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A Generalized Norm on Reproducing Kernel Hilbert Spaces and Its Applications

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Abstract: The aim of this article was to provide improved estimates for the (α, β)-norm of a bounded linear operator. In particular, our results enabled the determination of new upper bounds involving both the Berezin number and the Berezin norm of bounded linear operators that act on reproducing kernel Hilbert spaces. Through our analysis, we hoped to enhance the understanding of the properties and behavior of such operators and contribute to the development of new mathematical tools for their characterization and application.

Keywords: reproducing kernel Hilbert space; Berezin number; Berezin norm; inequality

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1. Introduction

The numerical radius and Berezin number of an operator have been widely studied in various fields, including engineering, quantum computing, quantum mechanics, numerical analysis, and differential equations, due to their numerous applications. In this article, we focused on characterizing the Berezin number and the Berezin norm. To achieve this goal, we introduced several key concepts and properties of bounded linear operators on a Hilbert space. Our aim was to provide a comprehensive overview of the relevant background material and to establish a solid foundation for our subsequent analysis of these important operator measures.

Consider a complex Hilbert space $\mathcal{E}$ equipped with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Let $\mathcal{L}(\mathcal{E})$ denote the C^*-algebra consisting of all bounded linear operators on $\mathcal{E}$, including the identity operator I. An operator $T \in \mathcal{L}(\mathcal{E})$ is said to be positive if $\langle Tx, x \rangle \geq 0$ holds for all $x \in \mathcal{E}$, denoted by $T \geq 0$. For a positive bounded linear operator T, there exists a unique positive bounded linear operator $T^{1/2}$ such that $T = (T^{1/2})^2$. Moreover, we define the absolute value of T as $|T| = (T^*T)^{1/2}$. It is worth noting that $|T| \geq 0$.

Let $T \in \mathcal{L}(\mathcal{E})$ be a bounded linear operator, and consider the operator norm $\|T\|$, defined as $\|T\| = \sup \left\{ \|Tx\| \ ; \ x \in \mathcal{E}, \ \|x\| = 1 \right\}$, and the numerical radius of T, denoted by $\omega(T)$, defined as $\omega(T) = \sup \left\{ |\langle Tx, x \rangle| \ ; \ x \in \mathcal{E}, \ \|x\| = 1 \right\}$. It is straightforward to verify that $\omega(T) \leq \|T\|$, where the equality holds if T is a normal operator, i.e., $T^*T = TT^*$. Moreover, the operator norm and the numerical radius are equivalent, since $\frac{\|T\|}{2} \leq \omega(T) \leq \|T\|$

for any $T \in \mathcal{L}(\mathcal{E})$. Notably, we also have $\omega(T) \leq \omega(|T|)$. The operator T is normaloid if $\omega(T) = \|T\|$. Some properties of the numerical radius can be found in [1].

Sain et al. introduced in [2] the (α, β)-norm on the space $\mathcal{L}(\mathcal{E})$ of bounded linear operators on a complex Hilbert space $\mathcal{E}$. Throughout this paper, we will use the notation α and β to denote non-negative real scalars, satisfying $(\alpha, \beta) \neq (0,0)$. The (α, β)-norm is a mapping $\|\cdot\|_{\alpha,\beta} : \mathcal{L}(\mathcal{E}) \to \mathbb{R}^+$, defined as follows:

$$\|T\|_{\alpha,\beta} := \sup_{\|x\|=1, x\in\mathcal{E}} \sqrt{\alpha\|Tx\|^2 + \beta\big|\langle Tx, x\rangle\big|^2}.$$

This norm captures the joint effect of the operator norm and the numerical radius of T, with the relative weight of each component determined by the parameters α and β. Note that $\|T\|_{\alpha,\beta}$ is indeed a norm on $\mathcal{L}(\mathcal{E})$ since it satisfies the three properties of a norm: non-negativity, homogeneity, and triangle inequality. Additionally, we can observe that $\|T\|_{0,1} = \omega(T)$ and $\|T\|_{1,0} = \|T\|$, taking $\alpha = 0$, $\beta = 1$, and $\alpha = 1$, $\beta = 0$, respectively, yield the same formulas as for the numerical radius and operator norm. Another notable instance of the (α, β)-norm occurs when we choose $\alpha = \beta = 1$, which leads to the modified Davis–Wielandt radius of the operator $T \in \mathcal{L}(\mathcal{E})$, denoted by $d\omega^*(T)$ (see [3]). In [2], it was established that:

$$\sqrt{\alpha+\beta}\,\omega(T) \leq \|T\|_{\alpha,\beta} \leq \sqrt{\alpha\|T\|^2 + \beta\omega^2(T)} \leq \sqrt{\alpha+4\beta}\,\omega(T). \tag{1}$$

Let Λ be a non-empty set and $\mathcal{F}(\Lambda)$ be the set of all complex-valued functions defined on Λ. A subset $\mathcal{E}_\Lambda$ of $\mathcal{F}(\Lambda)$ is called a reproducing kernel Hilbert space (RKHS) on Λ if it is a Hilbert space (with identity operator I_Λ) and if the linear evaluation functional $E_\mu : \mathcal{E}_\Lambda \to \mathbb{C}$ defined by $E_\mu(f) = f(\mu)$ is bounded for every $\mu \in \Lambda$. By applying the Riesz representation theorem, we can establish the existence of a unique vector $k_\mu \in \mathcal{E}_\Lambda$ for each $\mu \in \Lambda$ such that $f(\mu) = E_\mu(f) = \langle f, k_\mu\rangle$ holds for all $f \in \mathcal{E}_\Lambda$. It should be noted that the function k_μ is commonly referred to as the reproducing kernel for the point μ. On the other hand, the function $K : \Lambda \times \Lambda \to \mathbb{C}$ is defined as $K(\mu, \nu) = k_\nu(\mu)$ and is known as the reproducing kernel for $\mathcal{E}_\Lambda$. It is worth mentioning that we can express $K(\mu, \nu)$ as the inner product of k_ν and k_μ, that is,

$$K(\mu, \nu) = \langle k_\nu, k_\mu\rangle$$

for all $\mu, \nu \in \Lambda$. It should be noted that the collection of elements $\{k_\mu\,;\ \mu \in \Lambda\}$ is commonly known as the reproducing kernel of $\mathcal{E}_\Lambda$. Additionally, we use the notation $\widehat{k}_\mu = \frac{k_\mu}{\|k_\mu\|}$ for $\mu \in \Lambda$ to represent the normalized reproducing kernel of $\mathcal{E}_\Lambda$. Another important point to mention is that the set $\{\widehat{k}_\mu\,;\ \mu \in \Lambda\}$ is a complete set in $\mathcal{E}_\Lambda$.

The Berezin symbol or Berezin transform of $T \in \mathcal{L}(\mathcal{E}_\Lambda)$ is a bounded function $\widetilde{T} : \Lambda \to \mathbb{C}$, which was introduced by Berezin [4,5]. It is defined as $\widetilde{T}(\mu) = \langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle$ for $\mu \in \Lambda$. If $T = T^*$ is self-adjoint, then $\widetilde{T}(\mu) \in \mathbb{R}$. Furthermore, if T is a positive operator, then $\widetilde{T}(\mu) \geq 0$. The Berezin symbol is a useful tool in the study of Toeplitz and Hankel operators on the Hardy and Bergman spaces and has been extensively investigated. It plays a crucial role in various problems in analysis and is known to uniquely determine the corresponding operator. A comprehensive understanding of the Berezin symbol can be found in several references, including [6–9] and their cited sources.

The Berezin set and Berezin number of an operator T are defined as the set of all Berezin symbols $\widetilde{T}(\mu)$ for $\mu \in \Lambda$ and the supremum of $|\widetilde{T}(\mu)|$ over all $\mu \in \Lambda$, respectively. Specifically,

$$\mathbf{Ber}(T) := \left\{\widetilde{T}(\mu)\,;\ \mu \in \Lambda\right\} \text{ and } \mathbf{ber}(T) := \sup_{\mu\in\Lambda} |\widetilde{T}(\mu)| = \sup_{\mu\in\Lambda} \big|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\big|.$$

We demonstrate that $\mathbf{ber}(\cdot)$ is a norm on $\mathcal{L}(\mathcal{E}_\Lambda)$ by straightforward calculations. Moreover, we establish the inequalities $0 \leq \mathbf{ber}(T) \leq \omega(T) \leq \|T\|$ for all $T \in \mathcal{L}(\mathcal{E}_\Lambda)$. However, Karaev [10] has shown that $\frac{\|T\|}{2} \leq \mathbf{ber}(T)$ does not hold for every $T \in \mathcal{L}(\mathcal{E}_\Lambda)$. Notably, we observe that $\mathbf{ber}(I_\Lambda) = 1$ and $|\langle Tk_\mu, k_\mu\rangle| \leq \mathbf{ber}(T)\|k_\mu\|^2$ hold for all reproducing kernels k_μ. Similar to $\mathbf{ber}(T)$, we found in [11] the following concept:

$$c_{\mathbf{ber}}(T) := \inf\{|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle| \,;\, \mu \in \Lambda\}.$$

The Berezin norm of an operator $T \in \mathcal{L}(\mathcal{E}_\Lambda)$ is defined by

$$\|T\|_{\mathbf{ber}} := \sup\left\{|\langle T\widehat{k}_\mu, \widehat{k}_\nu\rangle| \,;\, \mu, \nu \in \Lambda\right\}.$$

The statement refers to two normalized reproducing kernels, denoted as $\widehat{k}_\mu$ and $\widehat{k}_\nu$, belonging to the space $\mathcal{E}_\Lambda$ (as defined in [12]). It is important to note that the norm $\|\cdot\|_{\mathbf{ber}}$ does not necessarily satisfy the submultiplicativity property. Additionally, it should be noted that the equality

$$\|T\|_{\mathbf{ber}} = \|T\|_{\widetilde{\mathbf{ber}}} := \sup_{\mu \in \Lambda} \|T\widehat{k}_\mu\|$$

may not hold for all T in $\mathcal{L}(\mathcal{E}_\Lambda)$ (as discussed in [13]). A notable observation is that the Berezin norm satisfies the inequalities

$$\mathbf{ber}(T) \leq \|T\|_{\mathbf{ber}} \leq \|T\|_{\widetilde{\mathbf{ber}}} \leq \|T\|, \quad \forall\, T \in \mathcal{L}(\mathcal{E}_\Lambda). \tag{2}$$

It is worth noting that the inequalities (2) can be strict in general. Nevertheless, Bhunia et al. proved in [14] that if $T \in \mathcal{L}(\mathcal{E}_\Lambda)$ is a positive operator, then

$$\mathbf{ber}(T) = \|T\|_{\mathbf{ber}}. \tag{3}$$

It is important to emphasize that the equality (3) may not hold for self-adjoint operators, in general, as demonstrated in [14].

This article aimed to provide new estimates for the (α, β)-norm of a bounded operator, as well as new upper bounds involving the Berezin number and Berezin norm of bounded linear operators on reproducing kernel Hilbert spaces.

The article is organized as follows: Section 2 contains several lemmas that are necessary to prove our main results. In Section 3, we introduce a bounded linear operator $T \in \mathcal{L}(\mathcal{E})$ and provide a new numerical value for it:

$$p(T) = \sup\left\{\sqrt{\|Tx\|^2 - |\langle Tx, x\rangle|^2} \,;\, x \in \mathcal{E},\ \|x\| = 1\right\}.$$

In Theorem 1, we introduce an alternative expression for the (α, β)-norm, which enabled us to derive the following improved estimate:

$$\sqrt{\alpha+\beta}\,\omega(T) \leq \|T\|_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\,\omega(T) + \sqrt{\alpha}\,p(T).$$

Next, we aimed to generalize the concept of the (α, β)-norm to bounded linear operators acting on reproducing kernel Hilbert spaces $\mathcal{L}(\mathcal{E}_\Lambda)$, which we defined as follows:

$$\|T\|^{\mathbf{ber}}_{\alpha,\beta} = \sup_{\mu \in \Lambda} \sqrt{\alpha\|T\widehat{k}_\mu\|^2 + \beta|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2}.$$

In addition, we defined a numerical quantity associated with a bounded linear operator T on a reproducing kernel Hilbert space $\mathcal{E}_\Lambda$ as follows:

$$p_{\mathbf{ber}}(T) = \sup\left\{\sqrt{\|T\widehat{k}_\mu\|^2 - |\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2} \,;\, \widehat{k}_\mu \in \mathcal{E}_\Lambda\right\}.$$

We present a number of results regarding the (α, β)-norm. Specifically, we proved the following two results. Firstly, for every $T \in \mathcal{L}(\mathcal{E}_\Lambda)$, we established the following inequality:

$$\sqrt{\alpha+\beta}\,\mathbf{ber}(T) \leq \|T\|^{\mathbf{ber}}_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\,\mathbf{ber}(T) + \sqrt{\alpha}p_{\mathbf{ber}}(T).$$

Secondly, we showed that for $T, S \in \mathcal{L}(\mathcal{E}_\Lambda)$, the following two assertions are equivalent:

(i) $\|T+S\|^{\mathbf{ber}}_{\alpha,\beta} = \|T\|^{\mathbf{ber}}_{\alpha,\beta} + \|S\|^{\mathbf{ber}}_{\alpha,\beta}$.

(ii) There exists a sequence $\{\mu_n\}$ in Λ such that

$$\lim_{n\to\infty}\left(\alpha\langle S\widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle + \beta\langle \widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle\right) = \|T\|^{\mathbf{ber}}_{\alpha,\beta}\,\|S\|^{\mathbf{ber}}_{\alpha,\beta},$$

where $\widehat{k}_{\mu_n}$ is the normalized reproducing kernel of $\mathcal{E}_\Lambda$ at μ_n for every n.

2. Fundamental Lemmas

This section contains important lemmas that were essential for achieving our objectives in this paper. These lemmas establish fundamental properties of complex Hilbert spaces that we used extensively in our proofs. Throughout this section, we use the notation $\mathcal{E}$ to refer to a complex Hilbert space with an inner product denoted by $\langle\cdot,\cdot\rangle$ and a corresponding norm denoted by $\|\cdot\|$.

Let us start by introducing a key lemma that was instrumental in our analysis. The following lemma, proposed by Kittaneh and Moradi in [15], refines the well-known Cauchy–Schwarz inequality and provided a foundation for our subsequent analysis.

Lemma 1. *For any $x, y \in \mathcal{E}$, we have*

$$|\langle x,y\rangle|^2 \leq |\langle x,y\rangle|\|x\|\|y\| + \frac{1}{2}\left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right) \leq \|x\|^2\|y\|^2.$$

Remark 1. *We can derive a useful inequality from Lemma 1 that relates the inner product of two vectors to their norms. Specifically, for all $x, y \in \mathcal{E}$, we have*

$$\left|\langle x,y\rangle\right|^2 \leq \frac{1}{3}\|x\|^2\|y\|^2 + \frac{2}{3}\left|\langle x,y\rangle\right|\|x\|\|y\|. \tag{4}$$

Buzano demonstrated an interesting inequality, which is presented in the following lemma and proven in [16].

Lemma 2. *Suppose that $a, b, e \in \mathcal{H}$ and e has a unit norm. Then, we have*

$$|\langle a,e\rangle\ \langle e,b\rangle| \leq \frac{1}{2}(\|a\|\ \|b\| + |\langle a,b\rangle|).$$

This inequality was used in our subsequent analysis to derive an important result.

In the next lemma, we present a useful inequality from [17] that relates the inner products of three vectors x, y, and z in a Hilbert space $\mathcal{E}$:

Lemma 3. *For any $x, y, z \in \mathcal{E}$ and $\nu \in [0,1]$, we have*

$$\nu\left|\langle x,y\rangle\right|^2 + (1-\nu)\left|\langle x,z\rangle\right|^2 \leq \|x\|^2\left(\max\left\{\nu\|y\|^2, (1-\nu)\|z\|^2\right\} + \sqrt{\nu(1-\nu)}\left|\langle y,z\rangle\right|\right).$$

We used this inequality in our subsequent analysis to prove a specific result.

3. Main Results

In this section, we present our main results concerning the (α, β)-norm of Hilbert space operators and a new (α, β)-norm of operators in reproducing Kernel Hilbert spaces.

Our first main goal was to provide an improvement to the recent inequality proposed by Sain et al. To achieve this, we first define the numerical value of an operator $T \in \mathcal{L}(\mathcal{E})$ as follows:
$$p(T) = \sup\left\{\sqrt{\|Tx\|^2 - \left|\langle Tx, x\rangle\right|^2}\ ;\ x \in \mathcal{E},\ \|x\| = 1\right\}.$$
The value can be shown to be well-defined by observing that for any $x \in \mathcal{E}$ with $\|x\| = 1$, we have:

$$0 \le \sqrt{\|Tx\|^2 - \left|\langle Tx, x\rangle\right|^2} = \|Tx - \langle Tx, x\rangle x\| \le \|Tx\| + \left|\langle Tx, x\rangle\right| \le \|T\| + \omega(T) \le 2\|T\|.$$

It is straightforward to see that the function $p(\cdot)$ is a semi-norm on $\mathcal{E}$, satisfying the property that $p(T - \lambda I) = p(T)$ for all $T \in \mathcal{L}(\mathcal{E})$ and $\lambda \in \mathbb{C}$. Moreover, we can show that $p(T) \le 2\|T\|$ holds for any $T \in \mathcal{L}(\mathcal{E})$. Before we proceed, we need to establish the following lemma:

Lemma 4. *Let $x, y \in \mathcal{E}$ with $\|y\| = 1$ and $\alpha, \beta \ge 0$. Then, the equality*
$$\alpha\|x\|^2 + \beta|\langle x, y\rangle|^2 = \|\sqrt{\alpha}x + (\sqrt{\alpha+\beta} - \sqrt{\alpha})\langle x, y\rangle y\|^2 \tag{5}$$
holds.

Proof. We have the following calculations:
$$\begin{aligned}
&\|\sqrt{\alpha}x + (\sqrt{\alpha+\beta} - \sqrt{\alpha})\langle x, y\rangle y\|^2 \\
&= \alpha\|x\|^2 + 2(\sqrt{\alpha^2 + \alpha\beta} - \alpha)|\langle x, y\rangle|^2 + (\sqrt{\alpha+\beta} - \sqrt{\alpha})^2|\langle x, y\rangle|^2\|y\|^2 \\
&= \alpha\|x\|^2 + (2\sqrt{\alpha^2 + \alpha\beta} - 2\alpha + \alpha + \beta - 2\sqrt{\alpha^2 + \alpha\beta} + \alpha)|\langle x, y\rangle|^2 \\
&= \alpha\|x\|^2 + \beta|\langle x, y\rangle|^2.
\end{aligned}$$
Therefore, the equality of the statement is true. □

Theorem 1. *The (α, β)-norm on $\mathcal{L}(\mathcal{E})$ satisfies the following equality:*
$$\|T\|_{\alpha,\beta} = \sup_{\|x\|=1, x\in\mathcal{E}} \|\sqrt{\alpha}Tx + (\sqrt{\alpha+\beta} - \sqrt{\alpha})\langle Tx, x\rangle x\|. \tag{6}$$

Proof. By replacing x with Tx and y with x in relation (5), we obtain the following inequality:
$$\alpha\|Tx\|^2 + \beta|\langle Tx, x\rangle|^2 = \|\sqrt{\alpha}Tx + (\sqrt{\alpha+\beta} - \sqrt{\alpha})\langle Tx, x\rangle x\|^2,$$
when $\|x\| = 1$. Taking the supremum over all unit vectors x in $\mathcal{E}$, we obtain the relation of the statement. □

Corollary 1. *The (α, β)-norm on $\mathcal{L}(\mathcal{E})$ satisfies the following inequality:*
$$\max\{\sqrt{\alpha}\|T\|, \sqrt{\beta}\omega(T)\} \le \|T\|_{\alpha,\beta} \le \sqrt{\alpha+\beta}\,\omega(T) + \sqrt{\alpha}\,p(T). \tag{7}$$

Proof. The first inequality is evident from the definition of the (α,β)-norm. Using the triangle inequality on the right-hand side of the equality in Theorem 1, we deduce that, when $\|x\|=1$,

$$\begin{aligned}\|\sqrt{\alpha}Tx+(\sqrt{\alpha+\beta}-\sqrt{\alpha})\langle Tx,x\rangle x\| &= \|\sqrt{\alpha+\beta}\langle Tx,x\rangle x+\sqrt{\alpha}(Tx-\langle Tx,x\rangle x)\| \\ &\le \sqrt{\alpha+\beta}|\langle Tx,x\rangle|+\sqrt{\alpha}\|Tx-\langle Tx,x\rangle x\| \\ &= \sqrt{\alpha+\beta}|\langle Tx,x\rangle|+\sqrt{\alpha}\sqrt{\|Tx\|^2-|\langle Tx,x\rangle|^2} \\ &\le \sqrt{\alpha+\beta}\omega(T)+\sqrt{\alpha}p(T).\end{aligned}$$

Taking the supremum over all unit vectors x in $\mathcal{E}$, we obtain the required relation. □

Remark 2. *From relations (1) and (7), we deduce*

$$\sqrt{\alpha+\beta}\omega(T)\le\|T\|_{\alpha,\beta}\le\sqrt{\alpha+\beta}\omega(T)+\sqrt{\alpha}p(T). \tag{8}$$

Comparing the upper bounds of inequalities (1) and (8), we notice that any of them can be greater than the other. For $\alpha=\beta=1$ in relation (8), we obtain an estimation for the modified Davis–Wielandt radius of an operator $T\in\mathcal{L}(\mathcal{E})$ [18,19]; thus,

$$\sqrt{2}\omega(T)\le dw^*(T)\le\sqrt{2}\omega(T)+p(T).$$

However, we have $\frac{\|T\|}{2}\le\omega(T)$ and $p(T)\le2\|T\|$ for every $T\in\mathcal{L}(\mathcal{E})$; hence, we deduce that $\omega(T)\le dw^*(T)\le(4+\sqrt{2})\omega(T)$. Therefore, the numerical radius norm is equivalent to the modified Davis–Wielandt radius.

Remark 3. *If we take two real numbers $\alpha,\beta>0$ and $\nu=\frac{\alpha}{\alpha+\beta}$ in Lemma 3, then we obtain*

$$\alpha|\langle x,y\rangle|^2+\beta|\langle x,z\rangle|^2\le\|x\|^2\Big(\max\{\alpha\|y\|^2,\beta\|z\|^2\}+\sqrt{\alpha\beta}|\langle y,z\rangle|\Big). \tag{9}$$

For non-zero vector $x\in\mathcal{E}$, by replacing y with $\dfrac{x}{\|x\|}$ in relation (9), we deduce

$$\alpha\|x\|^2+\beta|\langle x,z\rangle|^2\le\|x\|^2\max\{\alpha,\beta\|z\|^2\}+\sqrt{\alpha\beta}\|x\||\langle x,z\rangle|. \tag{10}$$

By replacing x with Tx and z with x in relation (10), we obtain the following inequality:

$$\alpha\|Tx\|^2+\beta|\langle Tx,x\rangle|^2\le\|Tx\|^2\max\{\alpha,\beta\|x\|^2\}+\sqrt{\alpha\beta}\|Tx\||\langle Tx,x\rangle|.$$

Taking the supremum over all unit vectors x in $\mathcal{E}$, we obtain the following inequality for the (α,β)-norm:

$$\|T\|_{\alpha,\beta}\le\sqrt{\|T\|^2\max\{\alpha,\beta\}+\sqrt{\alpha\beta}\|T\|\omega(T)}. \tag{11}$$

From relation (1), we have the upper bound of $\|T\|_{\alpha,\beta}$ as $\sqrt{\alpha\|T\|^2+\beta\omega^2(T)}$, and from relation (11) the upper bound $\sqrt{\|T\|^2\max\{\alpha,\beta\}+\sqrt{\alpha\beta}\|T\|\omega(T)}$. Comparing the two upper bounds, we find

$$\sqrt{\alpha\|T\|^2+\beta\omega^2(T)}\le\sqrt{\|T\|^2\max\{\alpha,\beta\}+\sqrt{\alpha\beta}\|T\|\omega(T)}, \tag{12}$$

for all $T\in\mathcal{L}(\mathcal{E})$ and $\alpha,\beta\ge0$.

Theorem 2. *Let $T \in \mathcal{L}(\mathcal{E})$ and let $\alpha\beta \neq 0$.*
If we have $\|T\|_{\alpha,\beta} = \sqrt{\|T\|^2 \max\{\alpha,\beta\} + \sqrt{\alpha\beta}\|T\|\omega(T)}$, then T is normaloid.

Proof. Using inequality (12) and obeying condition (i),
$\|T\|_{\alpha,\beta} = \sqrt{\|T\|^2 \max\{\alpha,\beta\} + \sqrt{\alpha\beta}\|T\|\omega(T)}$; then, we deduce

$$\|T\|_{\alpha,\beta} = \sqrt{\alpha\|T\|^2 + \beta\omega^2(T)}.$$

However, taking into account ([2], Theorem 2.3), we obtain the statement. □

Theorem 3. *Let $T \in \mathcal{L}(\mathcal{E})$ and let $\alpha, \beta \geq 0$. The following inequality holds:*

$$\|T\|_{\alpha,\beta} \leq \sqrt{\frac{\alpha+\beta}{3}\|T\|^2 + \frac{2\|T\|}{3}(\alpha\|T\| + \beta\omega(T))}. \tag{13}$$

Proof. Utilizing inequality (4) for both pairs of vectors (x,y) and (x,z) yields

$$|\langle x,y\rangle|^2 \leq \frac{1}{3}\|x\|^2\|y\|^2 + \frac{2}{3}|\langle x,y\rangle|\|x\|\|y\|$$

and

$$|\langle x,z\rangle|^2 \leq \frac{1}{3}\|x\|^2\|z\|^2 + \frac{2}{3}|\langle x,z\rangle|\|x\|\|z\|,$$

for all $x,y,z \in \mathcal{E}$. Multiplying the first inequality by α and the second by β and adding these relations, we obtain

$$\alpha|\langle x,y\rangle|^2 + \beta|\langle x,z\rangle|^2 \leq \frac{\|x\|^2}{3}\left(\alpha\|y\|^2 + \beta\|z\|^2\right) + \frac{2}{3}\|x\|(\alpha|\langle x,y\rangle|\|y\| + \beta|\langle x,z\rangle|\|z\|) \tag{14}$$

for all $x,y,z \in \mathcal{E}$ and for every $\alpha,\beta \geq 0$. For a non-zero vector $x \in \mathcal{E}$, by replacing y with $\frac{x}{\|x\|}$ in relation (14), we deduce

$$\alpha\|x\|^2 + \beta|\langle x,z\rangle|^2 \leq \frac{\|x\|^2}{3}\left(\alpha + \beta\|z\|^2\right) + \frac{2}{3}\|x\|(\alpha\|x\| + \beta|\langle x,z\rangle|\|z\|). \tag{15}$$

By replacing x with Tx and z with x in relation (15), we obtain the following inequality:

$$\begin{aligned}\alpha\|Tx\|^2 + \beta|\langle Tx,x\rangle|^2 &\leq \frac{\|Tx\|^2}{3}\left(\alpha + \beta\|x\|^2\right) + \frac{2}{3}\|Tx\|(\alpha\|Tx\| + \beta|\langle Tx,x\rangle|\|x\|) \\ &\leq \frac{\|T\|^2\|x\|^2}{3}\left(\alpha + \beta\|x\|^2\right) + \frac{2}{3}\|T\|\|x\|\left(\alpha\|T\|\|x\| + \beta\omega(T)\|x\|^2\right).\end{aligned}$$

Taking the supremum over all unit vectors x in $\mathcal{E}$, we obtain the following inequality for the (α,β)-norm:

$$\|T\|^2_{\alpha,\beta} \leq \frac{\alpha+\beta}{3}\|T\|^2 + \frac{2\|T\|}{3}(\alpha\|T\| + \beta\omega(T)).$$

Consequently, the inequality of the statement is true. □

To shift our focus to the new norm of RKHS and its related inequalities, we introduce $(\mathcal{E}_\Lambda, \langle\cdot,\cdot\rangle)$ as an RKHS on a set Λ with an associated norm $\|\cdot\|$. We proceed by introducing the (α,β)-Berezin norm on $\mathcal{L}(\mathcal{E}_\Lambda)$.

Definition 1. *For non-negative real scalars α and β satisfying $(\alpha,\beta)\neq(0,0)$, let us consider the mapping $\|\cdot\|^{\boldsymbol{ber}}_{\alpha,\beta}:\mathcal{L}(\mathcal{E}_\Lambda)\to\mathbb{R}^+$ defined as follows:*

$$\|T\|^{\boldsymbol{ber}}_{\alpha,\beta}=\sup_{\mu\in\Lambda}\sqrt{\alpha\|T\widehat{k}_\mu\|^2+\beta\left|\langle T\widehat{k}_\mu,\widehat{k}_\mu\rangle\right|^2},$$

where $\widehat{k}_\mu$ is the reproducing kernel associated with $\mathcal{E}_\Lambda$.

Next, we provide a specific example to demonstrate how to compute the (α,β)-norm.

Example 1. *Consider the reproducing kernel Hilbert space $\mathbb{C}^2$ on the set $\{1,2\}$ with standard orthonormal basis vectors e_1 and e_2 as kernel functions, defined by $e_i(j)=\delta_{i,j}$ for $i,j\in 1,2$. Here, $\delta_{i,j}$ denotes the Kronecker delta function, which takes two arguments and returns 1 if the arguments are equal, and 0 otherwise. The reproducing kernel for $\mathbb{C}^2$ is simply $\delta_{i,j}$.*

As a finite-dimensional space with dimension 2, $\mathbb{C}^2$ is equipped with the usual Euclidean norm. Let $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ be a 2×2 matrix. The $(1,2)$-Berezin norm of T is equal to

$$\|T\|^{\boldsymbol{ber}}_{1,2}=\sup_{i\in\{1,2\}}\sqrt{\|Te_i\|^2_{\mathbb{C}^2}+2\left|\langle Te_i,e_i\rangle_{\mathbb{C}^2}\right|^2}.$$

We clearly have

$$\|Te_1\|^2_{\mathbb{C}^2}=1\quad\text{and}\quad\|Te_2\|^2_{\mathbb{C}^2}=2.$$

Additionally,

$$\langle Te_1,e_1\rangle=1\quad\text{and}\quad\langle Te_2,e_2\rangle=1.$$

Therefore, we have: $\|T\|^{\boldsymbol{ber}}_{1,2}=2$.

Theorem 4. *The norm $\|T\|_{\widetilde{\boldsymbol{ber}}}$ satisfies the following inequality:*

$$\alpha\|T\|^2_{\boldsymbol{ber}}+\beta c^2_{\boldsymbol{ber}}(T)\leq\left(\max\{\alpha,\beta\}+\sqrt{\alpha\beta}\right)\|T\|^2_{\widetilde{\boldsymbol{ber}}}\tag{16}$$

for all $\alpha,\beta\geq 0$ and $T\in\mathcal{L}(\mathcal{E}_\Lambda)$.

Proof. By replacing x with $T\widehat{k}_\mu$, y with $\widehat{k}_\nu$, and z with $\widehat{k}_\mu$ in relation (9), we deduce

$$\begin{aligned}\alpha\left|\langle T\widehat{k}_\mu,\widehat{k}_\nu\rangle\right|^2+\beta\left|\langle T\widehat{k}_\mu,\widehat{k}_\mu\rangle\right|^2&\leq\|T\widehat{k}_\mu\|^2\left(\max\left\{\alpha\|\widehat{k}_\nu\|^2,\beta\|\widehat{k}_\mu\|^2\right\}+\sqrt{\alpha\beta}\left|\langle\widehat{k}_\nu,\widehat{k}_\mu\rangle\right|\right)\\&\leq\|T\widehat{k}_\mu\|^2\left(\max\left\{\alpha,\beta\right\}+\sqrt{\alpha\beta}\right).\end{aligned}$$

Taking the supremum over all μ,ν in Λ, we obtain the relation of the statement. □

For $T\in\mathcal{L}(\mathcal{E}_\Lambda)$, we define the numerical value of the operator T:

$$p_{\mathbf{ber}}(T)=\sup\left\{\sqrt{\|T\widehat{k}_\mu\|^2-\left|\langle T\widehat{k}_\mu,\widehat{k}_\mu\rangle\right|^2}\ ;\ \widehat{k}_\mu\in\mathcal{E}_\Lambda\right\}.$$

Another result in this paper related to the (α,β)-norm is as follows.

Theorem 5. *The (α,β)-norm on $\mathcal{L}(\mathcal{E}_\Lambda)$ satisfies the following inequality:*

$$\max\{\sqrt{\alpha}\|T\|_{\widetilde{\boldsymbol{ber}}},\sqrt{\beta}\,\boldsymbol{ber}(T)\}\leq\|T\|^{\boldsymbol{ber}}_{\alpha,\beta}\leq\sqrt{\|T\|^2_{\widetilde{\boldsymbol{ber}}}\max\left\{\alpha,\beta\right\}+\sqrt{\alpha\beta}\|T\|_{\widetilde{\boldsymbol{ber}}}\,\boldsymbol{ber}(T)}.\tag{17}$$

Proof. The first inequality is evident from the definition of the (α, β)-norm. For the second inequality, by replacing x with $T\widehat{k}_\mu$ and z with $\widehat{k}_\mu$ in relation (10), we find

$$\alpha\|T\widehat{k}_\mu\|^2 + \beta\big|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\big|^2 \leq \|T\widehat{k}_\mu\|^2 \max\{\alpha, \beta\} + \sqrt{\alpha\beta}\|T\widehat{k}_\mu\|\big|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\big|.$$

Taking the supremum over all μ in Λ, we prove the relation of the statement. □

Theorem 6. *The (α, β)-norm on $\mathcal{L}(\mathcal{E}_\Lambda)$ satisfies the following inequality:*

$$\|T\|^{\boldsymbol{ber}}_{\alpha,\beta} \leq \sqrt{\frac{\alpha+\beta}{3}\|T\|^2_{\widetilde{\boldsymbol{ber}}} + \frac{2\|T\|_{\widetilde{\boldsymbol{ber}}}}{3}\Big(\alpha\|T\|_{\widetilde{\boldsymbol{ber}}} + \beta\,\boldsymbol{ber}(T)\Big)}. \tag{18}$$

Proof. By replacing x with $T\widehat{k}_\mu$ and z with $\widehat{k}_\mu$ in relation (15), we find

$$\alpha\|T\widehat{k}_\mu\|^2 + \beta\Big|\Big\langle T\widehat{k}_\mu, \widehat{k}_\mu\Big\rangle\Big|^2 \leq \frac{\|T\widehat{k}_\mu\|^2}{3}\Big(\alpha + \beta\|\widehat{k}_\mu\|^2\Big) + \frac{2}{3}\|T\widehat{k}_\mu\|\Big(\alpha\|T\widehat{k}_\mu\| + \beta\Big|\Big\langle T\widehat{k}_\mu, \widehat{k}_\mu\Big\rangle\Big|\|\widehat{k}_\mu\|\Big).$$

Taking the supremum over all μ in Λ, we prove the relation of the statement. □

Theorem 7. *The (α, β)-norm on $\mathcal{L}(\mathcal{E}_\Lambda)$ satisfies the following equality:*

$$\|T\|^{\boldsymbol{ber}}_{\alpha,\beta} = \sup_{\mu\in\Lambda}\|\sqrt{\alpha}T\widehat{k}_\mu + (\sqrt{\alpha+\beta} - \sqrt{\alpha})\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\widehat{k}_\mu\|. \tag{19}$$

Proof. By replacing x with $T\widehat{k}_\mu$ and y with $\widehat{k}_\mu$ in relation (5), we obtain the following equality:

$$\alpha\|T\widehat{k}_\mu\|^2 + \beta|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 = \|\sqrt{\alpha}T\widehat{k}_\mu + (\sqrt{\alpha+\beta} - \sqrt{\alpha})\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\widehat{k}_\mu\|^2,$$

knowing that $\|\widehat{k}_\mu\| = 1$. Taking the supremum over all μ in Λ, we obtain the relation of the statement. □

Corollary 2. *The (α, β)-norm on $\mathcal{L}(\mathcal{E}_\Lambda)$ satisfies the following inequality:*

$$\sqrt{\alpha+\beta}\,\boldsymbol{ber}(T) \leq \|T\|^{\boldsymbol{ber}}_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\,\boldsymbol{ber}(T) + \sqrt{\alpha}p_{\boldsymbol{ber}}(T). \tag{20}$$

Proof. Using the triangle inequality on the right-hand side of the equality from Theorem 7 and applying the Cauchy–Schwarz inequality, we deduce

$$\begin{aligned}
\alpha\big|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\big|^2 + \beta\big|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\big|^2 &\leq \alpha\|T\widehat{k}_\mu\|^2 + \beta|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 \\
&= \|\sqrt{\alpha}T\widehat{k}_\mu + (\sqrt{\alpha+\beta} - \sqrt{\alpha})\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\widehat{k}_\mu\| \\
&\leq \sqrt{\alpha+\beta}|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle| + \sqrt{\alpha}\|T\widehat{k}_\mu - \langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\widehat{k}_\mu\| \\
&= \sqrt{\alpha+\beta}|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle| + \sqrt{\alpha}\sqrt{\|T\widehat{k}_\mu\|^2 - \big|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\big|^2} \\
&\leq \sqrt{\alpha+\beta}\,\mathbf{ber}(T) + \sqrt{\alpha}p_{\mathbf{ber}}(T).
\end{aligned}$$

Taking the supremum over all μ in Λ, we obtain the relation of the statement. □

To establish the following result, we needed to make use of Buzano's inequality, which is a generalization of the Cauchy–Schwarz inequality and can be found in [16]. The Buzano inequality is stated as follows (see [16]):

Lemma 5. *Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$. Then,*

$$|\langle a, e\rangle\ \langle e, b\rangle| \leq \frac{1}{2}(\|a\|\ \|b\| + |\langle a, b\rangle|).$$

Theorem 8. *Let $T \in \mathcal{L}(\mathcal{E}_\Lambda)$. Then,*

$$\|T\|_{\alpha,\beta}^{\mathbf{ber}} \leq \left\| \alpha T^* T + \frac{\beta}{4}(T^* T + T T^*) \right\|_{\mathbf{ber}} + \frac{\beta}{2}\, \mathbf{ber}(T^2).$$

Proof. Let $\mu \in \Lambda$ and $\widehat{k}_\mu$ be the normalized reproducing kernel of $\mathcal{E}_\Lambda$. Applying Lemma 5 with $a = T\widehat{k}\mu$, $b = T^*\widehat{k}_\mu$, and $e = \widehat{k}_\mu$ and utilizing the arithmetic-geometric mean inequality yields

$$\begin{aligned}
|\langle T\widehat{k}_\mu, \widehat{k}_\mu \rangle|^2 &\leq \frac{1}{2}(\|T\widehat{k}_\mu\|\, \|T^*\widehat{k}_\mu\| + |\langle T^2\widehat{k}_\mu, \widehat{k}_\mu\rangle|) \\
&= \frac{1}{2}|\langle T^2\widehat{k}_\mu, \widehat{k}_\mu\rangle| + \frac{1}{2}\langle T^*T\widehat{k}_\mu, \widehat{k}_\mu\rangle^{1/2}\langle TT^*\widehat{k}_\mu, \widehat{k}_\mu\rangle^{1/2} \\
&\leq \frac{1}{2}|\langle T^2\widehat{k}_\mu, \widehat{k}_\mu\rangle| + \frac{1}{4}(\langle T^*T\widehat{k}_\mu, \widehat{k}_\mu\rangle + \langle TT^*\widehat{k}_\mu, \widehat{k}_\mu\rangle) \\
&= \frac{1}{2}|\langle T^2\widehat{k}_\mu, \widehat{k}_\mu\rangle| + \frac{1}{4}\langle (T^*T + TT^*)\widehat{k}_\mu, \widehat{k}_\mu\rangle \\
&\leq \frac{1}{2}\,\mathbf{ber}(T^2) + \frac{1}{4}\langle (T^*T + TT^*)\widehat{k}_\mu, \widehat{k}_\mu\rangle.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\alpha\|T\widehat{k}_\mu\|^2 + \beta|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 &\leq \frac{\beta}{2}|\langle T^2\widehat{k}_\mu, \widehat{k}_\mu\rangle| + \left\langle \left(\frac{\beta}{4}(T^*T + TT^*) + \alpha T^*T\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \\
&\leq \frac{\beta}{2}\,\mathbf{ber}(T^2) + \mathbf{ber}\left(\frac{\beta}{4}(T^*T + TT^*) + \alpha T^*T\right).
\end{aligned}$$

Since $\frac{\beta}{4}(T^*T + TT^*) + \alpha T^*T \geq 0$, using (3), we obtain

$$\alpha\|T\widehat{k}_\mu\|^2 + \beta|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 \leq \frac{\beta}{2}\,\mathbf{ber}(T^2) + \left\|\frac{\beta}{4}(T^*T + TT^*) + \alpha T^*T\right\|_{\mathbf{ber}}.$$

By computing the supremum of the last inequality over all possible values of μ in the set Λ, we obtain the desired inequality. □

As a consequence, we have the following:

Corollary 3. *Let $T \in \mathcal{L}(\mathcal{E}_\Lambda)$. Then,*

$$\begin{aligned}
\mathbf{ber}^2(T) &\leq \inf_{\alpha,\beta}\left[\frac{1}{\alpha+\beta}\left(\left\|\alpha T^*T + \frac{\beta}{4}(T^*T + TT^*)\right\|_{\mathbf{ber}} + \frac{\beta}{2}\,\mathbf{ber}(T^2)\right)\right] \\
&\leq \frac{1}{2}\,\mathbf{ber}(T^2) + \frac{1}{4}\|T^*T + TT^*\|_{\mathbf{ber}}.
\end{aligned}$$

Proof. The combination of Corollary 2 and Theorem 8 results in the following statement:

$$\mathbf{ber}^2(T) \leq \frac{1}{\alpha+\beta}\left(\left\|\alpha T^*T + \frac{\beta}{4}(T^*T + TT^*)\right\|_{\mathbf{ber}} + \frac{\beta}{2}\,\mathbf{ber}(T^2)\right).$$

The first inequality in Corollary 3 can be obtained by finding the infimum over both β and α. The remaining inequality can then be derived by considering the specific values of $\beta = 1$ and $\alpha = 0$. □

It is well-known that for every $T \in \mathcal{L}(\mathcal{E}_\Lambda)$, we have

$$\|T\|_{\alpha,\beta}^{\mathbf{ber}} \leq \sqrt{\alpha\|T\|_{\widetilde{\mathbf{ber}}}^2 + \beta\,\mathbf{ber}^2(T)}, \tag{21}$$

for every α, β.

The bounds of the (α, β)-berezin norm were the subject of study in the above theorems. Now, we investigate the conditions under which equality holds. Our analysis commences with a theorem that identifies operators T satisfying the equation $\|T\|_{\alpha,\beta}^{\mathbf{ber}} = \sqrt{\alpha\|T\|^2_{\widetilde{\mathbf{ber}}} + \beta\,\mathbf{ber}^2(T)}$.

Theorem 9. *The following conditions are equivalent for $T \in \mathcal{L}(\mathcal{E}_\Lambda)$, assuming that $\alpha\beta \neq 0$:*

(1) $\|T\|_{\alpha,\beta}^{\mathbf{ber}} = \sqrt{\alpha\|T\|^2_{\widetilde{\mathbf{ber}}} + \beta\,\mathbf{ber}^2(T)}$.

(2) *There exists a sequence $\{\mu_j\}$ in Λ such that*

$$\lim_{j\to\infty} \|T\widehat{k}_{\mu_j}\| = \|T\|_{\widetilde{\mathbf{ber}}} \quad \text{and} \quad \lim_{j\to\infty} \left|\langle T\widehat{k}_{\mu_j}, \widehat{k}_{\mu_j}\rangle\right| = \mathbf{ber}(T),$$

where $\widehat{k}_{\mu_j}$ denotes the normalized reproducing kernel of $\mathcal{E}_\Lambda$ evaluated at μ_j.

Proof. "(2) $\Rightarrow$ (1)". Suppose there exists a sequence μ_j in Λ such that the following limits hold:

$$\lim_{j\to\infty} \|T\widehat{k}_{\mu_j}\| = \|T\|_{\widetilde{\mathbf{ber}}} \quad \text{and} \quad \lim_{j\to\infty} \left|\langle T\widehat{k}_{\mu_j}, \widehat{k}_{\mu_j}\rangle\right| = \mathbf{ber}(T). \tag{22}$$

Here, $\widehat{k}_{\mu_j}$ represents the normalized reproducing kernel of $\mathcal{E}_\Lambda$ when evaluated at μ_j. So, by using (22), we observe that

$$\begin{aligned}
\|T\|_{\alpha,\beta}^{\mathbf{ber}} &= \sup_{\mu\in\Lambda} \sqrt{\alpha\|T\widehat{k}_\mu\|^2 + \beta\left|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle\right|^2} \\
&\geq \lim_{j\to\infty} \sqrt{\alpha\|T\widehat{k}_{\mu_j}\|^2 + \beta\left|\langle T\widehat{k}_{\mu_j}, \widehat{k}_{\mu_j}\rangle\right|^2} \\
&= \sqrt{\alpha\|T\|^2_{\widetilde{\mathbf{ber}}} + \beta\,\mathbf{ber}^2(T)}.
\end{aligned}$$

By utilizing (21), we establish the assertion (1), as required.

"(1) $\Rightarrow$ (2)". From the definition of $\|T\|_{\alpha,\beta}^{\mathbf{ber}}$, it follows that there exists a sequence ν_n in Λ such that

$$\|T\|_{\alpha,\beta}^{\mathbf{ber}} = \lim_{n\to\infty} \sqrt{\alpha\|T\widehat{k}_{\nu_n}\|^2 + \beta\left|\langle T\widehat{k}_{\nu_n}, \widehat{k}_{\nu_n}\rangle\right|^2},$$

where $\widehat{k}_{\nu_n}$ denotes the normalized reproducing kernel of $\mathcal{E}_\Lambda$ evaluated at ν_n.

It is evident that both sequences $\{\left|\langle T\widehat{k}_{\nu_n}, \widehat{k}_{\nu_n}\rangle\right|\}$ and $\{\|T\widehat{k}_{\nu_n}\|\}$ are bounded in the set of real numbers. Therefore, there exists a subsequence $\{\nu_{n_j}\}$ of ν_n such that both $\{|\langle T\widehat{k}_{\nu_{n_j}}, \widehat{k}_{\nu_{n_j}}\rangle|\}$ and $\{\|T\widehat{k}_{\nu_{n_j}}\|\}$ converge. Therefore, considering condition (1), we have

$$\begin{aligned}
\alpha\|T\|^2_{\widetilde{\mathbf{ber}}} + \beta\,\mathbf{ber}^2(T) &= \left(\|T\|_{\alpha,\beta}^{\mathbf{ber}}\right)^2 \\
&= \lim_{j\to\infty} \left(\alpha\|T\widehat{k}_{\nu_{n_j}}\|^2 + \beta\left|\langle T\widehat{k}_{\nu_{n_j}}, \widehat{k}_{\nu_{n_j}}\rangle\right|^2\right) \\
&= \alpha \lim_{j\to\infty} \|T\widehat{k}_{\nu_{n_j}}\|^2 + \beta \lim_{j\to\infty} \left|\langle T\widehat{k}_{\nu_{n_j}}, \widehat{k}_{\nu_{n_j}}\rangle\right|^2 \\
&\leq \alpha \lim_{j\to\infty} \|T\widehat{k}_{\nu_{n_j}}\|^2 + \beta\,\mathbf{ber}^2(T) \\
&\leq \alpha\|T\|^2_{\widetilde{\mathbf{ber}}} + \beta\,\mathbf{ber}^2(T).
\end{aligned}$$

As a result, we can conclude that

$$\lim_{j\to\infty} \|T\widehat{k}_{\nu_{n_j}}\| = \|T\|^2_{\widetilde{\mathbf{ber}}},$$

and we can similarly demonstrate that

$$\lim_{j\to\infty} \left|\langle T\widehat{k}_{\nu_{n_j}}, \widehat{k}_{\nu_{n_j}}\rangle\right| = \mathbf{ber}(T).$$

Thus, by selecting $\mu_j = \nu_{n_j}$ for all j, we verify the desired assertion (2), and the proof is thereby concluded. □

In the next theorem, we provide a necessary and sufficient condition for the equality of the triangle inequality related to the norm $\|\cdot\|^{\mathbf{ber}}_{\alpha,\beta}$. This theorem is important, as it helped us understand the behavior of operators in the associated normed space and allowed us to determine when the inequality is strict.

Theorem 10. *Let $T, S \in \mathcal{L}(\mathcal{E}_\Lambda)$. Then, the following assertions are equivalent:*

(i) $\|T+S\|^{\mathbf{ber}}_{\alpha,\beta} = \|T\|^{\mathbf{ber}}_{\alpha,\beta} + \|S\|^{\mathbf{ber}}_{\alpha,\beta}$.

(ii) There exists a sequence $\{\mu_n\}$ in Λ such that

$$\lim_{n\to\infty}\Big(\alpha\langle S\widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle + \beta\langle \widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle\Big) = \|T\|^{\mathbf{ber}}_{\alpha,\beta}\|S\|^{\mathbf{ber}}_{\alpha,\beta},$$

where $\widehat{k}_{\mu_n}$ is the normalized reproducing kernel of $\mathcal{E}_\Lambda$ at μ_n for every n.

Proof. Notice first that if $T = 0$ or $S = 0$, then the equivalence between (i) and (ii) is clear. Assume that $T \neq 0$ and $S \neq 0$.

"$(i) \Rightarrow (ii)$". Let $\|T+S\|^{\mathbf{ber}}_{\alpha,\beta} = \|T\|^{\mathbf{ber}}_{\alpha,\beta} + \|S\|^{\mathbf{ber}}_{\alpha,\beta}$. There exists a sequence $\{\mu_n\}$ in Λ such that

$$\lim_{n\to\infty}\Big(\alpha\|(T+S)\widehat{k}_{\mu_n}\|^2 + \beta|\langle (T+S)\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle|^2\Big) = \big(\|T\|^{\mathbf{ber}}_{\alpha,\beta} + \|S\|^{\mathbf{ber}}_{\alpha,\beta}\big)^2. \tag{23}$$

On the other hand, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\alpha\|(T+S)\widehat{k}_{\mu_n}\|^2 + \beta|\langle (T+S)\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle|^2 \\
&= \alpha\|T\widehat{k}_{\mu_n}\|^2 + 2\mathfrak{Re}\Big(\alpha\langle S\widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle\Big) + \alpha\|S\widehat{k}_{\mu_n}\|^2 + \beta|\langle T\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle|^2 \\
&\quad + 2\mathfrak{Re}\Big(\beta\langle \widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle\Big) + \beta|\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle|^2 \\
&= \alpha\|T\widehat{k}_{\mu_n}\|^2 + \beta|\langle T\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle|^2 + \alpha\|S\widehat{k}_{\mu_n}\|^2 + \beta|\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle|^2 \\
&\quad + 2\mathfrak{Re}\Big(\alpha\langle S\widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle + \beta\langle \widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle\Big) \\
&\leq \big(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\big)^2 + \big(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\big)^2 + 2\Big|\alpha\langle S\widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle + \beta\langle \widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle\Big|.
\end{aligned}$$

Moreover, by applying the Cauchy–Schwarz inequality, we observe that

$$\begin{aligned}
&\alpha\|(T+S)\widehat{k}_{\mu_n}\|^2+\beta|\langle(T+S)\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\\
&\le \left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2+\left(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2+2\Big(\alpha\|T\widehat{k}_{\mu_n}\|\,\|S\widehat{k}_{\mu_n}\|+\beta|\langle T\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|\,|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|\Big)\\
&\le \left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2+\left(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2\\
&\qquad+2\sqrt{\alpha\|T\widehat{k}_{\mu_n}\|^2+\beta|\langle T\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2}\,\sqrt{\alpha\|S\widehat{k}_{\mu_n}\|^2+\beta|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2}\\
&\le \left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2+\left(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2+2\|T\|^{\mathbf{ber}}_{\alpha,\beta}\sqrt{\alpha\|S\widehat{k}_{\mu_n}\|^2+\beta|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2}\\
&\le \left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2+\left(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2+2\|T\|^{\mathbf{ber}}_{\alpha,\beta}\,\|S\|^{\mathbf{ber}}_{\alpha,\beta}\\
&=\left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}+\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2.
\end{aligned}$$

Letting n tend to $+\infty$ and using (23) implies that

$$\lim_{n\to\infty}\left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\sqrt{\alpha\|S\widehat{k}_{\mu_n}\|^2+\beta|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2}\right)=\|T\|^{\mathbf{ber}}_{\alpha,\beta}\|S\|^{\mathbf{ber}}_{\alpha,\beta}.$$

This yields

$$\lim_{n\to\infty}\sqrt{\alpha\|S\widehat{k}_{\mu_n}\|^2+\beta|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2}=\|S\|^{\mathbf{ber}}_{\alpha,\beta}.$$

Similarly, we obtain

$$\lim_{n\to\infty}\sqrt{\alpha\|T\widehat{k}_{\mu_n}\|^2+\beta|\langle T\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2}=\|T\|^{\mathbf{ber}}_{\alpha,\beta}. \tag{24}$$

This immediately shows that

$$\lim_{n\to\infty}\left|\alpha\langle S\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle+\beta\langle\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle\right|=\|T\|^{\mathbf{ber}}_{\alpha,\beta}\,\|S\|^{\mathbf{ber}}_{\alpha,\beta}.$$

Thus, the desired assertion is proved.

"$(ii)\Rightarrow(i)$". Suppose that there exists a sequence $\{\mu_n\}$ in Λ such that

$$\lim_{n\to\infty}\left(\alpha\langle S\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle+\beta\langle\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle\right)=\|T\|^{\mathbf{ber}}_{\alpha,\beta}\,\|S\|^{\mathbf{ber}}_{\alpha,\beta}, \tag{25}$$

where $\widehat{k}_{\mu_n}$ is the normalized reproducing kernel of $\mathcal{E}_\Lambda$ at μ_n for every n. By proceeding as above, we prove that for every $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\left|\alpha\langle S\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle+\beta\langle\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle\right|^2\\
&\le\left(\alpha\|T\widehat{k}_{\mu_n}\|^2+\beta|\langle T\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\right)\left(\alpha\|S\widehat{k}_{\mu_n}\|^2+\beta|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\right)\\
&\le\left(\alpha\|T\widehat{k}_{\mu_n}\|^2+\beta|\langle T\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\right)\left(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2\\
&\le\left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2\left(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2.
\end{aligned}$$

Thus, by letting n tend to $+\infty$ in the above inequalities and using (25), we obtain

$$\lim_{n\to\infty}\left(\alpha\|T\widehat{k}_{\mu_n}\|^2+\beta|\langle T\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\right)=\left(\|T\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2. \tag{26}$$

Using similar arguments as above, we infer that

$$\lim_{n\to\infty}\left(\alpha\|S\widehat{k}_{\mu_n}\|^2+\beta|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\right)=\left(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\right)^2. \tag{27}$$

On the other hand, according to (25), we obviously have

$$\lim_{n\to\infty}\mathfrak{Re}\Big(\alpha\langle S\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle+\beta\langle\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle\Big)=\|T\|^{\mathbf{ber}}_{\alpha,\beta}\,\|S\|^{\mathbf{ber}}_{\alpha,\beta}. \tag{28}$$

Hence, an application of (26), (27), and (28) proves that

$$\begin{aligned}\big(\|T\|^{\mathbf{ber}}_{\alpha,\beta}+\|S\|^{\mathbf{ber}}_{\alpha,\beta}\big)^2 &= \lim_{n\to\infty}\Big(\alpha\|T\widehat{k}_{\mu_n}\|^2+\beta|\langle T\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\Big)\\ &\quad+2\lim_{n\to\infty}\mathfrak{Re}\Big(\alpha\langle S\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle+\beta\langle\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle\Big)\\ &\quad+\lim_{n\to\infty}\Big(\alpha\|S\widehat{k}_{\mu_n}\|^2+\beta|\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\Big)\\ &=\lim_{n\to\infty}\Big(\alpha\|(T+S)\widehat{k}_{\mu_n}\|^2+\beta|\langle (T+S)\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle|^2\Big)\\ &\le\big(\|T+S\|^{\mathbf{ber}}_{\alpha,\beta}\big)^2\le\big(\|T\|^{\mathbf{ber}}_{\alpha,\beta}+(\|S\|^{\mathbf{ber}}_{\alpha,\beta}\big)^2.\end{aligned}$$

Therefore, we deduce that $\|T+S\|^{\mathbf{ber}}_{\alpha,\beta}=\|T\|^{\mathbf{ber}}_{\alpha,\beta}+\|S\|^{\mathbf{ber}}_{\alpha,\beta}$ as required. □

As special cases, we derive two important consequences of the theorem. These results highlight the significance of the (α,β)-norm in characterizing the behavior of operators in the associated normed space.

Corollary 4. *Let $T,S\in\mathcal{L}(\mathcal{E}_\Lambda)$. Then, the following assertions are equivalent:*

(i) $\boldsymbol{ber}(T+S)=\boldsymbol{ber}(T)+\boldsymbol{ber}(S)$.

(ii) *There exists a sequence $\{\mu_n\}$ in Λ such that*

$$\lim_{n\to\infty}\langle\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n},\widehat{k}_{\mu_n}\rangle=\boldsymbol{ber}(T)\,\boldsymbol{ber}(S),$$

where $\widehat{k}_{\mu_n}$ is the normalized reproducing kernel of $\mathcal{E}_\Lambda$ at μ_n for every n.

Proof. This follows by letting $(\alpha,\beta)=(0,1)$ in Theorem 10. □

Corollary 5. *Let $T,S\in\mathcal{L}(\mathcal{E}_\Lambda)$. Then, the following assertions are equivalent:*

(i) $\|T+S\|_{\widetilde{\boldsymbol{ber}}}=\|T\|_{\widetilde{\boldsymbol{ber}}}+\|S\|_{\widetilde{\boldsymbol{ber}}}$.

(ii) *There exists a sequence $\{\mu_n\}$ in Λ such that*

$$\lim_{n\to\infty}\langle S\widehat{k}_{\mu_n},T\widehat{k}_{\mu_n}\rangle=\|T\|_{\widetilde{\boldsymbol{ber}}}\,\|S\|_{\widetilde{\boldsymbol{ber}}},$$

where $\widehat{k}_{\mu_n}$ is the normalized reproducing kernel of $\mathcal{E}_\Lambda$ at μ_n for every n.

Proof. This follows by letting $(\alpha,\beta)=(1,0)$ in Theorem 10. □

4. Conclusions

The reproduction of kernel Hilbert spaces (RKHSs) has applications in many fields, including statistics, approximation theory, and group representation theory. It is known that the Berezin number has been studied for the Toeplitz and Hankel operators on Hardy and Bergman spaces. We found more characterizations based on inequalities for the Berezin number and the Berezin norm in a series of papers.

This article aimed to provide new estimates for the (α,β)-norm of a bounded operator, as well as new upper bounds involving the Berezin number and Berezin norm of bounded linear operators on RKHSs.

We presented an alternative formula for the (α,β)-norm on $\mathcal{L}(\mathcal{E}_\Lambda)$, as well as various estimates. Additionally, we provided a characterization of the (α,β)-norm when T is normaloid, and in Theorem 3 we found a new upper bound for the (α,β)-norm of an operator T. We introduced the (α,β)-Berezin norm and studied certain properties that characterize this norm.

In Section 3, we introduced a bounded linear operator $T \in \mathcal{L}(\mathcal{E})$ and provided a new numerical value for it:

$$p(T) = \sup\left\{\sqrt{\|Tx\|^2 - |\langle Tx, x\rangle|^2}\ ;\ x \in \mathcal{E},\ \|x\| = 1\right\}.$$

Next, we aimed to generalize the concept of the (α,β)-norm to bounded linear operators acting on RKHSs $\mathcal{L}(\mathcal{E}_\Lambda)$, which we defined as follows:

$$\|T\|^{\mathbf{ber}}_{\alpha,\beta} = \sup_{\mu\in\Lambda}\sqrt{\alpha\|T\widehat{k}_\mu\|^2 + \beta|\langle T\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2}.$$

We presented a number of results regarding the (α,β)-norm.

Finally, we showed that for $T, S \in \mathcal{L}(\mathcal{E}_\Lambda)$, the following two assertions are equivalent:

(i) $\|T+S\|^{\mathbf{ber}}_{\alpha,\beta} = \|T\|^{\mathbf{ber}}_{\alpha,\beta} + \|S\|^{\mathbf{ber}}_{\alpha,\beta}$.

(ii) There exists a sequence $\{\mu_n\}$ in Λ such that

$$\lim_{n\to\infty}\left(\alpha\langle S\widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle + \beta\langle \widehat{k}_{\mu_n}, T\widehat{k}_{\mu_n}\rangle\langle S\widehat{k}_{\mu_n}, \widehat{k}_{\mu_n}\rangle\right) = \|T\|^{\mathbf{ber}}_{\alpha,\beta}\,\|S\|^{\mathbf{ber}}_{\alpha,\beta},$$

where $\widehat{k}_{\mu_n}$ is the normalized reproducing kernel of $\mathcal{E}_\Lambda$ at μ_n for every n.

This paper introduced ideas that could potentially initiate further research in this field. Subsequently, we plan to explore additional connections of the (α,β)-Berezin norm and investigate its potential for generalization.

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