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New Aspects on the Solvability of a Multidimensional Functional Integral Equation with Multivalued Feedback Control

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Abstract: The current study demonstrates the existence of solutions to a multidimensional functional integral equation with multivalued feedback. We seek solutions for the multidimensional functional problem that is defined, continuous, and bounded on the semi-infinite interval. Our proof is based on the technique associated with measures of noncompactness by a given modulus of continuity in the space in $BC(\mathbb{R}_+)$. Also, some sufficient conditions are investigated to demonstrate the asymptotic stability of the solutions to that multidimensional functional equation. Additionally, we give an example and some particular cases to illustrate our outcomes.

Keywords: multidimensional integral equation; existence results; measure of noncompactness; Darbo fixed-point theorem; multivalued feedback control

MSC: 45G10; 47H09; 45M99

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1. Introduction

Fixed point theorems are an excellent tool [1–3] to discuss the solvability of problems of differential equations and inclusions, which have been investigated in several monographs and literature, especially those with multi-valued boundary conditions (see [4–9] and the references therein).

Gasiński et al. [4] studied nonlinear second-order differential inclusions including the ordinary vector p-Laplacian, a multivalued maximal monotone operator, and nonlinear multivalued boundary conditions. They use a broad, unified framework that considers boundary value problems' evolutionary variational inequalities and gradient systems.

In a separable Banach space, a semi-linear differential inclusion that includes a Caputo fractional derivative was examined by Kamenskii et al. [7]. They used the generalized translation multi-valued operator approach and certain fixed point theorems to demonstrate that this inclusion has a mild solution with a multivalued constraint by using the the multi-valued operators' Kakutani–Ky Fan fixed-point theorem [8].

The focus of the research [6] has been on whether there is a solution to the problem of a hybrid differential inclusion of the second type involving two multi-valued maps. Nonlocal multi-valued integral boundary conditions are also taken into consideration.

El-Sayed et al. [9], investigated a fractional-order nonlinear Riemann–Liouville hybrid delay differential inclusion with a nonlinear set-valued nonlocal integral condition. After establishing the existence and uniqueness of the two set-valued functions, the solutions' continuous dependency on these two sets of selections gives the result in $C(I, \mathbb{R})$.

In this article, we prove the existence and asymptotic stability of solutions for a class of a multidimensional functional equations with multivalued feedback,

$$\begin{aligned} x(\tau) &= f(\tau, x(\varphi(\tau))) \\ &\times g\left(\tau, x(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau, \varsigma, u_1(\varsigma), x(\varsigma))d\varsigma, \dots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau, \varsigma, u_k(\varsigma), x(\varsigma))d\varsigma\right), \end{aligned} \quad (1)$$

$$u_i(\tau) \in V_i(\tau, u_i(\tau), x(\tau)), \qquad i = 1, 2, \dots, k, \ \tau \in J = [0, +\infty) \quad (2)$$

where h_i, $i = 1, 2, \dots, k$ and f satisfy some conditions and u_i, $i = 1, 2, \dots, k$ are the control variables.

The importance of dealing with problems involving control variables is because unforeseen factors continually upset ecosystems in the actual world, which can lead to changes in biological characteristics like survival rates.

The disturbance functions are what we mean by control variables when we refer to them. By developing an appropriate Lyapunov function (Lyapunov functional), Chen in [10] was able to achieve certain averaging criteria for endurance and non-autonomous feedback-controlled LotkaVolterra system's global attractivity. There is a class of feedback-controlled nonlinear functional-integral equations that are asymptotically stable and are globally attractive due to Nasertayoob, using the measure of noncompactness in conjunction with Darbo's fixed point theorem [11]. Moreover, under appropriate circumstances, in [12], the author investigated whether a nonlinear neutral delay population system with feedback control has a positive periodic solution.

In this investigation, we have multivalued feedback control, and we introduce a modified definition for the technique based on the method of measures of noncompactness [13–15] to study the existence of solutions to the multidimensional functional Equation (1) in $BC(\mathbb{R}_+)$. The main tools in our studies are the fixed-point theorem of the Darbo type [16].

J. Banaś successfully used the method connected to a measure of noncompactness in the Banach space $BC(\mathbb{R}_+)$ to discover the existence of asymptotically stable solutions to many integral and quadratic integral equations (see [14,15]). Additionally, see [17–21] for information on the solvability of certain problems in the half-line axis.

The rest of the article is structured as follows: In Section 1. We outline some results and previous work to clarify our motivation and innovation. Section 2 states and demonstrates an existence result for a multidimensional functional equation with multi-delays through a direct application of Darbo's fixed point theorem [16]. Additionally, the asymptotic stability of the solution to our problem is studied. Finally, in Section 3, we will provide an illustration of our main result with an example and discuss some special cases of the studied problem.

In what follows, let us recall the definition of the measure of noncompactness in $BC(\mathbb{R}_+)$. Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}_+)$ and a positive number $T > 0$. For $x \in X$ and $\epsilon > 0$, we denote by $w(x, \epsilon)$ the modulus of continuity of the function x [22],

$$\text{i.e.,} \quad \omega^T(x, \epsilon) = \sup \{ |x(t) - x(s)| : t, s \in [0, T], \ |t - s| \leq \epsilon \},$$

and

$$\omega^T(X, \epsilon) = \sup \{ \omega(x, \epsilon) : x \in X \},$$

$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega(X, \epsilon), \quad \omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

Moreover, we put

$$\beta(X) = \lim_{T \to \infty} \left\{ \sup_{x \in X} \{ \sup\{|x(t)| : t \geq T\} \} \right\}.$$

Finally, let us define the function μ on the family $\mathfrak{m}_{BC(\mathbb{R}_+)}$ with the formula

$$\mu(X) = \omega_0(X) + \beta(X),$$

for a fixed $\epsilon > 0$ and arbitrary $x \in X$.

In the next section, we shall introduce a modified definition of the measure of non-compactness to establish the solvability of the multidimensional problem.

2. Fixed Point Results

Consider the multidimensional functional Equation (1) according to the following criteria:

(i) Let $V_i(\tau, u_i, x) : J \times \mathbb{R}_+ \times \mathbb{R}_+ \to 2^{\mathbb{R}_+}$, $(i = 1, 2, \ldots, k)$ meet the following conditions:

(a) The set $V_i(\tau, u_i, x)$ is a on-empty, closed, and convex subset for all $(\tau, u_i, x) \in J \times \mathbb{R}_+ \times \mathbb{R}_+$.

(b) $V_i(\tau, \cdot, \cdot)$ is upper semicontinuous in $u_i, x \in \mathbb{R}_+$ for each $\tau \in J$.

(c) $V_i(\cdot, u_i(\cdot), x(\cdot))$ is measurable in $\tau \in J$ for each $u_i, x \in \mathbb{R}_+$.

(d) There exist two measurable and bounded functions $\sigma_i : [0, 1] \to \mathbb{R}$, with norm $\|\sigma_i\|$, such that

$$|V_i(\tau, u_i(\tau), x(\tau))| = sup\{|v| : v \in V_i(t, u_i(\tau), x(\tau))\} \leq \sigma_i(\tau), \quad \tau \in J,$$

with $\sigma = \max\{\|\sigma_i\|\}$.

Remark 1. *From assumption (i), we are able to conclude that the set of selections* S_{V_i} $(i = 1, 2, \ldots, k)$ *of the set-valued function* V_i *is nonempty, and there exists a Carathéodory function* $v_i \in V_i$ *(see [23]) which is measurable in* $\tau \in J$, *for all* $u_i, x \in \mathbb{R}$ *and continuous in* $u_i, x \in \mathbb{R}$, *for all* $\tau \in J$

$$|v_i(\tau, u_i(\tau), x(\tau))| \leq \sigma_i(\tau),$$

and satisfies the implicit equation

$$u_i(\tau) \;=\; v_i(\tau, u_i(\tau), x(\tau)), \qquad i = 1, 2, \ldots, k, \; \tau \in J. \tag{3}$$

Therefore, any solution of problem (1)–(3) is a solution of problem (1) and (2).

(ii) φ, ϕ_i, $\psi_i : J \to J$, for $i = 1, 2, \ldots, k.$, are continuous, such that

$$\varphi(\tau),\; \phi_i(\tau),\; \psi_i(\tau) \leq \tau.$$

(iii) $g : J \times \mathbb{R}^k \to \mathbb{R}$ is a a continuous function, and there exists a positive constant l such that

$$|g(\tau, x_1, x_2, \ldots, x_k) - g(\tau, y_1, y_2, \ldots, y_k)| \leq l \sum_{i=1}^{k} |x_i - y_i|,$$

for each $\tau \in \mathbb{R}_+$, and for all $x_i, y_i \in R$. Moreover, the function $t \to g(t, 0, 0, , \ldots, 0)$ belongs to the space $BC(\mathbb{R}_+)$, and we have

$$|g(\tau, x_1, x_2, \ldots, x_k)| \leq l \sum_{i=1}^{k} |x_i| + M, \text{ where } M = \sup_{t \in I} |\{g(t, 0, 0, , \ldots, 0)| : \tau \in \mathbb{R}_+\} < +\infty.$$

(iv) $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exists a positive constant η such that

$$|f(\tau, x) - f(\tau, y)| \leq \eta \; |x - y|,$$

for each $\tau \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}$. Moreover, the function $t \to f(\tau, 0)$ belongs to the space $BC(\mathbb{R}_+)$, and we have

$$|f(\tau,x)| \leq \eta\, |x-y| + N, \text{ where } N = \sup_{\tau\in J} |\{f(\tau,0)| : \tau \in R_+\} < +\infty.$$

(v) $h_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, $(i = 1,2,3,..,k)$ are a Carathéodory function and there are functions $a_i, b_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, that are measurable and bounded, with

$$|h_i(\tau,\varsigma,u_i,x)| \leq a_i(\tau,\varsigma) + b_i(\tau,\varsigma)[|u_i(\varsigma)| + |x(\varsigma)|], \ \tau,\varsigma \in [0,+\infty)$$

and

$$\sup_{\tau\in[0,T]} \int_0^\tau a_i(\tau,\varsigma)\, d\varsigma = a_i(\tau), \quad \lim_{\tau\to\infty} \int_0^\tau a_i(\tau,\varsigma)\, d\varsigma = 0,$$
$$\sup_{\tau\in[0,T]} \int_0^\tau b_i(\tau,\varsigma)\, d\varsigma = b_i(\tau), \quad \lim_{\tau\to\infty} \int_0^\tau b_i(\tau,\varsigma)\, d\varsigma = 0.$$

where $a = \sup\{a_i(\tau) : \tau \in J\}$ and $b = \sup\{b_i(\tau) : \tau \in J\}$.

(vi) There exists $r > 0$, such that

$$\eta\, lkb\, r^3 + (N\, b + n(a + b\,\sigma)l\, k\, r^2 + (n\, M + lkN\,(a + b\,\sigma) - 1)r + NM = 0.$$

Definition 1. *For the modulus of continuity of the function x, in the case of a multidimensional functional equation, we introduce the following relations:*

$$\omega^T(x,\epsilon) = \sup\ \{ \sum_{i=1}^{k} |\, x(\phi_i(t)) - x(\phi_i(s))\,| : \ t,s \in [0,T], \ \ |\, t - s\,| \leq \epsilon \}$$

and

$$\omega^T(X,\epsilon) = \sup\ \{ \omega^T(x,\epsilon) : x \in X \}.$$

Next, let us put

$$\omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X,\epsilon), \quad \omega_0(X) = \lim_{T\to\infty} \omega^T(X,\epsilon).$$

Remark 2. *Assumption (i) gives the result that $\phi_i(t)$ is continuous and $\phi_i(t) \leq t$, for $i = 1,2,3$. From assumptions (ii) and (iii), we obtain*

$$|g(t,x_1,x_2,\ldots,x_k) - g(t,0,0,\ldots,0)| \leq l \sum_{i=1}^{k} |x_i|,$$
$$|g(t,x_1,x_2,\ldots,x_k)| \leq |g(t,0,0,\ldots,0)| + l \sum_{i=1}^{k} |x_i|,$$

and

$$|f(t,x) - f(t,0)| \leq \eta|x|$$
$$|f(t,x)| \leq |f(t,0)| + \eta|x|$$

Theorem 1. *Assume (i)–(vi) hold. If*

$$\eta\ \big[\, M + l\, k\, r[a + b\,(\sigma + r)]\big] + l\,[N + \eta\, r]\, k\,[a + b\,(\sigma + r)] < 1,$$

then the multidimensional functional problem (1) and (2) has at least one solution $x \in BC(\mathbb{R}_+)$.

Proof. Let the ball B_r be defined by

$$B_r = \{x \in BC(\mathbb{R}_+) : \ \|x\| \leq r\}.$$

Define the operator $\mathbb{F}$ by

$$\begin{aligned}\mathbb{F}x(\tau) &= f(\tau, x(\varphi(\tau))) \\ &\times g\bigg(\tau, x(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau, \varsigma, u_1(\varsigma), x(\varsigma))ds, \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau, \varsigma, u_k(\varsigma), x(\varsigma))d\varsigma\bigg).\end{aligned}$$

Since by (iii) and (iv), functions $f(\tau, x)$ and $g(\tau, x_1, x_2, \ldots, x_k)$ are continuous. Thus, it follows that $\mathbb{F}x \in C(J, \mathbb{R})$.

We will demonstrate that $\mathbb{F}B_r \subset B_r$ exists for some $r > 0$, and then

$$\begin{aligned}
& | \mathbb{F}x(\tau) | \\
= & \bigg| f(\tau, x(\varphi(\tau))) \\
\times & \; g\bigg(\tau, x(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau, \varsigma, u_1(\varsigma), x(\varsigma))d\varsigma, \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau, \varsigma, u_k(\varsigma), x(\varsigma))d\varsigma\bigg)\bigg| \\
\leq & \; |f(\tau, x(\varphi(\tau)))| \\
\times & \; \bigg|g\bigg(\tau, x(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau, \varsigma, u_1(\varsigma), x(\varsigma))d\varsigma, , \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau, \varsigma, u_k(\varsigma), x(\varsigma))d\varsigma\bigg)\bigg| \\
\leq & \; [f(\tau, 0)| + n|x(\varphi(\tau))|] \big[|g(\tau, 0, 0, \ldots, 0)| + l \sum_{i=1}^{k} |x(\phi_i(\tau))| \int_0^{\psi_i(\tau)} |h_i(\tau, \varsigma, u_i(\varsigma), x(\varsigma))| d\varsigma\big] \\
\leq & \; [|f(\tau, 0)| + n|x(\varphi(\tau))|] \big[|g(\tau, 0, 0, \ldots, 0)| \\
+ & \; l \sum_{i=1}^{k} |x(\phi_i(\tau))| \int_0^{\psi_i(\tau)} [a_i(\tau, \varsigma) + b_i(\tau, s)(|u_i(\varsigma)| + |x(\varsigma)|)] d\varsigma\big] \\
\leq & \; [|f(\tau, 0)| + n|x(\varphi(\tau))|] \big[|g(\tau, 0, 0, \ldots, 0)| \\
+ & \; l \sum_{i=1}^{k} |x(\phi_i(\tau))| \int_0^{\tau} [a_i(\tau, \varsigma) + b_i(\tau, \varsigma)(|v_i(\varsigma, u_i, x(\varsigma))| + \|x\|)] d\varsigma\big] \\
\leq & \; [N + n\|x\|] \big[M + l \sum_{i=1}^{k} \|x\| [a_i + b_i(|\sigma_i(\varsigma)| + \|x\|]\big] \\
\leq & \; [N + \eta \, r][M + l \, r \, k(a + b \, (\sigma + \, r))] = r.
\end{aligned}$$

Putting assumption (vi) into consideration, using the estimate above, we deduce that $\mathbb{F}$ makes the ball B_r turn into itself, since there is the solution $r = r_0 > 0$ of the equation

$$\eta \; lkb \, r^3 + (N \, b + n(a + b \, \sigma)l \, k \, r^2 + (n \, M + lkN \, (a + b \, \sigma) - 1)r + NM = 0,$$

with $\eta \left[\, M + l \, k \, r[a + b \, (\sigma + r)]\right] + l \, [N + \eta \, r] \, k \, [a + b \, (\sigma + r)] < 1$.

We now demonstrate that $\mathbb{F}$ is continuous on the ball B_r. To accomplish this, let us fix $\epsilon > 0$ and choose $x, y \in B_r$ such that $\|x - y\| \leq \epsilon$. Next, we obtain for $\tau \in \mathbb{R}_+$

$$
\begin{aligned}
&|\ \mathbb{F}x(\tau) - \mathbb{F}y(\tau)\ | \\
&= \Big| f(\tau, x(\varphi(\tau))) \\
&\times g\Big(\tau, x(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau,\varsigma,u_1,x(\varsigma))d\varsigma, \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau,\varsigma,u_k,x(\varsigma))d\varsigma\Big) \\
&- f(\tau, y(\varphi(\tau))) \\
&\times g\Big(\tau, y(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau,\varsigma,u_1(\varsigma),y(\varsigma))d\varsigma, \ldots, y(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau,\varsigma,u_k(\varsigma),y(s))d\varsigma\Big)\Big| \\
&\leq |f(\tau, x(\varphi(\tau))) - f(\tau, y(\varphi(\tau)))| \\
&\times\ \Big| g\Big(\tau, x(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau,\varsigma,u_1(\varsigma),x(\varsigma))d\varsigma, \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau,\varsigma,u_k(\varsigma),x(\varsigma))d\varsigma\Big)\Big| \\
&+ \Big| f(\tau, y(\varphi(\tau)))| \\
&\times g\Big(\tau, x(\phi_1(\tau) \int_0^{\psi_1(\tau)} h_1(\tau,\varsigma,u_1(\varsigma),x(\varsigma))d\varsigma, \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau,\varsigma,u_k(\varsigma),x(\varsigma))d\varsigma\Big) \\
&- g\Big(\tau, y(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau,\varsigma,u_1(\varsigma),y(\varsigma))d\varsigma, \ldots, y(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau,\varsigma,u_k(\varsigma),y(\varsigma))d\varsigma\Big)\Big| \\
&\leq\ \eta\ |x(\varphi(\tau)) - y(\varphi(\tau))| [\ |g(\tau,0,0,\ldots,0)| + l \sum_{i=1}^{k} |x(\phi_i(\tau))| \int_0^{\psi_i(\tau)} |h_i(\tau,\varsigma,u_i(\varsigma),x(\varsigma))| d\varsigma] \\
&+ l\ [|f(\tau,0)| + \eta |y(\varphi(\tau))|] \\
&\times [\sum_{i=1}^{k} \big| x(\phi_i(\tau)) \int_0^{\psi_i(\tau)} h_i(\tau,\varsigma,u_i(\varsigma),x(\varsigma))d\varsigma - y(\phi_i(\tau))) \int_0^{\psi_i(\tau)} h_i(\tau,\varsigma,u_i(\varsigma),y(\varsigma))d\varsigma\big| \\
&\leq \eta\ |x(\varphi(\tau)) - y(\varphi(\tau))|\ ] \big[(M + l \sum_{i=1}^{k} |x(\phi_i(\tau))|) \int_0^{\psi_i(\tau)} [a_i(\tau,\varsigma) + b_i(\tau,\varsigma)(|u_i(\varsigma)| + |x(\varsigma)|)] d\varsigma\big] \\
&+ l\ [N + \eta \|y\|] \Big[\sum_{i=1}^{k} \big| x(\phi_i(\tau)) - y(\phi_i(\tau)) \big| \int_0^{\psi_i(\tau)} |h_i(\tau,\varsigma,u_i(\varsigma),x(\varsigma))| d\varsigma \\
&+ \sum_{i=1}^{k} |y(\phi_i(\tau))| \int_0^{\psi_i(\tau)} |\ h_i(\tau,\varsigma,u_i(\varsigma),x(\varsigma)) - h_i(\tau,\varsigma,u_i(\varsigma),y(\varsigma))\ | d\varsigma\Big] \\
&\leq \eta\ |x(\varphi(\tau)) - y(\varphi(\tau))|\ ] \\
&\times \big[(M + l\ \sum_{i=1}^{k} |x(\phi_i(\tau)|) \int_0^{\psi_i(\tau)} [a_i(\tau,\varsigma) + b_i(\tau,\varsigma)(|V_i(\tau,u_i(\tau),x(\tau))| + \|x\|)] d\varsigma\big] \\
&+ l\ [N + \eta \|y\|] \Big[\sum_{i=1}^{k} \big| x(\phi_i(\tau)) - y(\phi_i(\tau)) \big| \int_0^{\psi_i(\tau)} [a_i(\tau,\varsigma) + b_i(\tau,\varsigma)(|V_i(\tau,u_i(\tau),x(\tau))| + \|x\|)] d\varsigma \\
&+ \sum_{i=1}^{k} \|y_i\| \int_0^{\tau} |\ h_i(\tau,\varsigma,u_i(\varsigma),x(\varsigma)) - h_i(\tau,,u_i(\varsigma),y(\varsigma))\ | d\varsigma\Big].
\end{aligned}
$$

Take into account the next two cases:

(i^*) Choose $T > 0$ such that for $\tau \geq T$ the given inequalities are true:

$$
r \int_0^{\tau} [a_i(\tau,\varsigma) + b_i(\tau,\varsigma)(\|\sigma_i\| + \|x\|)] d\varsigma \leq \epsilon_1 \ \text{ and } \ r \int_0^{\tau} [a_i(\tau,\varsigma) + b_i(\tau,\varsigma)(\|\sigma_i\| + \|y\|)] d\varsigma \leq \epsilon_2.
$$

Then

$$
\begin{aligned}
&|\ \mathbb{F}x(\tau)-\mathbb{F}y(\tau)\ |\leq \eta\ \|x-y\|\ ]\big[(M+l\,k\,r\int_0^{\tau}[a_i(\tau,\varsigma)+b_i(\tau,\varsigma)(\|\sigma_i\|+\ r)]d\varsigma\big] \\
&+l\ [N+\eta\ r]k\|x-y\|\int_0^{\tau}[a_i(\tau,\varsigma)+b_i(\tau,\varsigma)\ (\|\sigma_i\|+\ r]d\varsigma \\
&+k\,r\int_0^{\tau}\big[(a_i(\tau,\varsigma)+b_i(\tau,\varsigma))[2\|\sigma_i\|+\|x\|+\|y\|]\ \big]d\varsigma\Big] \\
&\leq \big(\eta\ \big[M+lkr[a_i+b_i(\|\sigma_i\|+r)]\big]+l[N+\eta\ r]k[a_i+b_i(\|\sigma_i\|+r]\big)\,\epsilon+\epsilon_1+\epsilon_2\leq \epsilon+\epsilon_1+\epsilon_2.
\end{aligned}
$$

(*ii**) For $\tau \leq T$. Define the function $\omega^T(h_i,\epsilon)$, $(i=1,2,..,k.)$ where, for $\epsilon>0$, we denote

$$
\omega^T(h_i,\epsilon)=\sup\{|h_i(\tau,\varsigma,u_i,x(\varsigma))-h_i(\tau,\varsigma,u_i,y(\varsigma))|:\ \tau,\varsigma\in[0,T],\quad \|x-y\|\leq\epsilon\}.
$$

Note that function h is uniformly continuous; we conclude that $\omega^T(h_i,\epsilon)\to 0$ as $\epsilon\to 0$, $(i=1,2,\dots,k)$. Therefore, using the above estimate in this case, we are able to conclude that

$$
\begin{aligned}
|\ \mathbb{F}x(\tau)-\mathbb{F}y(\tau)\ |\leq \eta\ \|x-y\|\big[\ M+l\ k\ r[a+b\ (\sigma+r)]\big] \\
+l\ [N+\eta\ r]k\Big[\|x-y\|[a+b\ (\sigma+r)]\ +k\ r\ \omega^T(h_i,\epsilon)\Big] \\
\leq \big[\eta\ \big[\ M+l\ k\ r[a+b\ (\sigma+r)]\big]+l\ [N+\eta\ r]\ k\ [a+b\ (\sigma+r)]\big]\epsilon\ \leq\epsilon.
\end{aligned}
$$

Finally, from the two cases (*i**) and (*ii**), considering the previous information, we come to the conclusion that the operator $\mathbb{F}$ continuously maps the ball B_r into itself.

Now take a nonempty subset of B_r, denoted by X. Fix $\epsilon>0$ and choosing $x\in X$ and $\tau_1,\tau_2\in J$ such that $|\tau_2-\tau_1|\leq\epsilon$. Without losing generality, we can suppose that $\tau_1\leq\tau_2$. Then

$$
\begin{aligned}
|\ \mathbb{F}x(\tau_2)-\mathbb{F}x(\tau_1)\ | &= \Big|f(\tau_2,x(\varphi((\tau_2)))\ g(\tau_2,x(\phi_1(\tau_2))\int_0^{\psi_1(\tau_2)}h_1(\tau_2,\varsigma,u_1(\varsigma),x(\varsigma))d\varsigma, \\
&\quad \dots,x(\phi_k(\varsigma)))\int_0^{\psi_k(t_2)}h_k(\tau_2,\varsigma,u_k(\varsigma),x(\varsigma))d\varsigma) \\
&- f(\tau_1,x(\varphi((\tau_1)))\ g(\tau_1,x(\phi_1(\tau_1))\int_0^{\psi_1(\tau_1)}h_1(\tau_1,\varsigma,u_1(\varsigma),x(\varsigma))d\varsigma, \\
&\quad \dots,x(\phi_k(\varsigma)))\int_0^{\psi_k(\tau_1)}h_k(\tau_1,\varsigma,u_k(\varsigma),x(\varsigma))d\varsigma)\Big| \\
&\leq \Big|f(\tau_2,x(\varphi((\tau_2)))\ g(\tau_2,x(\phi_1(\tau_2))\int_0^{\psi_1(\tau_2)}h_1(\tau_2,\varsigma,u_1(\varsigma),x(\varsigma))d\varsigma, \\
&\quad \dots,x(\phi_k(\varsigma)))\int_0^{\psi_k(\tau_2)}h_k(\tau_2,\varsigma,u_k(\varsigma),x(\varsigma))d\varsigma) \\
&- f(\tau_1,x(\varphi((\tau_1))))\ g(\tau_2,x(\phi_1(\tau_2))\int_0^{\psi_1(\tau_2)}h_1(\tau_2,\varsigma,u_1(\varsigma),x(\varsigma))d\varsigma, \\
&\quad \dots,x(\phi_k(\varsigma)))\int_0^{\psi_k(\tau_2)}h_k(\tau_2,\varsigma,u_k(\varsigma),x(\varsigma))d\varsigma)\Big|
\end{aligned}
$$

$$
\begin{aligned}
+ & \Big| f(\tau_1, x(\varphi((\tau_1))) \, g(\tau_2, x(\phi_1(t_2)) \int_0^{\psi_1(\tau_2)} h_1(\tau_2, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ldots, x(\phi_k(\varsigma))) \int_0^{\psi_k(\tau_2)} h_k(\tau_2, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma) \\
- & f(\tau_1, x(\varphi((\tau_1))) \, g(\tau_1, x(\phi_1(\tau_1)) \int_0^{\psi(\tau_1)} h_1(\tau_1, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ldots, x(\phi_k(\varsigma))) \int_0^{\psi_k(\tau_1)} h_k(\tau_1, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma) \Big| \\
\leq & |f(\tau_2, x(\varphi((\tau_2))) - f(\tau_1, x(\varphi((\tau_1)))||g(\tau_2, x(\phi_1(\tau_2)) \int_0^{\psi_1(\tau_2)} h_1(\tau_2, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau_2, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma)| \\
+ & |f(\tau_1, x(\varphi((\tau_1)))| \Big[|g(\tau_2, x(\phi_1(\tau_2)) \int_0^{\psi_1(\tau_2)} h_1(\tau_2, v, u_1(v), x(\varsigma)) ds, \\
& \ldots, x(\phi_k(\tau_2))) \int_0^{\psi_k(\tau_2)} h_k(\tau_2, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma) \\
& - g(\tau_1, x(\phi_1(\tau_1)) \int_0^{\psi_1(\tau_1)} h_1(\tau_1, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ldots, x(\phi_k(\tau_1))) \int_0^{\psi_k(\tau_1)} h_k(\tau_1, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma)| \Big].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
| \mathbb{F}x(\tau_2) - \mathbb{F}x(\tau_1) | \leq & \ [|f(\tau_2, x(\varphi((\tau_2))) - f(\tau_1, x(\varphi((\tau_2)))| + |f(\tau_1, x(\varphi((\tau_2))) - f(\tau_1, x(\varphi((\tau_1)))|] \\
\times & \ [|g(\tau_2, 0, 0, \ldots, 0)| + l \sum_{i=1}^{k} |x(\phi_i(\tau_2))| \int_0^{\psi_i(\tau_2)} |h_i(\tau_2, \varsigma, u_i(\varsigma), x(\varsigma))| d\varsigma] \\
+ & \ [|f(\tau_1, 0)| + \eta |x(\varphi((\tau_1))|] \Big[|g(\tau_2, x(\phi_i((\tau_2))) \int_0^{\psi_1(\tau_2)} h_1(\tau_2, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ \ldots, x(\phi_k(\tau_2))) \int_0^{\psi_k(\tau_2)} h_k(\tau_2, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma \\
- & \ g(\tau_1, x(\phi_i((\tau_2))) \int_0^{\psi_1(\tau_2)} h_1(\tau_2, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ \ldots, x(\phi_k(\tau_2))) \int_0^{\psi_k(\tau_2)} h_k(\tau_2, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma)| \\
+ & \ g(\tau_1, x(\phi_i((\tau_2))) \int_0^{\psi_1(\tau_2)} h_1(\tau_2, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ \ldots, x(\phi_k(\tau_2))) \int_0^{\psi_k(\tau_2)} h_k(\tau_2, (\varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma) \\
- & \ g(\tau_1, x(\phi_i(\tau_1)) \int_0^{\psi_1(\tau_1)} h_1(\tau_1, \varsigma, u_1(\varsigma), x(\varsigma)) d\varsigma, \\
& \ \ldots, x(\phi_k(\tau_1))) \int_0^{\psi_k(\tau_1)} h_k(\tau_1, \varsigma, u_k(\varsigma), x(\varsigma)) d\varsigma)| \Big]
\end{aligned}
$$

Now, using condition (iii)–(v), we obtain

$$
\begin{aligned}
\leq\ & [\theta_f(\delta) + \eta|x(\varphi(\tau_2)) - x(\varphi(\tau_1))|] \\
\times\ & \big[M + l\sum_{i=1}^{k}|x(\phi_i(\tau_2))|\int_0^{\psi_i(\tau_2)}[a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)|(|u_i(\varsigma)| + |x(\varsigma)|)]d\varsigma\big] \\
+\ & [N + \eta|x(\varphi(t_1))|]\,[\theta_g(\delta) + l\sum_{i=1}^{k}[|x(\phi_i((\tau_2)\int_0^{\psi_i(\tau_2)} h_i(\tau_2,\varsigma,u_i(\varsigma),x(\varsigma))d\varsigma \\
-\ & x(\phi_i((\tau_1)\int_0^{\psi_i(\tau_1)} h_i(\tau_1,\varsigma,u_i(\varsigma),x(\varsigma))d\varsigma|] \\
\leq\ & [\theta_f(\delta) + \eta|x(\varphi(\tau_2)) - x(\varphi(\tau_1))|] \\
\times\ & \big[\,M + l\sum_{i=1}^{k}|x(\phi_i(\tau_2))|\int_0^{\psi_i(\tau_2)}[a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|u_i(\varsigma)| + |x(\varsigma))|)]d\varsigma\big]\,[N + \eta|x(\varphi(\tau_1))|] \\
\times\ & \big[\theta_g(\delta) + l\ \sum_{i=1}^{k}|x(\phi_i(\tau_2))|\int_0^{\psi_i(\tau_2)}[h_i(\tau_2,\varsigma,u_i(\varsigma),x(\varsigma)) - h_i(\tau_1,\varsigma,u_i(\varsigma),x(\varsigma))]d\varsigma \\
+\ & l\ \sum_{i=1}^{k}|x(\phi_i(\tau_2)) - x(\phi_i(\tau_1))|\int_0^{\psi_i(\tau_1)}|h_i(\tau_1,\varsigma,u_i(\varsigma),x(\varsigma))|d\varsigma \\
+\ & l\ \sum_{i=1}^{k}\big|x(\phi_i(\tau_2))\big|\int_{\psi_(\tau_1)}^{\psi_i(\tau_2)}|h_i(\tau_1,\varsigma,u_i(\varsigma),x(\varsigma))|d\varsigma\big].
\end{aligned}
$$

Now, from condition (ii), we have

$$
\begin{aligned}
\leq\ & [\theta_f(\delta) + \eta|x(\varphi(\tau_2)) - x(\varphi(\tau_1))|] \\
\times\ & \big[\,M + l\sum_{i=1}^{k}|x(\phi_i(\tau_2))|\int_0^{\tau_2}[a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|u_i(\varsigma)| + |x(\varsigma))|)]d\varsigma\big] \\
+\ & [N + \eta|x(\varphi(\tau_1))|]\Big[\theta_g(\delta) + l\sum_{i=1}^{k}|x(\phi_i(\tau_2))|\int_0^{\psi_i(\tau_2)}\omega_{h_i}^T(x,\epsilon)\ d\varsigma \\
+\ & l\sum_{i=1}^{k}|x(\phi_i(\tau_2)) - x(\phi_i(\tau_1))|\int_0^{\tau_1}[a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|u_i(\varsigma)| + |x(\varsigma))|)]d\varsigma \\
+\ & l\sum_{i=1}^{k}\big|x(\phi_i(\tau_1))\big|\int_{\tau_1}^{\tau_2}\ [a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|u_i(\varsigma)| + |x(\varsigma))|)]d\varsigma\Big] \\
\leq\ & [\theta_f(\delta) + \eta|x(\varphi(\tau_2)) - x(\varphi(\tau_1))|] \\
\times\ & \big[\,M + l\sum_{i=1}^{k}\|x\|\int_0^{\tau_2}[a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|v_i(\tau,u_i(\varsigma),x(\varsigma)| + \|x\|)]d\varsigma\big] \\
+\ & [N + \eta\|x\|]\,[\theta_g(\delta) + l\ k\ r\ \omega_{h_i}^T(x,\epsilon) \\
+\ & l\ \sum_{i=1}^{k}|x(\phi_i(\tau_2)) - x(\phi_i(\tau_1))|\int_0^{\tau_1}[a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)|(|v_i(\tau,u_i(\varsigma),x(\varsigma)| + \|x\|)]d\varsigma \\
+\ & l\ \sum_{i=1}^{k}\|x\|\int_{\tau_1}^{\tau_2}\ [a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|v_i(\tau,u_i(\varsigma),x(\varsigma)| + \|x\|)]d\varsigma
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad [\theta_f(\delta) + \eta|x(\varphi(\tau_2)) - x(\varphi(\tau_1))|] \\
&\times \quad \Big[\, M + l\sum_{i=1}^{k} \|x\| \int_0^{\tau_2} [a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|\sigma_i(\varsigma)| + \|x\|)]d\varsigma\Big] \\
&+ \quad [N + \eta\|x\|]\big[\theta_g(\delta) + l\,k\, r\omega^T_{h_i}(x,\epsilon) \\
&+ \quad l\sum_{i=1}^{k} |x(\phi_i(\tau_2)) - x(\phi_i(\tau_1))| \int_0^{\tau_1} [a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)|(|\sigma_i(\varsigma)| + \|x\|)]d\varsigma \\
&+ \quad l\sum_{i=1}^{k} \|x\| \int_{\tau_1}^{\tau_2} [a_i(\tau_2,\varsigma) + b_i(\tau_2,\varsigma)(|\sigma_i(\varsigma)| + \|x\|)]d\varsigma \\
&\leq \quad [\theta_f(\delta) + \eta\, \omega^T(x, \omega^T(\varphi,\epsilon))]\,[\, M + l\, r\, k\, [a + b(\sigma + \ r)]] \\
&+ \quad [N + \eta\, r]\big[\theta_g(\delta) + l\sum_{i=1}^{k} r\,\omega^T_{h_i}(x,\epsilon) + l\sum_{i=1}^{k} \omega^T(x, \omega^T(\phi_i,\epsilon))\,[a_i + b_i(\sigma_i + \ r)]\big],
\end{aligned}
$$

where we used

$$
\begin{aligned}
&\theta_f(\delta) = \sup\{|f(\tau_2,x) - f(\tau_1,x)| :\ \tau_1,\tau_2 \in I,\ \tau_1 < \tau_2, |\tau_2 - \tau_1| < \delta,\ |x| \leq r\}, \\
&\theta_g(\delta) = \sup\{|g(\tau_2,x_1,..,x_k) - g(\tau_1,x_1..,x_k)| : \tau_1,\tau_2 \in I,\ \tau_1 < \tau_2, |\tau_2 - \tau_1| < \delta,\ |x| \leq r\}.
\end{aligned}
$$

Therefore, we arrive at the following estimate

$$
\begin{aligned}
\omega^T(\mathbb{F}x,\epsilon) \quad &\leq \quad [\theta_f(\delta) + \eta\, \omega^T(x,\omega^T(\varphi,\epsilon))]\,[M + l\, r\, k[a + b(\sigma + r)]] \\
&+ \quad [N + \eta r]\big[\theta_g(\delta) + l\sum_{i=1}^{k} r\omega^T_{h_i}(x,\epsilon) + l\sum_{i=1}^{k} \omega^T(x,\omega^T(\phi_i,\epsilon))[a_i + b_i(\sigma_i + r)]\big].
\end{aligned}
$$

Consequently, given that f and g have uniform continuity, and from conditions (iii) and (iv), we may say $\theta_g(\delta)\ \theta_f(\delta) \to 0,$ as $\delta \to 0$. Additionally, it is clear that $\omega^T(\phi_i,\epsilon) \to 0,\ (i = 1,2,..,k)$ as $\epsilon \to 0$. Thus,

$$
w^T_0(\mathbb{F}X) \quad \leq \quad \big[\eta\ (\ M + lrk(a + b\ (\sigma + r))) + l\ k\ [N + \eta\ r](a + b\ (\sigma + r))\big]\ w^T_0(X).
$$

Consequently, we obtain

$$
w_0(\mathbb{F}X) \quad \leq \quad \big[\eta\ (\ M + lrk(a + b\ r)) + l\ k\ [N + \eta\ r](a + b\ r)\big]\ w_0(X). \tag{4}
$$

In the following, we take a non-empty set $X \subset B_r$.Then for any $x,\ y \in X$, and fixed $\tau \geq 0$, we obtain

$$
\begin{aligned}
&|\ \mathbb{F}x(\tau) - \mathbb{F}y(\tau)\ | \\
&\leq \eta|x(\varphi(\tau)) - y(\varphi(\tau))||\big[(M + \sum_{i=1}^{k} l\|x\| \int_0^{\tau} [a_i + b_i\|x\|]ds\big] + l[N + \eta\|y\|] \\
&\times \Big[\sum_{i=1}^{k} l\ |x(\phi_i(\tau)) - y(\phi_i(\tau))| \int_0^{\tau} [a_i + b_i\|x\|]d\varsigma \\
&+ \sum_{i=1}^{k} \|y\| \int_0^{\tau} |\ h_i(\tau,\varsigma,u_i(\varsigma),x(\varsigma)) - h_i(\tau,\varsigma,u_i(\varsigma),y(\varsigma))\ |d\varsigma\Big].
\end{aligned}
$$

Hence, it is simple to arrive at the following inequality:

$$
\begin{aligned}
\operatorname{diam}(\mathbb{F}X)(\tau) \leq\ &\eta\ \operatorname{diam}X(\tau)\,[\ M + l\ k\ r|[a + b\ (\sigma + r)]] \\
&+\ l\ k\ [N + \eta\ r]\big[\operatorname{diam}X(\tau)[a + b\ (\sigma + r)]\ + k\ r\omega^T(h,\epsilon)\big].
\end{aligned}
$$

Now, we deduce this estimate:

$$\lim_{\tau\to\infty}\sup\ \mathrm{diam}\mathbb{F}X(\tau)$$
$$\leq\ (\eta\ \big[\ M + l\ k\ r|[a + b\ (\sigma + r)]\big] + l\ k\ [N + \eta\ r][a + b\ (\sigma + r)])\lim_{\tau\to\infty}\sup\ \mathrm{diam}X(\tau)$$

$$\lim_{\tau\to\infty}\sup\ \mathrm{diam}\mathbb{F}X(\tau) \leq c\ \lim_{\tau\to\infty}\sup\ \mathrm{diam}X(\tau), \tag{5}$$

where we indicate

$$c = (\eta\ \big[\ M + l\ k\ r|[a + b\ (\sigma + r)]\big] + l\ k\ [N + \eta\ r][a + b\ (\sigma + r)]).$$

From (vi), it is obvious that $c < 1$.

Finally, we obtain the following inequality by connecting (4) and (5) with the definition of the measure of noncompactness provided in [13]

$$\mu(\mathbb{F}X) \leq c\ \mu(X). \tag{6}$$

Now, taking into account the condition that $c = \eta\ \big[\ M + l\ k\ r[a + b\ (\sigma + r)]\big] + l\ k\ [N + \eta\ r][a + b\ (\sigma + r)]) < 1$ and Darbo's fixed point theorem [16], we conclude that the operator $\mathbb{F}$ has a fixed point x in the ball B_r. Then, x is a solution to the multidimensional problem (1) and (2). □

Asymptotic Stability

Now, we are able to deduce the next result from the evidence of Theorem 1.

Corollary 1. *The solution $x \in BC(R_+)$ to a multidimensional functional Equation (1) with multivalued feedback control (2) is asymptotically stable.*

Proof. Considering that the ball B_r contains the image of the space $BC(\mathbb{R}_+)$ under the operator $\mathbb{F}$, we conclude that B_r contains the set Fix$\mathbb{F}$ of all the fixed points of $\mathbb{F}$. It is evident that all solutions of the multidimensional problem (1) and (2) are included in the set Fix$\mathbb{F}$. On the other hand, we determine that the family $ker\mu$ includes the set Fix$\mathbb{F}$ [22]. Now, consider the description of sets that belong to $ker\mu$ (the kernel $ker\mu$ of this measure includes the nonempty and bounded subsets X of $BC(\mathbb{R}_+)$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$, and the thickness of the bundle they produce reduces to zero at infinity [20]). We obtain the conclusion that all of the multidimensional problems' (1) and (2) solutions are globally and asymptotically stable. □

3. Discussions and Example

We now discuss some particular cases, which are useful for the theory of qualitative analysis of some functional integral equations and important for some models and real problems.

(1) Let $f(\tau, x) = 1,\ \tau \in J$; then, the quadratic multidimensional functional Equation (1) is in the form

$$x(\tau) = g\bigg(\tau, x(\phi_1(\tau))\int_0^{\psi_1(\tau)} h_1(\tau,\varsigma,x(\varsigma))\,d\varsigma, \ldots, x(\phi_k(\tau))\int_0^{\psi_k(\tau)} h_k(\tau,\varsigma,x(\varsigma))\,d\varsigma\bigg), \tag{7}$$

in the absence of the control variable u_i, then, under the assumptions of Theorem 1, the multidimensional functional Equation (7) has at least one asymptotically stable solution $x \in BC(\mathbb{R}_+)$.

(2) Recently, some cubic functional integral equations have received much attention, particularly [24–28]. The investigation of these cubic problems can be considered as extended results obtained for some quadratic integral equations.

Furthermore, the solvability and asymptotic stability for a cubic functional integral equation involving a control variable have been investigated in [29]. Now, letting $f(\tau, x) = x$, then, the multidimensional functional Equation (1) is reduced to a cubic multidimensional functional equation with feedback control

$$\begin{aligned} x(\tau) &= x(\tau) \\ &\times g\left(\tau, x(\phi_1(\tau)) \int_0^{\psi_1(\tau)} h_1(\tau, \varsigma, u_1(\varsigma), x(\varsigma))\, d\varsigma, \ldots, x(\phi_k(\tau))) \int_0^{\psi_k(\tau)} h_k(\tau, \varsigma, u_k(\varsigma), x(\varsigma))\, d\varsigma\right), \end{aligned} \tag{8}$$

$$u_i(\tau) \in V_i(\tau, u_i(\tau), x(\tau)), \qquad i = 1, 2, \ldots, k,\ \tau \in J. \tag{9}$$

The cubic integral inclusion (8) and (9) has at least one asymptotically stable solution, $x \in BC(\mathbb{R}_+)$, according to the hypotheses of Theorem 1.

(3) Let $f(\tau, x) = x(\tau),\ g(\tau, w_1, w_2, \ldots, w_k) = w_1 + w_2 + \ldots + w_k,\ i = 1, 2, \ldots, k,\ \tau \in J,$; then, we have the multi-term cubic functional integral equation

$$\begin{aligned} x(\tau) &= x(\tau). \sum_{i=1}^{k} x(\phi_i(\tau)) \int_0^{\psi_i(\tau)} h_i(\tau, \varsigma, u_i(\varsigma), x(\varsigma))\, d\varsigma, \\ & u_i(\tau) \in V_i(\tau, u_i(\tau), x(\tau)), \end{aligned} \tag{10}$$

involving the control variables u_i. Then, under the assumptions of Theorem 1, the multidimensional functional Equation (10) has at least one asymptotically stable solution $x \in BC(R_+)$. Moreover, when $f(\tau, x) = 1$, we have the multi-term quadratic functional integral equation

$$x(\tau) = \sum_{i=1}^{k} x(\phi_i(\tau)) \int_0^{\psi_i(\tau)} h_i(\tau, \varsigma, u_i(\varsigma), x(\varsigma))\, d\varsigma,$$

with multivalued feedback control

$$u_i(\tau) \in V_i(\tau, u_i(\tau), x(\tau)),\ i = 1, 2, \ldots, k,\ \tau \in J.$$

(4) Let $f(\tau, x) = x(\tau),\ g(\tau, w_1, w_2, \ldots, w_k) = \sum_{i=1}^{k} w_i,\ \psi_i(\tau) = \tau$ and $h_i(\tau, \varsigma, x(\varsigma), u_i(\varsigma)) = \frac{\tau}{\tau+\varsigma} x(\varsigma) u_i(\varsigma)$, $i = 1, 2, \ldots, k,\ \tau \in J,$ then we have the multi-term cubic functional integral equation of the Chandrasekhar type

$$x(\tau) = x(\tau). \sum_{i=1}^{k} x(\phi_i(\tau)) \int_0^{\tau} \frac{\tau}{\tau + \varsigma} x(\varsigma) u_i(\varsigma)\, d\varsigma,\ u_i(\tau) \in V_i(\tau, u_i(\tau), x(\tau)), \tag{11}$$

involving the control variables u_i. Then, under the assumptions of Theorem 1, the multidimensional functional Equation (11) has at least one asymptotically stable solution $x \in BC(\mathbb{R}_+)$. Moreover, when $f(\tau, x) = 1$, we have the multi-term quadratic functional integral equation of the Chandrasekhar type

$$x(\tau) = \sum_{i=1}^{k} x(\phi_i(\tau)) \int_0^{\tau} \frac{\tau}{\tau + \varsigma} x(\varsigma) u_i(\varsigma)\, d\varsigma,$$

with multivalued feedback control

$$u_i(\tau) \in V_i(\tau, u_i(\tau), x(\tau)),\ i = 1, 2, \ldots, k,\ \tau \in J.$$

Example

Consider the next multidimensional functional problem

$$
\begin{aligned}
x(\tau) &= \frac{\tau}{1+\tau^2}\ \arctan(\tau + x(\tau)) \\
&\times \sum_{i=1}^{k} \frac{1}{1+\tau}\ \cos\left(\tau + x_i(\tau)\int_0^\tau [\frac{2\tau - s}{1+\tau^4} + \frac{\tau(|u_i(\varsigma)| + |x_i(\varsigma)|)}{2\pi(\tau^2+1)(\varsigma+1)}]d\varsigma\right), \qquad (12)
\end{aligned}
$$

$$
u_i(\tau) \in 0.2\, u_i(\tau) + \frac{1}{800}e^{-\tau}\sin(\tau) + e^{\frac{-3}{2}\tau}\, x(\tau), \quad \tau \geq 0.
$$

Consider that this multidimensional functional equation is a particular case of Equation (1) with

$$
\begin{aligned}
f(\tau, x(\tau)) &= \frac{\tau}{1+\tau^2}\ \arctan(\tau + x(\tau)), \\
g(t, x_1(\tau), \ldots, x_k(\tau)) &= \sum_{i=1}^{k} \frac{1}{1+t}\ \cos(\tau + x_i(\tau)), \\
h_i(\tau, \varsigma, u_i(\varsigma), x(\varsigma)) &= \frac{2\tau - \varsigma}{1+\tau^4} + \frac{\tau(|u_i(\varsigma)| + |x(\varsigma)|)}{2\pi(\tau^2+1)(\varsigma+1)}.
\end{aligned}
$$

Obviously, the function f is continuous. For any $x, y \in \mathbb{R}$ and $\tau \in [0,1]$

$$
|f(\tau, x(\tau)) - f(\tau, y(\tau))| \leq \frac{1}{2}|x(\tau) - y(\tau)|.
$$

This shows that condition (iv) has been met with $N = \frac{\pi}{4}$ and $\eta = \frac{1}{2}$, where $f(\tau, 0) = \frac{\tau}{1+\tau^2}\ \arctan(\tau)$. However, we also have

$$
\begin{aligned}
|g(\tau, x_1(\tau), \ldots,, x_k(\tau)) - g(\tau, y_1(\tau), \ldots, y_k(\tau))| &\leq \sum_{i=1}^{k} \frac{1}{1+\tau}\ |x_i(\tau) - y_i(\tau)| \\
&\leq \sum_{i=1}^{k} \frac{|x_i(\tau) - y_i(\tau)|}{2}\ ,
\end{aligned}
$$

where $l_i = \frac{1}{2}$ and $g_i(\tau, 0) = \frac{1}{1+\tau}\ \sin(\tau)$ with $M_i = \frac{1}{2}$. Observe further that assumption (v) is satisfied by the function $h_i(\tau, s, u_i, x)$, where

$$
|h_i(\tau, s, u_i(s), x(s))| \leq \frac{2\tau - s}{1+t^4} + \frac{\tau(|u_i(s)| + |x(s)|)}{2\pi(\tau^2+1)(s+1)}.
$$

Consequently, we can put $a_i(\tau, s) = \frac{\tau(2\tau - s)}{2(1+\tau^4)}$ and $b_i(\tau, s) = \frac{\tau}{2\pi(\tau^2+1)(s+1)}$. To confirm assumption (v), observe that

$$
\lim_{\tau\to\infty}\int_0^\tau a_i(\tau, s) = \lim_{\tau\to\infty}\int_0^\tau \frac{\tau(2\tau - s)}{2(1+\tau^4)}ds = \lim_{\tau\to\infty}\frac{3\tau^3}{4\tau^4+4} = 0,
$$

and

$$
\lim_{\tau\to\infty}\int_0^\tau b_i(\tau, s) = \lim_{\tau\to\infty}\int_0^\tau \frac{\tau}{2\pi(\tau^2+1)(s+1)}ds = \lim_{\tau\to\infty}\frac{\tau\ln(\tau+1)}{2\pi\cdot(\tau^2+1)} = 0.
$$

Moreover, we have $a_i = 0.14246919..$ and $b_i = 0.0906987..$

Finally, let us focus on the cubic equation of Theorem 1, which has the following root:

$$
r_1 = -8.65\ \ 81, \quad r_2 = 0.431989, \quad \text{and} \quad r_3 = 4.79223,
$$

it is easily seen that the root $r_0 = 0.831459$ of the previous equation satisfies the inequality

$$\eta\ \big[\ M + l\ k\ r[a + b\ (\sigma + r)]\big] + l\ [N + \eta\ r]\ k\ [a + b\ (\sigma + r)] \simeq 0.048061056 < 1.$$

As a result, all requirements of Theorem 1 are met. Thus, we draw the conclusion that Equation (12) has at least one solution in the space $BC(\mathbb{R}_+)$.

4. Conclusions

Many researchers have investigated the solvability and asymptotic stability for different types of integral equations or systems of integral equations in various classes of functions, for example, [30–32]. The main technique used in these papers is Darbo's fixed point theorem via the Concept of Measure of Noncompactness [17–22,33–35].

Moreover, the existence results of differential and integral equations involving some constraints or control variables have been established and discussed in [10–12,29,36,37].

In [29,38], the existence of asymptotic stable solutions was established in the real half-axis.

In this work, we extend these results and present a comprehensive study of multidimensional functional integral equations that involves multivalued feedback control. This study established the existence and the asymptotic stability of the solutions for (1) and (2) on the real half-line by using the technique associated with a measure of noncompactness. Our discussion is located in the class of bounded continuous functions $BC(\mathbb{R}_+)$. Finally, we introduce some remarks and an illustrative example.

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