow \beta}{\vdash \Sigma[\Delta]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

We prove the elimination of (Mix) by induction on $c(\alpha)$ and $k + m$. Let $k = 0$ or $m = 0$. Then, $\Delta \Rightarrow \alpha$ or $\Sigma[\alpha]^n \Rightarrow \beta$ is an instance of (Id). Hence, $\Delta = \alpha$ or $\Sigma[\alpha] = \beta$. Therefore, the conclusion is just the right or left premise of (Mix). Suppose that $k > 0$ and $m > 0$. Assume the last rules for deriving the left and right premises of (Mix) are (R_l) and (R_r), respectively. We have the following cases:

(1) At least one of (R_l) and (R_r) is a structural rule. We have the following cases:

(1.1) (R_l) is (Ctr). Let the derivation end with

$$\frac{\dfrac{\Delta[\Theta, \Theta] \Rightarrow \alpha}{\Delta[\Theta] \Rightarrow \alpha}\,(\mathrm{Ctr}) \quad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\Theta]]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\frac{\dfrac{\Delta[\Theta, \Theta] \Rightarrow \alpha \quad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\Theta, \Theta]]^n \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\Delta[\Theta]]^n \Rightarrow \beta}\,(\mathrm{Ctr})^n$$

where $(\mathrm{Ctr})^n$ means n times application of (Ctr).

(1.2) (R_l) is (Wek). Let the derivation end with

$$\frac{\dfrac{\Delta[\Theta_i] \Rightarrow \alpha}{\Delta[\Theta_1, \Theta_2] \Rightarrow \alpha}\,(\mathrm{Wek}) \quad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\Theta_1, \Theta_2]]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta[\Theta_i] \Rightarrow \alpha \quad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\Theta_i]]^n \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\Delta[\Theta_1, \Theta_2]]^n \Rightarrow \beta}\,(\mathrm{Wek})^n$$

where $i = 1, 2$ and $(\mathrm{Wek})^n$ means n times application of (Wek).

(1.3) (R_l) is $(\mathrm{Dual}_{\circ\bullet})$. Let $\Delta = (\Delta_1, \bullet\Delta_2)$, $\alpha = \perp$ and the derivation end with

$$\dfrac{\dfrac{\circ\Delta_1, \Delta_2 \Rightarrow \perp}{\Delta_1, \bullet\Delta_2 \Rightarrow \perp}\,(\mathrm{Dual}_{\circ\bullet}) \qquad \Sigma[\perp]^n \Rightarrow \beta}{\Sigma[\Delta_1, \bullet\Delta_2]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

The above subtree can be transformed into

$$\dfrac{\dfrac{\circ\Delta_1, \Delta_2 \Rightarrow \perp}{\Delta_1, \bullet\Delta_2 \Rightarrow \perp}\,(\mathrm{Dual}_{\circ\bullet})}{\Sigma[\Delta_1, \bullet\Delta_2]^n \Rightarrow \beta}\,(\perp)$$

(1.4) (R_l) is $(\mathrm{Dual}_{\circ\bullet})$. The proof is quite analogous to (1.3).

(1.5) (R_r) is (Ctr). Let $\Sigma[\alpha]^n = \Theta[\Pi[\alpha]^{n_1}][\alpha]^{n_2}$ and the derivation end with

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\Theta[\Pi[\alpha]^{n_1}, \Pi[\alpha]^{n_1}][\alpha]^{n_2} \Rightarrow \beta}{\Theta[\Pi[\alpha]^{n_1}][\alpha]^{n_2} \Rightarrow \beta}\,(\mathrm{Ctr})}{\Theta[\Pi[\Delta]^{n_1}][\Delta]^{n_2} \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \alpha \quad \Theta[\Pi[\alpha]^{n_1}, \Pi[\alpha]^{n_1}][\alpha]^{n_2} \Rightarrow \beta}{\Theta[\Pi[\Delta]^{n_1}, \Pi[\Delta]^{n_1}][\Delta]^{n_2} \Rightarrow \beta}\,(\mathrm{Mix})}{\Theta[\Pi[\Delta]^{n_1}][\Delta]^{n_2} \Rightarrow \beta}\,(\mathrm{Ctr})$$

(1.6) (R_r) is (Wek). Let $\Sigma[\alpha]^n = \Theta[\Sigma_1[\alpha]^{n_1}, \Sigma_2[\alpha]^{n_2}][\alpha]^{n_3}$ and the derivation end with

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\Theta[\Sigma_i[\alpha]^{n_i}][\alpha]^{n_3} \Rightarrow \beta}{\Theta[\Sigma_1[\alpha]^{n_1}, \Sigma_2[\alpha]^{n_2}][\alpha]^{n_3} \Rightarrow \beta}\,(\mathrm{Wek})}{\Theta[\Sigma_1[\Delta]^{n_1}, \Sigma_2[\Delta]^{n_2}][\alpha]^{n_3} \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \alpha \quad \Theta[\Sigma_i[\alpha]^{n_i}][\alpha]^{n_3} \Rightarrow \beta}{\Theta[\Sigma_i[\Delta]^{n_i}][\alpha]^{n_3} \Rightarrow \beta}\,(\mathrm{Mix})}{\Theta[\Sigma_1[\Delta]^{n_1}, \Sigma_2[\Delta]^{n_2}][\alpha]^{n_3} \Rightarrow \beta}\,(\mathrm{Wek})$$

(1.7) (R_r) is $(\mathrm{Dual}_{\circ\bullet})$. Let the derivation end with

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\circ\Sigma_1[\alpha]^{n_1}, \Sigma_2[\alpha]^{n_2} \Rightarrow \perp}{\Sigma_1[\alpha]^{n_1}, \bullet\Sigma_2[\alpha]^{n_2} \Rightarrow \perp}\,(\mathrm{Dual}_{\circ\bullet})}{\Sigma_1[\Delta]^{n_1}, \bullet\Sigma_2[\Delta]^{n_2} \Rightarrow \perp}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \alpha \qquad \circ\Sigma_1[\alpha]^{n_1}, \Sigma_2[\alpha]^{n_2} \Rightarrow \perp}{\circ\Sigma_1[\Delta]^{n_1}, \Sigma_2[\Delta]^{n_2} \Rightarrow \perp}\,(\mathrm{Mix})}{\Sigma_1[\Delta]^{n_1}, \bullet\Sigma_2[\Delta]^{n_2} \Rightarrow \perp}\,(\mathrm{Dual}_{\circ\bullet})$$

(1.8) (R_r) is $(\mathrm{Dual}_{\circ\bullet})$. The proof is quite analogous to (1.7).
(2) Both (R_l) and (R_r) are logical rules.
(2.1) α is not principal in (R_l). We have the following cases.
(2.1.1) (R_l) is $(\bot)$. Let the derivation end with

$$\dfrac{\dfrac{\Theta \Rightarrow \bot}{\Delta[\Theta] \Rightarrow \alpha}\,(\bot) \qquad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\Theta]]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

The above subtree can be transformed into

$$\dfrac{\Theta \Rightarrow \bot}{\Sigma[\Delta[\Theta]]^n \Rightarrow \beta}\,(\bot)$$

(2.1.2) (R_l) is a left logical rule. We apply (Mix) to the premise(s) of (R_l) and $\Sigma[\alpha]^n \Rightarrow \beta$, and then apply (R_l). Take $(R_l) = (\Diamond\mathrm{L})$ as an example. Other cases are addressed similarly. Let the derivation end with

$$\dfrac{\dfrac{\Delta[\circ\gamma] \Rightarrow \alpha}{\Delta[\Diamond\gamma] \Rightarrow \alpha}\,(\Diamond\mathrm{L}) \qquad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\Diamond\gamma]]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta[\circ\gamma] \Rightarrow \alpha \qquad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\circ\gamma]]^n \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\Delta[\Diamond\gamma]]^n \Rightarrow \beta}\,(\Diamond\mathrm{L})^n$$

where $(\Diamond\mathrm{L})^n$ means n times application of $(\Diamond\mathrm{L})$.
(2.2) α is not principal in (R_r). We have the following cases:
(2.2.1) (R_r) is $(\bot)$. Let the derivation end with

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\Theta[\alpha]^{n_1} \Rightarrow \bot}{\Sigma[\Theta[\alpha]^{n_1}][\alpha]^{n_2} \Rightarrow \beta}\,(\bot)}{\Sigma[\Theta[\Delta]^{n_1}][\Delta]^{n_2} \Rightarrow \beta}\,(\mathrm{Mix})$$

If $n_1 = 0$, then we obtain$\Sigma[\Delta[\Theta]]^n \Rightarrow \beta$ from $\Theta \Rightarrow \bot$ by $(\bot)$. Let $n_1 > 0$. By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \alpha \qquad \Theta[\alpha]^{n_1} \Rightarrow \bot}{\Theta[\Delta]^{n_1} \Rightarrow \bot}\,(\mathrm{Mix})}{\Sigma[\Theta[\Delta]^{n_1}][\Delta]^{n_2} \Rightarrow \beta}\,(\bot)$$

(2.2.2) (R_r) is a right logical rule. Apply (Mix) to $\Delta \Rightarrow \alpha$ and the premise(s) of (R_r) and then apply (R_r). Take $(R_r) = (\wedge\mathrm{R})$ as an example. Let the derivation end with

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\Sigma_1[\alpha]^{n_1} \Rightarrow \beta_1 \qquad \Sigma_2[\alpha]^{n_2} \Rightarrow \beta_2}{\Sigma_1[\alpha]^{n_1}, \Sigma_2[\alpha]^{n_2} \Rightarrow \beta_1 \wedge \beta_2}\,(\wedge\mathrm{R})}{\Sigma_1[\Delta]^{n_1}, \Sigma_2[\Delta]^{n_2} \Rightarrow \beta_1 \wedge \beta_2}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \alpha \qquad \Sigma_1[\alpha]^{n_1} \Rightarrow \beta_1}{\Sigma_1[\Delta]^{n_1} \Rightarrow \beta_1}\,(\mathrm{Mix}) \qquad \dfrac{\Delta \Rightarrow \alpha \qquad \Sigma_2[\alpha]^{n_2} \Rightarrow \beta_1}{\Sigma_2[\Delta]^{n_2} \Rightarrow \beta_1}\,(\mathrm{Mix})}{\Sigma_1[\Delta]^{n_1}, \Sigma_2[\Delta]^{n_2} \Rightarrow \beta_1 \wedge \beta_2}\,(\wedge\mathrm{R})$$

(2.2.3) (R_r) is a left logical rule. We apply (Mix) to $\Delta \Rightarrow \alpha$ and the premise(s) of (R_r), and then apply (R_r). Take $(R_r) = (\neg\mathrm{L})$ as an example. Let the derivation end with

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\Sigma[\alpha]^n \Rightarrow \xi}{\neg\xi, \Sigma[\alpha]^n \Rightarrow \bot}\,(\neg\mathrm{L})}{\neg\xi, \Sigma[\Delta]^n \Rightarrow \bot}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \alpha \qquad \Sigma[\alpha]^n \Rightarrow \xi}{\Sigma[\Delta]^n \Rightarrow \xi}\,(\mathrm{Mix})}{\neg\xi, \Sigma[\Delta]^n \Rightarrow \bot}\,(\neg\mathrm{L})$$

(2.3) α is principal in both (R_l) and (R_r). The proof proceeds by induction on the complexity of α. We have the following cases:

(2.3.1) $\alpha = \alpha_1 \wedge \alpha_2$. Let the derivation end with

$$\dfrac{\dfrac{\Delta_1 \Rightarrow \alpha_1 \qquad \Delta_2 \Rightarrow \alpha_2}{\Delta_1, \Delta_2 \Rightarrow \alpha_1 \wedge \alpha_2}\,(\wedge\mathrm{R}) \qquad \dfrac{\Sigma[\alpha_1, \alpha_2][\alpha_1 \wedge \alpha_2]^{n-1} \Rightarrow \beta}{\Sigma[\alpha_1 \wedge \alpha_2][\alpha_1 \wedge \alpha_2]^{n-1} \Rightarrow \beta}\,(\wedge\mathrm{L})}{\Sigma[\Delta_1, \Delta_2]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\Delta_1 \Rightarrow \alpha_1 \qquad \dfrac{\Delta_2 \Rightarrow \alpha_2 \qquad \dfrac{\Delta_1, \Delta_2 \Rightarrow \alpha_1 \wedge \alpha_2 \qquad \Sigma[\alpha_1, \alpha_2][\alpha_1 \wedge \alpha_2]^{n-1} \Rightarrow \beta}{\Sigma[\alpha_1, \alpha_2][\Delta_1, \Delta_2]^{n-1} \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\alpha_1, \Delta_2][\Delta_1, \Delta_2]^{n-1} \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\Delta_1, \Delta_2]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

(2.3.2) $\alpha = \neg\xi$. Let the derivation end with

$$\dfrac{\dfrac{\xi, \Delta \Rightarrow \bot}{\Delta \Rightarrow \neg\xi}\,(\neg\mathrm{R}) \qquad \dfrac{\Sigma[\neg\xi]^{n-1} \Rightarrow \xi}{\neg\xi, \Sigma[\neg\xi]^{n-1} \Rightarrow \bot}\,(\neg\mathrm{L})}{\Sigma[\Delta]^{n-1}, \Delta \Rightarrow \bot}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \neg\xi \qquad \Sigma[\neg\xi]^{n-1} \Rightarrow \xi}{\Sigma[\Delta]^{n-1} \Rightarrow \xi}\,(\mathrm{Mix}) \qquad \xi, \Delta \Rightarrow \bot}{\Sigma[\Delta]^{n-1}, \Delta \Rightarrow \bot}\,(\mathrm{Mix})$$

(2.3.3) $\alpha = \Diamond\xi$. Let the derivation end with

$$\dfrac{\dfrac{\Delta \Rightarrow \xi}{\circ\Delta \Rightarrow \Diamond\xi}\,(\Diamond\mathrm{R}) \qquad \dfrac{\Sigma[\circ\xi][\Diamond\xi]^{n-1} \Rightarrow \beta}{\Sigma[\Diamond\xi][\Diamond\xi]^{n-1} \Rightarrow \beta}\,(\Diamond\mathrm{L})}{\Sigma[\circ\Delta]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\Delta \Rightarrow \xi \qquad \dfrac{\circ\Delta \Rightarrow \Diamond\xi \qquad \Sigma[\circ\xi][\Diamond\xi]^{n-1} \Rightarrow \beta}{\Sigma[\circ\xi][\circ\Delta]^{n-1} \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\circ\Delta]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

(2.3.4) $\alpha = \blacklozenge\xi$. The proof is quite analogous to (2.3.3).

(2.3.5) $\alpha = \blacksquare\xi$. The derivation ends with

$$\dfrac{\dfrac{\circ\Delta \Rightarrow \xi}{\Delta \Rightarrow \blacksquare\xi}\,(\blacksquare\mathrm{R}) \quad \dfrac{\Sigma[\xi][\blacksquare\xi]^{n-1} \Rightarrow \beta}{\Sigma[\circ\blacksquare\xi][\blacksquare\xi]^{n-1} \Rightarrow \beta}\,(\blacksquare\mathrm{L})}{\Sigma[\circ\Delta][\Delta]^{n-1} \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\circ\Delta \Rightarrow \xi \quad \dfrac{\Delta \Rightarrow \blacksquare\xi \quad \Sigma[\xi][\blacksquare\xi]^{n-1} \Rightarrow \beta}{\Sigma[\xi][\Delta]^{n-1} \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\circ\Delta][\Delta]^{n-1} \Rightarrow \beta}\,(\mathrm{Mix})$$

(2.3.6) $\alpha = \Box\xi$. The proof is quite analogous to (2.3.5). □

Now, let $\mathsf{GK}_t^{\ddagger}$ be $\mathsf{GK}_t^{\dagger}$, eliminating (Cut). Next, we shall prove that $\mathsf{GK}_t \vdash s$ implies $\mathsf{GK}_t^{*} \vdash s$.

Remark 2. *We clearly have* $\mathsf{GK}_t^{*} \vdash \neg\neg\alpha \Leftrightarrow \alpha$*. Furthermore,* $\mathsf{GK}_t^{*} \vdash \alpha \Rightarrow \beta$ *iff* $\mathsf{GK}_t^{*} \vdash \neg\beta \Rightarrow \neg\alpha$*. The regular proof is omitted.*

Lemma 12. $\mathsf{GK}_t^{*} \vdash \alpha[\neg\neg\beta] \Leftrightarrow \alpha[\beta]$*, where* $\alpha[-]$ *is a formula context.*

Proof. Suppose $d(\alpha[-]) = 0$. Then, $\vdash \neg\neg\beta \Leftrightarrow \beta$. Suppose $d(\alpha[-]) > 0$. Assume $\alpha[-] = \neg\chi[-]$. By induction hypothesis, $\vdash \chi[\neg\neg\beta] \Leftrightarrow \chi[\beta]$. By Remark 2, $\vdash \neg\chi[\neg\neg\beta] \Leftrightarrow \neg\chi[\beta]$. Assume $\alpha[-] = \alpha_1[-] \wedge \alpha_2$ or $\alpha_1 \wedge \alpha_2[-]$. We prove only the former case. According to induction hypothesis, $\vdash \alpha_1[\neg\neg\beta] \Leftrightarrow \alpha_1[\beta]$. By ($\wedge$R) and ($\wedge$L), $\vdash \alpha_1[\neg\neg\beta] \wedge \alpha_2 \Leftrightarrow \alpha_1[\beta] \wedge \alpha_2$. Assume $\alpha[-] = \Diamond\chi[-]$ or $\blacklozenge\chi[-]$.We prove only the former case. According to induction hypothesis, $\vdash \chi[\neg\neg\beta] \Leftrightarrow \chi[\beta]$. By ($\Diamond$R) and ($\Diamond$L), $\vdash \Diamond\chi[\neg\neg\beta] \Leftrightarrow \Diamond\chi[\beta]$. Assume $\alpha[-] = \Box\chi[-]$ or $\blacksquare\chi[-]$. We prove only the former case. According to induction hypothesis, $\vdash \chi[\neg\neg\beta] \Leftrightarrow \chi[\beta]$. By ($\Box$R) and ($\Box$L), $\vdash \Box\chi[\neg\neg\beta] \Leftrightarrow \Box\chi[\beta]$. □

Lemma 13. *The following hold in* GK_t^{*}*:*

(1) *if* $\mathsf{GK}_t^{*} \vdash \Gamma[\alpha[\neg\neg\beta]] \Rightarrow \chi$*, then* $\mathsf{GK}_t^{*} \vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$*.*
(2) *if* $\mathsf{GK}_t^{*} \vdash \Gamma \Rightarrow \alpha[\neg\neg\beta]$*, then* $\mathsf{GK}_t^{*} \vdash \Gamma \Rightarrow \alpha[\beta]$*.*

Proof. Assume $\vdash_k \Gamma[\alpha\{\neg\neg\beta\}] \Rightarrow \chi$ and $\vdash_k \Gamma \Rightarrow \alpha[\neg\neg\beta]$. We prove (1) and (2) simultaneously by induction on $k \geq 0$. Suppose $k = 0$. By Lemma 12, (1) and (2) hold. Suppose $k > 0$. For (1), suppose $\Gamma[\alpha[\neg\neg\beta]] \Rightarrow \chi$ is derived by a rule (R). We have the following cases:

(1.1) (R) is a structural rule. We have the following cases:

(1.1.1) $(R) = $ (Ctr). Let $\vdash_{k-1} \Sigma[\Delta, \Delta] \Rightarrow \chi$ and $\vdash_k \Sigma[\Delta] \Rightarrow \chi$, where $\Sigma[\Delta] = \Gamma[\alpha[\neg\neg\beta]]$. Suppose $\alpha[\neg\neg\beta]$ does not occur in Δ. By (Ctr), $\vdash \Gamma[\alpha[\neg\neg\beta]] \Rightarrow \chi$. Suppose $\Delta = \Delta[\alpha[\neg\neg\beta]]$. By induction hypothesis, $\vdash \Sigma[\Delta[\alpha[\beta]], \Delta[\alpha[\beta]]] \Rightarrow \chi$. By (Ctr), $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$.

(1.1.2) $(R) = $ (Wek). Let $\vdash_{k-1} \Sigma[\Delta_i] \Rightarrow \chi$ and $\vdash_k \Sigma[\Delta_1, \Delta_2] \Rightarrow \chi$, where $\Sigma[\Delta_1, \Delta_2] = \Gamma[\alpha[\neg\neg\beta]]$. Suppose $\alpha[\neg\neg\beta]$ does not occur in Δ_i. By (Ctr), $\vdash \Gamma[\alpha[\neg\neg\beta]] \Rightarrow \chi$. Suppose $\Delta_i = \Delta_i[\alpha[\neg\neg\beta]]$. According to induction hypothesis, $\vdash \Sigma[\Delta_i[\alpha[\beta]]] \Rightarrow \chi$. By (Wek), $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$.

(1.1.3) $(R) = (\mathrm{Dual}_{\circ\bullet})$ or $(\mathrm{Dual}_{\bullet\circ})$.We prove only the former. Let $\vdash_{k-1} \circ\Delta_1, \Delta_2 \Rightarrow \chi$ and $\vdash_k \Delta_1, \bullet\Delta_2 \Rightarrow \chi$, where $\Delta_1, \bullet\Delta_2 = \Gamma[\alpha[\neg\neg\beta]]$. By induction hypothesis and $(\mathrm{Dual}_{\circ\bullet})$, $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$.

(1.2) (R) is a logical rule. If (R) is one of the rules $(\neg\mathrm{R}_1)$, $(\neg\mathrm{R}_2)$, $(\neg\neg\mathrm{R})$, $(\Diamond\mathrm{R})$, $(\blacklozenge\mathrm{R})$, $(\Box\mathrm{R})$ and $(\blacksquare\mathrm{R})$, we obtain immediately $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$ by induction hypothesis and the rule (R). We have the following remaining cases:

(1.2.1) $(R) = (\bot)$. Let $\vdash_{k-1} \Delta \Rightarrow \bot$ and $\vdash_k \Sigma[\Delta] \Rightarrow \chi$, where $\Sigma[\Delta] = \Gamma[\alpha[\neg\neg\beta]]$. Suppose $\alpha[\neg\neg\beta]$ does not occur in Δ. By $(\bot)$, $\vdash \Gamma[\alpha[\neg\neg\beta]] \Rightarrow \chi$. Suppose $\Delta = \Delta[\alpha[\neg\neg\beta]]$. By induction hypothesis, $\vdash \Delta[\alpha[\beta]] \Rightarrow \bot$. By $(\bot)$, $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$.

(1.2.2) $(R) = (\wedge\mathrm{L})$. Let $\vdash_{k-1} \Sigma[\xi_1, \xi_2] \Rightarrow \beta$ and $\vdash_{k-1} \Sigma[\xi_1 \wedge \xi_2] \Rightarrow \beta$, where $\Sigma[\xi_1 \wedge \xi_2] = \Gamma[\alpha[\neg\neg\beta]]$. Suppose $\alpha[\neg\neg\beta] = \xi_1 \wedge \xi_2$, and then $\xi_1 = \xi_1[\neg\neg\beta]$ or $\xi_2 = \xi_2[\neg\neg\beta]$. In each case, by induction hypothesis and $(\wedge\mathrm{L})$, we obtain $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$. If $\alpha[\neg\neg\beta] \neq \xi_1 \wedge \xi_2$, then we obtain immediately $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$ by induction hypothesis and $(\wedge\mathrm{L})$.

(1.2.3) $(R) = (\neg\mathrm{L}_1)$. Let $\vdash_{k-1} \Sigma \Rightarrow \neg\xi$ and $\vdash_k \xi, \Sigma \Rightarrow \bot$, where $\Gamma[\alpha[\neg\neg\beta]] = \xi, \Sigma$ and $\chi = \bot$. Suppose $\xi = \alpha[\neg\neg\beta]$. By $\vdash_{k-1} \Sigma \Rightarrow \neg\alpha[\neg\neg\beta]$ and induction hypothesis, $\vdash \Sigma \Rightarrow \neg\alpha[\beta]$. By $(\neg\mathrm{L}_1)$, $\vdash \alpha[\beta], \Sigma \Rightarrow \bot$. Suppose $\Sigma = \Sigma[\alpha[\neg\neg\beta]]$. By $\vdash_{k-1} \Sigma[\alpha[\neg\neg\beta]] \Rightarrow \neg\xi$ and induction hypothesis, $\vdash \Sigma[\alpha[\beta]] \Rightarrow \neg\xi$. By $(\neg\mathrm{L}_1)$, $\vdash \xi, \Sigma[\alpha[\beta]] \Rightarrow \bot$.

(1.2.4) $(R) = (\neg\mathrm{L}_2)$. Let $\vdash_{k-1} \Sigma \Rightarrow \xi$ and $\vdash_k \neg\xi, \Sigma \Rightarrow \bot$, where $\Gamma[\alpha[\neg\neg\beta]] = \neg\xi, \Sigma$ and $\xi \neq \neg\zeta$ for any formula ζ. Suppose $\alpha[\neg\neg\beta] = \neg\xi$. Suppose $\alpha[-] = -$ or $\alpha[-] = \neg[-]$. Then, $\neg\xi = \neg\neg\beta$ or $\neg\xi = \neg\neg\neg\beta$. Then, $\xi = \neg\beta$ or $\xi = \neg\neg\beta$, which is impossible. Hence, $\xi = \xi[\neg\neg\beta]$. By $\vdash_{k-1} \Sigma \Rightarrow \xi[\neg\neg\beta]$ and induction hypothesis, $\vdash \Sigma \Rightarrow \xi[\beta]$. By $(\neg\mathrm{L}_2)$, $\vdash \neg\xi[\beta], \Sigma \Rightarrow \bot$. Suppose $\Sigma = \Sigma[\alpha[\neg\neg\beta]]$. By $\vdash_{k-1} \Sigma[\alpha[\neg\neg\beta]] \Rightarrow \xi$ and induction hypothesis, $\vdash \Sigma[\alpha[\beta]] \Rightarrow \xi$. By $(\neg\mathrm{L}_2)$, $\vdash \neg\xi, \Sigma[\alpha[\beta]] \Rightarrow \bot$.

(1.2.5) $(R) = (\neg\neg\mathrm{L})$. Let $\vdash_{k-1} \Sigma[\xi] \Rightarrow \chi$ and $\vdash_k \Sigma[\neg\neg\xi] \Rightarrow \chi$, where $\Gamma[\alpha[\neg\neg\beta]] = \Sigma[\neg\neg\xi]$. Let $\neg\neg\xi = \alpha[\neg\neg\beta]$. Assume $\alpha[-] = -$ or $\alpha[-] = \neg[-]$. Then, $\neg\neg\xi = \neg\neg\beta$ or $\neg\neg\xi = \neg\neg\neg\beta$. Then, $\xi = \beta$ or $\xi = \neg\beta$. Then, $\vdash \Sigma[\beta] \Rightarrow \chi$ or $\vdash \Sigma[\neg\beta] \Rightarrow \chi$. Assume $\xi = \xi[\neg\neg\beta]$. By $\vdash_{k-1} \Sigma[\xi[\neg\neg\beta]] \Rightarrow \chi$ and induction hypothesis, $\vdash \Sigma[\xi[\beta]] \Rightarrow \chi$. By $(\neg\neg\mathrm{L})$, $\vdash \Sigma[\neg\neg\xi[\beta]] \Rightarrow \chi$. Let $\neg\neg\xi \neq \alpha[\neg\neg\beta]$. By induction hypothesis and $(\neg\neg\mathrm{L})$, $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$.

(1.2.6) $(R) = (\Diamond\mathrm{L})$ or $(\blacklozenge\mathrm{L})$. We prove only the former. Let $\vdash_{k-1} \Sigma[\circ\xi] \Rightarrow \chi$ and $\vdash_k \Sigma[\Diamond\xi] \Rightarrow \chi$, where $\Gamma[\alpha[\neg\neg\beta]] = \Sigma[\Diamond\xi]$. Suppose $\alpha[\neg\neg\beta] = \Diamond\xi$. Then, $\xi = \xi[\neg\neg\beta]$. According to induction hypothesis, $\vdash \Sigma[\circ\xi[\beta]] \Rightarrow \chi$. By $(\Diamond\mathrm{L})$, $\vdash \Sigma[\Diamond\xi[\beta]] \Rightarrow \chi$. Suppose $\alpha[\neg\neg\beta] \neq \Diamond\xi$. According to induction hypothesis and $(\Diamond\mathrm{L})$, $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$.

(1.2.7) $(R) = (\Box\mathrm{L})$ or $(\blacksquare\mathrm{L})$. We prove only the former. Let $\vdash_{k-1} \Sigma[\xi] \Rightarrow \chi$ and $\vdash_k \Sigma[\bullet\Box\xi] \Rightarrow \chi$, where $\Gamma[\alpha[\neg\neg\beta]] = \Sigma[\bullet\Box\xi]$. Suppose $\alpha[\neg\neg\beta] = \Box\xi$. Then, $\xi = \xi[\neg\neg\beta]$. According to induction hypothesis, $\vdash \Sigma[\xi[\beta]] \Rightarrow \chi$. By $(\Box\mathrm{L})$, $\vdash \Sigma[\bullet\Box\xi[\beta]] \Rightarrow \chi$. Suppose $\alpha[\neg\neg\beta] \neq \Box\xi$. According to induction hypothesis and $(\Box\mathrm{L})$, $\vdash \Gamma[\alpha[\beta]] \Rightarrow \chi$.

For (2), suppose $\Gamma \Rightarrow \alpha[\neg\neg\beta]$ is derived by a rule (R). We have the following cases:

(2.1) (R) is (Ctr) or (Wek). In each case, we obtain $\vdash \Gamma \Rightarrow \alpha[\beta]$ by induction hypothesis and (R).

(2.2) (R) is a logical rule. Note that $\alpha[\neg\neg\beta] \neq \bot$. Then, (R) cannot be $(\neg\mathrm{L}_1)$ or $(\neg\mathrm{L}_2)$. If (R) is one of the rules $(\wedge\mathrm{L})$, $(\neg\neg\mathrm{L})$, $(\Diamond\mathrm{L})$, $(\blacklozenge\mathrm{L})$, $(\Box\mathrm{L})$ and $(\blacksquare\mathrm{L})$, by induction hypothesis and (R), we obtain $\vdash \Gamma \Rightarrow \alpha[\beta]$. We have the following remaning cases:

(2.2.1) $(R) = (\bot)$. Let $\vdash_{k-1} \Delta \Rightarrow \bot$ and $\vdash_k \Gamma[\Delta] \Rightarrow \alpha[\neg\neg\beta]$. From $\vdash \Delta \Rightarrow \bot$ by $(\bot)$, we have $\vdash \Gamma[\Delta] \Rightarrow \alpha[\beta]$.

(2.2.2) $(R) = (\wedge\mathrm{R})$. Let $\vdash_{k-1} \Gamma_1 \Rightarrow \chi_1$; $\vdash_{k-1} \Gamma_2 \Rightarrow \chi_2$ and $\vdash_k \Gamma_1, \Gamma_2 \Rightarrow \chi_1 \wedge \chi_2$, where $\Gamma = (\Gamma_1, \Gamma_2)$ and $\alpha[\neg\neg\beta] = \chi_1 \wedge \chi_2$. Then, $\chi_1 = \chi_1[\neg\neg\beta]$ or $\chi_2 = \chi_2[\neg\neg\beta]$. Then, $\vdash \Gamma_1, \Gamma_2 \Rightarrow \alpha[\beta]$.

(2.2.3) $(R) = (\neg\mathrm{R}_1)$. Let $\vdash_{k-1} \neg\alpha[\neg\neg\beta], \Gamma \Rightarrow \bot$ and $\vdash_k \Gamma \Rightarrow \alpha[\neg\neg\beta]$. By induction hypothesis, $\vdash \neg\alpha[\beta], \Gamma \Rightarrow \bot$. By $(\neg\mathrm{R}_1)$, $\vdash \Gamma \Rightarrow \alpha[\beta]$.

(2.2.4) $(R) = (\neg\mathrm{R}_2)$. Let $\vdash_{k-1} \xi, \Gamma \Rightarrow \bot$ and $\vdash_k \Gamma \Rightarrow \neg\xi$, where $\alpha[\neg\neg\beta] = \neg\xi$ and $\xi \neq \neg\delta$ for any formula δ. Clearly, $\alpha[-] \neq -$ and $\alpha[-] \neq \neg[-]$. Then, $\xi = \xi[\neg\neg\beta]$. By induction hypothesis, $\vdash \xi[\beta], \Gamma \Rightarrow \bot$. By $(\neg\mathrm{R}_2)$, $\vdash \Gamma \Rightarrow \neg\xi[\beta]$.

(2.2.5) (R) is one of the rules $(\Diamond\mathrm{R})$, $(\Box\mathrm{R})$, $(\blacklozenge\mathrm{R})$ and $(\blacksquare\mathrm{R})$. We prove only the case $(R) = (\Diamond\mathrm{R})$. Let $\vdash_{k-1} \Sigma \Rightarrow \xi$ and $\vdash_k \circ\Sigma \Rightarrow \Diamond\xi$, where $\Gamma = \circ\Sigma$ and $\Diamond\xi = \alpha[\neg\neg\beta]$. Then, $\xi = \xi[\neg\neg\beta]$. According to induction hypothesis and $(\Diamond\mathrm{R})$, $\vdash \circ\Sigma \Rightarrow \Diamond\xi[\beta]$. □

Lemma 14. *The following hold in* GK_t^**:*

(1) *if* $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \alpha$, *then* $\mathsf{GK}_t^* \vdash \neg\alpha, \Gamma \Rightarrow \bot$.
(2) *if* $\mathsf{GK}_t^* \vdash \alpha, \Gamma \Rightarrow \bot$, *then* $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \neg\alpha$.

Proof. (1) Assume $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \alpha$. Suppose $\alpha = \neg\chi$. By $(\neg L_1)$ and $(\neg\neg L)$, $\mathsf{GK}_t^* \vdash \neg\neg\chi, \Gamma \Rightarrow \bot$. Suppose $\alpha \neq \neg\chi$ for any formula χ. By $(\neg L_2)$, $\mathsf{GK}_t^* \vdash \neg\alpha, \Gamma \Rightarrow \bot$.

(2) Assume $\mathsf{GK}_t^* \vdash \alpha, \Gamma \Rightarrow \bot$. Suppose $\alpha = \neg\chi$. By $(\neg R_1)$ and $(\neg\neg R)$, $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \neg\neg\chi$. Suppose $\alpha \neq \neg\chi$ for any formula χ. By $(\neg R_2)$, $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \neg\alpha$. □

Lemma 15. *If* $\mathsf{GK}_t^{\ddagger} \vdash \Gamma \Rightarrow \alpha$, *then* $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \alpha$.

Proof. By Lemma 14, $(\neg L)$ and $(\neg R)$ are admissible in GK_t^*. □

Lemma 16. *If* $\mathsf{GK}_t^* \vdash kol(\Gamma) \Rightarrow kol(\alpha)$, *then* $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \alpha$.

Proof. Let $\mathsf{GK}_t^* \vdash kol(\Gamma) \Rightarrow kol(\alpha)$. By Lemma 13, the double negation $\neg\neg$ in $kol(\Gamma) \Rightarrow kol(\alpha)$ are eliminable. Hence, $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \alpha$. □

Theorem 7. *If* $\mathsf{GK}_t \vdash \Gamma \Rightarrow \alpha$, *then* $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \alpha$.

Proof. Let $\mathsf{GK}_t \vdash \Gamma \Rightarrow \alpha$. By Theorem 5, $\mathsf{GK}_t^{\dagger} \vdash kol(\Gamma) \Rightarrow kol(\alpha)$. By Theorem 6, $\mathsf{GK}_t^{\ddagger} \vdash ko(\Gamma) \Rightarrow ko(\alpha)$. By Lemma 15, $\mathsf{GK}_t^* \vdash kol(\Gamma) \Rightarrow kol(\alpha)$. By Lemma 16, $\mathsf{GK}_t^* \vdash \Gamma \Rightarrow \alpha$. □

By Theorem 7, every derivation in GK_t can be transformed into a cut-free derivation in GK_t^*. This result provides the cut elimination of GK_t.

5. Strictly Positive Formulas

In this section, as in [17], we introduce strictly positive formulas and show that tense logics axiomatized by a set of such formulas have cut-free sequent calculi. The fundamental idea is that each strictly positive formula can be transformed into a structural rule, which is added to GK_t without affecting the cut elimination.

Definition 8 (cf. [17])**.** *The strictly positive formulas ('sp-formulas' for short) are defined recursivly as follows:*

$$\mathsf{SP} \ni \psi ::= p \mid \bot \mid \top \mid (\psi_1 \wedge \psi_2) \mid \Diamond\psi \mid \blacklozenge\psi$$

where $p \in \mathsf{Var}$ *and* SP *are the set of sp-formulas. A* sp-axiom *is a formula* $\varphi \to \psi$, *where* $\varphi, \psi \in \mathsf{SP}$.

An expression $\Pi[-_1]\ldots[-_n]$ is called a generalized context if it is built from n positions $-_1, \ldots, -_n$ by only structural operators. For $\Theta_1, \ldots, \Theta_n \in \mathcal{FS}$, $\Pi[\Theta_1]\ldots[\Theta_n]$ is the formula structure arised from $\Pi[-_1]\ldots[-_n]$ by replacing each $-_i$ with Θ_i for $1 \leq i \leq n$.

Definition 9. *For any sp-formula* $\psi \in \mathsf{SP}$ *and* $var(\psi) = \{q_1, \ldots, q_n\}$, *let* $\Pi_\psi[-_1]\ldots[-_n]$ *be the generalized context arised from* ψ *by (i) replacing each* q_i *with* $-_i$ *for* $1 \leq i \leq n$, *and (ii) replacing each occurrence of* $\wedge, \Diamond$ *or* $\blacklozenge$ *by the comma,* $\circ$ *or* $\bullet$, *respectively. If* $var(\psi) = \varnothing$, *we obtain* Π_ψ *only by (ii).*

Let $\varphi \to \chi$ *be a sp-axiom, where* $var(\varphi) = \{q_1, \ldots, q_n\}$ *and* $var(\chi) = \{p_1, \ldots, p_m\}$. *We define the structural rule* $R_{\varphi,\chi}$ *as follows:*

$$\frac{\Pi[\Xi_\chi[\Theta_1]\ldots[\Theta_m]] \Rightarrow \gamma}{\Pi[\Xi_\varphi[\Psi_1]\ldots[\Psi_n]] \Rightarrow \gamma}(R_{\varphi,\beta})$$

where $\Pi[-]$ *is a context,* $\Theta_1, \ldots, \Theta_m, \Psi_1, \ldots, \Psi_n \in \mathcal{FS}$ *and* $\gamma \in \mathcal{F}$.

Many tense logics are axiomatizable by sp-axioms. Table 1 provides some examples of formulas, the corresponding sp-axioms and structural rules.

Table 1. Some sp-axioms and structural rules.

Formulas	sp-Axioms	Structural Rules
$\Box q \to \Diamond q$	$\top \to \Diamond\top$	$\dfrac{\Theta[\circ\top] \Rightarrow \alpha}{\Theta[\top] \Rightarrow \alpha}$
$\Box q \to q$	$q \to \Diamond q$	$\dfrac{\Theta[\circ\Pi] \Rightarrow \alpha}{\Theta[\Pi] \Rightarrow \alpha}$
$\Diamond q \to \Box q$	$\blacklozenge\Diamond q \to q$	$\dfrac{\Theta[\Pi] \Rightarrow \alpha}{\Theta[\bullet\circ\Pi] \Rightarrow \alpha}$
$q \to \Box\Diamond q$	$\blacklozenge q \to \Diamond q$	$\dfrac{\Theta[\circ\Pi] \Rightarrow \alpha}{\Theta[\bullet\Pi] \Rightarrow \alpha}$
$\Box q \to \Box\Box q$	$\Diamond\Diamond q \to \Diamond q$	$\dfrac{\Theta[\circ\Pi] \Rightarrow \alpha}{\Theta[\circ\circ\Pi] \Rightarrow \alpha}$
$\Diamond q \to \Box\Diamond q$	$\blacklozenge\Diamond q \to \Diamond q$	$\dfrac{\Theta[\circ\Pi] \Rightarrow \alpha}{\Theta[\bullet\circ\Pi] \Rightarrow \alpha}$

Let S be a set of sp-axioms and $\mathfrak{R}(S)$ the set of all rules $(R_{\varphi,\psi})$, where $\varphi \to \psi \in S$. The tense logic S is $\mathsf{K}_t \oplus S$, and the sequent calculus GS is the extension of GK_t with rules in $\mathfrak{R}(S)$. Every sp-axiom in S is a simple Sahlqvist formula (cf. e.g., [2]), and so S is characterized by the frame class $\mathsf{Fr}(\mathsf{S})$.

Theorem 8. *For any set of sp-axioms S, $\mathsf{GS} \vdash \Gamma \Rightarrow \alpha$ iff $\mathsf{Fr}(\mathsf{S}) \models \Gamma \Rightarrow \alpha$.*

Proof. The left-to-right direction is obvious since all rules in $\mathfrak{R}(S)$ preserve validity in $\mathsf{Fr}(\mathsf{S})$. The other direction is shown as the proof of Theorem 4. Note that, if $\varphi \to \psi \in S$, then $\mathsf{GS} \vdash\, \Rightarrow \varphi \to \psi$ by the structural rule $(R_{\varphi,\psi})$. □

Let GS^*, $\mathsf{GS}^\dagger$ and $\mathsf{GS}^\ddagger$ be extensions of GK_t^*, $\mathsf{GK}_t^\dagger$ and $\mathsf{GK}_t^\ddagger$ with rules $\mathfrak{R}(S)$, respectively. Repeating the cut elimination proof provided in Section 4, it follows that these sequent calculi are equivalent.

Lemma 17. *If $\mathsf{GS}^* \vdash \Gamma \Rightarrow \alpha$, then $\mathsf{GS} \vdash \Gamma \Rightarrow \alpha$.*

Proof. The proof is analogous to Lemma 7. □

Theorem 9. $\mathsf{GS} \vdash \Gamma \Rightarrow \alpha$ *iff* $\mathsf{GS}^\dagger \vdash kol(\Gamma) \Rightarrow kol(\alpha)$.

Proof. The proof is analogous to Theorem 5. Structural rules in $\mathfrak{R}(S)$ pass the proof. □

Theorem 10. *If $\mathsf{GS}^\dagger \vdash \Gamma \Rightarrow \chi$, then $\mathsf{GS}^\dagger \vdash \Gamma \Rightarrow \chi$ without any applications of* (Mix).

Proof. The proof proceeds as the proof of Theorem 10 by adding cases of rules in $\mathfrak{R}(S)$. Suppose the derivation ends with

$$\dfrac{\dfrac{\Delta[\Xi_{\chi_2}[\Pi_1]\ldots[\Pi_m]] \Rightarrow \alpha}{\Delta[\Xi_{\chi_1}[\Theta_1]\ldots[\Theta_h]] \Rightarrow \alpha}\,(R_{\chi_1,\chi_2}) \quad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Delta[\Xi_{\chi_1}[\Theta_1]\ldots[\Theta_h]]^n \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta[\Xi_{\chi_2}[\Pi_1]\ldots[\Pi_m]] \Rightarrow \alpha \quad \Sigma[\alpha]^n \Rightarrow \beta}{\Sigma[\Xi_{\chi_2}[\Pi_1]\ldots[\Pi_m]]^n \Rightarrow \beta}\,(\mathrm{Mix})}{\Sigma[\Delta[\Xi_{\chi_1}[\Theta_1]\ldots[\Theta_h]]^n \Rightarrow \beta}\,(R_{\chi_1,\chi_2})^n$$

where $(R_{\chi_1,\chi_2})^n$ means n times application of (R_{χ_1,χ_2}).

Suppose the derivation ends with

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\Theta[\Xi_{\chi_2}[\Pi_1[\alpha]^{n_1}\ldots\Pi_m[\alpha]^{n_m}]][\alpha]^{n_{m+1}} \Rightarrow \beta}{\Theta[\Xi_{\chi_1}[\Theta_1[\alpha]^{k_1}\ldots\Theta_j[\alpha]^{k_j}]][\alpha]^{k_{j+1}} \Rightarrow \beta}\,(R_{\chi_1,\chi_2})}{\Theta[\Xi_{\chi_1}[\Theta_1[\Delta]^{k_1}\ldots\Theta_j[\Delta]^{k_j}]][\Delta]^{k_{j+1}} \Rightarrow \beta}\,(\mathrm{Mix})$$

By induction hypothesis, the above subtree can be transformed into

$$\dfrac{\dfrac{\Delta \Rightarrow \alpha \qquad \Theta[\Xi_{\chi_2}[\Pi_1[\alpha]^{n_1}\ldots\Pi_m[\alpha]^{n_m}]][\alpha]^{n_{m+1}} \Rightarrow \beta}{\Theta[\Xi_{\chi_2}[\Pi_1[\Delta]^{n_1}\ldots\Pi_m[\Delta]^{n_m}]][\Delta]^{n_{m+1}} \Rightarrow \beta}\,(\mathrm{Mix})}{\Theta[\Xi_{\chi_1}[\Theta_1[\Delta]^{k_1}\ldots\Theta_j[\Delta]^{k_j}]][\Delta]^{k_{j+1}} \Rightarrow \beta}\,(R_{\chi_1,\chi_2})$$

This completes the proof. □

Lemma 18. *The following hold in* GS^**:*

(1) *if* $\mathsf{GS}^* \vdash \Gamma[\varphi[\neg\neg\psi]] \Rightarrow \chi$, *then* $\mathsf{GS}^* \vdash \Gamma[\varphi[\psi]] \Rightarrow \chi$.
(2) *if* $\mathsf{GS}^* \vdash \Gamma \Rightarrow \varphi[\neg\neg\psi]$, *then* $\mathsf{GS}^* \vdash \Gamma \Rightarrow \varphi[\psi]$.

Proof. The proof is analogous to Lemma 13. Note that rules in $\mathfrak{R}(S)$ pass the proof. □

Lemma 19. *The following hold:*

(1) *if* $\mathsf{GS}^\ddagger \vdash \Gamma \Rightarrow \alpha$, *then* $\mathsf{GS}^* \vdash \Gamma \Rightarrow \alpha$.
(2) *if* $\mathsf{GS}^* \vdash kol(\Gamma) \Rightarrow kol(\alpha)$, *then* $\mathsf{GS}^* \vdash \Gamma \Rightarrow \alpha$.

Proof. The proofs are quite similar to Lemmas 15 and 16. □

Theorem 11. *If* $\mathsf{GS} \vdash \Gamma \Rightarrow \alpha$, *then* $\mathsf{GS}^* \vdash \Gamma \Rightarrow \alpha$.

Proof. Let $\mathsf{GS} \vdash \Gamma \Rightarrow \alpha$. By Theorem 17, $\mathsf{GS}^\dagger \vdash kol(\Gamma) \Rightarrow kol(\alpha)$. By Theorem 10, $\mathsf{GS}^\ddagger \vdash kol(\Gamma) \Rightarrow kol(\alpha)$. By Lemma 19, $\mathsf{GS}^* \vdash kol(\Gamma) \Rightarrow kol(\alpha)$, and so $\mathsf{GS}^* \vdash \Gamma \Rightarrow \alpha$. □

It follows that GS for each sp-axioms set S admits cut elimination. This provides a modular result in the study of proof theory for tense logic.

6. Concluding Remarks

The present work generalizes the Gentzen proof theory for intuitionistic tense logics provided in [17] to classical tense logics. We present a cut-free single-succedent Gentzen sequent calculus for the minimal tense logic K_t. Then, we show that all sp-axioms can be transformed into structural rules, which allow obtaining cut-free sequent calculi uniformly. There are still some questions for further exploration. First, we may consider formulas that are equivalent to sp-axioms, and describe tense logics axiomatized by these sp-axioms. Second, we could consider the proof-theoretic approach to the decidability of these tense logics axiomatized by sp-axioms. Third, the approach provided in the present work can be explored further for the proof of some results on the finite model property in tense logics. Finally, the sequent calculi we have developed can be appropriately extended to multimodal logics.

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