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Analysis of Water Infiltration under Impermeable Dams by Analytical and Boundary Element Methods in Complex

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Abstract: The boundary element method (BEM) is used by applying Cauchy's formula to the boundary of the water movement domain under a dam. By approximating the border with a polygon through linear interpolation, the relationships between the complex velocities on each edge of the polygon are analytically deduced. For the case of the flow domain described by a semi- circular closed contours, the numerical values of the velocity are computed and compared with those obtained only analytically. Conclusions on the analytical and numerical context are drawn.

Keywords: potential theory; BEM; complex analysis; estimations

MSC: 65N38; 30-04; 76-05; 31-00; 68W50

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1. Introduction

Analysis of water infiltration under impermeable dams has important applications, and it deals with the study of how water seeps into the ground through permeable channels and how this affects the groundwater recharge and the stormwater management. Water infiltration is the most important way to replenish groundwater in the dam land. Groundwater is a vital source of freshwater for many ecosystems and human activities.

Permeable channels are unlined or lined with flexible materials that allow part of the runoff to infiltrate through their boundaries while conveying the rest of the runoff. Permeable channels can reduce the volume of runoff and lessen the effort needed to control it in the downstream basin. However, water infiltration also affects the hydraulic behavior of the flow in permeable channels, such as the water depth, velocity, and discharge. Therefore, analytic methods are needed to model and predict the infiltration capacity of permeable channels under different flow conditions and channel characteristics.

Cases of planar movement of groundwater are very frequently encountered in practice. They can be solved by exact methods, for simple cases, or through approximation methods, using simplified assumptions and numerical approximations (FEM, BEM). The movement of groundwater is a potential movement, and, if it is also planar, its study can be completed with the help of analytic functions of a complex variable (see [1,2]).

In a general case, for a potential flow, with the velocity potential φ the equation of equipotential lines is $\varphi = const$ and the velocity is $u = \dfrac{\partial \varphi}{\partial x}$, $v = \dfrac{\partial \varphi}{\partial y}$. For a harmonic function φ, the complex potential could be considered through Cauchy-Riemann conditions. Denoting the analytic function that expresses the complex potential with $f(z) = \varphi + i\psi$, then $\dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \psi}{\partial y}$, $\dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$ and it's derivative

$$w(z) = \frac{df}{dz} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - i\frac{\partial \varphi}{\partial y} = u - iv,$$

the complex velocity is expressed, and ψ is the current function with $\psi = const$ the equation of current lines.

Among the classic types of plane fluid flows that can be expressed by a complex potential we can enumerate: uniform flow; stagnation point flow; source and sink; vortex. The complex potential $f(z) = Ue^{-i\alpha}z$ corresponds to uniform flow at speed U in a direction making an angle α with the x-axis. The stream function is $\psi = V(y\cos\alpha - x\sin\alpha)$. Here we are interested in finding the velocity field $\mathbf{v} = (u(x,y), v(x,y)) = V(\cos\alpha, \sin\alpha)$. Furthermore, the complex potential $f(z) = \dfrac{k}{2}z^2$ corresponds to stagnation point flow with strength $k \geq 0$.

For an arbitrary point (a,b) in the complex plane and $c = a + ib$, the complex potential $f(z) = \dfrac{Q}{2\pi}\log(z-c)$ represents a source of strength $Q > 0$ and a sink for $Q < 0$.

A vortex of strength C at the origin is represented by the complex potential $f(z) = \dfrac{-iC}{2\pi}\log z$. This is again a multi-valued function, but we consider the principal form. For $C > 0$, rotation is anticlockwise, and for $C < 0$ rotation is clockwise.

Terzaghi's theory is a classical method for analyzing seepage in earth dams. It involves the assumption of steady-state, two-dimensional flow through an isotropic, homogeneous soil medium. The theory uses Darcy's law and provides a basic understanding of seepage patterns and flow velocities (see [3–7]). In some cases, simplified analytical solutions can be employed to estimate seepage quantities. We can assume a potential flow, meaning that the flow is irrotational and the streamlines are parallel to the dam surface.

The paper is organized as follows. In Section 2, analytical methods for water infiltration under an impermeable dam are discussed, and two cases are highlighted: semicircular and half plane water infiltration zone. The main results are then given in Section 3, in which Cauchy's formula and the polygon decomposition of a curve are used for the analysis of water infiltration under an impermeable Dam. The solution of the system obtained leads to complex potential and complex velocity. At the end of the section, the numerical solution for the semicircular boundary is graphically represented. Section 4 end the paper.

2. Analytical Methods for Water Infiltration under an Impermeable Dam

The mathematical physics governing water infiltration under impermeable dams can be described by Darcy's Law, the Richards and Laplace's equation (see [3–7]). We consider a planar potential flow with x the spatial variable and elevation y.

Darcy's Law relates the flow rate of water through a porous medium to the hydraulic gradient and the hydraulic conductivity. It can be written as: then $q = -kA\nabla h$ or $q = -k(\nabla h + \gamma\nabla y)$, where: q is the volumetric flow rate of water per unit area (debit); K is the hydraulic conductivity of the soil; A is the cross-sectional area; ∇h is the hydraulic gradient in the direction of flow; h is the hydraulic head (water table elevation); and γ is the specific gravity of water.

The Richards Equation extends Darcy's Law by also considering the unsaturated flow of water through porous media. It considers the change in water content and hydraulic head with respect to time and space. The one-dimensional form of the Richards Equation for water infiltration can be expressed as:

$$\frac{\partial\theta}{\partial \mathrm{t}} = \nabla\cdot(k\nabla \mathrm{h}) + \mathrm{S} \text{ or } \frac{\partial\theta}{\partial \mathrm{t}} = \frac{\partial}{\partial \mathrm{y}}[\mathrm{k}(\theta)\frac{\partial \mathrm{h}}{\partial \mathrm{y}}] + \mathrm{S},$$

where: θ is the volumetric water content of the soil and $k(\theta)$ is the unsaturated hydraulic conductivity of the soil, which is a function of θ; S represents source/sink terms, such as precipitation or extraction. This equation describes the movement of water into and through the soil as a function of time, hydraulic conductivity, and the hydraulic head gradient.

The continuity equation ensures the conservation of mass during water flow. It's expression, $\partial\theta/\partial t + \nabla\cdot(\theta q) = 0$, with $(\nabla\cdot)$ the divergence operator. It can be written, in the case of water infiltration, as $\nabla \cdot \mathrm{q} = \nabla \cdot (-k\nabla h) = \mathrm{S}$, where S is the source/sink term representing any external water sources or sinks.

Laplace's equation is derived from combining Darcy's law and the continuity equation. For steady-state conditions without any external sources or sinks $(S = 0)$, it simplifies to $\nabla^2 h = 0$. This equation describes the potential distribution of hydraulic head throughout the soil domain.

Boundary conditions are essential to solving the problem. For impermeable dam surfaces, the hydraulic head at the dam surface is constant and equal to the elevation of the dam crest. Furthermore, the hydraulic head is known or prescribed at the ground surface, often determined by the initial conditions or external factors. Depending on the specific problem, appropriate boundary conditions need to be assigned at the lateral boundaries of the computational domain, such as no flow or a specified hydraulic head. Boundary conditions and initial conditions must be specified to solve the Richards Equation. The boundary conditions typically involve the hydraulic head or water content at the soil surface and at the interface with the impermeable dam. Solving the Richards Equation numerically, often using numerical methods such as finite difference or finite element techniques, allows for the prediction of water infiltration patterns and the assessment of factors such as seepage rates, water table variations, and potential risks to dam stability caused by water infiltration.

A linearized problem of the two-dimensional steady potential flow could be used for analyzing the water infiltration under an impermeable dam using complex analysis and BEM (see [8,9]), considering a closed domain for the water flow. To mathematically describe the potential complex of water infiltration under impermeable dams, we can use the concept of potential theory, in which case the governing equation for the potential function φ and for the stream function ψ, is Laplace's equation:

$$\nabla^2\varphi = \partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 = 0; \nabla^2\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0.$$

Suppose that an impermeable dam separates two accumulations of water, according to Figure 1. In one of the accumulations (the one on the left) upstream, the water rises to the level h_1, and in the other one (the one on the right) downstream, the water rises to the level h_2. The curves AB and CD (dam sole) are waterproof lines. Using the notation q for the debit, the conditions imposed on them are

$$\psi|_{AB} = c,\ c = \text{ const } \ \psi|_{CD} = c + q, \tag{1}$$

usually $c = 0$.

The curves BC and DA are power lines. As the potential calculus (both the complex and the real) is given through its derivatives, we can impose the boundary conditions by adding an arbitrary constant to the velocity potential (same on all the parts of the border). So we will put

$$\varphi|_{DA} = -kh_1,\ \varphi|_{BC} = -kh_2. \tag{2}$$

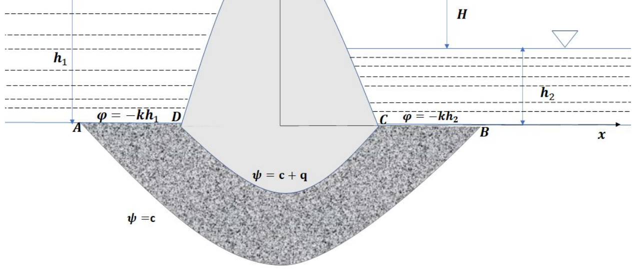

Figure 1. Water infiltration under an impermeable Dam.

We have mentioned that we do not know the value of the constant q in advance, which is why this method is often not used. Finding this constant is particularly important because the debit characterizes the flow of water infiltrated under the dam. Indeed, noting with $\mathbf{n}$ the external normal vector at the boundary, the infiltration rate is

$$\int_A^D \mathbf{v} \cdot (-\mathbf{n})\, ds = -\int_A^D \frac{\partial \varphi}{\partial n} ds = \int_A^D \frac{\partial \psi}{\partial s}\, ds = \psi|_D - \psi|_A = q. \tag{3}$$

In a problem of infiltration, the debit describes the seepage flow, meaning the amount of water that seeps or is lost through soil permeability or through cracks in the dam in this context ([10]). The seepage rate depends on the hydraulic pressure difference and the hydraulic characteristics of the medium through which the water flows, as stated before. If the dam is not impermeable, the infiltration area is considered as $A = LH$ with L length of the dam and the complex potential take the form $f(z) = C\left(e^{i\theta} z + \log z\right)$ with $C = \dfrac{k\gamma H^2}{4i}$ and θ the angle of inclination of the water surface. In Figure 2, the infiltration speed and the seepage flow are depicted.

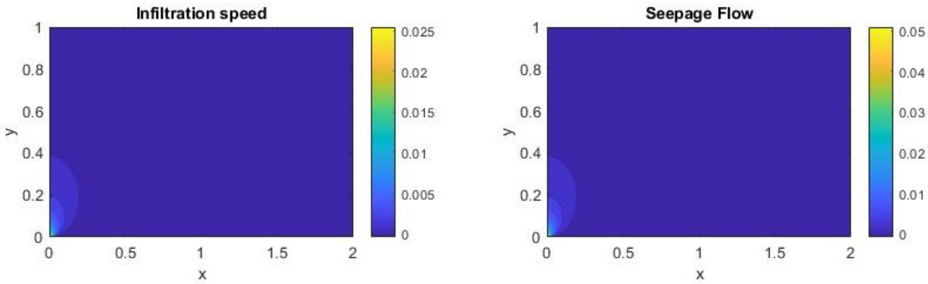

Figure 2. Infiltration speed and seepage flow: L = 2, H = 1, k = 0.1, $\theta = 0$.

In Figures 3 and 4 the potential and the stream function dependence of spatial variables is plotted, in case with inclination of water surface and without.

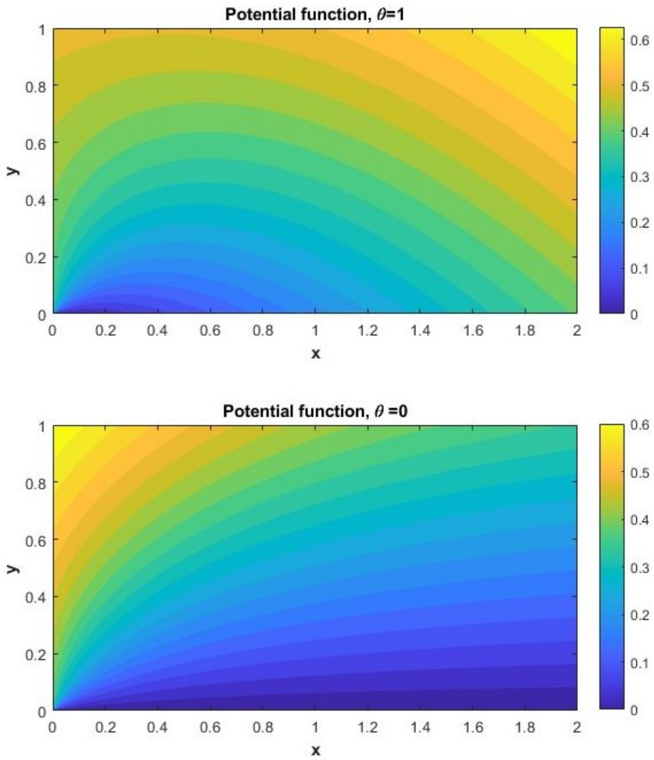

Figure 3. Potential function for different inclination angles (L = 2, H = 1).

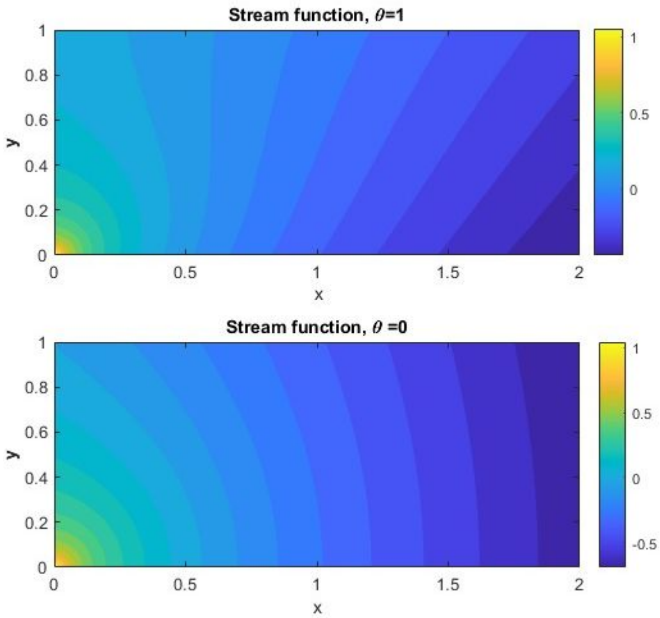

Figure 4. Stream function for different inclination angles (L = 2, H = 1, k = 0.1).

2.1. Semicircular Water Infiltration Zone

We consider now the case in which the flow of the infiltrated fluid is in a semicircular domain.

We assume that the waterproof lines are

$$\begin{aligned} AB : z = &\ R_2 e^{i\theta},\ \theta \in [-\pi, 0], R_2 > R_1, \\ DC : z = &\ R_1 e^{i\theta},\ \theta \in [-\pi, 0], \end{aligned} \tag{4}$$

and the power lines

$$\begin{aligned} AD :&\ y = 0,\ x \in [-R_2, -R_1], \\ CB :&\ y = 0,\ x \in [R_1, R_2]. \end{aligned} \tag{5}$$

Choosing the complex potential under the form

$$f(z) = \frac{k(h_2 - h_1)}{\pi} i \log z + k(h_2 - 2h_1), C = -2k(h_2 - h_1) > 0, \tag{6}$$

using the principal form of the logarithm, we have boundary conditions (2) as well as the conditions

$$\psi|_{DC} = \frac{k(h_2 - h_1)}{\pi} \log R_1,\ \psi|_{AB} = \frac{k(h_2 - h_1)}{\pi} \log R_2. \tag{7}$$

The infiltrated water flow rate is

$$q = \psi|_{DC} - \psi|_{AB} = \frac{k(h_2 - h_1)}{\pi} \log \frac{R_1}{R_2}. \tag{8}$$

The complex velocity is

$$u - iv = -\frac{ikH}{\pi z},\ H = h_1 - h_2. \tag{9}$$

In Figure 5, the velocity field for $h_1 - h_2 = 1, R_2 = 2, R_1 = 1, k = 1$ parameters is presented.

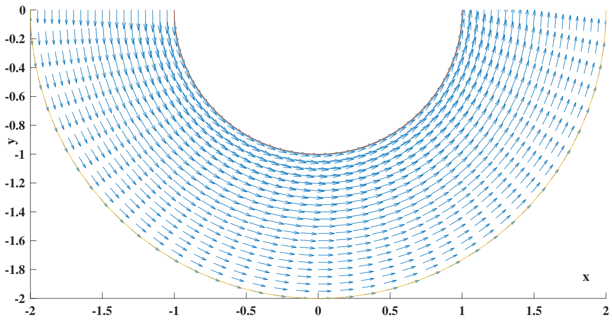

Figure 5. Velocity field for semicircular case.

2.2. The Lower Half Plane Water Infiltration Zone

The infiltrated fluid flows in the half-plane $y < 0$ and the affixes of the points in this half-plane are $z = re^{i\theta},\ 0 < r < \infty,\ -\pi < \theta < 0$.

We assume that the segment $y = 0, x \in (-R, R)$ represents the waterproof sole of the dam, where $\psi = c$ and the half-lines $y = 0, x \in (-\infty, -R)$ and $y = 0, x \in (R, \infty)$ represent power lines, on which $\varphi = -kh_1 + c$ respectively $\varphi = -kh_2 + c$. From where

$$v|_{y\to 0-, x\in[-R,R]} = -\frac{\partial\psi}{\partial x} = 0 \tag{10}$$

$$u|_{y=0, x\in(-\infty,-R)} = u|_{y=0, x\in(R,\infty)} = \frac{\partial\varphi}{\partial x} = 0 \tag{11}$$

because the function $\frac{ia}{\sqrt{z^2 - R^2}}, a > 0$, defined on the inferior half-plane is vanishing at infinity and has the following values on the boundary:

$$\frac{ia}{\sqrt{z^2 - R^2}} = \begin{cases} \frac{ia}{\sqrt{x^2 - R^2}}, y = 0,\ x \in (R, \infty), \\ \frac{-a}{\sqrt{R^2 - x^2}}, y \to 0-,\ x \in (-R, R), \\ \frac{-ia}{\sqrt{x^2 - R^2}}, y = 0,\ x \in (-\infty, -R), \end{cases} \tag{12}$$

we shall consider for the complex velocity the following expression

$$w(z) = u(x, y) - iv(x, y) = \frac{ia}{\sqrt{z^2 - R^2}} \tag{13}$$

from where we obtain the complex potential

$$f(z) = ia \log\left(z + \sqrt{z^2 - R^2}\right) + K, K = k(h_2 - 2h_1) \tag{14}$$

that on the real axis becomes

$$f(z) = \begin{cases} ia \log(x + \sqrt{x^2 - R^2}), y = 0, x \in (R, \infty), \\ ia \log R - \arg\left(x - i\sqrt{R^2 - x^2}\right), y \to 0-, x \in (-R, R), \\ ia \log\left(-x + \sqrt{x^2 - R^2}\right) + a\pi, y = 0, x \in (-\infty, -R). \end{cases} \tag{15}$$

In Figure 6, for water infiltration under a dam with segment $[-R, R]$ as an impermeable base, the velocity field for parameters $h_1 - h_2 = 1, R = 1$ is presented.

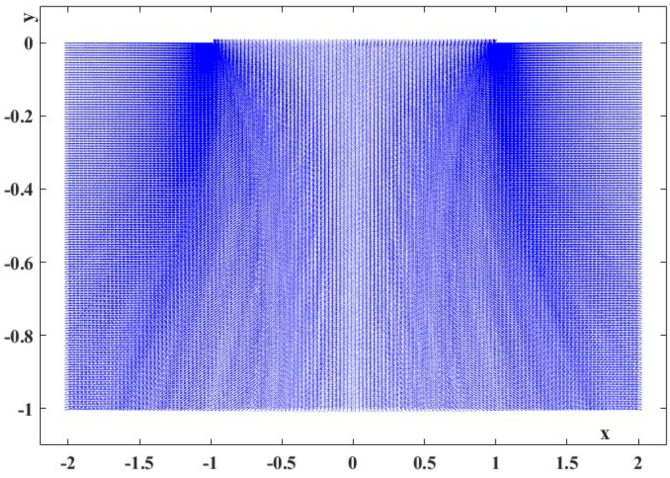

Figure 6. Velocity field for half-plane case.

In the vicinity of the points $z = -R$ and $z = R$ the speed suddenly changes direction and becomes infinite. To visualize this, in Figure 7 the streamlines of the velocity on the dependence of $r \in [0, R_2]$ and $\theta \in [-\pi, 0]$ are plotted, which describe part of the half-plane. Also, for the same case, the potential function and stream function are depicted in in Figure 8.

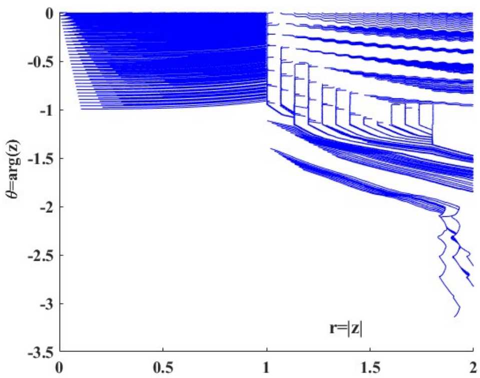

Figure 7. Velocity streamlines for half-plane case.

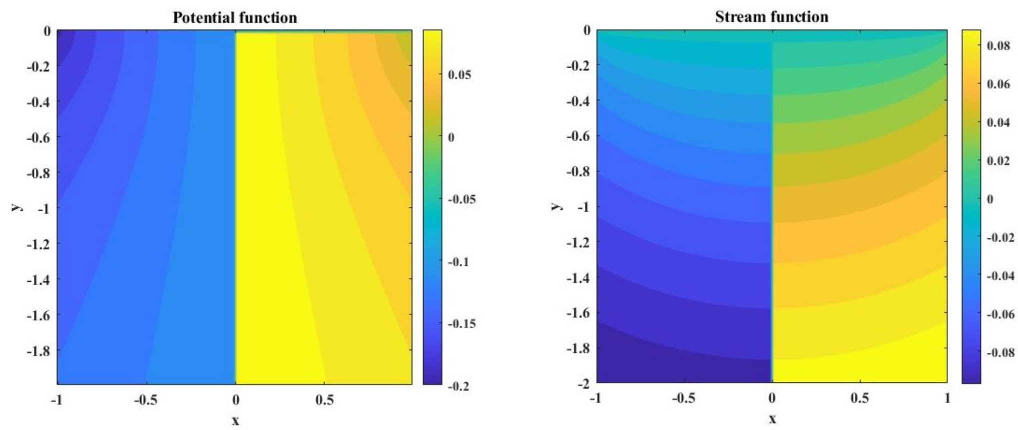

Figure 8. Potential function and stream function for half-plane case.

3. BEM in Complex Analysis

Water infiltration can affect the stability of an impermeable dam. BEM (see [11–15]) can help evaluate the uplift pressure distribution on the dam foundation, which is caused by water infiltration. By modeling the dam and its surrounding soil as boundary elements, BEM can analyze the uplift pressures and assess their impact on the dam's stability. Likewise, the seepage patterns and rates of water infiltration under an impermeable dam could be computed at various locations by discretizing the dam and the surrounding soil region into boundary elements and solving the Dirichlet problem expressed in Figure 1.

For each point z_0 under the boundary $\Gamma = AB \cup BC \cup CD \cup DA$, we shall use the Cauchy's formula, $f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{f(z)}{z-z_0}dz$, $z_0 \in D$, the domain determined by Γ, to compute the value of $f(z_0)$ through the values of the function on the boundary of the domain, Γ.

Within the boundary element method, the boundary Γ is approximated by a polygon consisting of $N = 2n + 2m$ sections $\Gamma_j = [z_j, z_{j+1}]$, $j = 1, 2, \ldots, N$; having the peaks $z_j = x_j + iy_j$, called nodes or control points located on Γ. With linear interpolation for $f(\zeta)$

$$f(z)|_{\Gamma_j} = f(z_j)\frac{z - z_{j+1}}{z_j - z_{j+1}} + f(z_{j+1})\frac{z_j - z}{z_j - z_{j+1}} \tag{16}$$

from where

$$\int_\Gamma \frac{f(z)}{z - z_0}dz = \sum_{j=1}^{N}\left[f(z_j)\int_{\Gamma_j}\frac{z - z_{j+1}}{z_j - z_{j+1}}\frac{dz}{z - z_0} + f(z_{j+1})\int_{\Gamma_j}\frac{z_j - z}{z_j - z_{j+1}}\frac{dz}{z - z_0}\right]. \tag{17}$$

Analytically computing the integrals of (17) results:

$$\begin{aligned} f(z) &= \sum_{j=1}^{N} f(z_j)g_j(z), \\ 2\pi i\, g_j(z) &= \frac{z - z_{j-1}}{z_j - z_{j-1}}\log\left(\frac{z - z_j}{z - z_{j-1}}\right) + \frac{z - z_{j+1}}{z_j - z_{j+1}}\log\left(\frac{z - z_{j+1}}{z - z_j}\right) \end{aligned} \tag{18}$$

relations in which $z_0 \in D$ was renamed with z, that is the variable of functions $(g_j(z))_{j\in\{1,\ldots,N\}}$ and f. For each $z := z_k = x_k + iy_k$, $j = 1, 2, \ldots, N$, we introduce the coefficients $G_{kj} = g_j(z_k)$. These coefficients are calculated using (18) except of $G_{(j-1)j}$, G_{jj}, $G_{(j+1)j}$ which have the expressions:

$$\begin{aligned} 2\pi i G_{(j-1)j} &= 2\pi i \lim_{z\to z_{j-1}} g_j(z) = \frac{z_{j-1} - z_{j+1}}{z_j - z_{j+1}}\log\frac{z_{j-1} - z_{j+1}}{z_{j-1} - z_j}, \\ 2\pi i G_{jj} &= 2\pi i \lim_{z\to z_j} g_j(z) = \log\frac{z_j - z_{j+1}}{z_j - z_{j-1}} \\ 2\pi i G_{(j+1)j} &= 2\pi i \lim_{z\to z_{j+1}} g_j(z) = \frac{z_{j+1} - z_{j-1}}{z_j - z_{j-1}}\log\frac{z_{j+1} - z_j}{z_{j+1} - z_{j-1}}. \end{aligned} \tag{19}$$

From (18) we obtain the algebraic system

$$f(z_k) = \sum_{j=1}^{N} f(z_j)G_{kj}. \tag{20}$$

With notations

$$\begin{aligned} &M_{kj} = \operatorname{Re} G_{kj}, \quad N_{kj} = \operatorname{Im} G_{kj}, \\ &\varphi_k = \operatorname{Re} f(z_k), \quad \psi_k = \operatorname{Im} f(z_k), \end{aligned} \tag{21}$$

separating the real terms from the imaginary ones in (20)

$$\begin{aligned}&\sum_{j=1}^{N}\left(\delta_{kj}-M_{kj}\right)\varphi_j+\sum_{j=1}^{N}N_{kj}\psi_j=0,k=1,2,\ldots,N\\&-\sum_{j=1}^{N}N_{kj}\varphi_j+\sum_{j=1}^{N}\left(\delta_{kj}-M_{kj}\right)\psi_j=0,k=1,2,\ldots,N\end{aligned}\tag{22}$$

with δ_{ij} the Kronocker symbol, that us one for $i=j$.

The equations of the system (22) are not independent, and below we will choose among them accordingly a number of $N=2n+2m$ independent equations. The choosing criterion for the equations is the following: we will look for the elements of the matrix with which the vector of the unknowns is multiplied to have diagonal elements of the form $1-M_{jj}$. Accordingly, the diagonal of the matrix of unknown coefficients will be dominant, and the matrix will be well conditioned. The system becomes

$$\begin{aligned}&\sum_{j=1}^{n}\left(\delta_{lj}-M_{lj}\right)\varphi_j+\sum_{j=n+1}^{n+m}N_{lj}\psi_j-\sum_{j=n+m+1}^{2n+m}M_{lj}\varphi_j+\sum_{j=2n+m+1}^{2n+2m}N_{lj}\psi_j+(\sum_{j=1}^{n}N_{lj})q=\\&=kH\sum_{j=2n+m+1}^{2n+2m}M_{lj},\qquad l=1,\ldots,n\end{aligned}\tag{23}$$

$$\begin{aligned}&-\sum_{j=1}^{n}N_{lj}\varphi_j+\sum_{j=n+1}^{n+m}\left(\delta_{lj}-M_{lj}\right)\psi_j-\sum_{j=n+m+1}^{2n+m}N_{lj}\varphi_j-\sum_{j=2n+m+1}^{2n+2m}M_{lj}\psi_j-(\sum_{j=1}^{n}M_{lj})q=\\&=kH\sum_{j=2n+m+1}^{2n+2m}N_{lj},\qquad l=n+1,\ldots,n+m\end{aligned}\tag{24}$$

$$\begin{aligned}&-\sum_{j=1}^{n}M_{lj}\varphi_j+\sum_{j=n+1}^{n+m}N_{lj}\psi_j+\sum_{j=n+m+1}^{2n+m}\left(\delta_{lj}-M_{lj}\right)\varphi_j+\sum_{j=2n+m+1}^{2n+2m}N_{lj}\psi_j+(\sum_{j=1}^{n}N_{lj})q=\\&=kH\sum_{j=2n+m+1}^{2n+2m}M_{lj},l=n+m+1,\ldots,2n+m\end{aligned}\tag{25}$$

$$\begin{aligned}&-\sum_{j=1}^{n}N_{kj}\varphi_j-\sum_{j=n+1}^{n+m}M_{kj}\psi_j-\sum_{j=n+m+1}^{2n+m}N_{kj}\varphi_j-\sum_{j=2n+m+1}^{2n+2m}\left(\delta_{kj}-M_{kj}\right)\psi_j-(\sum_{j=1}^{n}M_{lj})q=\\&=kH\sum_{j=2n+m+1}^{2n+2m}N_{lj},l=2n+m+1,\ldots,2n+2m.\end{aligned}\tag{26}$$

Using the Cauchy theorem $\int_\Gamma f(z)dz=0$ and based on (16) approximation, one obtain

$$\sum_{j=1}^{2n+2m}\int_{\Gamma_j}\left[f(z_j)\frac{z-z_{j+1}}{z_j-z_{j+1}}+f(z_{j+1})\frac{z_j-z}{z_j-z_{j+1}}\right]dz=0\tag{27}$$

that is

$$\sum_{j=1}^{N}\left(\int_{\Gamma_j}\frac{f(z_{j+1})z_j-f(z_j)z_{j+1}}{z_j-z_{j+1}}dz+\int_{\Gamma_j}\frac{f(z_j)-f(z_{j+1})}{z_j-z_{j+1}}zdz\right)=0$$

and after integration

$$\sum_{j=1}^{N}f(z_j)(z_j-z_{j+1})+f(z_{j+1})(z_j-z_{j+1})=0,z_{N+1}=z_1$$

from where, separating the imaginary part and taking into account the boundary conditions described in Table 1, results

$$\sum_{j=1}^{n}(y_{j+1}-y_{j-1})\varphi_j+\sum_{j=n+1}^{n+m}(x_{j+1}-x_{j-1})\psi_j+\sum_{j=n+m+1}^{2n+m}(y_{j+1}-y_{j-1})\varphi_j+ \\ \sum_{j=2n+m+1}^{2n+2n}(x_{j+1}-x_{j-1})\psi_j+q\sum_{j=1}^{n}(x_{j+1}-x_{j-1})=kH\sum_{j=2n+m+1}^{2n+2n}(y_{j+1}-y_{j-1}) \tag{28}$$

equation that closes the system (23)–(26) in a matriceal form $AX=B$ with the vector of the unknowns expressed by

$$X=[(\varphi_j)_{j=1,\dots,n},(\psi_j)_{j=n+1,\dots,n+m},(\varphi_j)_{j=n+m+1,\dots,2n+m},(\psi_j)_{j=2n+m+1,\dots,2n+2m},q]^t,$$

where

$$\begin{aligned} B_l&=kH\sum_{j=2n+m+1}^{2n+2m}M_{lj},\ l\in\{1,\dots,n\},\\ B_l&=kH\sum_{j=2n+m+1}^{2n+2m}N_{lj},\ l\in\{n+1,\dots,n+m\},\\ B_l&=kH\sum_{j=2n+m+1}^{2n+2m}M_{lj},\ l\in\{n+m+1,\dots,2n+m\},\\ B_l&=kH\sum_{j=2n+m+1}^{2n+2m}N_{lj},l\in\{2n+m+1,\dots,2n+2m\}. \end{aligned} \tag{29}$$

Using (19), that for $j=1$ and $j=N$ becomes

$$2\pi iG_{11}=\log\left(\frac{z_1-z_2}{z_1-z_N}\right),\ 2\pi iG_{NN}=\log\left(\frac{z_N-z_1}{z_N-z_{N-1}}\right) \\ 2\pi iG_{1N}=\frac{z_N-z_2}{z_1-z_2}\log\left(\frac{z_N-z_2}{z_N-z_1}\right),\ 2\pi iG_{N1}=\frac{z_1-z_N}{z_1-z_{N-1}}\log\left(\frac{z_1-z_N}{z_1-z_{N-1}}\right), \tag{30}$$

and $(21)_1$ leads to

$$\begin{aligned} N_{11}&=-\frac{1}{2\pi}\log\left|\frac{z_1-z_2}{z_1-z_N}\right|;M_{11}=\frac{1}{2\pi}\arg\left(\frac{z_1-z_2}{z_1-z_N}\right)\\ N_{jj}&=-\frac{1}{2\pi}\log\left|\frac{z_j-z_{j+1}}{z_j-z_{j-1}}\right|;M_{jj}=\frac{1}{2\pi}\arg\left(\frac{z_j-z_{j+1}}{z_{ji}-z_{j-1}}\right),\ j\in\{2,\dots,N-1\}\\ N_{NN}&=-\frac{1}{2\pi}\log\left|\frac{z_N-z_1}{z_N-z_{N-1}}\right|;M_{NN}=\frac{1}{2\pi}\arg\left(\frac{z_N-z_1}{z_N-z_{N-1}}\right). \end{aligned} \tag{31}$$

Furthermore, for $j\in\{2,\dots,N-1\}$

$$\begin{aligned} G_{(j-1)j}&=\frac{-i}{2\pi}\frac{z_{j-1}-z_{j+1}}{z_j-z_{j+1}}\log\left(\frac{z_{j-1}-z_{j+1}}{z_{j-1}-z_j}\right),\\ G_{(j+1)j}&=\frac{-i}{2\pi}\frac{z_{j+1}-z_{j-1}}{z_j-z_{j-1}}\log\left(\frac{z_{j+1}-z_j}{z_{j+1}-z_{j-1}}\right),\\ G_{kj}&=\frac{-i}{2\pi}\frac{z_k-z_{j-1}}{z_j-z_{j-1}}\log\left(\frac{z_k-z_j}{z_k-z_{j-1}}\right)+\\ &\quad\frac{(-i)}{2\pi}\frac{z_k-z_{j+1}}{z_j-z_{j+1}}\log\left(\frac{z_k-z_{j+1}}{z_k-z_j}\right),k\neq j-1,j,j+1, \end{aligned} \tag{32}$$

further, the complex velocity in a point $z_0 \in D$ with $\Gamma = \partial D$ is determined from

$$w(z_0) = \frac{df}{dz}(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^2}dz \tag{33}$$

and using (16)

$$w(z_0) = \sum_{j=1}^{2n+2m} h_j(z_0)\, f(z_j) \tag{34}$$

with

$$\begin{aligned} h_j(z_0) &= \frac{1}{2\pi i}\int_{z_j}^{z_{j+1}}\left[f(z_j)\frac{z-z_{j+1}}{z_j-z_{j+1}} + f(z_{j+1})\frac{z_j - z}{z_j - z_{j+1}}\right]\frac{dz}{(z-z)^2} = \\ &= \frac{1}{2\pi i}\int_{z_j}^{z_{j+1}}\frac{z-z_{j+1}}{z_j-z_{j+1}}\frac{dz}{(z-z_0)^2} + \frac{1}{2\pi i}\int_{z_{j-1}}^{z_j}\frac{z_{j-1}-z}{z_{j-1}-z_j}\frac{dz}{(z-z_0)^2} = \\ &= \frac{1}{2\pi i}\left(\frac{1}{z_j - z_{j-1}}\log\left(\frac{z_j - z_0}{z_{j-1}-z_0}\right) - \frac{1}{z_{j+1}-z_j}\log\left(\frac{z_{j+1}-z_0}{z_j - z_0}\right)\right). \end{aligned} \tag{35}$$

Table 1. Boundary conditions specifying.

Boundary Γ	**Nodes z_j**	**Unknowns**	**Conditions**
AB: waterproof bed	$j = 1, \ldots, n$	φ_j	$\psi_j = q$
BC: feeding surface	$j = n+1, \ldots, n+m$	ψ_j	$\varphi_j = 0$
CD: bottom dam	$j = n+m+1, \ldots, 2n+m$	φ_j	$\psi_j = 0$
DA: feeding surface	$j = 2n+m+1, \ldots, 2n+2m$	ψ_j	$\varphi_j = -kH$

Semicircular Water Infiltration Zone

For the semicircular boundary, we shall compare the results with those obtained through analytical methods.

As expressed in Table 2, for the nodes in Cartesian coordinates, we have

$$\begin{aligned} &z_j \in AB,\ x_j = R_2\cos\left(\frac{(n+1-j)\pi}{n}\right),\ \ y_j = -R_2\sin\left(\frac{(n+1-j)\pi}{n}\right) \\ &z_j \in BC,\ \ x_j = R_2 - \frac{(R_2-R_1)(j-n-1)}{m},\ y_j = 0 \\ &z_j \in CD,\ x_j = R_1\cos\left(\frac{(j-n-m-1)\pi}{n}\right), y_j = -R_1\sin\left(\frac{(j-n-m-1)\pi}{n}\right) \\ &z_j \in DA,\ \ x_j = -R_1 - \frac{(R_2-R_1)(j-2n-m-1)}{m},\ y_j = 0. \end{aligned} \tag{36}$$

We consider $z_0 \in D$, with $x_0 = r\cos(\theta_0)$, $y_0 = r\sin(\theta_0)$, case in which $R_1 < r = |z_0| < R_2$, $\theta_0 \in [-\pi, 0]$.

Table 2. Boundary parametrization for semicircular case.

Boundary Γ	**Nodes z_j**	**Nodes Values**
AB	$j = 1, \ldots, n$	$z_j = R_2\exp(-i(n+1-j)\pi/n)$
BC	$j = n+1, \ldots, n+m$	$z_j = R_2 - \frac{(R_2-R_1)(j-n-1)}{m}$
CD	$j = n+m+1, \ldots, 2n+m$	$z_j = R_1\exp(-i(j-n-m-1)\pi/n)$
DA	$j = 2n+m+1, \ldots, 2n+2m$	$z_j = -R_1 - \frac{(R_2-R_1)(j-2n-m-1)}{m}$

In Figure 9, the velocity field for parameters $h_1 - h_2 = 1, R_2 = 2, R_1 = 1, k = 1$ in case of a semicircular water infiltration zone, is presented.

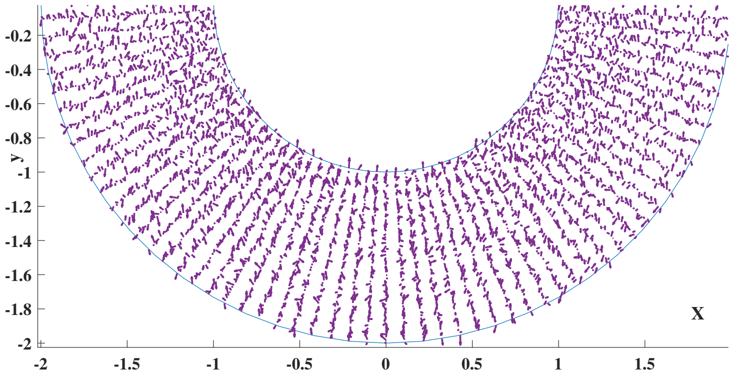

Figure 9. Velocity field for a semicircular water infiltration zone, numerical result.

4. Conclusions

The infiltration process of check dams is complex and is affected by many factors such as rainfall, soil characteristics, land cover, slope, and evapotranspiration. Analytical methods based on potential theory and complex analysis can help to simulate and predict the process of water infiltration from dams, allowing for more precise and effective control than empirical models (see [16]). This can provide a theoretical basis for the optimal use of soil water and the design of permeable channels for stormwater management (see [17]).

Different types of complex potentials expressed in literature were used to simulate the water infiltration, and the quantities derived from knowing the potential were determined using the original MatLab R2022b codes. A detailed determination of complex potential for particular cases, such as semicircular crowns and lower half-plane water infiltration zones were considered. The velocity field for both cases was precised.

Based on the analytical calculus made through the boundary element method in Section 3, it is possible to obtain the numerical values for the complex potential and the stream function as well as for the flow rate (debit constant q) using the (23)–(26) system in the matriceal form. The solution could be used if the boundary Γ of the flow domain is known, such as in the semicircular case. Using the two functions, the complex velocity is analytically obtained through Cauchy's formula.The system could be used for any smooth, closed curve describing the boundary of the flow domain when the parameterization of the curve is known.

If only the correspondences of the points (x_j, y_j) are known through experimental measurements then curve fitting could be used to improve the accuracy of the results more than obtained with the linearization used in (16). The accuracy of the original solution of the Dirichlet problem proposed in Figure 1 obtained in Section 3 can be improved by replacing the linear interpolation, at least with Spline functions whose analytical form can be used. Other studies, such as ([18]), are made using artificial neural networks instead of BEM ([19]).

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