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Stability Results for the Darboux Problem of Conformable Partial Differential Equations

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Abstract: In this paper, we investigate the Darboux problem of conformable partial differential equations (DPCDEs) using fixed point theory. We focus on the existence and Ulam–Hyers–Rassias stability (UHRS) of the solutions to the problem, which requires finding solutions to nonlinear partial differential equations that satisfy certain boundary conditions. Using fixed point theory, we establish the existence and uniqueness of solutions to the DPCDEs. We then explore the UHRS of the solutions, which measures the sensitivity of the solutions to small perturbations in the equations. We provide three illustrative examples to demonstrate the effectiveness of our approach.

Keywords: generalized conformable derivative; Darboux problem; Ulam–Hyers–Rassias stability

MSC: 34A08; 26A33; 47H10

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1. Introduction

Fractional calculus (FC) is a fascinating and dynamic branch of mathematical analysis that focuses on studying the properties and applications of fractional derivatives and integrals. These noninteger order operators offer a powerful way to model complex physical, chemical, and engineering systems that cannot be easily described using traditional calculus techniques. In particular, FC has found applications in fields ranging from fluid mechanics, electromagnetism, and signal processing to finance, biology, and medicine. One of the key advantages of FC is its ability to describe nonlocal and memory-dependent phenomena, making it a powerful tool for modeling and analyzing complex systems in both time and space domains. As research in this field continues to grow, we can expect to see even more exciting applications and innovations in the years to come (see [1–3]).

In the past decade, a groundbreaking concept known as the fractional conformable derivative (FCD) has emerged as a transformative tool in the realm of FC, revolutionizing the investigation of nonregular solutions. The introduction of the FCD by Khalil et al. (see [4]) has brought about a profound shift in the understanding and application of fractional derivatives. By possessing properties akin to their integer-order counterparts, the FCD has opened up new avenues for modeling and analyzing intricate systems across diverse disciplines. The study of conformable derivatives has attracted considerable attention, with numerous researchers exploring their definitions, properties, and applications. The work of Khalil et al. has laid the foundation for the understanding of the FCD, highlighting its efficacy in capturing the behavior of complex systems that elude traditional calculus approaches. This novel approach has found application in a wide range of fields, including physics, engineering, biology, and finance. Further advancements in conformable calculus have been documented in a series of seminal publications. For instance, ref. [5] delved into the exploration of controllability in a class of conformable differential systems, shedding light on the efficient manipulation of these systems. Meanwhile, ref. [6] focused on the investigation of nonlinear evolution equations within a Wick-type stochastic environment, incorporating conformable derivatives to account for the inherent uncertainties. In [7], the researchers successfully established the existence of solutions to the conformable diffusion equation, enriching our understanding of diffusion processes influenced by conformable calculus. Furthermore, ref. [8] explored the notion of stability in the Ulam sense for conformable differential equations, presenting crucial insights into the behavior and predictability of such equations. These noteworthy contributions underscore the growing significance of the FCD and conformable calculus, as researchers strive to unravel its full potential and push the boundaries of its applications. As the scientific community continues to delve into the intricacies of conformable derivatives, we anticipate further groundbreaking developments and novel insights in the coming years, propelling us towards a deeper understanding of complex systems through the lens of fractional calculus.

In 1940, Ulam posed the question of stability for functional equations at Wisconsin University (see [9]). The Ulam–Hyers stability was first established by Hyers in 1941 in the context of Banach spaces (see [10]). This type of stability is now referred to as Ulam–Hyers stability. In 1978, Rassias [11] extended the Ulam–Hyers stability (UHS) to include functions of multiple variables. The monographs [12,13] present a comprehensive overview of the UHS and UHRS of various functional equations. Recently, the study of Ulam's problem has been extended to include a wide range of functional equations, such as symmetrical differential equations, integral equations, integro-differential equations, partial differential equations, and other types of equations (see [8,14–22]). For example, in [15], the authors studied the UHRS of pseudoparabolic partial differential equations, while in [19], the UHS of pantograph fractional stochastic differential equations was investigated. However, to the best of our knowledge, there is no existing work on the HHRS of the DPCDEs. Building upon the research conducted by [8], our article aims to generalize the UHRS for PCDEs. The main contributions of our work can be summarized as follows:

1. Existence and uniqueness of the solution: We provide a rigorous proof of the existence and uniqueness of the solution for the DPCDEs.
2. UHRS of the DPCDEs: Our study delves into the UHRS of the DPCDEs. We explore the behavior and stability characteristics of solutions to the DPCDEs under perturbations, taking into account the principles and methodologies established in the UHRS framework.

The organization of our paper is as follows: Section 2 provides the necessary preliminaries, setting the foundation for the subsequent analyses. In Section 3, we delve into the investigation of the existence, uniqueness, and UHRS of the DPCDEs. To illustrate the practical relevance and applicability of the obtained results, Section 4 showcases three carefully selected examples. Finally, in Section 5, we summarize our contributions and discuss directions for future research.

2. Basic Definitions and Tools

In this section, we introduce and define some key terms and concepts that are essential for understanding the subsequent discussions and analyses presented in this paper [4,5,7,23,24].

Definition 1. *Let $\phi : [w,d) \longrightarrow \mathbb{R}$. The generalized conformable derivative of ϕ is defined by*

$$T_w^{\delta,\psi_w}\phi(y) = \lim_{\sigma\to 0}\frac{\phi(y+\sigma\psi_w(y,\delta)) - \phi(y)}{\sigma}, \tag{1}$$

for every $y > w$, where $\delta \in (0,1)$, and $\psi_w(y,\delta)$ is continuous and nonnegative with

$$\psi_w(y,1) = 1,$$

$$\psi_w(.,\delta_1) \neq \psi_w(.,\delta_2), \text{ where } \delta_1 \neq \delta_2, \text{ and } \delta_1,\delta_2 \in (0,1].$$

If $T_w^{\delta,\psi_w}\phi(y)$ exists, for every $y \in (w,a)$, for some $a > w$, $\lim_{y \longrightarrow w^+} T_w^{\delta,\psi_w}\phi(y)$ exists; therefore,

$$T_w^{\delta,\psi_w}\phi(w) := \lim_{y \longrightarrow w^+} T_w^{\delta,\psi_w}\phi(y).$$

Remark 1. *We assume that $\psi_w(y,\delta) > 0$, for all $y > w$, and $\frac{1}{\psi_w}(.,\delta)$ is locally integrable.*

Definition 2. *For $\delta \in (0,1)$, the conformable fractional integral of ϕ is defined by*

$$I_w^{\delta,\psi_w}\phi(y) = \int_w^y \frac{\phi(l)}{\psi_w(l,\delta)}dl. \tag{2}$$

Remark 2. *Let $l \in \mathbb{R}^*$. If*

$$h(z) := \mathbb{E}_\delta^{\psi_w}(l,z,w) = e^{l\int_w^z \frac{1}{\psi_w(x,\delta)}dx},$$

then

$$T_w^{\delta,\psi_w}h(z) = lh(z), \quad \text{and} \quad I_w^{\delta,\psi_w}h(z) = \frac{1}{l}(h(z)-1).$$

The objective of this investigation is to explore and assess the stability properties of the system described by the following set of equations

$$T_{c_1}^{\theta_1,\psi_{c_1}} T_{c_2}^{\theta_2,\psi_{c_2}} u(\lambda_1,\lambda_2) = f(\lambda_1,\lambda_2,u(\lambda_1,\lambda_2)), \tag{3}$$

for all $(\lambda_1,\lambda_2) \in J = [c_1,d_1] \times [c_2,d_2]$, with

$$\begin{cases} u(\lambda_1,c_2) = \varphi(\lambda_1), & \text{if } \lambda_1 \in [c_1,d_1] \\ u(c_1,\lambda_2) = \tilde{\varphi}(\lambda_2), & \text{if } \lambda_2 \in [c_2,d_2] \\ \varphi(c_1) = \tilde{\varphi}(c_2), \end{cases}$$

where $f \in C(J \times \mathbb{R},\mathbb{R})$ and $\varphi : [c_1,d_1] \to \mathbb{R}$, $\tilde{\varphi} : [c_2,d_2] \to \mathbb{R}$ are given absolutely continuous functions. Equation (3) is equivalent to the following equation

$$u(\lambda_1,\lambda_2) = \Phi(\lambda_1,\lambda_2) + \int_{c_1}^{\lambda_1}\int_{c_2}^{\lambda_2} \frac{f(t,s,u(t,s))}{\psi_{c_1}(t,\theta_1)\psi_{c_2}(s,\theta_2)}dsdt,$$

with

$$\Phi(\lambda_1,\lambda_2) = \varphi(\lambda_1) + \tilde{\varphi}(\lambda_2) - \varphi(c_1).$$

In this study, we proceed by considering a crucial assumption that plays a fundamental role in our analysis.
$\mathcal{H}_1$: There exists $\bar{K} > 0$, such that

$$|f(\lambda_1,\lambda_2,u_1) - f(\lambda_1,\lambda_2,u_2)| \leq \bar{K}|u_1 - u_2|, \tag{4}$$

for all $(\lambda_1,\lambda_2) \in J$, $u_1,u_2 \in \mathbb{R}$.

3. Stability Results

In this part, we present the definitions of the UHR and proceed to showcase our main results.

Definition 3. *Equation (3) is UHR stable with respect to (ϵ, π), with $\epsilon > 0$ and $\psi \in C(J, \mathbb{R})$ if there is $r > 0$, such that for each solution V of*

$$\left| T_{c_1}^{\theta_1, \psi_{c_1}} T_{c_2}^{\theta_2, \psi_{c_2}} V(\lambda_1, \lambda_2) - f(\lambda_1, \lambda_2, V(\lambda_1, \lambda_2)) \right| \leq \epsilon \pi(\lambda_1, \lambda_2), \tag{5}$$

$\forall\ (\lambda_1, \lambda_2) \in J$, there is a solution $U^*(\lambda_1, \lambda_2)$ to (3):

$$|V(\lambda_1, \lambda_2) - U^*(\lambda_1, \lambda_2)| \leq r\epsilon\pi(\lambda_1, \lambda_2),\ \forall(\lambda_1, \lambda_2) \in J.$$

Theorem 1. *Suppose that $\mathcal{H}_1$ holds. If $V \in AC(J, \mathbb{R})$ satisfies*

$$\left| T_{c_1}^{\theta_1, \psi_{c_1}} T_{c_2}^{\theta_2, \psi_{c_2}} V(\lambda_1, \lambda_2) - f(\lambda_1, \lambda_2, V(\lambda_1, \lambda_2)) \right| \leq \epsilon \pi(\lambda_1, \lambda_2), \tag{6}$$

$\forall\ (\lambda_1, \lambda_2) \in J$, where $\epsilon > 0$, and $\pi \in C(J, \mathbb{R})$ is nondecreasing with respect to λ_1 and λ_2; then, there is a unique solution U^* to (3), such that

$$|V(\lambda_1, \lambda_2) - U^*(\lambda_1, \lambda_2)| \leq \epsilon \frac{\bar{K} + \varrho}{\varrho} \int_{c_1}^{d_1} \frac{ds_1}{\psi_{c_1}(s_1, \theta_1)} \int_{c_2}^{d_2} \frac{ds_2}{\psi_{c_2}(s_2, \theta_2)} \beta(d_1, d_2)\pi(\lambda_1, \lambda_2), \forall(\lambda_1, \lambda_2) \in J,$$

for any positive constant ϱ, where

$$\beta(\lambda_1, \lambda_2) = \mathbb{E}_{\theta_1}^{\psi_{c_1}}\left(\sqrt{\bar{K} + \varrho}, \lambda_1, c_1\right) \times \mathbb{E}_{\theta_2}^{\psi_{c_2}}\left(\sqrt{\bar{K} + \varrho}, \lambda_2, c_2\right).$$

Proof. Let us consider the metric d on $C(J, \mathbb{R})$, given by:

$$d(\vartheta_1, \vartheta_2) = \sup_{(\lambda_1, \lambda_2) \in J} \frac{|\vartheta_1(\lambda_1, \lambda_2) - \vartheta_2(\lambda_1, \lambda_2)|}{\beta(\lambda_1, \lambda_2)\pi(\lambda_1, \lambda_2)}. \tag{7}$$

We have $\big(C(J, \mathbb{R}), d\big)$, which is a complete metric space. Let $\mathcal{A} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$, such that

$$(\mathcal{A}u)(\lambda_1, \lambda_2) := V(c_1, \lambda_2) + V(\lambda_1, c_2) - V(c_1, c_2) + \int_{c_1}^{\lambda_1} \int_{c_2}^{\lambda_2} \frac{f(t, s, u(t, s))}{\psi_{c_1}(t, \theta_1)\psi_{c_2}(s, \theta_2)} ds dt,\ \forall(\lambda_1, \lambda_2) \in J.$$

Let $u_1, u_2 \in C(J, \mathbb{R})$. By using $\mathcal{H}_1$, we obtain

$$\begin{aligned}
&|(\mathcal{A}u_1)(\lambda_1, \lambda_2) - (\mathcal{A}u_2)(\lambda_1, \lambda_2)| \\
&\leq \left| \int_{c_1}^{\lambda_1} \int_{c_2}^{\lambda_2} \frac{f(s_1, s_2, u_1(s_1, s_2)) - f(s_1, s_2, u_2(s_1, s_2))}{\psi_{c_1}(s_1, \theta_1)\psi_{c_2}(s_2, \theta_2)} ds_2 ds_1 \right| \\
&\leq \int_{c_1}^{\lambda_1} \int_{c_2}^{\lambda_2} \left| \frac{f(s_1, s_2, u_1(s_1, s_2)) - f(s_1, s_2, u_2(s_1, s_2))}{\psi_{c_1}(s_1, \theta_1)\psi_{c_2}(s_2, \theta_2)} \right| ds_2 ds_1 \\
&\leq \bar{K} \int_{c_1}^{\lambda_1} \int_{c_2}^{\lambda_2} \frac{|u_1(s_1, s_2) - u_2(s_1, s_2)|}{\psi_{c_1}(s_1, \theta_1)\psi_{c_2}(s_2, \theta_2)} ds_2 ds_1 \\
&\leq \bar{K} \int_{c_1}^{\lambda_1} \int_{c_2}^{\lambda_2} \frac{|u_1(s_1, s_2) - u_2(s_1, s_2)|}{\beta(s_1, s_2)\pi(s_1, s_2)} \frac{\beta(s_1, s_2)\pi(s_1, s_2)}{\psi_{c_1}(s_1, \theta_1)\psi_{c_2}(s_2, \theta_2)} ds_2 ds_1 \\
&\leq \bar{K} d(u_1, u_2) \int_{c_1}^{\lambda_1} \int_{c_2}^{\lambda_2} \frac{\beta(s_1, s_2)\pi(s_1, s_2)}{\psi_{c_1}(s_1, \theta_1)\psi_{c_2}(s_2, \theta_2)} ds_2 ds_1 \\
&\leq \bar{K} d(u_1, u_2)\pi(\lambda_1, \lambda_2) \int_{c_1}^{\lambda_1} \int_{c_2}^{\lambda_2} \frac{\beta(s_1, s_2)}{\psi_{c_1}(s_1, \theta_1)\psi_{c_2}(s_2, \theta_2)} ds_2 ds_1 \\
&\leq \bar{K} d(u_1, u_2)\pi(\lambda_1, \lambda_2) \int_{c_1}^{\lambda_1} \frac{\mathbb{E}_{\theta_1}^{\psi_{c_1}}\left(\sqrt{\bar{K} + \varrho}, s_1, c_1\right)}{\psi_{c_1}(s_1, \theta_1)} ds_1 \int_{c_2}^{\lambda_2} \frac{\mathbb{E}_{\theta_2}^{\psi_{c_2}}\left(\sqrt{\bar{K} + \varrho}, s_2, c_2\right)}{\psi_{c_2}(s_2, \theta_2)} ds_2.
\end{aligned} \tag{8}$$

By using Remark 2, we obtain

$$|(\mathcal{A}u_1)(\lambda_1,\lambda_2)-(\mathcal{A}u_2)(\lambda_1,\lambda_2)|\le\frac{\bar{K}}{\bar{K}+\varrho}d(u_1,u_2)\pi(\lambda_1,\lambda_2)\mathbb{E}^{\psi_{c_1}}_{\theta_1}\left(\sqrt{\bar{K}+\varrho},\lambda_1,c_1\right)\mathbb{E}^{\psi_{c_2}}_{\theta_2}\left(\sqrt{\bar{K}+\varrho},\lambda_2,c_2\right). \tag{9}$$

Then,

$$|(\mathcal{A}u_1)(\lambda_1,\lambda_2)-(\mathcal{A}u_2)(\lambda_1,\lambda_2)|\le\frac{\bar{K}}{\bar{K}+\varrho}d(u_1,u_2)\pi(\lambda_1,\lambda_2)\beta(\lambda_1,\lambda_2).$$

Therefore,

$$\frac{|(\mathcal{A}u_1)(\lambda_1,\lambda_2)-(\mathcal{A}u_2)(\lambda_1,\lambda_2)|}{\pi(\lambda_1,\lambda_2)\beta(\lambda_1,\lambda_2)}\le\frac{\bar{K}}{\bar{K}+\varrho}d(u_1,u_2). \tag{10}$$

It follows from (7) and (10) that

$$d(\mathcal{A}u_1,\mathcal{A}u_2)\le\frac{\bar{K}}{\bar{K}+\varrho}d(u_1,u_2).$$

Consequently, by establishing the contractiveness of $\mathcal{A}$, we can derive from (6) that

$$\begin{aligned}|V(\lambda_1,\lambda_2)-\mathcal{A}V(\lambda_1,\lambda_2)| &\le \epsilon\int_{c_1}^{\lambda_1}\int_{c_2}^{\lambda_2}\frac{\pi(s_1,s_2)}{\psi_{c_1}(s_1,\theta_1)\psi_{c_2}(s_2,\theta_2)}ds_2ds_1\\ &\le \epsilon\pi(\lambda_1,\lambda_2)\int_{c_1}^{d_1}\frac{ds_1}{\psi_{c_1}(s_1,\theta_1)}\int_{c_2}^{d_2}\frac{ds_2}{\psi_{c_2}(s_2,\theta_2)},\ \forall(\lambda_1,\lambda_2)\in J;\end{aligned}$$

then,

$$\frac{|V(\lambda_1,\lambda_2)-\mathcal{A}V(\lambda_1,\lambda_2)|}{\beta(\lambda_1,\lambda_2)}\le\epsilon\pi(\lambda_1,\lambda_2)\int_{c_1}^{d_1}\frac{ds_1}{\psi_{c_1}(s_1,\theta_1)}\int_{c_2}^{d_2}\frac{ds_2}{\psi_{c_2}(s_2,\theta_2)},\ \forall(\lambda_1,\lambda_2)\in J,$$

so that

$$d(V,\mathcal{A}V)\le\epsilon\int_{c_1}^{d_1}\frac{ds_1}{\psi_{c_1}(s_1,\theta_1)}\int_{c_2}^{d_2}\frac{ds_2}{\psi_{c_2}(s_2,\theta_2)}.$$

It follows from Theorem 2 in [18] that there is a solution U^* to (3) such that

$$d(V,U^*)\le\epsilon\frac{\bar{K}+\varrho}{\varrho}\int_{c_1}^{d_1}\frac{ds_1}{\psi_{c_1}(s_1,\theta_1)}\int_{c_2}^{d_2}\frac{ds_2}{\psi_{c_2}(s_2,\theta_2)},$$

so that

$$|V(\lambda_1,\lambda_2)-U^*(\lambda_1,\lambda_2)|\le\epsilon\frac{\bar{K}+\varrho}{\varrho}\int_{c_1}^{d_1}\frac{ds_1}{\psi_{c_1}(s_1,\theta_1)}\int_{c_2}^{d_2}\frac{ds_2}{\psi_{c_2}(s_2,\theta_2)}\beta(d_1,d_2)\pi(\lambda_1,\lambda_2),$$

for all $(\lambda_1,\lambda_2)\in J$. □

In order to investigate the Ulam stability of Equation (3), we present the following notable results.

Theorem 2. *Suppose that $\mathcal{H}_1$ holds. If $V\in AC(J,\mathbb{R})$ satisfies*

$$\left|T^{\theta_1,\psi_{c_1}}_{c_1}T^{\theta_2,\psi_{c_2}}_{c_2}V(\lambda_1,\lambda_2)-f(\lambda_1,\lambda_2,V(\lambda_1,\lambda_2))\right|\le\epsilon, \tag{11}$$

$\forall\ (\lambda_1,\lambda_2)\in J$, where $\epsilon>0$; then, there is a unique solution U^* to (3), such that

$$|V(\lambda_1,\lambda_2)-U^*(\lambda_1,\lambda_2)|\le\epsilon\frac{\bar{K}+\varrho}{\varrho}\int_{c_1}^{d_1}\frac{ds_1}{\psi_{c_1}(s_1,\theta_1)}\int_{c_2}^{d_2}\frac{ds_2}{\psi_{c_2}(s_2,\theta_2)}\beta(d_1,d_2),\forall(\lambda_1,\lambda_2)\in J,$$

for any positive constant ϱ, where

$$\beta(\lambda_1,\lambda_2) = \mathbb{E}_{\theta_1}^{\psi_{c_1}}\left(\sqrt{\bar{K}+\varrho},\lambda_1,c_1\right) \times \mathbb{E}_{\theta_2}^{\psi_{c_2}}\left(\sqrt{\bar{K}+\varrho},\lambda_2,c_2\right).$$

Proof. The proof is similar to Theorem 1. □

Remark 3. *An important observation to highlight is that the outcomes presented in [18] align with our findings when $\theta_1 = \theta_2 = 1$ within the current context.*

4. Illustrative Examples

In this section, we provide three illustrative examples to corroborate the major results outlined in Section 3.

Example 1. *We consider Equation (3) for $c_1 = c_2 = 0$, $d_1 = d_2 = 1$, $\theta_1 = 1$, $\theta_2 = 0.5$, $\psi_{c_2}(s,\theta_2) = s^{1-\theta_2}$, and $f(v_1,v_2,r) = {v_1}^3 v_2 \sin(r)$.*
We have

$$\left|{v_1}^3 v_2 \sin(r_1) - {v_1}^3 v_2 \sin(r_2)\right| \leq |r_1 - r_2|,\ \forall\ (v_1,v_2) \in [0,1]\times[0,1],\ r_1,r_2 \in \mathbb{R}.$$

Then, $\bar{K} = 1$. Suppose that V satisfies

$$\left|T_{c_1}^{\theta_1,\psi_{c_1}} T_{c_2}^{\theta_2,\psi_{c_2}} V(\lambda_1,\lambda_2) - f(\lambda_1,\lambda_2,V(\lambda_1,\lambda_2))\right| \leq 0.1(\lambda_1+\lambda_2+2), \tag{12}$$

for all $(\lambda_1,\lambda_2) \in [0,1]\times[0,1]$. Here, $\epsilon = 0.1$, and $\pi(\lambda_1,\lambda_2) = \lambda_1+\lambda_2+2$. It follows from Theorem 1 that there is a solution U^* to the equation, and $L > 0$, such that

$$|V(\lambda_1,\lambda_2) - U^*(\lambda_1,\lambda_2)| \leq 0.1L(\lambda_1+\lambda_2+2),\ \forall\ (\lambda_1,\lambda_2) \in [0,1]\times[0,1].$$

The exact solution U^* and the approximate solution V are plotted in Figure 1.

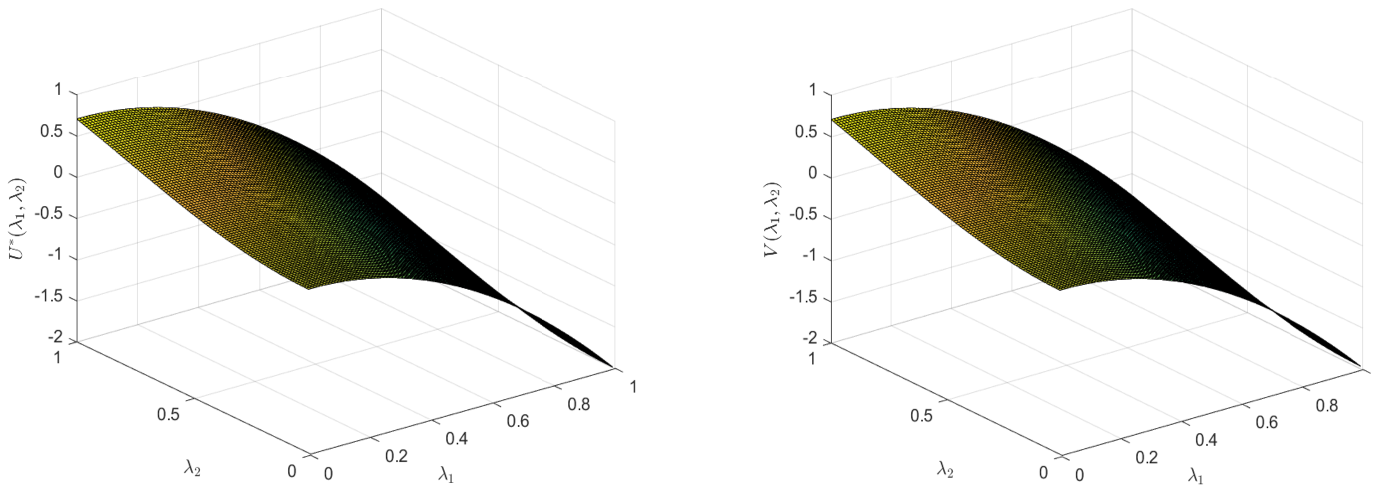


Figure 1. Side-by-side comparison of the exact solution (**left**) and the approximate solution (**right**) for Example 1, with $\theta_1 = 1$, $\theta_2 = 0.5$, $\varphi(\lambda) = -2\lambda^2$, and $\tilde{\varphi}(\lambda) = \sin^2(\lambda)$, on the domain $[0,1]\times[0,1]$.

Example 2. *We consider Equation (3) for $c_1 = c_2 = 0$, $d_1 = d_2 = 2$, $\theta_1 = 0.8$, $\theta_2 = 0.6$, $\psi_{c_1}(s,\theta_1) = s^{1-\theta_1}$, $\psi_{c_2}(s,\theta_2) = s^{1-\theta_2}$ and $f(v_1,v_2,r) = v_1 {v_2}^2 \cos(r)$.*
We have

$$\left|v_1 {v_2}^2 \cos(r_1) - v_1 {v_2}^2 \cos(r_2)\right| \leq 8|r_1 - r_2|,\ \forall\ (v_1,v_2) \in [0,2]\times[0,2],\ r_1,r_2 \in \mathbb{R}.$$

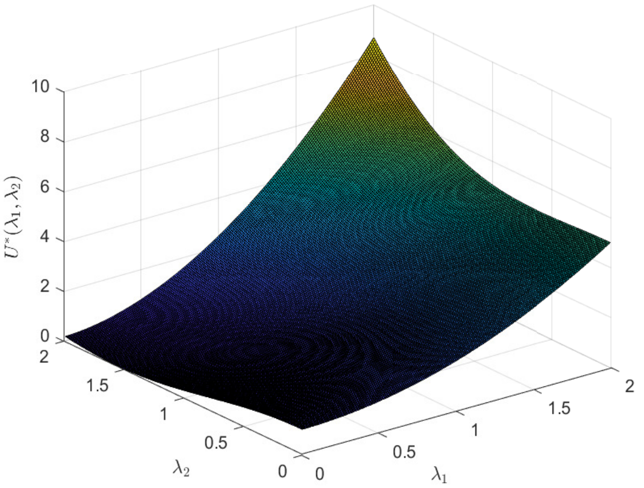
Then, $\bar{K} = 8$*. Suppose that V satisfies*

$$\left| T_{c_1}^{\theta_1,\psi_{c_1}} T_{c_2}^{\theta_2,\psi_{c_2}} V(\lambda_1,\lambda_2) - f(\lambda_1,\lambda_2,V(\lambda_1,\lambda_2)) \right| \leq 0.01(\lambda_1^2+\lambda_2^2+5), \tag{13}$$

for all $(\lambda_1,\lambda_2) \in [0,2]\times[0,2]$*. Here,* $\epsilon = 0.01$*, and* $\pi(\lambda_1,\lambda_2) = (\lambda_1^2+\lambda_2^2+5)$*. It follows from Theorem 1 that there is a solution* U^* *to the equation, and* $L > 0$*, such that*

$$|V(\lambda_1,\lambda_2) - U^*(\lambda_1,\lambda_2)| \leq 0.01L(\lambda_1^2+\lambda_2^2+5),\ \forall\ (\lambda_1,\lambda_2) \in [0,2]\times[0,2].$$

The exact solution U^* *and the approximate solution V are plotted in Figure 2.*



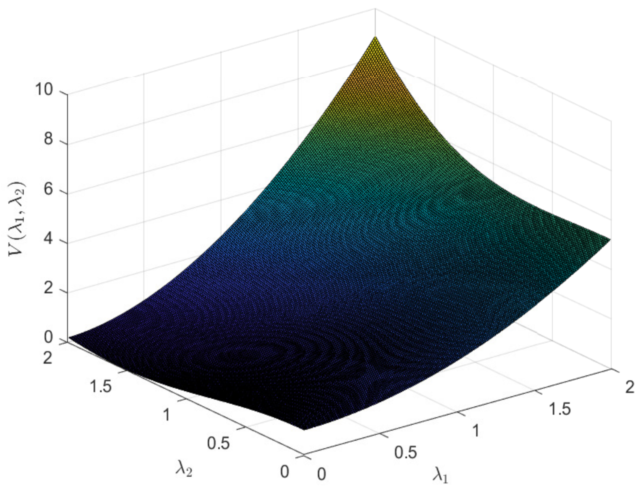


Figure 2. Comparison of the precise solution (on the **left**) and the approximated solution (on the **right**) for Example 2, considering $\theta_1 = 0.8$, $\theta_2 = 0.6$, $\varphi(\lambda) = 1+\lambda^2$, and $\tilde{\varphi}(\lambda) = \cos^2(\lambda)$ on the interval $[0,2]\times[0,2]$.

Example 3. *We consider Equation (3) for* $c_1 = c_2 = 0$*,* $d_1 = d_2 = 3$*,* $\theta_1 = 0.4$*,* $\theta_2 = 0.6$*,* $\psi_{c_1}(s,\theta_1) = s^{1-\theta_1}$*,* $\psi_{c_2}(s,\theta_2) = s^{1-\theta_2}$*, and* $f(v_1,v_2,r) = \cos(v_1)v_2 r$*.*
We have

$$|\cos(v_1)v_2r_1 - \cos(v_1)v_2r_2| \leq 3|r_1-r_2|,\ \forall\ (v_1,v_2) \in [0,3]\times[0,3],\ r_1,r_2 \in \mathbb{R}.$$

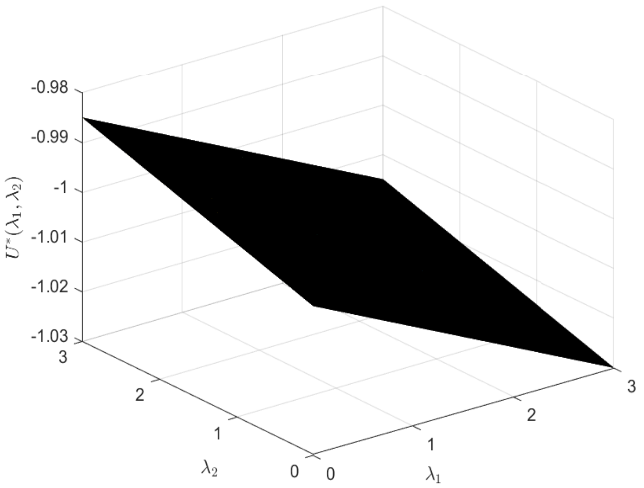
Then, $\bar{K} = 3$*. Suppose that V satisfies*

$$\left| T_{c_1}^{\theta_1,\psi_{c_1}} T_{c_2}^{\theta_2,\psi_{c_2}} V(\lambda_1,\lambda_2) - f(\lambda_1,\lambda_2,V(\lambda_1,\lambda_2)) \right| \leq 0.01, \tag{14}$$

for all $(\lambda_1,\lambda_2) \in [0,3]\times[0,3]$*. Here,* $\epsilon = 0.01$*. It follows from Theorem 2 that there is a solution* U^* *to the equation, and* $L > 0$*, such that*

$$|V(\lambda_1,\lambda_2) - U^*(\lambda_1,\lambda_2)| \leq 0.01L,\ \forall\ (\lambda_1,\lambda_2) \in [0,3]\times[0,3].$$

The exact solution U^* *and the approximate solution V are plotted in Figure 3.*



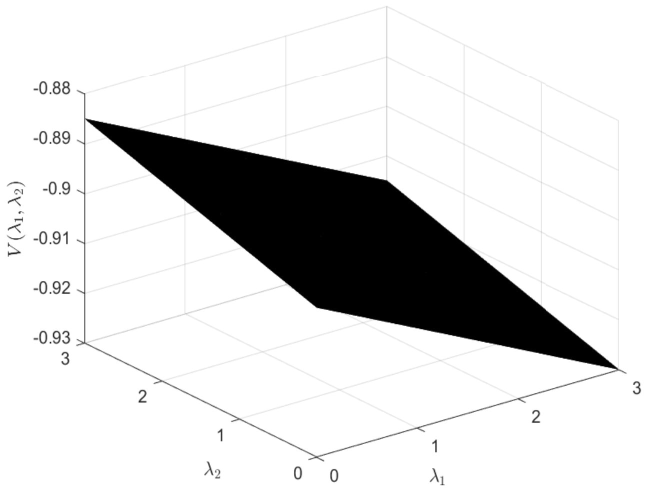


Figure 3. The exact solution (**left**) and the approximate solution (**right**) for Example 3, with $\theta_1 = 0.4$, $\theta_2 = 0.6$, $\varphi(\lambda) = -1 + \frac{1}{200}\lambda$, and $\tilde{\varphi}(\lambda) = -1 - \frac{1}{100}\lambda$, on the interval $[0,3] \times [0,3]$, displayed side by side for easy comparison.

5. Conclusions

In conclusion, this paper delved into a comprehensive investigation of the existence, uniqueness, and UHRS for the DPCDEs. Using the Banach fixed-point theorem, we established the existence and uniqueness of solutions to the DPCDEs that satisfy the prescribed boundary conditions. Furthermore, our exploration of the UHRS for the DPCDEs shed light on the robustness and resilience of the solutions under perturbations. By considering the appropriate stability concepts and utilizing analytical tools, we quantified the stability properties of the solutions. The inclusion of three illustrative examples in this paper serves to solidify and showcase the main obtained results. We can generalize our work by using the operators given in [25–34].

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