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An Exponentiated Skew-Elliptic Nonlinear Extension to the Log–Linear Birnbaum–Saunders Model with Diagnostic and Residual Analysis

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Abstract: In this paper, we propose a nonlinear regression model with exponentiated skew-elliptical errors distributed, which can be fitted to datasets with high levels of asymmetry and kurtosis. Maximum likelihood estimation procedures in finite samples are discussed and the information matrix is deduced. We carried out a diagnosis of the influence for the nonlinear model. To analyze the sensitivity of the maximum likelihood estimators of the model's parameters to small perturbations in distribution assumptions and parameter estimation, we studied the perturbation schemes, the case weight, and the explanatory and response variables of perturbations; we also carried out a residual analysis of the deviance components. Simulation studies were performed to assess some properties of the estimators, showing the good performance of the proposed estimation procedure in finite samples. Finally, an application to a real dataset is presented.

Keywords: Birnbaum–Saunders distribution; maximum likelihood; skewed power-normal model; skewed-elliptical sinh alpha-power distribution; influence diagnostic; nonlinear regression model

MSC: 62E15; 62E20

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1. Introduction

The linear model theory has been widely studied in the literature, for both symmetric and asymmetric models. In either case, one of the main problems occurs when the distribution of the residuals presents high levels of asymmetry and kurtosis. Various works have addressed this situation, notably Cancho et al. [1] and Martínez-Flórez et al. [2]. A more complex situation occurs when the systematic trend of the response variable is nonlinear, and the error component presents high levels of asymmetry and kurtosis. The first approach used by many investigators is to consider a model with error-multiplying effects and apply a transformation to the response variable, such that the transformation applied to the systematic part results in a linear relation between the transformed response and the set of explanatory variables. In other cases, this linear relation is more easily reached by applying the transformation to the explanatory variable; these models are known, in practice, as intrinsically linear models. Although this practice is easily applied, it presents serious problems in the interpretation of the model's parameters, specifically with respect to the original (untransformed) variables. In other cases, transformation is impossible, and the only solution then is to use a nonlinear relation for the systematic component under certain assumptions regarding error distribution. There are few works in the literature on nonlinear cases, e.g., Cancho et al. [1], Martínez-Flórez et al. [2], and Lemonte and Cordeiro [3]. The situation becomes even more complex when working with survival data, and when a nonlinear relation has to be fitted; for such cases, the most useful works in the literature are those of Lemonte and Cordeiro [3], Lemonte [4], and Martínez-Flórez et al. [5]. Even in these works, the problem of fitting the data when they present high levels of asymmetry and kurtosis persists. There is, therefore, a need for new proposals for fitting nonlinear relations under error distribution assumptions with high levels of asymmetry and kurtosis. When the data distribution has tails heavier than the normal distribution, the family of elliptical distributions offers an alternative solution for fitting the dataset. This family corresponds specifically to symmetric-type distributions with a lower or higher kurtosis than the normal distribution; see, for example, Cambanis et al. [6], Fang et al. [7], Gupta and Varga [8], and Díaz-García and Leiva-Sánchez [9]. The probability density function (pdf) of this family of distributions is given by:

$$f(x) = cg\left(x^2\right), \tag{1}$$

for some non-negative functions $g(z)$, $z > 0$, such that $\int_0^\infty z^{-\frac{1}{2}} g(z) dz = 1/c$, with a normalizing constant c. The function $g(\cdot)$ is known as the generating function. We denote this as $X \sim EC(g)$. As a special case, this family contains the normal distribution when $g(z) = \exp^{-\frac{1}{2}z}$, which leads to $c = 1/\sqrt{2\pi}$.

Other cases of $X \sim EC(g)$ distributions are represented, for example, by the Pearson type VII, Student t_ν, Kotz, Cauchy, and normal distributions. The properties of this family can be explored in studies by Kelker [10], Cambanis et al. [6], Fang et al. [7], and Gupta and Varga [8] among others.

Although this model is a viable alternative for data with kurtosis that is either less or greater than that of the normal distribution, it is not suitable for asymmetric distributions.

Generalization of the elliptic family to the asymmetric case is represented by the pdf (see Lachos et al. [11])

$$h_Y(y;\lambda) = 2f(y)F(\lambda y), \quad y, \lambda \in \mathbb{R}, \tag{2}$$

where f is given in (1), F is its corresponding cumulative distribution function (cdf), and λ is an asymmetry parameter. This model is denoted by $Y \sim SE(g, \lambda)$. The cdf of this model is given by

$$H_Y(y,\lambda) = 2\int_{-\infty}^{y} f(t)F(\lambda t)dt. \tag{3}$$

Here, it can be seen that for $\lambda = 0$, the symmetrical elliptic family follows. A particular case of model (2) is the skew-normal (SN) distribution (see Azzalini [12]) when $f = \phi$ and $F = \Phi$. Therefore, we have the pdf

$$\phi_{SN}(y) = 2\phi(y)\Phi(\lambda y), \quad y \in \mathbb{R},$$

where λ is an asymmetry parameter. We denote this by $SN(\lambda)$. The cdf of the SN model is given by

$$\Phi_{SN}(y) = \Phi(y) - 2T(y;\lambda), \quad y \in \mathbb{R},$$

where $T(\cdot,\cdot)$ is Owen's function (See [13]). For $\lambda = 0$, the standard normal model is obtained. This distribution is widely used in different areas of data modeling with degrees of asymmetry in the range of $(-0.995, 0.995)$ and kurtosis in the range of $(3, 3.869)$.

This model has been extended to many areas of statistics; the following regression model, in particular, has been studied

$$Y_i = x_i^T \boldsymbol{\beta} + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

where Y_i is the i-th experimental unit, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)^T$ is an unknown parameter vector, $x_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^T$ is a p-dimensional vector with the values of the explanatory variables, and ϵ_i are independent and identically distributed random variables with $\epsilon_i \sim \text{SN}(0, \eta, \lambda)$, $i = 1, 2, \ldots, n$.

A more general case of regression models was studied by Cancho et al. [1], who introduced the nonlinear regression model with asymmetric errors; that is, the model

$$Y_i = \psi(\beta, x_i) + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

where Y_i is the response variable, $\psi(\cdot)$ is an injective continuous function that is twice differentiable with respect to the parameter vector β, x_i is a vector of values of an explanatory variable, and ϵ_i are independent and identically distributed random variables $SN(0, \eta, \lambda)$.

Another asymmetric type of distribution was studied by Durrans [14]; this is called the exponentiated distribution with pdf

$$\varphi_Z(z; \alpha) = \alpha f(z)\{F(z)\}^{\alpha-1}, \quad z \in \mathbb{R}, \ \alpha \in \mathbb{R}^+,$$

where F is an absolutely continuous cdf with pdf $f = dF$ and α is a shape parameter that controls the amount of asymmetry in the distribution. We use the notation $Z \sim EXP(\alpha)$. Case $F = \Phi(\cdot)$ is called the exponentiated normal distribution and is denoted as $EXPn(\alpha)$. This is an alternative asymmetric model with asymmetry in the range of $(-0.611, 0.900)$ and kurtosis in the range of (1.717, 4.355) (see Pewsey et al. [15]), as is the case with the SN distribution (Azzalini [12]).

Another asymmetric type of distribution that has been mainly used for modeling the lifetimes of certain structures under dynamic loads was introduced by Birnbaum and Saunders [16]. It is popularly known as the Birnbaum–Saunders (BS) distribution, and its pdf is given by

$$f_T(t) = \phi(a_t)\frac{t^{-3/2}(t+\tau)}{2\gamma\sqrt{\tau}}, \quad t > 0,$$

with $a_t = \frac{1}{\gamma}\left(\sqrt{\frac{t}{\tau}} - \sqrt{\frac{\tau}{t}}\right)$ where $\gamma > 0$ is the shape parameter and $\tau > 0$ is a scale parameter. We shall use the notation $T \sim BS(\gamma, \tau)$.

This model has been extended to a large number of distribution families. Initially, the extension of this model to symmetric elliptic distributions was proposed by Díaz-García and Leiva-Sánchez [9], while Castillo et al. [17] considered the asymmetric epsilon Birnbaum–Saunders model, and Gómez et al. [18] considered an extension based on the slash-elliptical family of distributions.

The BS model has also been used to study linear regression models, literally known as the log–Birnbaum–Saunders (log–BS) model (see Rieck and Nedelman [19]). In this type of model, it is assumed that $Y_i = log(T_i)$, where $T_i \sim BS(\gamma, \tau)$ for $i = 1, 2, \ldots, n$, and that the linear model errors follow a sinh-normal (SHN) distribution (see [19]), with a vector of parameters γ, 0, and 2. This is,

$$\varphi(\epsilon_i) = \frac{\frac{2}{\gamma}\cosh\left(\frac{\epsilon_i}{2}\right)}{2}\phi\left(\frac{2}{\gamma}\sinh\left(\frac{\epsilon_i}{2}\right)\right),$$

which is denoted by $\epsilon_i \sim \mathrm{SHN}(\gamma, 0, 2)$, $i = 1, 2, \ldots, n$.

More recently, Barros et al. [20] extended this model to error distributions with heavier tails, emphasizing the use of the Student-t distribution. They also conducted estimations and diagnostic studies for the model studied. Extensions for the SHN model using an asymmetric setup were found in the models studied by Leiva et al. [21], where a skewed-sinh-normal model was developed and used in a study on air pollution in the city of Santiago, Chile. Some other asymmetric extensions of the sinh-normal models were reported in Barros et al. [20], Lemonte and Cordeiro [3], and Santana et al. [22], where a study on the influence of observations was reported.

In nonlinear type models, few papers have been published for cases involving the BS distribution. Among these are the works on nonlinear log–BS models studied by Lemonte and Cordeiro [3] and the works on diagnosis and influence in the nonlinear log–BS skew-normal (ssinh) model, see Lemonte [4].

In this work, we propose a nonlinear model for datasets where the errors follow the skew-elliptical alpha-power distribution; the ranges of asymmetry and kurtosis are greater than those of SN (Azzalini [12]) and alpha-power models (Durrans [14]), and it contains the SN, exponentiated, and normal distributions as special cases. In other words, it is much more flexible than these models. We also include the inference of the model, a study of the estimation process, the variance–covariance matrix of the estimator vector, a diagnostic analysis of influence, and the residual analysis.

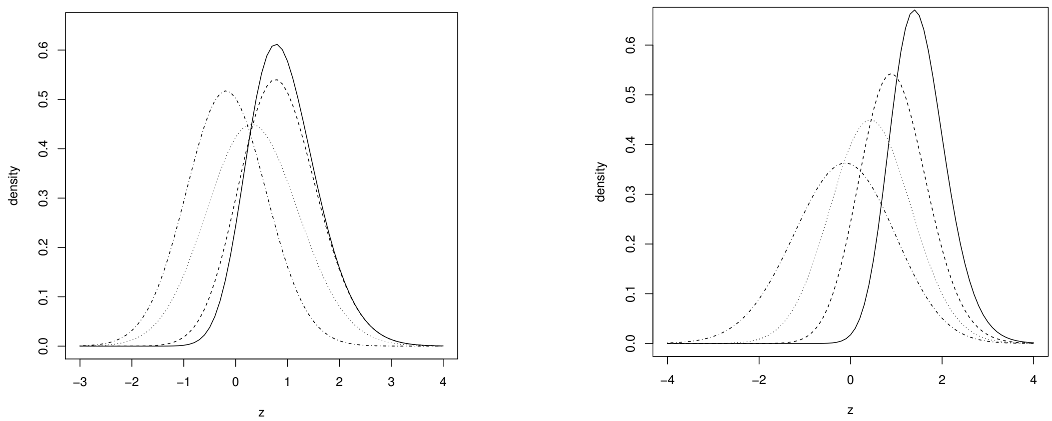
The paper is organized as follows. In Section 2, we present the nonlinear log–BS exponentiated skew-elliptic regression model, study its properties, estimate its parameters, and deduce the observed and expected information matrices. In Section 3, we present diagnostic and residual analyses for the proposed model. In Section 4, we perform a simulation study. In Section 5, a real dataset is analyzed using the proposed distribution to illustrate its applicability.

2. New Model

In this section, we present the exponentiated skew-elliptical (EXPSE) distribution, some of its properties, and the nonlinear transformation involved in the sinh-normal exponential skew-elliptical distribution. Subsequently, we present the nonlinear skew-elliptical log–Birnbaum–Saunders alpha-power regression model, study its properties and the parameter estimation process, and deduce the observed and expected information matrices.

The $EXP(\alpha)$ and $SN(\lambda)$ models were combined to obtain a new model studied by Martínez-Flórez et al. [23], which they call the exponentiated SN model. We will denote this by $EXPsn(\lambda, \alpha)$. They show that this model is more flexible (in terms of skewness and kurtosis) than the EXPn and SN models.

Special cases of the EXPsn model occur with $\alpha = 1$, so the SN model $\phi_{SN}(x)$, follows. On the other hand, with $\lambda = 0$, the model with pdf $\varphi_{\Phi}(x)$, which is the Durrans generalized normal model, follows. The ordinary standard normal model is also a special case that follows by taking $\alpha = 1$ and $\lambda = 0$, which is $\varphi(z; 0, 1) = \phi(x)$. Notice from Figure 1a,b below that α and λ affect both the asymmetry and kurtosis of the distribution; hence, the proposed model seems more flexible than the models proposed by Azzalini [12] and Durrans [14].



(**a**) (**b**)

Figure 1. Plots of the EXPsn distribution. (**a**) $\alpha = 1.5$ and $\lambda = -0.75$ (dotted dashed line), 0 (dotted line), 1 (dashed line), 1.75 (solid line), (**b**) $\lambda = 0.70$, $\alpha = 0.50$ (dotted-dashed line), 1.0 (dotted line), 2.0 (dashed line), and 5.0 (solid line).

The $EXPsn(\lambda, \alpha)$ model can be extended to a much more general family, which includes the $EXP(\alpha)$ and SE models given in (2) by the parameter λ and generator g.

Definition 1. *The pdf of the exponentiated skewed-elliptical distribution is given by*

$$\varphi(z;\lambda,\alpha) = \alpha h_Z(z;\lambda)\{H_Z(z;\lambda)\}^{\alpha-1}, \ z \in \mathbb{R}. \quad (4)$$

where $h(\cdot)$ is given in (2) and H is its cdf. We will denote it by $Z \sim EXPSE(g,\lambda,\alpha)$.

Special cases of the EXPSE model occur with $\alpha = 1$, so that the $SE(g,\lambda)$ model follows. On the other hand, with $\lambda = 0$, the EXP model of Durrans [14] follows.

The moments of the random variable Z do not have closed forms, but under a variable change, the r-th moment of the random variable Z can be written as follows:

$$E(Z^r) = \alpha \int_0^1 [H_Z^{-1}(z;\lambda)]^r z^{\alpha-1} dz,$$

where H_Z^{-1} is the inverse of the function $H_Z(z,\lambda)$. When $h_Z(z,\lambda)$ is the SN model of Azzalini [12], we then have the EXPsn model of parameters λ and α. The transformation $Y = \xi + \eta Z$ leads to the location–scale model of the EXPsn model, denoted by $EXPsn(\xi,\eta,\alpha)$. In other cases, the observation types lead to nonlinear transformations of the variable under study; while the transformation $Y = \text{arcsinh}(\gamma Z/2)\eta + \xi$ leads to the skewed sinh power-normal distribution, with the pdf given by

$$\varphi_{SEXPsn}(y) = \alpha \frac{\frac{2}{\gamma}\cosh\left(\frac{y-\xi}{\eta}\right)}{\eta} \phi_{SN}\left(\frac{2}{\gamma}\sinh\left(\frac{y-\xi}{\eta}\right)\right)\left\{\Phi_{SN}\left(\frac{2}{\gamma}\sinh\left(\frac{y-\xi}{\eta}\right)\right)\right\}^{\alpha-1},$$

which we denote by $Y \sim \text{SEXPsn}(\gamma,\xi,\eta;g,\lambda,\alpha)$. For more details on this model, see Martínez-Flórez et al. [24].

Notice that when $\lambda = 0$, $\alpha = 1$, and $\eta = 2$, we have the SHN model, so this special case can be assessed by testing the hypothesis $H_0: (\lambda,\alpha) = (0,1)$. On the other hand, with $\alpha = 1$, the skewed sinh-normal model follows; see Lemonte and Cordeiro [3].

The cdf of Y is given by

$$\mathbb{F}_{SEXPsn}(y;\lambda) = \left\{\Phi_{SN}\left[\frac{2}{\gamma}\sinh\left(\frac{y-\xi}{\eta}\right)\right]\right\}^{\alpha}.$$

From the random variable, $T = \exp(Y)$ follows the exponentiated BS asymmetric distribution, with parameters γ, $\beta = \exp(\xi)$, λ, and α.

Nonlinear Log–BS Model

The nonlinear log–BS model is defined by

$$Y_i = \psi_i(x_i,\boldsymbol{\beta}) + \epsilon_i,$$

where $\epsilon_i \sim \text{SEXPsn}(\gamma,0,2;\phi,\lambda,\alpha)$, $y_i = \log(t_i)$ is the logarithm of the observed lifetime, $\psi_i(x_i,\boldsymbol{\beta})$ is a specified nonlinear function, which depends on a p-dimensional vector of covariates (say x_i) and the regression coefficients $\boldsymbol{\beta} = (\beta_0,\beta_1,\ldots,\beta_p)'$, satisfying that it is a continuous and twice differentiable function in relation to $\boldsymbol{\beta}$.
Some properties are as follows:

1. The pdf and cdf:

$$\varphi(y_i) = \alpha \frac{\frac{2}{\gamma}\cosh\left(\frac{y_i-\psi(x_i,\beta)}{2}\right)}{2} \phi_{SN}\left(\frac{2}{\gamma}\sinh\left(\frac{y-\psi(x_i,\beta)}{2}\right)\right)\left\{\Phi_{SN}\left(\frac{2}{\gamma}\sinh\left(\frac{y_i-\psi(x_i,\beta)}{2}\right)\right)\right\}^{\alpha-1}, \quad (5)$$

which we denote by $Y_i \sim \text{SEXPsn}(\gamma,\psi(x_i,\beta),2;\phi,\lambda,\alpha)$, and

$$\mathbb{F}_{SEXPsn}(y_i;\lambda) = \left\{\Phi_{SN}\left(\frac{2}{\gamma}\sinh\left(\frac{y_i-\psi(x_i,\beta)}{2}\right)\right)\right\}^{\alpha}. \quad (6)$$

2. Percentiles:
If $U \sim U(0,1)$, following the uniform distribution, then the random variable

$$Y_i = \psi(x_i, \beta) + 2\Big[\text{arcsinh}\Big\{\frac{\gamma}{2}\Phi_{SN}^{-1}(U^{1/\alpha})\Big\}\Big]$$

is distributed according to the SEXPsn distribution, with parameters $\gamma, \psi(x_i, \beta), 2, \lambda$ and α, where Φ_{SN}^{-1} is the inverse of the skew-normal distribution.

3. Flexibility:
   (a) $\lambda = 0$ follows the power-normal (exponentiated-normal) nonlinear regression model.
   (b) $\alpha = 1$ follows the skew-normal nonlinear regression model.
   (c) $\lambda = 0$ and $\alpha = 1$ follow the normal nonlinear regression model.

   Hence, in terms of skewness and kurtosis, this model is more flexible than the power-normal, skew-normal, and normal nonlinear models.
4. Let $Y_i \sim SEXPsn(\gamma, \psi(x_i, \beta), 2; \phi, \lambda, \alpha)$. Then, for constants $\sigma_0 \in \mathbb{R}$ and $\sigma_1 \in \mathbb{R}^+$,

$$V = \sigma_0 + \sigma_1 Y \sim SEXPsn(\gamma, \sigma_0 + \sigma_1\psi(x_i, \beta), 2\sigma_1; \phi, \lambda, \alpha).$$

5. If $Y_i \sim SEXPsn(\gamma, \psi(x_i, \beta), 2; \phi, \lambda, \alpha)$, then

$$V = \frac{2}{\gamma}\sinh\left(\frac{Y_i - \psi(x_i, \beta)}{2}\right) \sim EXPSE(0, 1; \phi, \lambda, \alpha).$$

6. Let $Y_i \sim SEXPsn(\gamma, \psi(x_i, \beta), 2; \phi, \lambda, 1)$. Then,

$$V^2 = \frac{4}{\gamma^2}\sinh^2\left(\frac{Y_i - \psi(x_i, \beta)}{2}\right) \sim \chi_1^2.$$

7. Expectation and variance:

$$\mathbb{E}(Y_i) = \psi(x_i, \beta) + 2c_1(\gamma, \lambda, \alpha)$$

where

$$c_1(\gamma, \lambda, \alpha) = \int_{-\infty}^{\infty} arcsinh\Big(\frac{\gamma z}{2}\Big)\phi_{SN}(z)\{\Phi_{SN}(z)\}^{\alpha-1}dz$$

and

$$Var(Y_i) = 4Var(\gamma, \lambda, \alpha),$$

with $Var(\gamma, \lambda, \alpha)$ representing the variance of the random variable $W = arcsinh\Big(\frac{\gamma Z}{2}\Big)$, where $Z \sim EXPSE(\phi, \lambda, \alpha)$.

Defining $\xi_{i1} = \frac{2}{\gamma}\cosh\Big(\frac{y_i - \psi(x_i, \beta)}{2}\Big)$ and $\xi_{i2} = \frac{2}{\gamma}\sinh\Big(\frac{y_i - \psi(x_i, \beta)}{2}\Big)$ for $i = 1, 2, \ldots, n$, the log-likelihood function for the parameter $(\gamma, \beta, \lambda, \alpha)^\top$ for a random sample $Y_i \sim SEXPsn(\gamma, \psi(x_i, \beta), 2, \lambda, \alpha)$, $i = 1, \ldots, n$, up to an additive constant, is given by

$$\ell(\gamma, \boldsymbol{\beta}, \lambda, \alpha) = n\log(\alpha) + \sum_{i=1}^{n}\log(\xi_{i1}) + \sum_{i=1}^{n}\log(\phi_{SN}(\xi_{i2})) + (\alpha - 1)\sum_{i=1}^{n}\log(\Phi_{SN}(\xi_{i2})).$$

Denoting by $W_{SN} = \frac{\phi_{SN}(\xi_2)}{\Phi_{SN}(\xi_2)}$, $d_j = \frac{\partial\psi(\beta,x)}{\partial\beta_j}$, $W_{SN1} = \frac{\phi_{SN}(\sqrt{1+\lambda^2}\xi_2)}{\phi_{SN}(\xi_2)}$, and $W_{SN2} = \frac{\phi_{SN2}(\sqrt{1+\lambda^2}\xi_2)}{\Phi_{SN}(\xi_2)}$, we have the following score functions

$$U(\beta_j) = \sum_{i=1}^{n}\left\{-\frac{\xi_{i2}}{2\xi_{i1}}d_{ij} + \frac{1}{2}\xi_{i1}\xi_{i2}d_{ij} - \frac{\lambda}{\sqrt{2\pi}}\xi_{i1}d_{ij}W_{SN1} - \frac{\alpha - 1}{2}\xi_{i1}d_{ij}W_{SN}\right\}, \quad \text{for } j = 1, 2, \ldots, p,$$

$$U(\lambda) = \sum_{i=1}^{n}\left\{\sqrt{\frac{2}{\pi}}\xi_{i2}W_{SN1} - \sqrt{\frac{2}{\pi}}\frac{(\alpha - 1)}{(1 + \lambda^2)}W_{SN2}\right\}, \qquad U(\gamma) = \frac{1}{\gamma}\sum_{i=1}^{n}\left\{-1 + \xi_{i1}\xi_{i2} - \sqrt{\frac{2}{\pi}}W_{SN1} - (\alpha - 1)\xi_{i2}W_{SN}\right\},$$

$$U(\alpha) = \sum_{i=1}^{n} \left\{ \frac{1}{\alpha} + \log(\Phi_{SN}(\xi_{i2})) \right\}.$$

Setting these equations equal to zero, we obtain the score equations, whose solutions by iterative numerical methods lead to the maximum likelihood (ML) estimators.

The elements of the observed information matrix are given by $J(\boldsymbol{\theta}) = -H(\boldsymbol{\theta})$, where $H(\boldsymbol{\theta})$ is the Hessian matrix; that is, the second derivative, with respect to the parameters of the log-likelihood function. We will denote the elements of the information matrix by $j_{\gamma\gamma}, \; j_{\beta_j\gamma}, \ldots, j_{\alpha\alpha},$ which can be found in Appendix A.

From these results, we obtain the Fisher information matrix, $I(\boldsymbol{\theta})$, the elements of which are obtained by finding the expected elements of the observed information matrix, i.e.,

$$i_{\theta_j\theta_{j'}} = \mathbb{E}(j_{\theta_j\theta_{j'}}).$$

This information matrix is non-singular; thus, for large samples, we have that

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \sim N_{p+4}(\boldsymbol{\theta}, I(\boldsymbol{\theta})^{-1}).$$

Therefore, the inverse of $I(\boldsymbol{\theta})$ is the covariance matrix of the vector of the ML estimators of the model parameters.

3. Diagnostic Analysis

The verification of possible deviations from the assumptions made for the model, as well as the existence of extreme observations and some interferences that may affect the estimate parameters, can be studied using diagnostic methods similar to those employed by Cook [25] for the normal model. Usually, these methods can be performed by eliminating cases to assess the global influence and incorporating various types of perturbations to assess the local influence. We denote the perturbation vector as $\omega = (\omega_1, \omega_2, \ldots, \omega_n)'$.

Now, we implement the perturbation schemes for the response variable, the explanatory variable, and the weighting of the cases; we also analyze the deviance component residual to study possible departures from the model's assumptions.

3.1. Local Influence

The main object of the local influence method is to evaluate changes in the results of the analysis when small perturbations are incorporated into the model and/or the data. If these perturbations cause disproportionate effects, it may be an indication that the model is ill-fitted or that there may be serious departures from the assumptions made for it. We are now going to apply this technique to the nonlinear regression model. We will use the perturbed log-likelihood, as in Cook [25], to assess the local influence.

The influence of perturbation ω on the ML estimator can be evaluated based on the analysis of the distance by likelihood

$$LD(\omega) = 2\{L(\hat{\boldsymbol{\theta}}) - L(\hat{\boldsymbol{\theta}}_\omega)\}.$$

Cook [25] proposed studying the local behavior of $LD(\omega)$ around ω_0, using the normal curvature C_l in the unperturbed vector in one unit direction, where $\|l\| = 1$, considering the graph of $LD(\omega_0 + al)$ against a with $a \in \mathbb{R}$. This graph is called the projected line. Each projected line can be characterized by the normal curvature $C_l(\boldsymbol{\theta})$ around $a = 0$.

Cook shows that

$$C_l = 2\left| l'\Delta'\ddot{L}^{-1}\Delta l \right|,$$

with $\|l\| = 1$, where $\ddot{L}$ is the Hessian matrix and Δ is a matrix $(p+q) \times n$, which depends on the perturbation scheme used, whose elements are $\Delta_{ij} = \frac{\partial^2 \ell(\boldsymbol{\theta}|\omega)}{\partial\theta_j \partial\omega_i}, \;\; j = 1, 2, \ldots, p + q$ and $i = 1, 2, \ldots, n$, with all quantities evaluated at $\omega = \omega_0$ and $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$.

Let l_{max} be the direction of the maximum curvature, which is the direction that produces the greatest change in $\hat{\boldsymbol{\theta}}$. The most influential element of the data can be identified by the largest component of the vector l_{max}, corresponding to the largest eigenvalue of

$$B = -\Delta' \ddot{L}^{-1} \Delta,$$

(see Galea et al. [26]). If the interest is to evaluate the partial influence of a θ_1 subset of $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2')'$, then the normal curvature in the direction of the vector l is given by

$$C_l(\boldsymbol{\theta}_1) = 2\left| l'\Delta'(\ddot{L}^{-1} - B_1)\Delta l \right|,$$

with

$$B_1 = \begin{pmatrix} \mathbf{0}_{11} & \mathbf{0}_{12} \\ \mathbf{0}_{21} & \ddot{L}_{22}^{-1} \end{pmatrix}$$

where $\ddot{L}_{22}^{-1} = \frac{\partial^2 \ell(\boldsymbol{\theta}|\omega)}{\partial \boldsymbol{\theta}_2' \partial \boldsymbol{\theta}_2}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$. The graph of the eigenvector associated with the largest eigenvalue of the matrix $-\Delta'(\ddot{L}^{-1} - B_1)\Delta$ against the index of observations can reveal which observations are influencing $\hat{\boldsymbol{\theta}}_1$. Similarly, if the interest is on θ_2, then the normal curvature in the direction of vector l is given by

$$C_l(\boldsymbol{\theta}_2) = 2\left| l'\Delta'(\ddot{L}^{-1} - B_2)\Delta l \right|,$$

with

$$B_2 = \begin{pmatrix} \ddot{L}_{11}^{-1} & \mathbf{0}_{12} \\ \mathbf{0}_{21} & \mathbf{0}_{22} \end{pmatrix}$$

where $\ddot{L}_{11}^{-1} = \frac{\partial^2 \ell(\boldsymbol{\theta}|\omega)}{\partial \, \boldsymbol{\theta}_1' \partial \boldsymbol{\theta}_1}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$. The local influence of the observations on θ_2 can be evaluated considering the graph l_{max} for the matrix $-\Delta'(\ddot{L}^{-1} - B_2)\Delta$ against the index of observations.

The curvature in the direction of the i-th observation was suggested by Lesaffre and Verbeke [27]; that is, to calculate the curvature in the direction of l_i, where l_i is an $n \times 1$ vector of zeros with one in the i-th position. For Δ_i', denoting the i-th row of Δ, the total local influence of the i-th case is given by

$$C_i = 2\left| \Delta_i' \ddot{L}^{-1} \Delta_i \right|, \quad i = 1, 2, \ldots, n.$$

In the work by Verbeke and Molenberghs [28], it is proposed to consider cases as influential when $C_i \geq 2\overline{C}$, where $\overline{C} = \frac{1}{n}\sum_{i=1}^{n} C_i$.

Poon and Poon [29] proposed a second alternative for studying influential points; they introduced the conformal normal curvature, defined by

$$B_l = - \left. \frac{l'\Delta'(\ddot{L}^{-1})\Delta l}{\sqrt{tr\left[l'\Delta'(\ddot{L}^{-1})\Delta l\right]^2}} \right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}, \omega=\omega_0},$$

where $tr(A)$ is the trace of matrix A. Thus, the computation of B_l requires almost no more effort than the computation of C_l. Furthermore, the conformal normal curvature enjoys several interesting properties, among which we highlight:

1. The normal curvature, conforming in any direction to ω_0, is invariant under reparametrization.
2. In any direction l, $0 \leq |B_l| \leq 1$; therefore, B_l is a normalized measure, which will allow comparisons of the curvatures.

3.2. Local Influence for the Nonlinear Log–BS Exponentiated Skew-Normal Model

Let us define $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)'$ with $\boldsymbol{\theta}_1 = \beta$ and $\boldsymbol{\theta}_2 = (\gamma, \lambda, \alpha)'$.

3.2.1. Weighting Cases

For the nonlinear log–Birnbaum–Saunders exponentiated SN model, the perturbed log-likelihood function is given by

$$\ell(\boldsymbol{\theta}|\omega) = \sum_{i=1}^{n} \omega_i \ell_i(\boldsymbol{\theta}),$$

with $0 \leq \omega_i \leq 1$, for $i = 1, \ldots, n$, and $\omega_0 = (1, 1, \ldots, 1)'$ is the vector of no perturbations.

The matrix Δ is given by

$$\Delta = \begin{pmatrix} \Delta_\beta \\ \Delta_{\theta_2} \end{pmatrix},$$

where Δ_β is a matrix of size $p \times n$, and Δ_{θ_2} is a matrix of size $3 \times n$, with elements

$$\Delta_\beta = \mathbf{D} diag\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\}$$

where $a_i = -\frac{\xi_{i2}}{2\xi_{i1}} + \frac{1}{2}\xi_{i1}\xi_{i2} - \frac{\lambda}{\sqrt{2\pi}}\xi_{i1} W_{SN1} - \frac{(\alpha-1)}{2}\xi_{i1} W_{SN}$ with $\mathbf{D} = \{d_{ij}\} = \left\{\frac{\partial\psi(x_i,\beta)}{\partial\beta'}\right\}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p$ while

$$\Delta_{\theta_2} = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n),$$

where

$$b_i = \begin{pmatrix} \frac{1}{\gamma}\left\{-1 + \xi_{i1}\xi_{i2} - \sqrt{\frac{2}{\pi}} W_{SN1} - (\alpha-1)\xi_{i2} W_{SN}\right\} \\ \sqrt{\frac{2}{\pi}}\xi_{i2} W_{SN1} - \sqrt{\frac{2}{\pi}}\frac{(\alpha-1)}{(1+\lambda^2)} W_{SN2} \\ \frac{1}{\alpha} + \log(\Phi_{SN}(\xi_{i2})) \end{pmatrix}_{i=1,2,\ldots,n}$$

and $\hat{a}_i$ and $\hat{b}_i$ are the estimates of a_i and b_i for $i = 1, 2, \ldots, n$ which are obtained by replacing β_j, γ, λ and α, by the respective ML estimates $\hat{\beta}_j$, $\hat{\gamma}$, $\hat{\lambda}$ and $\hat{\alpha}$.

3.2.2. Perturbation in the Response Variable

Suppose y_i presents a perturbation of the form $y_{i\omega} = y_i + \omega_i S_y$, where S_y is a scale factor that can be estimated as the standard deviation of Y and $\omega_i \in \mathbb{R}$. Thus, the logarithm of the perturbed likelihood function takes the form

$$\ell(\boldsymbol{\theta}|\omega) = n\log(\alpha) + \sum_{i=1}^{n}\log(\xi_{i1\omega_1}) + \sum_{i=1}^{n}\log(\phi_{SN}(\xi_{i2\omega_1})) + (\alpha-1)\sum_{i=1}^{n}\log(\Phi_{SN}(\xi_{i2\omega_1})),$$

where $\xi_{i1\omega_1} = \frac{2}{\gamma}\cosh\left(\frac{y_{i\omega_1} - \psi(x_i,\beta)}{2}\right)$ and $\xi_{i2\omega_1} = \frac{2}{\gamma}\sinh\left(\frac{y_{i\omega_1} - \psi(x_i,\beta)}{2}\right)$ for $i = 1, 2, \ldots, n$.

The elements of the Δ array are $\Delta_\beta = S_y \mathbf{D} diag\{\hat{m}_i\}$, where

$$m_i = \left[\frac{1}{4\xi_{i2}^2}(\xi_{i2}^2 - \xi_{i1}^2) + \frac{(\xi_{i2}^2 + \xi_{i1}^2)}{2} - \frac{\lambda\xi_{i2}}{\sqrt{2\pi}} W_{SN1}[1 - \xi_{i1}^2\lambda^2] + \frac{\lambda^2\xi_{i1}}{\pi} W_{SN1}^2\right] - \frac{(\alpha-1)}{4}\left[W_{SN}[\xi_{2i} - \xi_{i1}^2(\xi_{i2} + W_{SN})] + \sqrt{\frac{2}{\pi}}\lambda\xi_{i1}^2 W_{SN2}\right]$$

while $\Delta_{\theta_2} = s_y(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n)$, where

$$c_i = \begin{pmatrix} \kappa_{1i} \\ \kappa_{2i} \\ \frac{\xi_{i1}}{2} W_{SN} \end{pmatrix}_{i=1,\ldots,n},$$

κ_{1i} and κ_{2i} are defined in Appendix B.

3.2.3. Perturbation in the Explanatory Variable

Let us now consider the case where an explanatory variable, x_q, presents an additive perturbation of the form $x_{iqw} = x_{iq} + \omega_i S_q$, where S_q is a scale factor that can be estimated by the standard deviation of x_q and $\omega_i \in \mathbb{R}$, $q \in \{1, 2, \ldots, p\}$. Thus, the logarithm of the perturbed likelihood function takes the form

$$\ell(\boldsymbol{\theta}|\omega) = n\log(\alpha) + \sum_{i=1}^{n}\log(\xi_{i1\omega_2}) + \sum_{i=1}^{n}\log(\phi_{SN}(\xi_{i2\omega_2})) + (\alpha-1)\sum_{i=1}^{n}\log(\Phi_{SN}(\xi_{i2\omega_2})),$$

where $\xi_{i1\omega_2} = \frac{2}{\gamma}\cosh\left(\frac{y_i - \psi(x_{i\omega_2},\beta)}{2}\right)$ and $\xi_{i2\omega_2} = \frac{2}{\gamma}\sinh\left(\frac{y_i - \psi(x_{i\omega_2},\beta)}{2}\right)$ for $i = 1, 2, \ldots, n$.

The array elements $\Delta\beta$ are $\Delta_{\beta_{ij}} = S_{ijw}\kappa_{3i} + S_{iw}S_{ij}\kappa_{4i}$ where κ_{3i} and κ_{4i} are defined in Appendix B and

$$S_{ijw} = \left.\frac{\partial^2\psi(x_{iw},\beta)}{\partial\omega_2\partial\beta_j}\right|_{\theta=\hat{\theta},\ \omega=\mathbf{0}}, \quad S_{iw} = \left.\frac{\partial\psi(x_{iw},\beta)}{\partial\omega_2}\right|_{\theta=\hat{\theta},\ \omega=\mathbf{0}}, \quad \text{and} \quad S_{ij} = \left.\frac{\partial\psi(x_{iw},\beta)}{\partial\beta_j}\right|_{\theta=\hat{\theta},\ \omega=\mathbf{0}}.$$

Even so, we have θ_2

$$\Delta_{\theta_2} = (\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_n),$$

where

$$d_i = \begin{pmatrix} \kappa_{5i} \\ \kappa_{6i} \\ -\frac{1}{2}d_{iw}\xi_{i1w}W_{SN} \end{pmatrix}_{i=1,\ldots,n},$$

κ_{5i}, κ_{6i}, and d_{iw} are defined in Appendix B.

3.3. Residual Analysis

To analyze the existence of influential observations and high leverage points that may be affecting parameter estimates, we define residual components and a matrix of generalized leverage for the nonlinear log–BS exponentiated skew-normal.

3.3.1. Residual Components

Considering γ, λ, and α as fixed (known) quantities, according to Galea et al. [26], the residual components are given by

$$r_{DC_i} = \text{sgn}(\hat{e}_i)\sqrt{2}\left[-\log\left(\cosh\left(\frac{\hat{e}_i}{2}\right)\right) + \frac{1}{2}\hat{\xi}_{i2}^2 - \log\{2\Phi(\lambda\hat{\xi}_{i2})\} - (\hat{\alpha}-1)\log\left\{\frac{\Phi_{SN}(0)}{\Phi_{SN}(\hat{\xi}_{i2})}\right\}\right]^{1/2},$$

$i = 1, 2, \ldots, n$, where sgn is the sign function and $\hat{e}_i = y_i - \psi(x_i, \hat{\boldsymbol{\beta}})$.

3.3.2. Standardized Residuals

The standardized residual components are given by

$$r^*_{DC_i} = \frac{r_{DC_i}}{\sqrt{1 - GL_{ii}}},$$

where GL_{ii} is the i-th element of the main diagonal of the matrix of generalized leverage (see Wei et al. [30]), as defined by

$$\mathbf{GL}(\boldsymbol{\theta}) = D_{\boldsymbol{\theta}}(-\ddot{L})^{-1}\ddot{L}_{\boldsymbol{\theta}y},$$

where $D_{\boldsymbol{\theta}} = \frac{\partial\mu}{\partial\boldsymbol{\theta}^T} = (\mathbf{D}, \mathbf{0})$, $\ddot{L}$ is the Hessian matrix and

$$\ddot{L}_{\boldsymbol{\theta}y} = \begin{pmatrix} \ddot{L}_{\boldsymbol{\beta}y} \\ \ddot{L}_{\boldsymbol{\theta}_2 y} \end{pmatrix},$$

with

$$\ddot{L}_{\boldsymbol{\beta} y_i} = \mathbf{Ddiag}\{\nu_i\}$$

where

$$\nu_i = \frac{1}{4}\left[2\xi_{i2}^2 + \frac{4}{\gamma^2} - \sqrt{\frac{2}{\pi}}\lambda\xi_{i2}W_{SN1} - \frac{1}{\gamma^2\xi_{i1}^2} + \sqrt{\frac{2}{\pi}}\lambda\xi_{i1}^2\xi_{i2}W_{SN} + \frac{2}{\pi}\lambda^2\xi_{i1}^2W_{SN1}^2\right]$$
$$-\frac{(\alpha-1)}{4}\left[\xi_{i2}W_{SN}(1-\xi_{i1}^2) + \sqrt{\frac{2}{\pi}}\lambda\xi_{i1}^2W_{SN2} - \xi_{i1}^2W_{SN}^2\right]$$

for $i = 1, 2, \ldots, n$, and

$$\ddot{L}_{\boldsymbol{\theta}_2 y_i} = \begin{pmatrix} \hat{\kappa}_{7i} \\ \hat{\kappa}_{8i} \\ \frac{1}{2}\hat{\xi}_{i1}\hat{W_{SN}} \end{pmatrix}_{i=1,\ldots,n},$$

where κ_{1i} and κ_{2i} are defined in Appendix B, $\boldsymbol{\theta}_2 = (\gamma, \lambda, \alpha)'$ and $\hat{\xi}_{i1}$, $\hat{\xi}_{i2}$ are the ML estimators for ξ_{i1} and ξ_{i2}.

4. Simulation Study

In this section, we present a simulation study in order to assess the properties of the ML estimators. We consider that $y_i \sim SEXPsn(\gamma, \psi_i, 2, \lambda, \alpha)$, for $i = 1, \ldots, n$ and $\psi_i = \beta_0 + \beta_1 x_i^{\beta_2}$, where x_i is drawn from the $U(0, 10)$ distribution. We consider two vectors for $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$: $(1.5, -0.5, 0.25)$ and $(-2, -1.5, 0.5)$; two values for γ: 0.5 and 1.8, and two vectors for (λ, α): $(-1, 0.8)$ and $(1, 1.25)$. In addition, we consider three sample sizes: 50, 100, and 200. Each of the 24 different cases was replicated 1000 times. For each replicate, we compute the ML estimator of $(\beta_0, \beta_1, \beta_2, \gamma, \lambda, \alpha)$ and the corresponding standard error. Table 1 summarizes the bias, the standard deviation of the estimators (SE_1), the mean of the estimated standard errors (SE_2), and the 95% coverage probabilities (CP), based on the asymptotic distribution for the ML estimators. We highlight that the bias is acceptable and is reduced when n is increased. Moreover, the terms SE_1 and SE_2 are closer when the sample size is increased, suggesting that the variances of the estimators are also well estimated. Finally, the CP terms being closer to the nominal values used for their construction suggest a good performance of the asymptotic normal distribution for the ML estimators. In short, the estimators have desirable properties even in finite samples.

Table 1. Simulation study for the ML estimators obtained from the $SEXPsn(\gamma, \psi_i, 2, \lambda, \alpha)$ model.

True Values				$n = 50$				$n = 100$				$n = 200$			
$\beta^\top$	γ	$(\lambda, \alpha)^\top$	param.	bias	SE1	SE2	CP	bias	SE1	SE2	CP	bias	SE1	SE2	CP
$\begin{bmatrix} 1.50 \\ -0.50 \\ 0.25 \end{bmatrix}$	0.5	$\begin{bmatrix} -1.0 \\ 0.8 \end{bmatrix}$	β_0	−0.168	0.372	0.269	0.940	−0.094	0.275	0.205	0.940	−0.038	0.066	0.055	0.949
			β_1	−0.069	0.648	0.467	0.927	−0.024	0.372	0.293	0.933	−0.013	0.120	0.100	0.947
			β_2	0.036	0.116	0.088	0.921	0.009	0.071	0.053	0.933	0.006	0.017	0.015	0.947
			γ	0.491	0.880	0.523	0.872	0.138	0.514	0.402	0.910	0.071	0.144	0.128	0.945
			λ	−1.232	2.974	1.932	0.994	−0.455	1.584	1.270	0.978	−0.188	0.561	0.493	0.972
			α	−0.532	0.847	0.518	0.877	−0.155	0.454	0.348	0.891	−0.073	0.127	0.110	0.950
		$\begin{bmatrix} 1.00 \\ 1.25 \end{bmatrix}$	β_0	0.210	2.414	1.690	0.917	0.057	1.331	1.020	0.939	0.033	0.307	0.270	0.946
			β_1	0.079	0.507	0.383	0.935	0.030	0.339	0.281	0.940	0.013	0.082	0.072	0.946
			β_2	−0.037	0.171	0.120	0.902	−0.011	0.114	0.086	0.931	−0.004	0.037	0.032	0.948
			γ	−0.433	0.715	0.441	0.909	−0.125	0.336	0.268	0.935	−0.068	0.098	0.084	0.947
			λ	−1.437	3.754	2.454	0.972	−0.464	2.218	1.782	0.964	−0.229	0.821	0.685	0.959
			α	0.816	2.157	1.347	0.894	0.269	1.241	1.006	0.909	0.113	0.438	0.395	0.947
	1.8	$\begin{bmatrix} -1.0 \\ 0.8 \end{bmatrix}$	β_0	0.208	3.244	2.259	0.933	0.071	2.144	1.688	0.938	0.045	0.692	0.613	0.949
			β_1	0.050	0.151	0.110	0.939	0.023	0.094	0.072	0.940	0.017	0.029	0.025	0.946
			β_2	−0.028	0.057	0.040	0.918	−0.010	0.036	0.029	0.938	−0.004	0.011	0.009	0.947
			γ	1.649	7.589	4.563	0.873	0.525	4.481	3.374	0.896	0.282	0.982	0.871	0.948
			λ	−1.188	3.541	2.186	0.972	−0.391	2.024	1.503	0.963	−0.218	0.541	0.491	0.957
			α	−0.467	1.388	0.877	0.904	−0.171	0.912	0.700	0.928	−0.078	0.221	0.194	0.946
		$\begin{bmatrix} 1.00 \\ 1.25 \end{bmatrix}$	β_0	0.185	2.794	1.866	0.901	0.095	1.821	1.457	0.907	0.048	0.598	0.523	0.946
			β_1	−0.054	0.146	0.104	0.938	−0.029	0.106	0.080	0.939	−0.015	0.031	0.027	0.946
			β_2	0.024	0.262	0.201	0.906	0.009	0.185	0.152	0.937	0.006	0.070	0.060	0.947
			γ	1.528	5.450	3.336	0.881	0.487	2.638	2.132	0.942	0.288	0.852	0.731	0.945
			λ	−1.273	4.803	2.862	0.975	−0.411	2.878	2.277	0.965	−0.223	0.778	0.679	0.965
			α	0.780	2.230	1.435	0.878	0.284	1.244	1.018	0.894	0.125	0.314	0.271	0.949
$\begin{bmatrix} -2.0 \\ -1.5 \\ 0.5 \end{bmatrix}$	0.5	$\begin{bmatrix} -1.0 \\ 0.8 \end{bmatrix}$	β_0	−0.217	3.783	2.819	0.910	−0.102	2.437	1.980	0.912	−0.037	0.652	0.577	0.945
			β_1	−0.283	0.757	0.544	0.939	−0.060	0.505	0.402	0.939	−0.029	0.155	0.138	0.950
			β_2	−0.074	0.760	0.521	0.915	−0.027	0.502	0.374	0.923	−0.011	0.136	0.121	0.947
			γ	0.437	0.826	0.497	0.894	0.163	0.436	0.348	0.943	0.074	0.149	0.125	0.947
			λ	−1.074	2.333	1.542	0.990	−0.379	1.344	1.044	0.976	−0.243	0.409	0.343	0.961
			α	−0.491	1.911	1.127	0.886	−0.140	1.117	0.856	0.890	−0.092	0.262	0.235	0.946
		$\begin{bmatrix} 1.00 \\ 1.25 \end{bmatrix}$	β_0	0.291	0.752	0.501	0.917	0.104	0.397	0.316	0.920	0.038	0.102	0.091	0.949
			β_1	0.283	1.584	1.204	0.919	0.084	0.977	0.733	0.928	0.050	0.294	0.253	0.949
			β_2	−0.061	0.872	0.633	0.903	−0.026	0.648	0.501	0.908	−0.009	0.219	0.192	0.949
			γ	−0.417	0.402	0.267	0.913	−0.151	0.251	0.207	0.935	−0.067	0.066	0.059	0.946
			λ	−1.036	2.627	1.618	0.993	−0.449	1.517	1.252	0.975	−0.181	0.494	0.447	0.967
			α	0.684	2.730	1.713	0.893	0.286	1.763	1.316	0.909	0.133	0.469	0.398	0.950
	1.8	$\begin{bmatrix} -1.0 \\ 0.8 \end{bmatrix}$	β_0	−0.348	4.158	3.050	0.914	−0.119	2.869	2.321	0.940	−0.060	0.974	0.874	0.949
			β_1	0.224	1.438	1.050	0.916	0.087	0.837	0.664	0.935	0.028	0.308	0.263	0.945
			β_2	0.056	0.247	0.172	0.922	0.028	0.138	0.104	0.932	0.010	0.046	0.041	0.947
			γ	−1.511	4.382	2.688	0.888	−0.514	2.596	2.062	0.920	−0.259	0.716	0.611	0.947
			λ	−1.374	3.618	2.220	0.991	−0.417	1.956	1.480	0.975	−0.211	0.510	0.441	0.972
			α	−0.451	2.141	1.422	0.878	−0.139	1.278	0.987	0.936	−0.078	0.368	0.312	0.949
		$\begin{bmatrix} 1.00 \\ 1.25 \end{bmatrix}$	β_0	−0.361	3.658	2.741	0.915	−0.111	2.546	2.098	0.918	−0.046	0.747	0.654	0.946
			β_1	−0.249	2.080	1.443	0.931	−0.060	1.478	1.120	0.935	−0.047	0.420	0.370	0.947
			β_2	0.081	0.727	0.526	0.921	0.027	0.443	0.337	0.922	0.012	0.147	0.131	0.947
			γ	1.476	5.614	3.382	0.885	0.546	3.224	2.541	0.902	0.277	0.835	0.756	0.950
			λ	−1.452	5.870	3.474	0.990	−0.436	2.751	2.237	0.979	−0.197	0.948	0.837	0.961
			α	0.873	3.933	2.545	0.880	0.289	2.597	1.932	0.887	0.135	0.651	0.576	0.949

5. Application

This section illustrates the SEXPsn model using a real data application. We consider the data of 202 athletes collected at the Australian Institute of Sport (AIS), which is available in the `sn` package [31] of the R software, [32]. The data are intended to explain the hematocrit (`Hc`) in terms of hemoglobin (`Hg`). Figure 2 shows the plot for `Hg` versus $\log(\texttt{Hc})$.

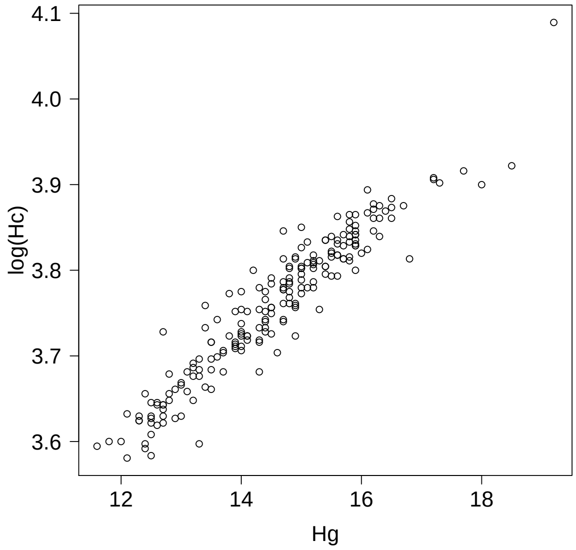


Figure 2. Plot for Hg versus log(Hc) in the AIS dataset.

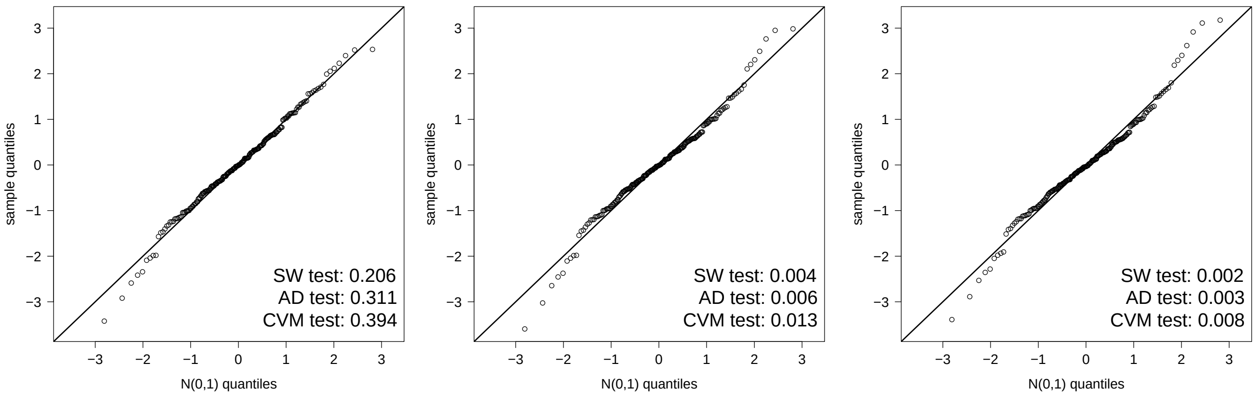
Based on the figure, we consider that $\log(\mathtt{Hc}_i) \sim SEXPsn(\gamma, \psi(\mathtt{Hg}_i, \boldsymbol{\beta}), 2, \lambda, \alpha)$, where we consider two proposals for the function ψ:

- A linear relation: $\psi(\mathtt{Hg}_i, \boldsymbol{\beta}) = \beta_0 + \beta_1 \times \mathtt{Hg}_i$; and
- A nonlinear relation: $\psi(\mathtt{Hg}_i, \boldsymbol{\beta}) = \beta_0 \times \mathtt{Hg}_i^{\beta_1}$.

In both cases, we also consider particular models, $\alpha = 1$ (ssinh model), $\alpha = 1$, and $\lambda = 0$ (nonlinear log–BS exponentiated, ssinh model). The results are presented in Table 2. Note that the nonlinear relation provides better results than the linear relations for the three models. To compare the fit of these models, we use the Akaike criterion (AIC) see [33]. According to this criterion, the model that best fits the data is the one with the lowest AIC value. The lowest AIC is achieved by the SEXPsn model with a nonlinear relation. Note that all parameters are significant in this model. Moreover, Figure 3 shows the rQR of the SEXPsn, ssinh, and sinh models using a nonlinear relation. Based on the three normality tests presented, those residuals are random samples from the standard normal distribution for the SEXPsn model, whereas those for the ssinh and sinh models are not. Therefore, the SEXPsn model is appropriate for this dataset, while the ssinh and sinh models are not.

Table 2. Estimates and standard errors (s.e.) for SEXPsn, ssinh, and sinh models in the AIS dataset.

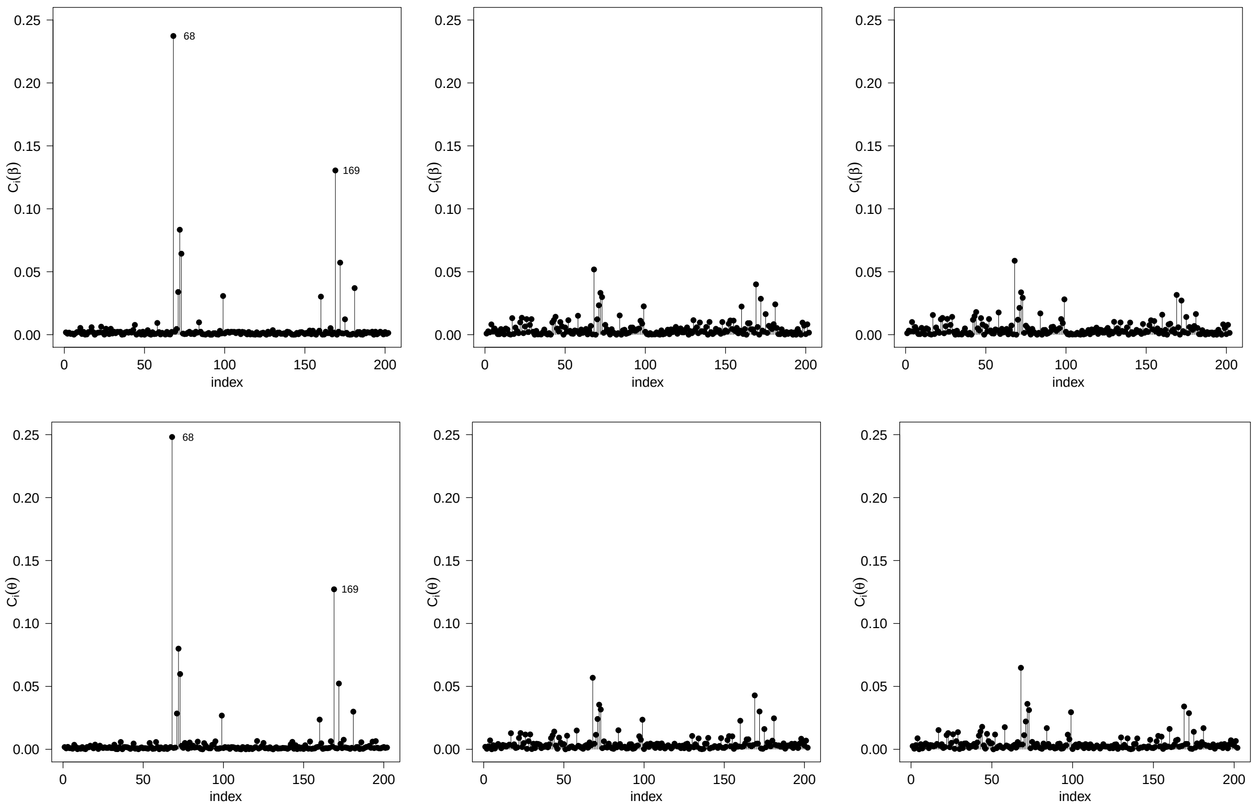
		Model					
		SEXPsn		ssinh		sinh	
Relation	**Parameter**	**Estimate**	**s.e.**	**Estimate**	**s.e.**	**Estimate**	**s.e.**
linear	β_0	2.9173	0.0193	2.9092	0.0212	2.8968	0.0199
	β_1	0.0596	0.0013	0.0593	0.0014	0.0592	0.0014
	γ	0.0318	0.0075	0.0296	0.0049	0.0262	0.0013
	λ	4.3979	1.8245	−0.7088	0.6070	0	-
	α	0.0565	0.0318	1	-	1	-
	log-likelihood	452.16		448.82		448.73	
	AIC	−894.31		−889.64		−891.46	
nonlinear	β_0	2.0602	0.0247	2.0217	0.0279	2.0343	0.0282
	β_1	0.2274	0.0044	0.2302	0.0051	0.2296	0.0052
	γ	0.0350	0.0090	0.0311	0.0037	0.0258	0.0013
	λ	5.3683	2.3188	0.9722	0.4327	0	-
	α	0.0489	0.0270	1	-	1	-
	log-likelihood	456.81		452.33		451.86	
	AIC	−903.61		−896.65		−897.71	



(**a**) SEXPsn (**b**) ssinh (**c**) sinh

Figure 3. QQ plots and p-values for three normality tests for rQR for SEXPsn, ssinh, and sinh models in the AIS dataset, using a nonlinear relation.

For this reason, the local influence analysis will be performed only for the SEXPsn model with the nonlinear regression. Figure 4 shows the local influence of the weight, response, and covariate perturbation for the partition $\boldsymbol{\psi}^\top = (\boldsymbol{\beta}^\top, \boldsymbol{\theta}^\top)$, where $\boldsymbol{\theta}^\top = (\gamma, \lambda, \alpha)$. Note that only observations 68 and 169 are potentially influential. Table 3 shows the relative changes (RC, in %) for the ML estimates in the model when observations 68 and 169 are dropped (separately and jointly). Note that the major changes are given for α. However, in all the cases, the significance for all parameters is maintained.



(**a**) weight (**b**) response (**c**) covariate

Figure 4. Index plots of C_i for $\boldsymbol{\beta}$ (**top**) and $\boldsymbol{\theta}$ (**bottom**) under the weight perturbation (**left**), response perturbation (**center**), and covariate perturbation (**right**) schemes for SEXPsn nonlinear regression in the AIS dataset.

Table 3. RCs (in %) in the ML estimates and the corresponding SEs for the indicated parameters and respective p-values for the SEXPsn nonlinear regression model in the AIS dataset when observations 68 and 169 are dropped.

Dropped Cases		Parameter				
		β_0	β_1	γ	λ	α
68	RC	0.16	0.38	0.56	3.87	21.22
	RCSE	5.91	6.25	7.44	6.15	25.23
	p-value	<0.0001	<0.0001	<0.0001	0.0130	<0.0001
169	RC	0.46	0.68	0.70	2.63	12.78
	RCSE	2.58	2.66	5.05	4.40	14.41
	p-value	<0.0001	<0.0001	<0.0001	0.0135	<0.0001
68, 169	RC	0.33	0.33	1.44	7.23	41.67
	RCSE	7.37	7.89	14.49	12.59	48.96
	p-value	<0.0001	<0.0001	<0.0001	0.0100	<0.0001

6. Conclusions

In the present work, we developed a new nonlinear regression model that is more flexible in terms of asymmetry and kurtosis than asymmetric nonlinear regression models known in the literature. This is an important contribution given the few models in the literature, and it extends the study of nonlinear models to cases of symmetric and asymmetric elliptic distributions. This new proposal also avoids the need for transformation to obtain an intrinsically linear model. We studied the model's properties and estimated its parameters; we deduced its information matrix and the asymptotic distribution of the ML estimator vector; moreover, we presented an analysis of its diagnostics and residuals.

More precisely, we presented the exponentiated skew-elliptical and the exponentiated skew-elliptical sinh-normal families of distributions and studied some of their properties. We proposed the nonlinear skew-elliptical log–BS exponentiated regression model and discussed some particular cases of this model known in the literature. We discussed the ML estimation procedures in finite samples for the parameters of the nonlinear skew-elliptical log–BS exponentiated regression model, deducing the observed and expected information matrices and covariance matrix of the estimated parameter vector. We studied the diagnostic analysis of influence under some perturbation schemes and also addressed the residual analysis of the deviance component. We carried out a simulation study to assess some properties of the estimators, showing the good performance of the proposed estimation procedure in finite samples. In the analysis of real data, the model produced a better fit than other models known in the literature, measured by the AIC criterion.

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Appendix A. The Elements of the Information Matrix

The entries in the information matrix $j_{\gamma\gamma}, j_{\beta_j\gamma}, \ldots, j_{\alpha\alpha}$ can be written as:

$$j_{\gamma\gamma} = -\frac{1}{\gamma^2}\sum_{i=1}^{n}\left\{1+\xi_{i2}^2(\xi_{i2}^2-3)+\sqrt{\frac{2}{\pi}}\lambda\xi_{i2}W_{SN1}[2-2\xi_{i2}^2(1+\lambda^2)]-\left[\xi_{i2}^2-\sqrt{\frac{2}{\pi}}\lambda\xi_{i2}W_{SN1}\right]^2\right\}$$
$$+\frac{\alpha-1}{\gamma^2}\sum_{i=1}^{n}\left\{\xi_{i2}(\xi_{i2}^2-3)W_{SN}-\sqrt{\frac{2}{\pi}}\lambda\xi_{i2}W_{SN2}+2(\xi_{i2}W_{SN})^2\right\},$$

$$j_{\beta_j\gamma} = -\frac{1}{2\gamma}\sum_{i=1}^{n}d_{ij}\left\{(\xi_{i1}\xi_{i2})(\xi_{i2}^2-2)+\left(\sqrt{\frac{2}{\pi}}\lambda\xi_{i1}W_{SN1}\right)(\lambda^2\xi_{i2}^2-1)-\xi_{i1}\xi_{i2}\left(\xi_{i2}-\sqrt{\frac{2}{\pi}}\lambda W_{SN1}\right)^2\right\}$$
$$+\frac{\alpha-1}{\gamma}\sum_{i=1}^{n}d_{ij}\left\{\left(\frac{\xi_{i1}}{2}W_{SN}\right)(\xi_{i2}^2-1)-\frac{\lambda\xi_{12}}{\sqrt{2\pi}}W_{SN2}+\frac{\xi_{i1}\xi_{i2}}{2}W_{SN}^2\right\},$$

$$j_{\lambda\gamma} = -\frac{1}{\gamma}\sum_{i=1}^{n}\left\{\sqrt{\frac{2}{\pi}}\xi_{i2}W_{SN1}[\xi_{i2}^2(1+\lambda^2-W_{SN})-1]+\frac{2\lambda}{\pi}[\xi_{i2}W_{SN1}]^2\right\}+\frac{\alpha-1}{\gamma}\sum_{i=1}^{n}\left\{\sqrt{\frac{2}{\pi}}\xi_{i2}W_{SN2}\left[2\xi_{i2}+\frac{\sqrt{2}}{1+\lambda^2}W_{SN}\right]\right\},$$

$$j_{\beta_j\beta_k} = -\sum_{i=1}^{n}\left\{-\frac{\xi_{i2}}{2\xi_{i1}}g_{i_{jk}}-\frac{[\xi_{i2}^2-\xi_{i1}^2]}{4\xi_{i1}^2}d_{ij}d_{ik}+\frac{\xi_{i1}}{2}\left(g_{i_{jk}}-\frac{d_{ik}}{2}\right)\left(\xi_{i2}-\sqrt{\frac{2}{\pi}}\lambda W_{SN1}\right)\right\}$$
$$-\sum_{i=1}^{n}\frac{\xi_{i1}}{4}d_{ij}d_{ik}\left([\xi_{i2}^2-1]-\sqrt{\frac{2}{\pi}}(\lambda^3+2\lambda)\xi_{i2}W_{SN1}-\left[\xi_{i2}-\sqrt{\frac{2}{\pi}}\lambda W_{SN1}\right]^2\right)$$
$$+(\alpha-1)\sum_{i=1}^{n}\left\{\frac{\xi_{i1}}{2}g_{i_{jk}}W_{SN}+\left[\frac{\xi_{i1}}{2}d_{ij}d_{ik}\right]\left(\frac{\xi_{i2}}{2\xi_{i1}}[\xi_{i1}^2-1]W_{SN}-\frac{\lambda\xi_{i1}}{\sqrt{2\pi}}W_{SN2}+\left[\frac{\xi_{i1}}{\sqrt{2\pi}\xi_{i1}}\right]^2\right)\right\},$$

$$j_{\lambda\beta_j} = -\frac{1}{2}\sum_{i=1}^{n}\xi_{i1}d_{ij}\left\{\sqrt{\frac{2}{\pi}}W_{SN1}[\xi_{i1}\xi_{i2}-1+\lambda^2\xi_{i2}^2]-\left(\xi_{i2}-\sqrt{\frac{2}{\pi}}\lambda W_{SN1}\right)\left(\sqrt{\frac{2}{\pi}}\xi_{i2}W_{SN1}\right)\right\}$$
$$+(\alpha-1)\sum_{i=1}^{n}\frac{\xi_{i1}}{\sqrt{2\pi}}d_{ij}\left(\xi_{i2}W_{SN2}+\frac{1}{1+\lambda^2}W_{SN}W_{SN2}\right),$$

$$j_{\lambda\lambda} = \frac{2}{\pi}\sum_{i=1}^{n}\left\{\lambda\xi_{i2}^3W_{SN1}+\sqrt{\frac{2}{\pi}}W_{SN1}^2\right\}-\sqrt{\frac{2}{\pi}}(\alpha-1)\sum_{i=1}^{n}\left\{\frac{2\lambda}{\sqrt{(1+\lambda^2)^3}}W_{SN3}-\frac{2\lambda}{1+\lambda^2}W_{SN2}+\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}W_{SN2}\right)^2\right\},$$

$$j_{\alpha\gamma} = \frac{1}{\gamma}\sum_{i=1}^{n}\xi_{i2}W_{SN},\qquad j_{\alpha\beta_j} = \frac{1}{2}\sum_{i=1}^{n}\xi_{i1}d_{ij}W_{SN},\qquad j_{\alpha\lambda} = \sum_{i=1}^{n}\sqrt{\frac{2}{\pi}}\frac{W_{SN2}}{1+\lambda^2},\qquad j_{\alpha\alpha} = \frac{n}{\alpha^2},$$

where $g_{i_{jk}}$ is the second partial derivative of $\psi(x_i,\beta)$ with respect to β_j and β_k.

Appendix B. Definitions of κ_{ji} for $j = 1, 2, \cdots, 8$

The expressions of the terms $\kappa_{1i}, \kappa_{2i}, \ldots, \kappa_{8i}$ can be written as:

$$\kappa_{1i} = \frac{1}{\gamma}\left[\xi_{i1}\xi_{i2}-\frac{\xi_{i1}}{\sqrt{2\pi}}W_{SN1}[1-\xi_{i2}^2\lambda^2]+\frac{\lambda\xi_{i1}\xi_{i2}}{\pi}W_{SN1}^2\right]-\frac{(\alpha-1)}{\gamma}\left[\frac{\xi_{i1}W_{SN}}{2}[1-\xi_{i2}^2-\xi_{i2}W_{SN}]+\frac{\lambda\xi_{i1}\xi_{i2}}{\sqrt{2\pi}}W_{SN2}\right],$$

$$\kappa_{2i} = \frac{1}{\sqrt{2\pi}}\xi_{i1}W_{SN1}[1-\xi_{i2}^2\lambda^2]-\frac{1}{\pi}\xi_{i1}W_{SN1}^2+\frac{\alpha-1}{\sqrt{2\pi}}\left[\xi_{i1}\xi_{i2}W_{SN2}+\frac{1}{1+\lambda^2}\xi_{i1}W_{SN}W_{SN2}\right],$$

$$\kappa_{3i} = \frac{1}{2}\left[\xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} - \xi_{i1}\left(\lambda\sqrt{\frac{2}{\pi}}W_{SN1} + (\alpha-1)W_{SN}\right)\right],$$

$$\kappa_{4i} = \frac{1}{4}\left\{\frac{4}{\gamma^2}\left(\frac{1}{\xi_{i1}^2} - 1\right) - 2\xi_{i2}^2 + \sqrt{\frac{2}{\pi}}\lambda W_{SN1}\left[\xi_{i2} + \xi_{i1}^2\left(\lambda^2\xi_{i2} - \sqrt{\frac{2}{\pi}}W_{SN1} + (\alpha-1)\right)\right]\right\}$$
$$+ \frac{1}{4}\{W_{SN}[(\alpha-1)\xi_{i2} - \xi_{i1}^2((\alpha-1)\xi_{i2} + W_{SN})]\},$$

$$\kappa_{5i} = \frac{1}{\gamma}d_{iw}\left[-\xi_{i1w}\xi_{i2w} + \frac{1}{\sqrt{2\pi}}\xi_{i1w}W_{SN1}[1 - \xi_{i2w}^2(2+\lambda^2)] - \frac{\lambda}{\pi}\xi_{i1w}\xi_{i2w}W_{SN1}^2\right]$$
$$- \frac{(\alpha-1)}{2\gamma}d_{iw}\left[\xi_{i1w}W_{SN}[\xi_{i2w}^2 - 1] - \sqrt{\frac{2}{\pi}}\lambda\xi_{i1w}\xi_{i2w}W_{SN2} + \xi_{i1w}\xi_{i2w}W_{SN}^2\right],$$

$$\kappa_{6i} = \frac{1}{\sqrt{2\pi}}d_{iw}\xi_{i1w}W_{SN1}\left[1 + \xi_{i2w}(1+\lambda^2) - \xi_{i2w} + \sqrt{\frac{2}{\pi}}W_{SN1}\right] - \frac{\alpha-1}{1+\lambda^2}\frac{1}{\sqrt{2\pi}}\left[d_{iw}\xi_{i1w}W_{SN2}[\xi_{i2w}(1+\lambda^2) + W_{SN}]\right],$$

and $d_{iw} = \frac{\partial\psi(x_{i\omega},\beta)}{\partial\omega}$,

$$\kappa_{7i} = \frac{1}{\gamma}\left[\xi_{i1}\xi_{i2}\left(1 + \frac{\lambda}{\sqrt{2\pi}}W_{SN1}\right) + \frac{\lambda}{\pi}\xi_{i1}W_{SN1}^2\right] - \frac{\alpha-1}{\gamma}\left[\frac{1}{2}\xi_{i1}W_{SN}(1 - \xi_{i2}^2 - \xi_{i2}W_{SN}) + \frac{\lambda}{\sqrt{2\pi}}\xi_{i1}\xi_{i2}W_{SN2}\right],$$

$$\kappa_{8i} = \frac{1}{\sqrt{2\pi}}\left[\xi_{i1}W_{SN1}(1 - \lambda^2\xi_i^2) - 2\lambda\sqrt{\frac{2}{\pi}}\xi_{i1}\xi_{i2}W_{SN1}^2\right] + \frac{\alpha-1}{1+\lambda^2}\frac{1}{\sqrt{2\pi}}[\xi_{i1}W_{SN2}[(1+\lambda^2)\xi_{i2} + W_{SN}]].$$

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