axioms

MDPI

Article

On Modified Interpolative Almost $\mathcal{E}-$Type Contraction in Partial Modular $b-$Metric Spaces

Dilek Kesik [1,†], Abdurrahman Büyükkaya [2,†] and Mahpeyker Öztürk [1,3,*,†]

[1] Department of Mathematics, Faculty of Science, Sakarya University, Sakarya 54000, Turkey; dikekkesik@gmail.com
[2] Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon 61000, Turkey; abdurrahman.giresun@hotmail.com
[3] Sakarya University Technology Developing Zones Manager Co., Sakarya 54000, Turkey
* Correspondence: mahpeykero@sakarya.edu.tr
† These authors contributed equally to this work.

Abstract: The current study attempts to identify a new generalized metric space structure, referred to as partial modular $b-$metric, that extends both partial modular metric space via $b-$metric space and explains the topological aspects of the new space implementing examples. In addition, a new contraction mapping referred to as modified interpolative almost $\mathcal{E}-$type contraction is determined, which is an interpretation of interpolative contraction bestowed with almost contraction and $\mathcal{E}-$contraction as well as a simulation function and a fixed point theorem that encompass such mappings in the context of partial modular $b-$metric space is demonstrated. In conclusion, an example and an application that endorse the main theorem's outcomes are offered.

Keywords: interpolative contraction; simulation function; partial modular $b-$metric space

MSC: 47H10; 54H25; 46A80

check for updates

Citation: Kesik, D.; Büyükkaya, A.; Öztürk, M. On Modified Interpolative Almost $\mathcal{E}-$Type Contraction in Partial Modular $b-$Metric Spaces. *Axioms* **2023**, *12*, 669. https://doi.org/10.3390/axioms12070669

Academic Editors: Kamsing Nonlaopon, Ali Shokri, Daniela Marian and Daniela Inoan

Received: 17 May 2023
Revised: 26 June 2023
Accepted: 5 July 2023
Published: 7 July 2023

1. Introduction

Fixed point theory in metric structure provides a broad occasion for researchers. The most notable outcome of this theory is Banach contraction maps, which were pioneered by S. Banach [1] and have undergone several generalizations and expansions.

The metric function and the metric space structure are effective due to obtaining different generalized distance functions by adding new features to this function, changing some of the metric ones, or both. Numerous experts have achieved new topological structures and brought them to the literature. One of the most basic generalizations of metric space obtained with a general triangle inequality regarding a constant greater or equal than one is the $b-$metric function, which was acquainted by primarily Bakhtin [2] in 1989 and then, mainly, by Czerwik [3,4] in 1993 and 1998.

In 1994, Matthews [5] offered up a partial metric space structure with an aspect of the self-distance of each point in space that may not be zero. Furthermore, this space has various application areas and allows researchers to study in both subsections of mathematics as well as many other fields, such as computer domain and semantics. Mustafa et al. [6] established a new generalized distance function called partial $b-$metric in 2013 by integrating the ideas of partial metric and the $b-$metric function. Subsequently, in 2014, Shukla [7] modified the concept of partial $b-$metric by changing the first definition's last assumption.

Any distance function's physical meaning is fundamentally understandable and easily described. However, in this regard, the idea of modular metric, which Chistyakov [8] suggested in 2010, has a distinct appeal from metric one, and numerous useful discoveries connected to this function have been produced since this year. Following that, the concept of modular metric has been expanded to the notion of modular $b-$metric claimed by Ege and Alaca [9] in 2018.

In addition, Hosseinzadeh and Parvaneh [10] developed an innovative generalized metric function entitled partial modular metric space and provided some new insights on this space, considering modular metric and partial metric functions. However, to remove inconsistency in non-zero self-distance and triangular inequality, Das et al. [11] modified the notion of partial modular metric in 2022.

As a result, more general metric function structures exist in addition to the above structures.

Throughout the research, the symbol $\mathbb{N}$ represents the set of all positive natural numbers, whereas $\mathbb{R}^+$ represents the set of all non-negative real numbers.

2. Preliminaries

In this section, we provide reminders of some fundamental concepts and characteristics that will be useful in the outcome of our work.

Definition 1. *Let $\rho : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a function on a non-empty set $\mathcal{M}$ and $\mathfrak{s} \geq 1$ be a real valued constant. Thereupon, for all $\chi, y, z \in \mathcal{M}$, we listed the below circumstances:*

(ρ_1) $\rho(\chi,\chi) = \rho(\chi, y) = \rho(y, y) \Leftrightarrow \chi = y$;
(ρ_2) $\rho(\chi,\chi) \leq \rho(\chi, y)$;
(ρ_3) $\rho(\chi, y) = \rho(y,\chi)$;
(ρ_4) $\rho(\chi, y) \leq \rho(\chi, z) + \rho(z, y) - \rho(z, z)$;
$(\rho_4{}')$ $\rho(\chi, y) \leq \mathfrak{s}[\rho(\chi, z) + \rho(z, y) - \rho(z, z)] + \frac{1-\mathfrak{s}}{2}(\rho(\chi,\chi) + \rho(y, y))$;
$(\rho_4{}'')$ $\rho(\chi, y) \leq \mathfrak{s}[\rho(\chi, z) + \rho(z, y)] - \rho(z, z)$.

Taking the above axioms into consideration, we conclude that

- *the axioms $(\rho_1 - \rho_4)$ are satisfied $\Rightarrow \rho$ is a partial metric function w.r.t. [5].*
- *the axioms $(\rho_1 - \rho_3, \rho_4{}')$ are satisfied $\Rightarrow \rho$ is a partial $b-$metric function w.r.t. [6].*
- *the axioms $(\rho_1 - \rho_3, \rho_4{}'')$ are satisfied $\Rightarrow \rho$ is a partial $b-$metric function w.r.t. [7].*

Chistyakov [8] identified the modular metric function as the one below.

Definition 2 ([8]). *Let $\omega : (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ be a function defined by $\omega_\lambda(\chi, y) = \omega(\lambda,\chi, y)$ on a non-empty set $\mathcal{M}$. If, for all $\chi, y, z \in \mathcal{M}$, the circumstances*

(ω_1) $\omega_\lambda(\chi, y) = 0, \forall\lambda > 0 \Leftrightarrow \chi = y$;
(ω_2) $\omega_\lambda(\chi, y) = \omega_\lambda(y,\chi), \forall\lambda > 0$;
(ω_3) $\omega_{\lambda+\mu}(\chi, y) \leq \omega_\lambda(\chi, z) + \omega_\mu(z, y), \forall\lambda, \mu > 0$

are fulfilled, then the function ω is called modular metric.

If we consider the below axiom with a constant $\mathfrak{s} \geq 1$ instead of (ω_3), then we achieve that ω is a modular $b-$metric acquainted by Ege and Alaca [9]:

$(\omega_3{}')$ $\omega_{\lambda+\mu}(\chi, y) \leq \mathfrak{s}\big[\omega_\lambda(\chi, z) + \omega_\mu(z, y)\big], \forall\lambda, \mu > 0$.

Further, it seems that the modular $b-$metric space and modular metric space coincide in case of $\mathfrak{s} = 1$, and the sets

$$\mathcal{M}^*_\omega = \{\chi \in \mathcal{M} : \exists\lambda = \lambda(\chi) > 0 \text{ such that } \omega_\lambda(\chi, \chi_0) < \infty\} \ (\chi_0 \in \mathcal{M})$$

are mentioned as an modular $b-$metric space (around χ_0).

Furthermore, for additional details, see [12–15].

Definition 3 ([10]). *Let $\varpi : (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ be a function defined by $\varpi_\lambda(\chi, y) = \varpi(\lambda,\chi, y)$ on a non-empty set $\mathcal{M}$. For all $\chi, y, z \in \mathcal{M}$, if the conditions*

(ϖ_1) $\varpi_\lambda(\chi,\chi) = \varpi_\lambda(\chi, y) = \varpi_\lambda(y, y) \Leftrightarrow \chi = y, \forall\lambda, \mu > 0$;
(ϖ_2) $\varpi_\lambda(\chi,\chi) \leq \varpi_\lambda(\chi, y), \forall\lambda > 0$;
(ϖ_3) $\varpi_\lambda(\chi, y) = \varpi_\lambda(y,\chi), \forall\lambda > 0$;

(ϖ_4) $\varpi_{\lambda+\mu}(\chi, y) \leq \varpi_\lambda(\chi, z) + \varpi_\mu(z, y) - \frac{\varpi_\lambda(\chi,\chi)+\varpi_\lambda(z,z)+\varpi_\mu(z,z)+\varpi_\lambda(y,y)}{2}$, $\forall \lambda, \mu > 0$
are provided, then the function ϖ is termed a partial modular metric.

Subsequently, the notation of the partial modular metric function was redefined by Das et al. [11] in 2022 by considering the below conditions instead of the (ϖ_1) and (ϖ_4) of Definition 3:

($\varpi_1{}'$) $\varpi_\lambda(\chi, y) = \varpi_\mu(\chi, y)$ and $\varpi_\lambda(\chi, \chi) = \varpi_\lambda(\chi, y) = \varpi_\lambda(y, y) \Leftrightarrow \chi = y, \forall \lambda, \mu > 0$.
($\varpi_4{}'$) $\varpi_{\lambda+\mu}(\chi, y) \leq \varpi_\lambda(\chi, z) + \varpi_\mu(z, y) - \varpi_\lambda(z, z), \forall \lambda, \mu > 0$.

We utilize the definition of partial modular metric as defined by Das et al. [11] throughout the study.

On the other hand, researchers have used some auxiliary functions to achieve more diverse results in fixed point theory. In this context, we use the new control functions identified by Khojasteh et al. [16] entitled simulation functions, as noted below.

Definition 4 ([16])**.** *A function $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation function provided that the subsequent requirements are met:*

(ξ_1) $\xi(0, 0) = 0$,
(ξ_2) $\xi(\iota, \nu) < \nu - \iota$ *for all* $\iota, \nu > 0$,
(ξ_3) *if* $\{\iota_\mathfrak{z}\}$, $\{\nu_\mathfrak{z}\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{\mathfrak{z}\to\infty} \iota_\mathfrak{z} = \lim_{\mathfrak{z}\to\infty} \nu_\mathfrak{z} > 0$

$$\limsup_{\mathfrak{z}\to\infty} \xi(\iota_\mathfrak{z}, \nu_\mathfrak{z}) < 0.$$

In the sequel, $\mathcal{Z}$ indicates the collection of all simulation functions. Further, from (ξ_2), it follows that $\xi(\iota, \nu) < 0$ for all $\iota \geq \nu > 0$.

Definition 5 ([16])**.** *A mapping $\mathcal{Y} : \mathcal{M} \to \mathcal{M}$ is referred to as $\mathcal{Z}$-contraction on a metric space $(\mathcal{M}, d)$, according to $\xi \in \mathcal{Z}$ if the inequality*

$$\xi(d(\mathcal{Y}\chi, \mathcal{Y}y), d(\chi, y)) \geq 0$$

is fulfilled, for all $\chi, y \in \mathcal{M}$.

Furthermore, choosing $\xi \in \mathcal{Z}$ as $\xi(\iota, \nu) = q\nu - \iota$ for all $\iota, \nu \in [0, \infty)$, the prominent contraction, entitled Banach contraction, is attained.

Remark 1 ([16])**.** *Presume that $\mathcal{Y}$ is a $\mathcal{Z}$-contraction. For all $\iota \geq \nu > 0$, the statement $\xi(\iota, \nu) < 0$ is true, and also, $d(\mathcal{Y}\chi, \mathcal{Y}y) < d(\chi, y)$. In turn, we deduce that $\mathcal{Z}$-contraction mapping is contractive, and eventually, continuous.*

Moreover, the notation of the simulation function has been generalized in many directions. For some of them, see [17–24].

Let Ψ^* be denoted as the set of all ψ self-mappings on $[1, +\infty)$ such that $\psi(\mathfrak{a}) = 1 \Leftrightarrow \mathfrak{a} = 1$, which own non-decreasing properties.

In 2018 and 2022, S. H. Cho [25,26] identified the specification of simulation functions, as stated beneath.

Definition 6 ([25,26])**.** *Consider η is a real-valued mapping on $[1, \infty)^2$, which satisfies the ensuing circumstances.*

(η_1) $\eta(1, 1) = 1$;
(η_2) $\eta(\iota, \nu) < \nu/\iota, \quad \forall \iota, \nu > 1$;
(η_2') $\eta(\iota, \nu) < \frac{\psi(\nu)}{\psi(\iota)}, \quad \forall \iota, \nu > 1$, *where* $\psi \in \Psi^*$;

(η_3) *for any sequence* $\{\iota_\mathfrak{z}\}, \{\nu_\mathfrak{z}\} \subset (1,\infty)$ *with* $\iota_\mathfrak{z} \leq \nu_\mathfrak{z}$, $\forall \mathfrak{z} = 1,2,3,...$

$$\lim_{\mathfrak{z}\to\infty} \iota_\mathfrak{z} = \lim_{\mathfrak{z}\to\infty} \nu_\mathfrak{z} > 1 \quad \Rightarrow \quad \limsup_{\mathfrak{z}\to\infty} \eta(\iota_\mathfrak{z}, \nu_\mathfrak{z}) < 1.$$

If the requirements $(\eta_1) - (\eta_2) - (\eta_3)$ are met, η is called an $\mathcal{L}$-simulation function. Besides, η is $\mathcal{L}_\psi-$simulation whenever the conditions of $(\eta_1) - (\eta_2') - (\eta_3)$ hold.

Also, note that $\eta(\iota,\iota) < 1$, $\forall \iota > 1$.

$\mathcal{L}$ stands for the class of all $\mathcal{L}$-simulation functions η, and $\mathcal{L}_\psi$ stands for the collection of all $\mathcal{L}_\psi-$simulation functions $\eta : [1,\infty)^2 \to \mathbb{R}$.

Example 1 ([25,26])**.** *The functions $\eta_k, \eta_\phi, \eta_\mathfrak{z} : [1,\infty) \times [1,\infty) \to \mathbb{R}$ that are identified below, belong to $\mathcal{L}_\psi$.*

(*i*) $\eta_k(\iota,\nu) = \frac{[\psi(\nu)]^k}{\psi(\iota)}$, $\forall \iota,\nu > 1; k \in (0,1)$;

(*ii*) $\eta_\phi(\iota,\nu) = \frac{\psi(\nu)}{\psi(\iota)\phi(\psi(\nu))}$, $\forall \iota,\nu \geq 1$, *where $\phi : [1,\infty) \to [1,\infty)$ is non-decreasing and lower semi-continuous self-mapping on $[1,\infty)$, that fulfills $\phi^{-1}(\{1\}) = 1$;*

(*iii*)

$$\eta_\mathfrak{p}(\iota,\nu) = \begin{cases} 1, & \text{if } (\iota,\nu) = (1,1), \\ \frac{\psi(\nu)}{2\psi(\iota)}, & \text{if } \nu < \iota, \\ \frac{[\psi(\nu)]^\lambda}{\psi(\iota)}, & \text{otherwise}, \end{cases}$$

$\forall \iota,\nu \geq 1$, *where $\lambda \in (0,1)$.*

Note that if $\psi(\iota) = \iota$ for $\iota \geq 1$, then $\eta_k, \eta_\phi, \eta_\mathfrak{p} \in \mathcal{L}$.

Jleli and Samet [27] proposed an intriguing idea termed $\mathfrak{O}-$contraction and verified the subsequent theorem in 2014.

Theorem 1 ([27])**.** *The self-mapping $\mathcal{Y}$ on a complete metric space $(\mathcal{M}, d)$, is referred to as $\mathfrak{O}-$contraction, which means that, a constant $k \in (0,1)$ exists such that the expression*

$$d(\mathcal{Y}\chi, \mathcal{Y}y) \neq 0 \quad \Rightarrow \quad \mathfrak{O}(d(\mathcal{Y}\chi, \mathcal{Y}y)) \leq [\mathfrak{O}(d(\chi, y))]^k$$

is provided for each $\chi, y \in \mathcal{M}$, while $\mathfrak{O} : (0,\infty) \to (1,\infty)$ is subject to the succeeding circumstances

$(\mathfrak{O}_1)$ *$\mathfrak{O}$ is non-decreasing;*

$(\mathfrak{O}_2)$ *for each sequence $\{\iota_\mathfrak{z}\} \subset (0,\infty)$, $\lim_{\mathfrak{z}\to\infty} \mathfrak{O}(\iota_\mathfrak{z}) = 1 \Leftrightarrow \lim_{\mathfrak{z}\to\infty} \iota_\mathfrak{z} = 0^+$;*

$(\mathfrak{O}_3)$ *the terms $r \in (0,1)$ and $q \in (0,\infty]$ exist such that $\lim_{\iota\to 0^+} \frac{\mathfrak{O}(\iota)-1}{\iota^r} = q$.*

Thereupon, $\mathcal{Y}$ owns a unique fixed point.

Next, Liu [28] added the following new criteria instead of the condition $(\mathfrak{O}_3)$ and reproved the same theorem under new conditions.

$(\mathfrak{O}_3{}')$ $\mathfrak{O}$ is continuous.

Define $\Theta = \{\mathfrak{O} : (0,\infty) \to (1,\infty) : \ \mathfrak{O} \text{ holds } (\mathfrak{O}_1), \ (\mathfrak{O}_2) \text{ and} (\mathfrak{O}_3{}')\}$.

E. Karapınar [29] (also, improved version [30]) recently proposed a new idea indicated as interpolative contraction and derived a fixed point theorem that included interpolative Kannan-type contraction mapping, as stated below:

Theorem 2 ([29,30])**.** *A self-mapping $\mathcal{Y} : \mathcal{M} \to \mathcal{M}$, on a complete metric space $(\mathcal{M}, d)$, meeting the inequality*

$$d(\mathcal{Y}\chi, \mathcal{Y}y) \leq \delta[d(\chi, \mathcal{Y}\chi)]^\alpha [d(y, \mathcal{Y}y)]^{1-\alpha}$$

with constants $\delta \in [0,1)$ and $\alpha \in (0,1)$, for all $\varkappa, y \in \mathcal{M} - Fix(\mathcal{Y})$, wherein $Fix(\mathcal{Y})$ signifies the set of fixed point of $\mathcal{Y}$, enjoys a unique fixed point.

3. Partial Modular $b-$Metric Spaces and Some Topological Characteristics

In this section, the construction of a new structure using a new distance function and the investigation of various topological properties are proposed.

Definition 7. *Consider a non-empty set $\mathcal{M}$ and a real-valued constant $\mathfrak{s} \geq 1$. A map $\tilde{\omega} : (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ is regarded as a partial modular $b-$metric (briefly $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$) if the subsequent terms are met: for all $\varkappa, y, z \in \mathcal{M}$,*

$(\tilde{\omega}_1)$ $\tilde{\omega}_\lambda(\varkappa,\varkappa) = \tilde{\omega}_\mu(\varkappa,\varkappa)$ *and* $\tilde{\omega}_\lambda(\varkappa,\varkappa) = \tilde{\omega}_\lambda(\varkappa,y) = \tilde{\omega}_\lambda(y,y) \Leftrightarrow \varkappa = y, \forall \lambda,\mu > 0;$
$(\tilde{\omega}_2)$ $\tilde{\omega}_\lambda(\varkappa,\varkappa) \leq \tilde{\omega}_\lambda(\varkappa,y), \forall \lambda > 0;$
$(\tilde{\omega}_3)$ $\tilde{\omega}_\lambda(\varkappa,y) = \tilde{\omega}_\lambda(y,\varkappa), \forall \lambda > 0;$
$(\tilde{\omega}_4)$ $\tilde{\omega}_{\lambda+\mu}(\varkappa,y) \leq \mathfrak{s}\big[\tilde{\omega}_\lambda(\varkappa,z) + \tilde{\omega}_\mu(z,y)\big] - \tilde{\omega}_\lambda(z,z), \quad \forall \lambda,\mu > 0.$

Then, $\mathcal{M}_{\tilde{\omega}}$ is a partial modular $b-$metric space, which is abbreviated with $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}\mathfrak{s}}$.

Definition 8. *In addition to the axioms $\tilde{\omega}_1$, $\tilde{\omega}_2$, and $\tilde{\omega}_3$, a $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$ $\tilde{\omega}$ on $\mathcal{M}$ is regarded to be convex if it meets the condition specified below:*

$(\tilde{\omega}_5)$ $\tilde{\omega}_{\lambda+\mu}(\varkappa,y) \leq \mathfrak{s}\left[\frac{\lambda}{\lambda+\mu}\tilde{\omega}_\lambda(\varkappa,z) + \frac{\mu}{\lambda+\mu}\tilde{\omega}_\mu(z,y)\right] - \frac{\lambda}{\lambda+\mu}\tilde{\omega}_\lambda(z,z),$

for all $\varkappa, y, z \in \mathcal{M}$ and for all $\lambda,\mu > 0$.

Definition 9. *Regard $\tilde{\omega}$ be a $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$ on a set $\mathcal{M}$. For a given $\varkappa_0 \in \mathcal{M}$, we set up*

- $\mathcal{M}_{\tilde{\omega}}(\varkappa_0) = \left\{\varkappa \in \mathcal{M} : \lim_{\lambda\to\infty} \tilde{\omega}_\lambda(\varkappa_0,\varkappa) = c\right\}$, *for some $c \geq 0$ and*
- $\mathcal{M}^*_{\tilde{\omega}}(\varkappa_0) = \{\varkappa \in \mathcal{M} : \exists \lambda = \lambda(\varkappa) > 0, \tilde{\omega}_\lambda(\varkappa_0,\varkappa) < \infty\}.$

Then, $\mathcal{M}_{\tilde{\omega}}$ and $\mathcal{M}^*_{\tilde{\omega}}$ are called $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}\mathfrak{s}}$ centered at $\varkappa_0$.

Every partial modular space is undoubtedly a $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}\mathfrak{s}}$ with the parameter $\mathfrak{s} = 1$, and every modular $b-$metric space is a $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}\mathfrak{s}}$ with the same parameter and zero self-distance. However, the contrary of these facts does not have to be accurate.

As a $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$ is a partial modular when $\mathfrak{s} = 1$, the class of $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}\mathfrak{s}}$ is larger than the class of partial modular metric spaces. We illustrate how a $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}\mathfrak{s}}$ on $\mathcal{M}^*_{\tilde{\omega}}$ may be neither a partial metric nor a modular $b-$metric.

Example 2. *Let $\mathcal{M} = \mathbb{R}$ and $\tilde{\omega} : (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ be characterized as*

$$\tilde{\omega}_\lambda(\varkappa,y) = e^{-\lambda}|\varkappa - y|^2$$

for all $\lambda > 0$ and for all $\varkappa, y \in \mathcal{M}$. Then, $\tilde{\omega}$ is a $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$ on $\mathcal{M}$. We currently possess, in fact,

- $(\tilde{\omega}_1) : \begin{cases} \tilde{\omega}_\lambda(\varkappa,\varkappa) = e^{-\lambda}|\varkappa-\varkappa|^2 = 0 = e^{-\mu}|\varkappa-\varkappa|^2 = \tilde{\omega}_\mu(\varkappa,\varkappa), \\ e^{-\lambda}|\varkappa-\varkappa|^2 = e^{-\lambda}|\varkappa-y|^2 = e^{-\lambda}|y-y|^2 = 0, \\ e^{-\lambda}|\varkappa-y|^2 = 0 \Rightarrow |\varkappa-y| = 0 \Rightarrow \varkappa = y. \end{cases}$
- $(\tilde{\omega}_2) : \tilde{\omega}_\lambda(\varkappa,\varkappa) = e^{-\lambda}|\varkappa-\varkappa|^2 \leq |\varkappa-y|^2 \leq e^{-\lambda}|\varkappa-y|^2 = \tilde{\omega}_\lambda(\varkappa,y)$, *for all $\lambda > 0$.*
- $(\tilde{\omega}_3) : \tilde{\omega}_\lambda(\varkappa,y) = e^{-\lambda}|\varkappa-y|^2 = e^{-\lambda}|y-\varkappa|^2 = \tilde{\omega}_\lambda(y,\varkappa)$, *for all $\lambda > 0$.*
- $(\tilde{\omega}_4)$: *By considering the inequality $(\varkappa-y)^2 \leq 2\Big[(\varkappa-z)^2 + (z-y)^2\Big]$, we have*

$$e^{-\lambda}|\varkappa-y|^2 \leq 2e^{-\lambda}\Big[|\varkappa-z|^2 + |z-y|^2\Big] = 2\Big[e^{-\lambda}.|\varkappa-z|^2 + e^{-\lambda}.|z-y|^2\Big],$$

that is to say $|\varkappa-y|^2 \leq 2\Big(|\varkappa-z|^2 + |z-y|^2\Big)$. Thereupon, we get

$$\begin{aligned}\tilde{\varpi}_{\lambda+\mu}(x,y) &= e^{-(\lambda+\mu)}|x-y|^2 = e^{-(\lambda+\mu)}|x-z+z-y|^2 - e^{-\lambda}|z-z|^2 \\ &\leq e^{-\lambda}.e^{-\mu}.2\Big[|x-z|^2+|z-y|^2\Big] - e^{-\lambda}|z-z|^2 \\ &= 2.e^{-\lambda}.e^{-\mu}|x-z|^2 + 2.e^{-\lambda}.e^{-\mu}|z-y|^2 - e^{-\lambda}|z-z|^2 \\ &\leq 2.e^{-\lambda}.1|x-z|^2 + 2.1.e^{-\mu}|z-y|^2 - e^{-\lambda}|z-z|^2 \\ &= 2.\big[\tilde{\varpi}_{\lambda}(x,z)+\tilde{\varpi}_{\mu}(z,y)\big] - \tilde{\varpi}_{\lambda}(z,z).\end{aligned}$$

$\tilde{\varpi}$ is a $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$ on $\mathcal{M}$ with $\mathfrak{s}=2$.

Example 3. *Consider $\mathcal{M}=\mathbb{R}$ and $\tilde{\varpi}:(0,\infty)\times\mathcal{M}\times\mathcal{M}\to[0,\infty]$ to be regarded as*

$$\tilde{\varpi}_{\lambda}(x,y) = \frac{|x-y|^q}{\lambda} + \max\{x,y\}, \quad q>1,$$

for all $\lambda>0$ and for all $x,y\in\mathcal{M}$. Then, $\tilde{\varpi}$ is a $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$ with $\mathfrak{s}=2^{q-1}$.

Definition 10. *Let $\mathcal{M}^*_{\tilde{\varpi}}$ be a $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}\mathfrak{s}}$ and $\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ be a sequence in $\mathcal{M}^*_{\tilde{\varpi}}$. Then:*

(*i*) *the sequence $\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ is termed $\tilde{\varpi}-$convergent to $x\in\mathcal{M}^*_{\tilde{\varpi}}$ if and only if $\tilde{\varpi}_{\lambda}(x_{\mathfrak{z}},x)\to\tilde{\varpi}_{\lambda}(x,x)$, as $\mathfrak{z}\to\infty$. Also, the point x is termed the $\tilde{\varpi}-$limit of $\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$.*

(*ii*) *$\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ is termed $\tilde{\varpi}-$Cauchy if*

$$\lim_{m,\mathfrak{z}\to\infty}\tilde{\varpi}_{\lambda}(x_m,x_{\mathfrak{z}}) = \lim_{m,\mathfrak{z}\to\infty}\tilde{\varpi}_{\lambda}(x_m,x_m) = \lim_{m,\mathfrak{z}\to\infty}\tilde{\varpi}_{\lambda}(x_{\mathfrak{z}},x_{\mathfrak{z}}).$$

(*iii*) *$\mathcal{M}^*_{\tilde{\varpi}}$ is entitled $\tilde{\varpi}-$complete if every Cauchy sequence in $\mathcal{M}^*_{\tilde{\varpi}}$ is $\tilde{\varpi}-$convergent to an element $x\in\mathcal{M}^*_{\tilde{\varpi}}$ wherein $\lim_{\mathfrak{z}\to\infty}\tilde{\varpi}_{\lambda}(x_{\mathfrak{z}},x)=\tilde{\varpi}_{\lambda}(x,x)$.*

(*iv*) *A function $\mathcal{Y}:\mathcal{M}^*_{\tilde{\varpi}}\to\mathcal{M}^*_{\tilde{\varpi}}$ is called $\tilde{\varpi}-$continuous in $\mathcal{M}^*_{\tilde{\varpi}}$ if the sequence $\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in N}\subset\mathcal{M}^*_{\tilde{\varpi}}$ satisfying $x_{\mathfrak{z}}\xrightarrow{\tilde{\varpi}}x$ as $\mathfrak{z}\to\infty$, whenever $\mathcal{Y}x_{\mathfrak{z}}\xrightarrow{\tilde{\varpi}}\mathcal{Y}x$.*

Lemma 1. *Every $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$ $\tilde{\varpi}$ defines a modular $b-$metric ω, where*

$$\omega_{\lambda}(x,y) = 2\tilde{\varpi}_{\lambda}(x,y) - \tilde{\varpi}_{\lambda}(x,x) - \tilde{\varpi}_{\lambda}(y,y). \tag{1}$$

Proof. Owing to the fact that $\tilde{\varpi}$ is a $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{m}}$, $\tilde{\varpi}$ fulfills $(\tilde{\varpi}_1)-(\tilde{\varpi}_4)$. We shall now demonstrate that the axioms $(\omega_1,\omega_2,\omega_3{}')$ of Definition 2 are met.

(ω_1) If $x=y$, from (1), the expression $\omega_{\lambda}(x,y)=0$, $\forall\lambda>0$ is attained. Suppose that $\omega_{\lambda}(x,y)=0$, $\forall\lambda>0$, then $2\tilde{\varpi}_{\lambda}(x,y)-\tilde{\varpi}_{\lambda}(x,x)-\tilde{\varpi}_{\lambda}(y,y)=0$, which yields that $2\tilde{\varpi}_{\lambda}(x,y)=\tilde{\varpi}_{\lambda}(x,x)+\tilde{\varpi}_{\lambda}(y,y)$. By $(\tilde{\varpi}_1)$, we achieve

$$2\tilde{\varpi}_{\lambda}(x,x)\leq 2\tilde{\varpi}_{\lambda}(x,y)=\tilde{\varpi}_{\lambda}(x,x)+\tilde{\varpi}_{\lambda}(y,y)\Rightarrow\tilde{\varpi}_{\lambda}(x,x)\leq\tilde{\varpi}_{\lambda}(y,y),$$

and similarly

$$2\tilde{\varpi}_{\lambda}(y,y)\leq 2\tilde{\varpi}_{\lambda}(x,y)=\tilde{\varpi}_{\lambda}(x,x)+\tilde{\varpi}_{\lambda}(y,y)\Rightarrow\tilde{\varpi}_{\lambda}(y,y)\leq\tilde{\varpi}_{\lambda}(x,x).$$

Consequently, we gain $\tilde{\varpi}_{\lambda}(x,y)=\tilde{\varpi}_{\lambda}(x,x)=\tilde{\varpi}_{\lambda}(y,y)$, which means that $x=y$.

(ω_2) $\omega_{\lambda}(x,y) = 2\tilde{\varpi}_{\lambda}(x,y)-\tilde{\varpi}_{\lambda}(x,x)-\tilde{\varpi}_{\lambda}(y,y) = 2\tilde{\varpi}_{\lambda}(y,x)-\tilde{\varpi}_{\lambda}(y,y)-\tilde{\varpi}_{\lambda}(x,x) = \omega_{\lambda}(y,x)$.

$(\omega_3{}')$ From $(\tilde{\omega}_1)$, we get $\tilde{\omega}_{\lambda+\mu}(\chi,\chi) = \tilde{\omega}_\lambda(\chi,\chi)$ and $\tilde{\omega}_{\lambda+\mu}(y,y) = \tilde{\omega}_\lambda(y,y)$, $\forall \chi, y \in \mathcal{M}$ and $\forall \lambda, \mu > 0$. Hence, we have

$$\tilde{\omega}_\lambda(\chi,\chi) \le \tilde{\omega}_\lambda(\chi,y) \Rightarrow 0 \le \tilde{\omega}_\lambda(\chi,y) - \tilde{\omega}_\lambda(\chi,\chi) \le \mathfrak{s}(\tilde{\omega}_\lambda(\chi,y) - \tilde{\omega}_\lambda(\chi,\chi)),$$

and

$$\tilde{\omega}_\lambda(y,y) \le \tilde{\omega}_\lambda(\chi,y) \Rightarrow 0 \le \tilde{\omega}_\lambda(\chi,y) - \tilde{\omega}_\lambda(y,y) \le \mathfrak{s}(\tilde{\omega}_\lambda(\chi,y) - \tilde{\omega}_\lambda(y,y)),$$

which implies that

$$2\tilde{\omega}_\lambda(\chi,y) - \tilde{\omega}_\lambda(\chi,\chi) - \tilde{\omega}_\lambda(y,y) \le \mathfrak{s}(2\tilde{\omega}_\lambda(\chi,y) - \tilde{\omega}_\lambda(\chi,\chi) - \tilde{\omega}_\lambda(y,y)).$$

Now, considering $(\tilde{\omega}_4)$ and the above inequality, we obtain

$$\begin{aligned}
&\omega_{\lambda+\mu}(\chi,y)\\
&= 2\tilde{\omega}_{\lambda+\mu}(\chi,y) - \tilde{\omega}_{\lambda+\mu}(\chi,\chi) - \tilde{\omega}_{\lambda+\mu}(y,y)\\
&= 2\tilde{\omega}_{\lambda+\mu}(\chi,y) - \tilde{\omega}_{\lambda}(\chi,\chi) - \tilde{\omega}_{\mu}(y,y)\\
&\le 2\big(\mathfrak{s}\big[\tilde{\omega}_\lambda(\chi,z) + \tilde{\omega}_\mu(z,y)\big] - \tilde{\omega}_\mu(z,z)\big) - \tilde{\omega}_\lambda(\chi,\chi) - \tilde{\omega}_\mu(y,y)\\
&= \big(2\mathfrak{s}\tilde{\omega}_\lambda(\chi,z) - \tilde{\omega}_\mu(\chi,\chi) - \tilde{\omega}_\lambda(z,z)\big) + \big(2\mathfrak{s}\tilde{\omega}_\mu(z,y) - \tilde{\omega}_\lambda(z,z) - \tilde{\omega}_\mu(y,y)\big)\\
&\le \mathfrak{s}\big(2\tilde{\omega}_\lambda(\chi,z) - \tilde{\omega}_\mu(\chi,\chi) - \tilde{\omega}_\lambda(z,z)\big) + \mathfrak{s}\big(2\tilde{\omega}_\mu(z,y) - \tilde{\omega}_\lambda(z,z) - \tilde{\omega}_\mu(y,y)\big)\\
&= \mathfrak{s}\big(\big(2\tilde{\omega}_\lambda(\chi,z) - \tilde{\omega}_\mu(\chi,\chi) - \tilde{\omega}_\lambda(z,z)\big) + \big(2\tilde{\omega}_\mu(z,y) - \tilde{\omega}_\lambda(z,z) - \tilde{\omega}_\mu(y,y)\big)\big)\\
&= \mathfrak{s}\big(\omega_\lambda(\chi,z) + \omega_\mu(z,y)\big).
\end{aligned}$$

Accordingly, it appears that the proof is completed. □

Lemma 2. *Consider $\tilde{\omega}$ is a partial modular metric on $\mathcal{M}$ and $\{\chi_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a sequence in $\mathcal{M}^*_{\tilde{\omega}}$.*

(*i*) *$\{\chi_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a $\tilde{\omega}-$Cauchy sequence in the $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ if and only if it is an $\omega-$Cauchy sequence in the modular $b-$metric space $\mathcal{M}^*_\omega$ induced by $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ $\tilde{\omega}$.*

(*ii*) *A $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ $\mathcal{M}^*_{\tilde{\omega}}$ is complete if and only if the modular $b-$metric space $\mathcal{M}^*_\omega$ induced by $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ $\tilde{\omega}$ is complete. Furthermore,*

$$\lim_{\mathfrak{z}\to\infty} \omega_\lambda(\chi_\mathfrak{z},\chi) = 0 \Leftrightarrow \lim_{\mathfrak{z}\to\infty}[2\tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi) - \tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi_\mathfrak{z}) - \tilde{\omega}_\lambda(\chi,\chi)] = 0$$

or

$$\lim_{\mathfrak{z}\to\infty} \omega_\lambda(\chi_\mathfrak{z},\chi) = 0 \Leftrightarrow \lim_{\mathfrak{z}\to\infty} \tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi) = \lim_{\mathfrak{z}\to\infty} \tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi_\mathfrak{z}) = \tilde{\omega}_\lambda(\chi,\chi), \forall \lambda > 0.$$

(*iii*) *$\{\chi_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is $\tilde{\omega}-$convergent to $\chi^* \in \mathcal{M}^*_{\tilde{\omega}}$ if and only if $\lim_{\mathfrak{z}\to\infty} \tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi^*) = \lim_{\mathfrak{z},m\to\infty} \tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi_m) = \tilde{\omega}_\lambda(\chi^*,\chi^*), \forall \lambda > 0$, as $\mathfrak{z}\to\infty$.*

Proof. We begin by emphasizing that every $\tilde{\omega}-$Cauchy sequence in the $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ $\mathcal{M}^*_{\tilde{\omega}}$ is an $\omega-$Cauchy sequence in modular $b-$metric space $\mathcal{M}^*_\omega$. Let $\{\chi_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ be a $\tilde{\omega}-$Cauchy sequence in $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ $\mathcal{M}^*_{\tilde{\omega}}$. A point $\hbar \in \mathbb{R}$ exists such that, for any $\varepsilon > 0$, there is a natural number $\mathfrak{z}_\varepsilon$ fulfilling $|\tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi_m) - \hbar| < \frac{\varepsilon}{4}$ for all $\mathfrak{z}, m \ge \mathfrak{z}_\varepsilon$. Thence,

$$
\begin{aligned}
&|\omega_\lambda(x_\mathfrak{z}, x_m)| \\
&= 2\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \tilde{\varpi}_\lambda(x_m, x_m) \\
&= |\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) - \hbar + \hbar - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) + \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) - \hbar + \hbar - \tilde{\varpi}_\lambda(x_m, x_m)| \\
&\leq |\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) - \hbar| + |\hbar - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})| + |\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) - \hbar| + |\hbar - \tilde{\varpi}_\lambda(x_m, x_m)| \\
&< \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = \varepsilon,
\end{aligned}
$$

for all $\mathfrak{z}, m \geq \mathfrak{z}_\varepsilon$. As a result $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is an $\omega-$Cauchy sequence in $\mathcal{M}^*_\omega$.

Subsequently, we attest that $\omega-$completeness of $\mathcal{M}^*_\omega$ entails $\tilde{\varpi}-$completeness of $\mathcal{M}^*_{\tilde{\varpi}}$. Indeed, if $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a $\tilde{\varpi}-$Cauchy sequence in $\mathcal{M}^*_{\tilde{\varpi}}$, according to above discussion, $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is an $\omega-$Cauchy sequence in $\mathcal{M}^*_\omega$, too. Due to the fact that modular $b-$metric space $\mathcal{M}^*_\omega$ is $\varpi-$complete, we infer that an element y belongs to $\mathcal{M}^*_\omega$ exists such that $\lim_{\mathfrak{z}\to\infty} \omega_\lambda(x_\mathfrak{z}, y) = 0$. Hence,

$$\lim_{\mathfrak{z}\to\infty}[2\tilde{\varpi}_\lambda(x_\mathfrak{z}, y) - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \tilde{\varpi}_\lambda(y, y)] = 0$$

which implies

$$\lim_{\mathfrak{z}\to\infty}[\tilde{\varpi}_\lambda(x_\mathfrak{z}, y) - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) + \tilde{\varpi}_\lambda(x_\mathfrak{z}, y) - \tilde{\varpi}_\lambda(y, y)] = 0.$$

Thereupon, $\lim_{\mathfrak{z}\to\infty}[\tilde{\varpi}_\lambda(x_\mathfrak{z}, y) - \tilde{\varpi}_\lambda(y, y)] = 0$. Further, we attain

$$\lim_{\mathfrak{z}\to\infty}[\tilde{\varpi}_\lambda(x_\mathfrak{z}, y) - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})] = 0.$$

Consequently, $\lim_{\mathfrak{z}\to\infty} \tilde{\varpi}_\lambda(x_\mathfrak{z}, y) = \tilde{\varpi}_\lambda(y, y) = \lim_{\mathfrak{z}\to\infty} \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})$. Also, from $(\tilde{\varpi}_2)$, we achieve

$$\tilde{\varpi}_\lambda(y, y) \leq \lim_{\mathfrak{z},m\to\infty} \tilde{\varpi}_\lambda(x_\mathfrak{z}, y) = \lim_{\mathfrak{z},m\to\infty} \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) \leq \lim_{\mathfrak{z},m\to\infty} \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m). \tag{2}$$

On the other hand, $\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) \leq \mathfrak{s}\left[\tilde{\varpi}_{\frac{\lambda}{2}}(x_\mathfrak{z}, y) + \tilde{\varpi}_{\frac{\lambda}{2}}(y, x_m)\right] - \tilde{\varpi}_{\frac{\lambda}{2}}(y, y)$. Letting $n, m \to \infty$

$$
\begin{aligned}
\lim_{\mathfrak{z},m\to\infty} \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) &\leq \lim_{\mathfrak{z},m\to\infty} \mathfrak{s}\tilde{\varpi}_{\frac{\lambda}{2}}(x_\mathfrak{z}, y) + \lim_{\mathfrak{z},m\to\infty} \mathfrak{s}\tilde{\varpi}_{\frac{\lambda}{2}}(y, x_m) - \lim_{\mathfrak{z},m\to\infty} \tilde{\varpi}_{\frac{\lambda}{2}}(y, y) \\
&= \tilde{\varpi}_\lambda(y, y).
\end{aligned} \tag{3}
$$

From (2) and (3), we obtain $\lim_{\mathfrak{z},m\to\infty} \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) = \tilde{\varpi}_\lambda(y, y)$; that is, $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a $\tilde{\varpi}-$convergent sequence in $\mathcal{M}^*_{\tilde{\varpi}}$.

It will be demonstrated that every $\omega-$Cauchy sequence that belongs to $\mathcal{M}^*_\omega$ is a $\tilde{\varpi}-$Cauchy sequence in $\mathcal{M}^*_{\tilde{\varpi}}$. Consider $\varepsilon = \frac{1}{2}$. Then, a natural number $\mathfrak{z}_0 \in \mathbb{N}$ exists such that $\omega_\lambda(x_\mathfrak{z}, x_m) < \frac{1}{2}$ for all $\mathfrak{z}, m \geq \mathfrak{z}_0$. Because

$$\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) \leq \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) \Rightarrow -\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) \leq -\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}),$$

which follows

$$\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) = 2\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) \leq 2\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})$$

and

$$
\begin{aligned}
\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) - \tilde{\varpi}_\lambda(x_{\mathfrak{z}_0}, x_{\mathfrak{z}_0}) &\leq 2\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) - \tilde{\varpi}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \tilde{\varpi}_\lambda(x_{\mathfrak{z}_0}, x_{\mathfrak{z}_0}) \\
&\leq \omega_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) < \tfrac{1}{2}.
\end{aligned}
$$

Hence, the inequality

$$\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) \leq \tilde{\omega}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) \leq \omega_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}_0}) + \tilde{\omega}_\lambda(x_{\mathfrak{z}_0}, x_{\mathfrak{z}_0}) < \frac{1}{2} + \tilde{\omega}_\lambda(x_{\mathfrak{z}_0}, x_{\mathfrak{z}_0})$$

indicates that the sequence $\{\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})\}$ is bounded in $\mathbb{R}$ and an element $\hbar \in \mathbb{R}$ exists such that a subsequence $\{\tilde{\omega}_\lambda(x_{n_k}, x_{n_k})\}$ of $\{\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})\}$ is convergent to $\hbar \in \mathbb{R}$, which means, $\lim_{k\to\infty} \tilde{\omega}_\lambda(x_{n_k}, x_{n_k}) = \hbar$.

Now, we prove that $\{\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is an $\omega-$Cauchy sequence in $\mathcal{M}^*_\omega$, for given $\varepsilon > 0$, there is a natural number $\mathfrak{z}_\varepsilon$ such that $\omega_\lambda(x_\mathfrak{z}, x_m) < \varepsilon$ for all $\mathfrak{z}, m \geq \mathfrak{z}_\varepsilon$. Thus, for all $\mathfrak{z}, m \geq \mathfrak{z}_\varepsilon$, we have $\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) \leq \tilde{\omega}_\lambda(x_\mathfrak{z}, x_m)$ and thereby,

$$\begin{aligned}\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \tilde{\omega}_\lambda(x_m, x_m) &\leq \tilde{\omega}_\lambda(x_\mathfrak{z}, x_m) - \tilde{\omega}_\lambda(x_m, x_m) \\ &\leq \omega_\lambda(x_\mathfrak{z}, x_m) < \varepsilon.\end{aligned}$$

Therefore, we achieve $\lim_{\mathfrak{z}\to\infty} \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) = \hbar$.

On the other hand, for all $\mathfrak{z}, m \geq \mathfrak{z}_\varepsilon$, we have

$$\begin{aligned}|\tilde{\omega}_\lambda(x_\mathfrak{z}, x_m) - \hbar| &= |\tilde{\omega}_\lambda(x_\mathfrak{z}, x_m) - \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) + \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \hbar| \\ &\leq |\tilde{\omega}_\lambda(x_\mathfrak{z}, x_m) - \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})| + |\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \hbar| \\ &\leq \omega_\lambda(x_\mathfrak{z}, x_m) + |\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \hbar|.\end{aligned}$$

Hence, $\lim_{\mathfrak{z},m\to\infty} \tilde{\omega}_\lambda(x_\mathfrak{z}, x_m) = \hbar$ and consequently, $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a $\tilde{\omega}-$Cauchy sequence in $\mathcal{M}^*_{\tilde{\omega}}$.

Conversely, consider that $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is an $\omega-$Cauchy sequence in $\mathcal{M}^*_\omega$. Then, $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a $\tilde{\omega}-$Cauchy sequence belongs to $\mathcal{M}^*_{\tilde{\omega}}$ and so it is convergent to a point $x \in \mathcal{M}^*_{\tilde{\omega}}$ with

$$\lim_{\mathfrak{z}\to\infty} \tilde{\omega}_\lambda(x_\mathfrak{z}, x) = \lim_{\mathfrak{z},m\to\infty} \tilde{\omega}_\lambda(x_\mathfrak{z}, x_m) = \tilde{\omega}_\lambda(x, x), \forall \lambda > 0.$$

For a given $\varepsilon > 0$, a natural number $\mathfrak{z}_\varepsilon$ exists such that $\tilde{\omega}_\lambda(x, x_\mathfrak{z}) - \tilde{\omega}_\lambda(x, x) < \frac{\varepsilon}{4}$ and

$$\tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \tilde{\omega}_\lambda(x, x) \leq \tilde{\omega}_\lambda(x_m, x_\mathfrak{z}) - \tilde{\omega}_\lambda(x, x) < \frac{\varepsilon}{4}.$$

Thereupon, we obtain

$$\begin{aligned}&|\omega_\lambda(x_\mathfrak{z}, x)| \\ &= |2\tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) - \tilde{\omega}_\lambda(x, x)| \\ &= |\tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) + \tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x, x)| \\ &= |\tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x, x) + \tilde{\omega}_\lambda(x, x) - \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z}) + \tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x, x)| \\ &\leq |\tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x, x)| + |\tilde{\omega}_\lambda(x, x) - \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})| + |\tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x, x)| \\ &< \tfrac{3\varepsilon}{4} < \varepsilon,\end{aligned}$$

whenever $\mathfrak{z} \geq \mathfrak{z}_\varepsilon$. Therefore, $\mathcal{M}^*_\omega$ is $\omega-$complete. Finally, $\lim_{\mathfrak{z}\to\infty} \omega_\lambda(x_\mathfrak{z}, x) = 0$ and so,

$$\lim_{\mathfrak{z}\to\infty}[\tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x_\mathfrak{z}, x_\mathfrak{z})] + \lim_{\mathfrak{z}\to\infty}[\tilde{\omega}_\lambda(x_\mathfrak{z}, x) - \tilde{\omega}_\lambda(x, x)] = 0.$$

Furthermore, we have the following statement for all $\mathfrak{z}, m \geq \mathfrak{z}_\varepsilon$

$$\begin{aligned}
&\lim_{\mathfrak{z},m\to\infty}[\tilde{\omega}_\lambda(\chi_\mathfrak{z}, \chi_m) - \tilde{\omega}_\lambda(\chi, \chi)] \\
&\leq \lim_{\mathfrak{z}\to\infty}\left[\mathfrak{s}\left[\tilde{\omega}_{\frac{\lambda}{2}}(\chi_\mathfrak{z}, \chi) + \tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi_m)\right] - \tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi) - \tilde{\omega}_\lambda(\chi, \chi)\right] \\
&\leq \lim_{\mathfrak{z}\to\infty}\left[\mathfrak{s}\tilde{\omega}_{\frac{\lambda}{2}}(\chi_\mathfrak{z}, \chi) - \tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi)\right] + \lim_{m\to\infty}\left[\mathfrak{s}\tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi_m) - \tilde{\omega}_\lambda(\chi, \chi)\right] = 0.
\end{aligned}$$

□

Given that the $b-$metric is discontinuous in general, the partial modular $b-$metric is not continuous. As a result, the ensuing critical lemma plays a vital role in establishing our key findings.

Lemma 3. *$\mathcal{M}^*_{\tilde{\omega}}$ is a $\mathcal{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ with the parameter $\mathfrak{s} > 1$. Presume that $\{\chi_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ and $\{y_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ are $\tilde{\omega}-$convergent to χ and y, respectively. The following expression is acquired.*

$$\begin{aligned}
\frac{1}{\mathfrak{s}^2}\tilde{\omega}_\lambda(\chi, y) - \frac{1}{\mathfrak{s}}\tilde{\omega}_\lambda(\chi, \chi) - \tilde{\omega}_\lambda(y, y) &\leq \lim_{\mathfrak{z}\to\infty}\inf\tilde{\omega}_\lambda(\chi_\mathfrak{z}, y_\mathfrak{z}) \leq \lim_{\mathfrak{z}\to\infty}\sup\tilde{\omega}_\lambda(\chi_\mathfrak{z}, y_\mathfrak{z}) \\
&\leq \mathfrak{s}\tilde{\omega}_\lambda(\chi, \chi) + \mathfrak{s}^2\tilde{\omega}_\lambda(y, y) + \mathfrak{s}^2\tilde{\omega}_\lambda(\chi, y).
\end{aligned}$$

In particular, if $\tilde{\omega}_\lambda(\chi, y) = 0$, then we have $\lim_{\mathfrak{z}\to\infty}\tilde{\omega}_\lambda(\chi_\mathfrak{z}, y_\mathfrak{z}) = 0$. Moreover, for each $z \in \mathcal{M}^*_{\tilde{\omega}}$, the subsequent expression is met

$$\begin{aligned}
\frac{1}{\mathfrak{s}}\tilde{\omega}_\lambda(\chi, z) - \tilde{\omega}_\lambda(\chi, \chi) &\leq \lim_{\mathfrak{z}\to\infty}\inf\tilde{\omega}_\lambda(\chi_\mathfrak{z}, z) \leq \lim_{\mathfrak{z}\to\infty}\sup\tilde{\omega}_\lambda(\chi_\mathfrak{z}, z) \\
&\leq \mathfrak{s}\tilde{\omega}_\lambda(\chi, \chi) + \mathfrak{s}\tilde{\omega}_\lambda(\chi, z).
\end{aligned}$$

Also, in case of $\tilde{\omega}_\lambda(\chi, \chi) = 0$, we get

$$\frac{1}{\mathfrak{s}}\tilde{\omega}_\lambda(\chi, z) \leq \lim_{\mathfrak{z}\to\infty}\inf\tilde{\omega}_\lambda(\chi_\mathfrak{z}, z) \leq \lim_{\mathfrak{z}\to\infty}\sup\tilde{\omega}_\lambda(\chi_\mathfrak{z}, z) \leq \mathfrak{s}\tilde{\omega}_\lambda(\chi, z).$$

Proof. Considering the statement $(\tilde{\omega}_4)$, we achieve

$$\begin{aligned}
\tilde{\omega}_\lambda(\chi, y) &\leq \mathfrak{s}\left[\tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi_\mathfrak{z}) + \tilde{\omega}_{\frac{\lambda}{2}}(\chi_\mathfrak{z}, y)\right] - \tilde{\omega}_{\frac{\lambda}{2}}(\chi_\mathfrak{z}, \chi_\mathfrak{z}) \\
&\leq \mathfrak{s}\tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi_\mathfrak{z}) + \mathfrak{s}\tilde{\omega}_{\frac{\lambda}{2}}(\chi_\mathfrak{z}, y) \\
&\leq \mathfrak{s}\tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi_\mathfrak{z}) + \mathfrak{s}\left[\mathfrak{s}\left(\tilde{\omega}_{\frac{\lambda}{4}}(\chi_\mathfrak{z}, y_\mathfrak{z}) + \tilde{\omega}_{\frac{\lambda}{4}}(y_\mathfrak{z}, y)\right) - \tilde{\omega}_{\frac{\lambda}{4}}(y_\mathfrak{z}, y_\mathfrak{z})\right] \\
&\leq \mathfrak{s}\tilde{\omega}_{\frac{\lambda}{2}}(\chi, \chi_\mathfrak{z}) + \mathfrak{s}^2\tilde{\omega}_{\frac{\lambda}{4}}(\chi_\mathfrak{z}, y_\mathfrak{z}) + \mathfrak{s}^2\tilde{\omega}_{\frac{\lambda}{4}}(y_\mathfrak{z}, y).
\end{aligned}$$

Since $\tilde{\omega}_\lambda(\chi, \chi) \leq \tilde{\omega}_\mu(\chi, \chi)$ for $\mu < \lambda$, we have

$$\tilde{\omega}_\lambda(\chi, y) \leq \mathfrak{s}\tilde{\omega}_\lambda(\chi, \chi_\mathfrak{z}) + \mathfrak{s}^2\tilde{\omega}_\lambda(\chi_\mathfrak{z}, y_\mathfrak{z}) + \mathfrak{s}^2\tilde{\omega}_\lambda(y_\mathfrak{z}, y),$$

which yields

$$\tilde{\omega}_\lambda(\chi, y) - \mathfrak{s}\tilde{\omega}_\lambda(\chi, \chi_\mathfrak{z}) - \mathfrak{s}^2\tilde{\omega}_\lambda(y_\mathfrak{z}, y) \leq \mathfrak{s}^2\tilde{\omega}_\lambda(\chi_\mathfrak{z}, y_\mathfrak{z}). \tag{4}$$

Likewise,

$$\tilde{\omega}_\lambda(\chi_\mathfrak{z}, y_\mathfrak{z}) \leq \mathfrak{s}\tilde{\omega}_\lambda(\chi_\mathfrak{z}, \chi) + \mathfrak{s}^2\tilde{\omega}_\lambda(\chi, y) + \mathfrak{s}^2\tilde{\omega}_\lambda(y, y_\mathfrak{z}). \tag{5}$$

If the lower limit is applied as $\mathfrak{z} \to \infty$ on both sides of inequality (4), considering $x_{\mathfrak{z}} \to x$ and $y_{\mathfrak{z}} \to y$, we acquire

$$\frac{1}{\mathfrak{s}^2}\tilde{\varpi}_\lambda(x,y) - \frac{1}{\mathfrak{s}}\tilde{\varpi}_\lambda(x,x) - \tilde{\varpi}_\lambda(y,y) \leq \lim_{\mathfrak{z}\to\infty}\inf\tilde{\varpi}_\lambda(x_{\mathfrak{z}},y_{\mathfrak{z}}).$$

Also, taking the upper limit as $\mathfrak{z} \to \infty$ in (5), we gain

$$\lim_{\mathfrak{z}\to\infty}\sup\tilde{\varpi}_\lambda(x_{\mathfrak{z}},y_{\mathfrak{z}}) \leq \mathfrak{s}\tilde{\varpi}_\lambda(x,x) + \mathfrak{s}^2\tilde{\varpi}_\lambda(x,y) + \mathfrak{s}^2\tilde{\varpi}_\lambda(y,y).$$

Since $\lim_{\mathfrak{z}\to\infty}\inf\tilde{\varpi}_\lambda(x_{\mathfrak{z}},y_{\mathfrak{z}}) \leq \lim_{\mathfrak{z}\to\infty}\sup\tilde{\varpi}_\lambda(x_{\mathfrak{z}},y_{\mathfrak{z}})$, we achieve the first desired result. If $\tilde{\varpi}_\lambda(x,y) = 0$, via the triangle inequality, we procure $\tilde{\varpi}_\lambda(x,x) = 0$, and $\tilde{\varpi}_\lambda(y,y) = 0$. Therefore, we have $\lim_{\mathfrak{z}\to\infty}\tilde{\varpi}_\lambda(x_{\mathfrak{z}},y_{\mathfrak{z}}) = 0$ since $\tilde{\varpi}_\lambda(x,y) = \tilde{\varpi}_\lambda(x,x) = \tilde{\varpi}_\lambda(y,y) = 0$. Moreover, for each $z \in \mathcal{M}^*_{\tilde{\varpi}}$, we obtain

$$\begin{aligned}\tilde{\varpi}_\lambda(x,z) &\leq \mathfrak{s}\Big[\tilde{\varpi}_{\frac{\lambda}{2}}(x,x_{\mathfrak{z}}) + \tilde{\varpi}_{\frac{\lambda}{2}}(x_{\mathfrak{z}},z)\Big] - \tilde{\varpi}_{\frac{\lambda}{2}}(x_{\mathfrak{z}},x_{\mathfrak{z}}) \\ &\leq \mathfrak{s}[\tilde{\varpi}_\lambda(x,x_{\mathfrak{z}}) + \tilde{\varpi}_\lambda(x_{\mathfrak{z}},z)],\end{aligned}$$

such that it follows that

$$\frac{1}{\mathfrak{s}}\tilde{\varpi}_\lambda(x,z) - \tilde{\varpi}_\lambda(x,x) \leq \lim_{\mathfrak{z}\to\infty}\inf\tilde{\varpi}_\lambda(x_{\mathfrak{z}},z).$$

Similarly

$$\begin{aligned}\tilde{\varpi}_\lambda(x_{\mathfrak{z}},z) &\leq \mathfrak{s}\Big[\tilde{\varpi}_{\frac{\lambda}{2}}(x_{\mathfrak{z}},x) + \tilde{\varpi}_{\frac{\lambda}{2}}(x,z)\Big] - \tilde{\varpi}_{\frac{\lambda}{2}}(x,x) \\ &\leq \mathfrak{s}[\tilde{\varpi}_\lambda(x_{\mathfrak{z}},x) + \tilde{\varpi}_\lambda(x,z)].\end{aligned}$$

Owing to $\tilde{\varpi}_\lambda(x,x) \leq \tilde{\varpi}_\mu(x,x)$ for $\mu < \lambda$ and taking the upper limit as $\mathfrak{z} \to \infty$ with $x_{\mathfrak{z}} \to x$, we conclude

$$\lim_{\mathfrak{z}\to\infty}\sup\tilde{\varpi}_\lambda(x_{\mathfrak{z}},z) \leq \mathfrak{s}\tilde{\varpi}_\lambda(x,x) + \mathfrak{s}\tilde{\varpi}_\lambda(x,z).$$

Further, since $\lim_{\mathfrak{z}\to\infty}\inf\tilde{\varpi}_\lambda(x_{\mathfrak{z}},z) \leq \lim_{\mathfrak{z}\to\infty}\sup\tilde{\varpi}_\lambda(x_{\mathfrak{z}},z)$, we achieve the desired result. □

The following outcomes are critical in our future thoughts, and the proofs can be completed with respect to [31,32].

Lemma 4. *Consider $\mathcal{M}^*_{\tilde{\varpi}}$ is a $\mathfrak{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$. Then, a sequence $\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ on $\mathcal{M}^*_{\tilde{\varpi}}$ is $0-\tilde{\varpi}-$Cauchy if $\lim_{\mathfrak{z},m\to\infty}\tilde{\varpi}_\lambda(x_{\mathfrak{z}},x_m)$. Moreover, $\mathcal{M}^*_{\tilde{\varpi}}$ is said to be $0-\tilde{\varpi}-$complete if for each $0-\tilde{\varpi}-$Cauchy sequence in $\mathcal{M}$, there is $u \in \mathcal{M}$, such that*

$$\lim_{\mathfrak{z}\to\infty}\tilde{\varpi}_\lambda(x_{\mathfrak{z}},x_m) = \lim_{\mathfrak{z},m\to\infty}\tilde{\varpi}_\lambda(x_{\mathfrak{z}},u) = \tilde{\varpi}_\lambda(u,u) = 0. \tag{6}$$

Lemma 5. *If $\mathfrak{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ $\mathcal{M}^*_{\tilde{\varpi}}$ is $\tilde{\varpi}-$complete, then it is $0-\tilde{\varpi}-$complete.*

Lemma 6. *Consider $\mathcal{M}^*_{\tilde{\varpi}}$ is a $\mathfrak{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$, $\mathcal{Y} : \mathcal{M}^*_{\tilde{\varpi}} \to \mathcal{M}^*_{\tilde{\varpi}}$ is a mapping and $\mathfrak{c} \in [0,1)$. If $\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ is a sequence in $\mathcal{M}^*_{\tilde{\varpi}}$, where $x_{\mathfrak{z}+1} = \mathcal{Y}x_{\mathfrak{z}}$ and*

$$\tilde{\varpi}_\lambda\big(x_{\mathfrak{z}},x_{\mathfrak{z}+1}\big) \leq \mathfrak{c}\tilde{\varpi}_\lambda\big(x_{\mathfrak{z}-1},x_{\mathfrak{z}}\big) \tag{7}$$

for each $\mathfrak{z} \in \mathbb{N}$, then the sequence $\{x_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ is $0-\tilde{\varpi}-$Cauchy.

4. Some Fixed Point Results in the Context of Partial Modular $b-$Metric Spaces

This section puts forward a new contraction mapping termed as modified interpolative almost $\mathcal{E}-$type contraction, as well as a new fixed point theorem using such mappings within the context of $\mathcal{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat \mathfrak{ms}}$.

Initially, let Δ_φ represent the set of all φ self-mappings on $[1, +\infty)$ satisfying the conditions

- φ is a non-decreasing mapping,
- $\varphi(\mathfrak{a}) \leq \mathfrak{a}$, for all $\mathfrak{a} > 0$.

Definition 11. *Consider that $\mathcal{M}^*_{\tilde{\omega}}$ is a $\mathcal{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat \mathfrak{ms}}$ with a parameter $\mathfrak{s} > 1$. A mapping $\mathcal{Y} : \mathcal{M}^*_{\tilde{\omega}} \to \mathcal{M}^*_{\tilde{\omega}}$ is referred to as a modified interpolative almost $\mathcal{E}-$type contraction if $\alpha, \beta \in (0,1)$ exist with $\alpha + \beta < \mathfrak{s}^p$ and $\alpha + 2\beta > \mathfrak{s}^p$ and also, $\eta \in \mathcal{L}_\Psi$, $\mathfrak{O} \in \Theta$ and $\varphi \in \Delta_\varphi$ such that*

$$\eta\Big(\mathfrak{O}\Big(\mathfrak{s}^{p+1}\tilde{\omega}_\lambda(\mathcal{Y}x, \mathcal{Y}y)\Big), \mathfrak{O}(\mathcal{C}(x, y))\Big) \geq 1, \tag{8}$$

where

$$\begin{aligned} \mathcal{C}(x, y) = {} & \alpha\tilde{\omega}_\lambda(x, y) + \beta|\tilde{\omega}_\lambda(x, \mathcal{Y}x) - \tilde{\omega}_\lambda(y, \mathcal{Y}y)| \\ & +(\mathfrak{s}^p - \alpha - \beta)\varphi\Big(\tfrac{1}{\mathfrak{s}}\tilde{\omega}_{2\lambda}(x, \mathcal{Y}y)\Big), \end{aligned} \tag{9}$$

for all distinct $x, y \in \mathcal{M}^*_{\tilde{\omega}} - Fix(\mathcal{Y})$ and for all $\lambda > 0$.

Theorem 3. *Let $\mathcal{M}^*_{\tilde{\omega}}$ be a $\tilde{\omega}-$complete $\mathcal{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat \mathfrak{ms}}$ with $\mathfrak{s} > 1$ and $\mathcal{Y} : \mathcal{M}^*_{\tilde{\omega}} \to \mathcal{M}^*_{\tilde{\omega}}$ be a modified interpolative almost $\mathcal{E}-$type contraction mapping. Then, $\mathcal{Y}$ admits exactly one fixed point.*

Proof. Assume $\omega_0 \in \mathcal{M}^*_{\tilde{\omega}}$ is an initial point and we shall construct $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ by:

$$x_{\mathfrak{z}+1} = \mathcal{Y}x_\mathfrak{z}, \quad \text{for all } \mathfrak{z} \in \mathbb{N}.$$

If there exists some $\mathfrak{z}_0 \in \mathbb{N}$ such that $x_{\mathfrak{z}_0} = x_{\mathfrak{z}_0+1}$, then $\mathfrak{z}_0$ becomes a fixed point of $\mathcal{Y}$. Consequently, we presume that $x_k \neq x_{k+1}$ for all $k \in \mathbb{N}$. By using (8) and (η_2'), we obtain

$$\begin{aligned} 1 & \leq \eta\big(\mathfrak{O}\big(\mathfrak{s}^{p+1}\tilde{\omega}_\lambda\big(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}x_{\mathfrak{z}+1}\big)\big), \mathfrak{O}\big[\mathcal{C}\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big)\big]\big) \\ & < \frac{\psi\big(\mathfrak{O}\big[\mathcal{C}\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big)\big]\big)}{\psi\big(\mathfrak{O}\big(\mathfrak{s}^{p+1}\tilde{\omega}_\lambda\big(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}x_{\mathfrak{z}+1}\big)\big)\big)}, \end{aligned}$$

that is,

$$\psi\Big(\mathfrak{O}\Big(\mathfrak{s}^{p+1}\tilde{\omega}_\lambda\big(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}x_{\mathfrak{z}+1}\big)\Big)\Big) < \psi\big(\mathfrak{O}\big[\mathcal{C}\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big)\big]\big).$$

Because $\mathfrak{O} \in \Theta$ and also given features of the function ψ, the above inequality gives

$$\mathfrak{s}^p\tilde{\omega}_\lambda\big(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+2}\big) \leq \mathfrak{s}^{p+1}\tilde{\omega}_\lambda\big(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+2}\big) < \mathcal{C}\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big), \tag{10}$$

where

$$\begin{aligned} \mathcal{C}\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big) & = \alpha\tilde{\omega}_\lambda\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big) + \beta\big|\tilde{\omega}_\lambda\big(x_\mathfrak{z}, \mathcal{Y}x_\mathfrak{z}\big) - \tilde{\omega}_\lambda\big(x_{\mathfrak{z}+1}, \mathcal{Y}x_{\mathfrak{z}+1}\big)\big| \\ & \quad + (\mathfrak{s}^p - \alpha - \beta)\varphi\Big(\tfrac{1}{\mathfrak{s}}\tilde{\omega}_{2\lambda}\big(x_\mathfrak{z}, \mathcal{Y}x_{\mathfrak{z}+1}\big)\Big) \\ & = \alpha\tilde{\omega}_\lambda\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big) + \beta\big|\tilde{\omega}_\lambda\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big) - \tilde{\omega}_\lambda\big(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+2}\big)\big| \\ & \quad + (\mathfrak{s}^p - \alpha - \beta)\varphi\Big(\tfrac{1}{\mathfrak{s}}\tilde{\omega}_{2\lambda}\big(x_\mathfrak{z}, x_{\mathfrak{z}+2}\big)\Big). \end{aligned}$$

From $(\tilde{\omega}_4)$, we have

$$\tilde{\omega}_{2\lambda}\big(x_\mathfrak{z}, x_{\mathfrak{z}+2}\big) \leq \mathfrak{s}\big[\tilde{\omega}_\lambda\big(x_\mathfrak{z}, x_{\mathfrak{z}+1}\big) + \tilde{\omega}_\lambda\big(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+2}\big)\big] - \tilde{\omega}_\lambda\big(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+1}\big).$$

Now, we utilize the representation $\kappa_\mathfrak{z}$ instead of $\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}+1})$. Thereupon, we conclude that

$$\mathcal{C}(x_\mathfrak{z}, x_{\mathfrak{z}+1}) = \alpha\kappa_\mathfrak{z} + \beta|\kappa_\mathfrak{z} - \kappa_{\mathfrak{z}+1}| + (\mathfrak{s}^p - \alpha - \beta)\varphi\left(\kappa_\mathfrak{z} + \kappa_{\mathfrak{z}+1} - \frac{\tilde{\varpi}_\lambda(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+1})}{s}\right)$$

and so, by using $\varphi \in \Delta_\varphi$, the inequality (10) becomes

$$\begin{aligned} \mathfrak{s}^p\kappa_{\mathfrak{z}+1} &< \alpha\kappa_\mathfrak{z} + \beta|\kappa_\mathfrak{z} - \kappa_{\mathfrak{z}+1}| + (\mathfrak{s}^p - \alpha - \beta)\varphi\left(\kappa_\mathfrak{z} + \kappa_{\mathfrak{z}+1} - \frac{\tilde{\varpi}_\lambda(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+1})}{s}\right) \\ &< \alpha\kappa_\mathfrak{z} + \beta|\kappa_\mathfrak{z} - \kappa_{\mathfrak{z}+1}| + (\mathfrak{s}^p - \alpha - \beta)(\kappa_\mathfrak{z} + \kappa_{\mathfrak{z}+1}). \end{aligned} \tag{11}$$

If we assume $\kappa_\mathfrak{z} \leq \kappa_{\mathfrak{z}+1}$, then, we deduce that $|\kappa_\mathfrak{z} - \kappa_{\mathfrak{z}+1}| = \kappa_{\mathfrak{z}+1} - \kappa_\mathfrak{z}$, thereby, from (11), we achieve that

$$\mathfrak{s}^p\kappa_{\mathfrak{z}+1} < \alpha\kappa_\mathfrak{z} + \beta(\kappa_{\mathfrak{z}+1} - \kappa_\mathfrak{z}) + (\mathfrak{s}^p - \alpha - \beta)(\kappa_\mathfrak{z} + \kappa_{\mathfrak{z}+1}),$$

and by simple calculations, we get $\kappa_{\mathfrak{z}+1} < \frac{\mathfrak{s}^p - 2\beta}{\alpha}\kappa_\mathfrak{z}$. Since $\alpha + 2\beta > \mathfrak{s}^p$, this causes a contradiction due to our assumption. Hence, we yield that $\kappa_{\mathfrak{z}+1} < \kappa_\mathfrak{z}$ such that $|\kappa_\mathfrak{z} - \kappa_{\mathfrak{z}+1}| = \kappa_\mathfrak{z} - \kappa_{\mathfrak{z}+1}$. By (11), we get

$$\mathfrak{s}^p\kappa_{\mathfrak{z}+1} < \alpha\kappa_\mathfrak{z} + \beta(\kappa_\mathfrak{z} - \kappa_{\mathfrak{z}+1}) + (\mathfrak{s}^p - \alpha - \beta)(\kappa_\mathfrak{z} + \kappa_{\mathfrak{z}+1})$$

which implies that

$$\kappa_{\mathfrak{z}+1} < \frac{\mathfrak{s}^p}{\alpha + 2\beta}\kappa_\mathfrak{z}.$$

Denoting $\frac{\mathfrak{s}^p}{\alpha+2\beta}$ by δ and as $\alpha + 2\beta > \mathfrak{s}^p$, we have $\tilde{\varpi}_\lambda(x_{\mathfrak{z}+1}, x_{\mathfrak{z}+2}) < \delta\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_{\mathfrak{z}+1})$ with $0 \leq \delta < 1$. Thus, by Lemma 6, $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a $0 - \varpi^{\rho_b}-$Cauchy sequence on the $\varpi^{\rho_b}-$complete $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$. Owing to Lemma 5, the space is also $0 - \varpi^{\rho_b}-$complete; it entails that $u \in \mathcal{M}^*_{\tilde{\varpi}}$ exists such that

$$\lim_{n,m\to+\infty}\tilde{\varpi}_\lambda(x_\mathfrak{z}, x_m) = \lim_{n\to+\infty}\tilde{\varpi}_\lambda(x_\mathfrak{z}, u) = \tilde{\varpi}_\lambda(u, u) = 0. \tag{12}$$

Thus, $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$, that implements

$$\eta\left(\mathfrak{O}\left(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda\left(\mathcal{Y}x_{\mathfrak{z}(l)}, \mathcal{Y}u\right)\right), \mathfrak{O}\left[\mathcal{C}\left(x_{\mathfrak{z}(l)}, u\right)\right]\right) \geq 1,$$

has a subsequence $\left\{x_{\mathfrak{z}(l)}\right\}$. Taking (η_2') with $\psi \in \Psi^*$, $\mathfrak{O} \in \Theta$ and $\varphi \in \Delta_\varphi$ into account, the above inequality gives

$$\begin{aligned} \mathfrak{s}^{p+1}\tilde{\varpi}_\lambda\left(x_{\mathfrak{z}(l)+1}, \mathcal{Y}u\right) \leq \mathcal{C}\left(x_{\mathfrak{z}(l)}, u\right) &= \alpha\tilde{\varpi}_\lambda\left(x_{\mathfrak{z}(l)}, u\right) + \beta\left|\tilde{\varpi}_\lambda\left(x_{\mathfrak{z}(l)}, \mathcal{Y}x_{\mathfrak{z}(l)}\right) - \tilde{\varpi}_\lambda(u, \mathcal{Y}u)\right| \\ &\quad + (\mathfrak{s}^p - \alpha - \beta)\varphi\left(\tfrac{1}{\mathfrak{s}}\tilde{\varpi}_{2\lambda}\left(x_{\mathfrak{z}(l)}, \mathcal{Y}u\right)\right) \\ &\leq \alpha\tilde{\varpi}_\lambda\left(x_{\mathfrak{z}(l)}, u\right) + \beta\left|\tilde{\varpi}_\lambda\left(x_{\mathfrak{z}(l)}, x_{\mathfrak{z}(l)+1}\right) - \tilde{\varpi}_\lambda(u, \mathcal{Y}u)\right| \\ &\quad + (\mathfrak{s}^p - \alpha - \beta)\tilde{\varpi}_\lambda\left(x_{\mathfrak{z}(l)}, \mathcal{Y}u\right). \end{aligned}$$

Thereby, letting $l \to +\infty$ and considering (12), we achieve

$$\mathfrak{s}^p \tilde{\varpi}_\lambda(u, \mathcal{Y}u) \leq \mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(u, \mathcal{Y}u) \leq \lim_{n \to +\infty} \mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u)$$
$$< \lim_{n \to +\infty} \mathcal{C}(x_\mathfrak{z}, u)$$
$$\leq (\mathfrak{s}^p - \alpha) \tilde{\varpi}_\lambda(u, \mathcal{Y}u) \leq \mathfrak{s}^p \tilde{\varpi}_\lambda(u, \mathcal{Y}u)$$

which implies

$$\lim_{l \to +\infty} \mathcal{C}\left(x_{\mathfrak{z}(l)}, u\right) = \mathfrak{s}^p \tilde{\varpi}_\lambda(u, \mathcal{Y}u). \tag{13}$$

Moreover, we presume that $x_\mathfrak{z} \neq u$, for infinitely many $\mathfrak{z} \in \mathbb{N}$, without losing generality. So, utilizing (8), we have

$$\eta\left(\mathfrak{O}\left(\mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u)\right), \mathfrak{O}[\mathcal{C}(x_\mathfrak{z}, u)]\right) \geq 1.$$

Likewise, by (η_2'), we obtain

$$\psi\left(\mathfrak{O}\left(\mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u)\right)\right) < \psi(\mathfrak{O}[\mathcal{C}(x_\mathfrak{z}, u)]).$$

Owing to $\psi \in \Psi^*$ and $\mathfrak{O} \in \Theta$, we procure

$$\mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u) < \mathcal{C}(x_\mathfrak{z}, u).$$

Further,

$$\mathfrak{s}^p \tilde{\varpi}_{2\lambda}(u, \mathcal{Y}u) \leq \mathfrak{s}^{p+1}[\tilde{\varpi}_\lambda(u, \mathcal{Y}x_\mathfrak{z}) + \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u)] - \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}x_\mathfrak{z})$$
$$\leq \mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(u, \mathcal{Y}x_\mathfrak{z}) + \mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u)$$
$$< \mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(u, \mathcal{Y}x_\mathfrak{z}) + \mathcal{C}(x_\mathfrak{z}, u).$$

In the above expression, taking the limit as n tends to ∞ by considering (12) and (13), we attain

$$\mathfrak{s}^p \tilde{\varpi}_\lambda(u, \mathcal{Y}u) \leq \lim_{n \to +\infty} \mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u) < \lim_{n \to +\infty} \mathcal{C}(x_\mathfrak{z}, u) = \mathfrak{s}^p \tilde{\varpi}_\lambda(u, \mathcal{Y}u).$$

Thereupon, we obtain $\lim_{n \to +\infty} \mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u) = \mathfrak{s}^p \tilde{\varpi}_\lambda(u, \mathcal{Y}u)$. Thus, letting

$$\iota_\mathfrak{z} = \mathfrak{O}\left(\mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}x_\mathfrak{z}, \mathcal{Y}u)\right)$$

and

$$\nu_\mathfrak{z} = \mathfrak{O}(\mathcal{C}(x_\mathfrak{z}, u)).$$

By (η_3), we have $\lim_{\mathfrak{z} \to \infty} \iota_\mathfrak{z} = \lim_{\mathfrak{z} \to \infty} \nu_\mathfrak{z} = \mathfrak{O}(\mathfrak{s}^p \tilde{\varpi}_\lambda(u, \mathcal{Y}u))$ such that $\limsup_{\mathfrak{z} \to \infty} \eta(\iota_\mathfrak{z}, \nu_\mathfrak{z}) < 1$. However, this causes a contradiction. Thus, $\tilde{\varpi}_\lambda(u, \mathcal{Y}u) = u = \tilde{\varpi}_\lambda(u, u)$ for all $\lambda > 0$, that is to say u is a fixed point of $\mathcal{Y}$.

As a final case, to obtain the uniqueness of fixed point, we need to accept that another fixed point $u^* \in \mathcal{M}^*_{\tilde{\varpi}}$ with $u \neq u^*$ exists such that $\mathcal{Y}u^* = u^*$. Utilizing (8), we write

$$1 \leq \eta\left(\mathfrak{O}\left(\mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}u, \mathcal{Y}u^*)\right), \mathfrak{O}[\mathcal{C}(u, u^*)]\right) < \frac{\psi(\mathfrak{O}[\mathcal{C}(u, u^*)])}{\psi\left(\mathfrak{O}\left(\mathfrak{s}^{p+1} \tilde{\varpi}_\lambda(\mathcal{Y}u, \mathcal{Y}u^*)\right)\right)},$$

owing to the fact that ψ and $\mathfrak{O}$ are nondecreasing functions, the above expression entails

$$\begin{aligned}\mathfrak{s}^{p+1}\tilde{\omega}_\lambda(\mathcal{Y}u,\mathcal{Y}u^*) &< \alpha\tilde{\omega}_\lambda(u,u^*)+\beta|\tilde{\omega}_\lambda(u,\mathcal{Y}u)-\tilde{\omega}_\lambda(u^*,\mathcal{Y}u^*)| \\ &\quad +(\mathfrak{s}^p-\alpha-\beta)\varphi\left(\tfrac{1}{\mathfrak{s}}\tilde{\omega}_{2\lambda}(u,\mathcal{Y}u^*)\right) \\ &< \alpha\tilde{\omega}_\lambda(u,u^*)+\beta|\tilde{\omega}_\lambda(u,u)-\tilde{\omega}_\lambda(u^*,u^*)| \\ &\quad +(\mathfrak{s}^p-\alpha-\beta)\tilde{\omega}_\lambda(u,u^*) \\ &< (\mathfrak{s}^p-\beta)\tilde{\omega}_\lambda(u,u^*),\end{aligned}$$

which causes a contradiction. We gain that u is a unique fixed point of $\mathcal{Y}$. □

4.1. Consequences

In this subsection, we initially recall the concept of $\mathcal{E}-$type or $\mathcal{E}-$contraction, which was put forward by Fulga and Proca [33] in 2017, involving the term

$$\mathcal{E}(\chi,y)=d(\chi,y)+|d(\chi,\mathcal{Y}\chi)-d(y,\mathcal{Y}y)|.$$

If we select $\alpha=\beta$ in Theorem 3, then we achieve the ensuing result.

Corollary 1. *Consider that $\mathcal{M}^*_{\tilde{\omega}}$ is a $\tilde{\omega}-$complete $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ with $\mathfrak{s}>1$ and $\mathcal{Y}:\mathcal{M}^*_{\tilde{\omega}}\to\mathcal{M}^*_{\tilde{\omega}}$ is a mapping. Presume that $\alpha\in(0,1)$ with $2\alpha<\mathfrak{s}^p<3\alpha$, $\eta\in\mathcal{L}_\Psi$, $\mathfrak{O}\in\Theta$ and $\varphi\in\Delta_\varphi$ exist such that*

$$\eta\left(\mathfrak{O}\left(\mathfrak{s}^{p+1}\tilde{\omega}_\lambda(\mathcal{Y}\chi,\mathcal{Y}y)\right),\mathfrak{O}\left[\alpha\mathcal{E}(\chi,y)+(\mathfrak{s}^p-2\alpha)\varphi\left(\frac{1}{\mathfrak{s}}\tilde{\omega}_{2\lambda}(\chi,\mathcal{Y}y)\right)\right]\right)\geq 1,$$

where

$$\mathcal{E}(\chi,y)=\tilde{\omega}_\lambda(\chi,y)+|\tilde{\omega}_\lambda(\chi,\mathcal{Y}\chi)-\tilde{\omega}_\lambda(y,\mathcal{Y}y)|, \tag{14}$$

for all distinct $\chi,y\in\mathcal{M}^*_{\tilde{\omega}}-Fix(\mathcal{Y})$ and for all $\lambda>0$. Thereupon, $\mathcal{Y}$ owns a unique fixed point.

If $s=1$, our results obtained from Theorem 3 are valid in the context of partial modular metric space.

Corollary 2. *Let ω^ρ_λ be a partial modular metric and $\mathcal{M}^*_{\omega^\rho}$ be a $\omega-$complete partial modular metric space, and $\mathcal{Y}:\mathcal{M}^*_{\omega^\rho}\to\mathcal{M}^*_{\omega^\rho}$ be a mapping. Presume that $\alpha,\beta\in(0,1)$ with $\alpha+\beta<1$ and $\alpha+2\beta>1$, $\eta\in\mathcal{L}_\Psi$ and $\mathfrak{O}\in\Theta$ exist such that*

$$\eta\left(\mathfrak{O}\left(\omega^\rho_\lambda(\mathcal{Y}\chi,\mathcal{Y}y)\right),\mathfrak{O}(\mathcal{C}_s(\chi,y))\right)\geq 1,$$

where

$$\mathcal{C}_s(\chi,y)=\alpha\omega^\rho_\lambda(\chi,y)+\beta\left|\omega^\rho_\lambda(\chi,\mathcal{Y}\chi)-\omega^\rho_\lambda(y,\mathcal{Y}y)\right|+(1-\alpha-\beta)\omega^\rho_{2\lambda}(\chi,\mathcal{Y}y),$$

for all distinct $\chi,y\in\mathcal{M}^*_{\omega^\rho}-Fix(\mathcal{Y})$ and for all $\lambda>0$. Thereupon, $\mathcal{Y}$ owns a unique fixed point.

Proof. As well as $s=1$, we consider $\varphi\in\Delta_\varphi$ as $\varphi(t)\leq t$ for $t>0$, then, we achieve the desired consequence. □

We procure the subsequent result if we choose $\alpha=\beta$ in Corollary 2.

Corollary 3. *Let ϖ_λ^ρ be a partial modular metric and $\mathcal{M}^*_{\varpi^\rho}$ be a $\varpi-$complete partial modular metric space. Also, $\mathcal{Y} : \mathcal{M}^*_{\varpi^\rho} \to \mathcal{M}^*_{\varpi^\rho}$ is a mapping. Further, let there exist $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, $\eta \in \mathcal{L}_\Psi$ and $\mathfrak{O} \in \Theta$ such that*

$$\eta(\mathfrak{O}(\tilde{\varpi}_\lambda(\mathcal{Y}\chi, \mathcal{Y}y)), \mathfrak{O}[\alpha\mathcal{E}(\chi, y) + (1-2\alpha)\tilde{\varpi}_{2\lambda}(\chi, \mathcal{Y}y)]) \geq 1,$$

where $\mathcal{E}(\chi, y)$ is defined as (14), for all distinct $\chi, y \in \mathcal{M}^*_{\varpi^\rho} - Fix(\mathcal{Y})$ and for all $\lambda > 0$. Thereupon, $\mathcal{Y}$ owns a unique fixed point.

Following that, we provide some additional corollaries concerning η depending on the selection of $\mathcal{L}_\Psi$.

Corollary 4. *Let $\mathcal{M}^*_{\tilde{\varpi}}$ be a $\tilde{\varpi}-$complete $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat \mathfrak{ms}}$ with $\mathfrak{s} > 1$ and $\mathcal{Y} : \mathcal{M}^*_{\tilde{\varpi}} \to \mathcal{M}^*_{\tilde{\varpi}}$ be a mapping. Presume that $k \in (0,1)$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta < \mathfrak{s}^p$ and $\alpha + 2\beta > \mathfrak{s}^p$ and also, $\psi \in \Psi^*$, $\mathfrak{O} \in \Theta$ and $\varphi \in \Delta_\varphi$ exist such that*

$$\psi\left(\mathfrak{O}\left(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{Y}\chi, \mathcal{Y}y)\right)\right) \leq [\psi(\mathfrak{O}(\mathcal{C}(\chi, y)))]^k$$

where $\mathcal{C}(\chi, y)$ is defined as (9), for all distinct $\chi, y \in \mathcal{M}^*_{\tilde{\varpi}} - Fix(\mathcal{Y})$ and for all $\lambda > 0$. Thereupon, $\mathcal{Y}$ owns a unique fixed point.

Proof. Assume that $\eta_k \in \mathcal{L}_\Psi$, that is, $\eta_k(\iota, \nu) = \frac{[\psi(\nu)]^k}{\psi(\iota)}$ with $k \in (0,1)$, the proof can be easily obtained. □

Corollary 5. *Let $\mathcal{M}^*_{\tilde{\varpi}}$ be $\tilde{\varpi}-$complete $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat \mathfrak{ms}}$ with $\mathfrak{s} > 1$ and $\mathcal{Y} : \mathcal{M}^*_{\tilde{\varpi}} \to \mathcal{M}^*_{\tilde{\varpi}}$ be a mapping. Presume that $k \in (0,1)$ and $\alpha \in (0,1)$ with $2\alpha < \mathfrak{s}^p < 3\alpha$ and $\alpha + 1 > \mathfrak{s}^p$ and also, $\psi \in \Psi^*$, $\mathfrak{O} \in \Theta$ and $\varphi \in \Delta_\varphi$ exist such that*

$$\psi\left(\mathfrak{O}\left(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{Y}\chi, \mathcal{Y}y)\right)\right) \leq [\psi(\mathfrak{O}(\mathcal{C}^*(\chi, y)))]^k$$

where

$$\begin{aligned} \mathcal{C}^*(\chi, y) &= \alpha\tilde{\varpi}_\lambda(\chi, y) + \alpha|\tilde{\varpi}_\lambda(\chi, \mathcal{Y}\chi) - \tilde{\varpi}_\lambda(y, \mathcal{Y}y)| \\ &\quad + (\mathfrak{s}^p - 2\alpha)\varphi\left(\tfrac{1}{\mathfrak{s}}\tilde{\varpi}_{2\lambda}(\chi, \mathcal{Y}y)\right), \end{aligned} \tag{15}$$

for all distinct $\chi, y \in \mathcal{M}^*_{\tilde{\varpi}} - Fix(\mathcal{Y})$ and for all $\lambda > 0$. Thereupon, $\mathcal{Y}$ owns a unique fixed point.

Proof. Considering $\alpha = \beta$ in the expression of $\mathcal{C}(\chi, y)$, the proof follows Corollary 4. □

Corollary 6. *Let $\mathcal{M}^*_{\tilde{\varpi}}$ be a $\tilde{\varpi}-$complete $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat \mathfrak{ms}}$ with $\mathfrak{s} > 1$ and $\mathcal{Y} : \mathcal{M}^*_{\tilde{\varpi}} \to \mathcal{M}^*_{\tilde{\varpi}}$ be a mapping. Presume that $\alpha, \beta \in (0,1)$ with $\alpha + \beta < \mathfrak{s}^p$ and $\alpha + 2\beta > \mathfrak{s}^p$ and also, $\psi \in \Psi^*$, $\mathfrak{O} \in \Theta$ and $\varphi \in \Delta_\varphi$ exist such that*

$$\psi\left(\mathfrak{O}\left(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{Y}\chi, \mathcal{Y}y)\right)\right) \leq \frac{\psi(\mathfrak{O}(\mathcal{C}(\chi, y)))}{\phi(\psi(\mathfrak{O}(\mathcal{C}(\chi, y))))},$$

where $\mathcal{C}(\chi, y)$ is defined as (9) and ϕ is a non-decreasing and lower semi-continuous self-mapping on $[1, \infty)$, satisfying $\phi^{-1}(\{1\}) = 1$, for all distinct $\chi, y \in \mathcal{M}^*_{\tilde{\varpi}} - Fix(\mathcal{Y})$ and for all $\lambda > 0$. Thereupon, $\mathcal{Y}$ owns a unique fixed point.

Proof. Contemplating the function η as $\eta_\phi \in \mathcal{L}_\Psi$, i.e., $\eta_\phi(\iota, \nu) = \frac{\psi(\nu)}{\psi(\iota)\phi(\psi(\nu))}$, the proof follows as in Theorem 3. □

Remark 2. *Corollary (6) can be redefined by considering $\alpha = \beta$ in the expression $\mathcal{C}(\chi, y)$. Moreover, as in Corollaries (2) and (3), by taking $s = 1$ in Corollaries (4) and (6), various consequences can be achieved in the context of partial modular metric space, too.*

Example 4. *Let $\mathcal{M}^*_{\tilde{\varpi}} = [0,1]$ and consider the partial modular b−metric by*

$$\tilde{\varpi}_\lambda(\chi, y) = \frac{|\chi - y|^2}{\lambda} + \max\{\chi, y\}$$

for all $\chi, y \in \mathcal{M}^*_{\tilde{\varpi}} - Fix(\mathcal{Y})$ and for all $\lambda > 0$. Bear in mind that $\mathcal{M}^*_{\tilde{\varpi}}$ is a $\tilde{\varpi}$−complete $\mathfrak{P}^{\mathfrak{artial}}_{\mathfrak{m}_\flat \mathfrak{ms}}$ with the parameter $\mathfrak{s} = 2$.

Moreover, let the mapping $\mathcal{Y} : \mathcal{M}^*_{\tilde{\varpi}} \to \mathcal{M}^*_{\tilde{\varpi}}$ be verified with $\mathcal{Y}\chi = \frac{\chi}{4}$. Now, we demonstrate the contractivity conditions of Corollary 1, that is, the conditions

$$\eta\Big(\mathfrak{D}\Big(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{Y}\chi, \mathcal{Y}y)\Big), \mathfrak{D}[\mathcal{C}^*(\chi, y)]\Big) \geq 1, \tag{16}$$

where $\mathcal{C}^*(\chi, y)$ as defined in (15), the constants $\alpha = \frac{1}{2} \in (0,1)$ and $\exists p > 0$ such that $\mathfrak{s}^p = \frac{11}{10}$, which satisfying the statement $2\alpha < \mathfrak{s}^p < 3\alpha$, and contemplating the $\eta = \eta_k$, i.e., $\eta(\iota, \nu) = \frac{[\psi(\nu)]^k}{\psi(\iota)}$ with $k = \frac{9}{10} \in (0,1)$ satisfied $\chi, y \in \mathcal{M}^*_{\tilde{\varpi}} - Fix(\mathcal{Y})$ and for all $\lambda > 0$. In reality, we also yield to maintain the criteria of the Corollary 5. For this, we select $\psi \in \Psi^*$ as $\psi(\mathfrak{a}) = \mathfrak{a}^2$ and further, the function $\mathfrak{D} : (0,\infty) \to (1,\infty)$ by $\mathfrak{D}(\mathfrak{a}) = e^{\mathfrak{a}}$.

Without disregarding the broader case, we believe that $\chi > y > \sqrt{\frac{10{,}887}{10{,}968}}\chi$.

Thereupon, denoting $f = \mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{Y}\chi, \mathcal{Y}y)$ and $g = \mathcal{C}^*(\chi, y)$, from (16), we achieve

$$\eta(\mathfrak{D}(f), \mathfrak{D}[g]) = \frac{[\psi(\mathfrak{D}(g))]^{\frac{9}{10}}}{\psi(\mathfrak{D}[f])} = \frac{[\psi(e^g)]^{\frac{9}{10}}}{\psi(e^f)} = \frac{(e^{2g})^{\frac{9}{10}}}{e^{2f}} = e^{2\left(\frac{9}{10}g - f\right)} \tag{17}$$

where

$$f = \mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{Y}\chi, \mathcal{Y}y) = \frac{21}{10}\frac{\left|\frac{\chi}{4} - \frac{y}{4}\right|^2}{\lambda} + \frac{21}{10}\max\left\{\frac{\chi}{4}, \frac{y}{4}\right\} = \frac{21(\chi - y)^2}{160\lambda} + \frac{21x}{40}$$

and, by considering $\varphi \in \Delta_\varphi$,

$$\begin{aligned} g = \mathcal{C}^*(\chi, y) &= \alpha\tilde{\varpi}_\lambda(\chi, y) + \alpha|\tilde{\varpi}_\lambda(\chi, \mathcal{Y}\chi) - \tilde{\varpi}_\lambda(y, \mathcal{Y}y)| \\ &\quad + (\mathfrak{s}^p - 2\alpha)\varphi\left(\tfrac{1}{\mathfrak{s}}\tilde{\varpi}_{2\lambda}(\chi, \mathcal{Y}y)\right), \\ &= \frac{1}{2}\left[\frac{|\chi-y|^2}{\lambda} + \max\{\chi, y\} + \left|\frac{|\chi-\frac{\chi}{4}|^2}{\lambda} + \max\{\chi, \tfrac{\chi}{4}\} - \frac{|y-\frac{y}{4}|^2}{\lambda} - \max\{y, \tfrac{y}{4}\}\right|\right] \\ &\quad + \left(\frac{11}{10} - 2\frac{1}{2}\right)\varphi\left(\frac{1}{2}\frac{|\chi-\frac{y}{4}|^2}{2\lambda} + \frac{1}{2}\max\{\chi, \tfrac{y}{4}\}\right) \\ &= \frac{1}{2}\left[\frac{|\chi-y|^2}{\lambda} + \chi + \left|\frac{\chi^2-y^2}{16\lambda} + \chi - y\right|\right] + \frac{|4x-y|^2}{640\lambda} + \frac{\chi}{20} \\ &= \frac{356\chi^2 - 648xy + 301y^2}{640\lambda} + \frac{21x - 10y}{20}. \end{aligned}$$

Thereby, (17) turns into

$$\eta(\mathfrak{O}(f),\mathfrak{O}[g]) = e^{2\left[\frac{9}{10}\left(\frac{356x^2-648xy+301y^2}{640\lambda}+\frac{21x-10y}{20}\right)-\frac{21(x-y)^2}{160\lambda}-\frac{21x}{40}\right]}$$
$$\geq e^{2\left(\frac{-5196x^2+10{,}968xy-5691y^2}{6400\lambda}+\frac{9(11x-10y)^2}{200\lambda}\right)}$$
$$\geq e^{2\left(\frac{10{,}968y^2-10{,}887x^2}{6400\lambda}+\frac{9(11x-10y)^2}{200\lambda}\right)} \geq 1,$$

that is to say that all the terms of Corollary 1 are fulfilled. It is obvious that $Fix(\mathcal{Y}) = \{0\}$. On the other hand, if we select the constant k much closer to point 1, then one can achieve a wider interval for the x and y.

4.2. An Application to Homotopy Theory

This section includes an application of homotopy theory that supports the validity of our results.

Theorem 4. *Regard $(\mathcal{M},\tilde{\varpi})$ as a $\tilde{\varpi}-$complete $\mathcal{P}^{\mathfrak{artial}}_{\mathfrak{m}_b\mathfrak{m}\mathfrak{s}}$, and Υ, Λ is an open and closed subset of $\mathcal{M}$, respectively. Consider $\mathcal{H} : \Lambda \times [0,1] \to \mathcal{M}$ to be an operator fulfilling the ensuing terms.*

(a) $x \neq \mathcal{H}(x,\iota)$ *for every* $x \in \Lambda \backslash \Upsilon$ *and* $\iota \in [0,1)$.

(b) *For all* $x, y \in \Lambda$ *and* $\iota, k \in [0,1)$, *we have*

$$\psi\left(\mathfrak{O}\left(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{H}(x,\iota),\mathcal{H}(y,\iota))\right)\right) \leq [\psi(\mathfrak{O}(\mathcal{C}(x,y)))]^k$$

where

$$\mathcal{C}(x,y) = \alpha\tilde{\varpi}_\lambda(x,y) + \alpha|\tilde{\varpi}_\lambda(x,\mathcal{H}(x,\iota)) - \tilde{\varpi}_\lambda(y,\mathcal{H}(y,\iota))|$$
$$+(\mathfrak{s}^p - 2\alpha)\varphi\left(\tfrac{1}{\mathfrak{s}}\tilde{\varpi}_{2\lambda}(x,\mathcal{H}(y,\iota))\right),$$

(c) $\psi : [0,1] \to \mathbb{R}$ *is continuous and holds the subsequent inequality*

$$\mathfrak{s}\tilde{\varpi}_\lambda(\mathcal{H}(x,\iota),\mathcal{H}(x,\iota^*)) \leq |\psi(\iota) - \psi(\iota^*)|$$

for all $\iota, \iota^* \in [0,1)$ *and* $\forall x \in \Lambda$.

$\mathcal{H}(\cdot,0)$ *admits a fixed point* $\Leftrightarrow$ $\mathcal{H}(\cdot,1)$ *admits a fixed point.*

Proof. Construct the ensuing set

$$\mathfrak{X} = \{\iota \in [0,1] : x = \mathcal{H}(x,\iota) \text{ for some } x \in \Upsilon\}.$$

$(\Rightarrow:)$ Presume that $\mathcal{H}(\cdot,0)$ enjoys a fixed point. Then, $\mathfrak{X}$ is non-empty; that is, $0 \in \mathfrak{X}$. It is necessary to verify that in $[0,1]$, $\mathfrak{X}$ is both open and closed. Utilizing the connectedness, $\mathfrak{X} = [0,1]$ is met. As a result, $\mathcal{H}(\cdot,1)$ enjoys a fixed point in Υ.

The closedness of $\mathfrak{X}$ in $[0,1]$ shall be indicated. Let $\{\iota_\mathfrak{z}\}_{n=1}^{\infty} \subseteq \mathfrak{X}$ with $\iota_\mathfrak{z} \to \iota \in [0,1]$ as $\mathfrak{z} \to \infty$. The aim is to show that ι belongs to $\mathfrak{X}$. Owing to $\iota_\mathfrak{z} \in \mathfrak{X}$ for $n = 1,2,3,\ldots$, $x_\mathfrak{z} \in \Upsilon$ with $x_\mathfrak{z} = \mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z})$ exists. Also, for $n,m \in \{1,2,3,\ldots\}$, we have

$$\begin{aligned}\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) &= \tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z}),\mathcal{H}(x_m,\iota_m)) \\ &\leq s\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z}),\mathcal{H}(x_\mathfrak{z},\iota_m)) + s\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_m),\mathcal{H}(x_m,\iota_m)).\end{aligned} \quad (18)$$

Also, from (b), we obtain

$$\begin{aligned}\psi(\mathfrak{O}(\mathfrak{s}\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_m),\mathcal{H}(x_m,\iota_m)))) &\leq \psi\big(\mathfrak{O}\big(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_m),\mathcal{H}(x_m,\iota_m))\big)\big)\\ &\leq [\psi(\mathfrak{O}(C(x_\mathfrak{z},x_m)))]^k \\ &\leq \psi(\mathfrak{O}(C(x_\mathfrak{z},x_m))),\end{aligned} \tag{19}$$

where

$$\begin{aligned}C(x_\mathfrak{z},x_m) = {}& \alpha\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) + \alpha|\tilde{\varpi}_\lambda(x_\mathfrak{z},\mathcal{H}(x_\mathfrak{z},\iota_m)) - \tilde{\varpi}_\lambda(x_m,\mathcal{H}(x_m,\iota_m))| \\ & +(\mathfrak{s}^p - 2\alpha)\varphi\Big(\tfrac{1}{\mathfrak{s}}\tilde{\varpi}_{2\lambda}(x_\mathfrak{z},\mathcal{H}(x_m,\iota_m))\Big)\\ \leq {}& \alpha\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) + \alpha\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z}),\mathcal{H}(x_\mathfrak{z},\iota_m))\\ & +(\mathfrak{s}^p - 2\alpha)\varphi\Big(\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) + \tilde{\varpi}_\lambda(x_m,\mathcal{H}(x_m,\iota_m)) - \tfrac{1}{\mathfrak{s}}\tilde{\varpi}_\lambda(x_m,x_m)\Big)\\ \leq {}& \alpha\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) + \alpha\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z}),\mathcal{H}(x_\mathfrak{z},\iota_m))\\ & +(\mathfrak{s}^p - 2\alpha)\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m).\end{aligned}$$

Thereby, by using the properties of ψ and $\mathfrak{O}$ and also, contemplating the above, the inequality (19) turns into

$$\mathfrak{s}\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_m),\mathcal{H}(x_m,\iota_m)) \leq (\mathfrak{s}^p - \alpha)\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) + \alpha\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z}),\mathcal{H}(x_\mathfrak{z},\iota_m)).$$

In turn, if we combine the last inequality with (18) and consider (c), we obtain

$$\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) \leq |\psi(\iota) - \psi(\iota_0)| + (\mathfrak{s}^p - \alpha)\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) + \frac{\alpha}{s}|\psi(\iota) - \psi(\iota_0)|,$$

which implies

$$\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) \leq \left(\frac{\alpha + s}{s(\alpha + 1 - \mathfrak{s}^p)}\right)|\psi(\iota) - \psi(\iota_0)|.$$

By the convergence of $\{\iota_\mathfrak{z}\}_{\mathfrak{z}\in\mathbf{N}}$ with $n, m \to \infty$, we attain

$$\lim_{\mathfrak{z},m\to\infty}\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) = 0.$$

This means that $\{x_\mathfrak{z}\}_{\mathfrak{z}\in\mathbf{N}}$ is a $\tilde{\varpi}-$Cauchy sequence in $\mathcal{M}$. Due to the $\tilde{\varpi}-$completeness of $(\mathcal{M},\tilde{\varpi})$, $x^* \in \Lambda$ exists such that

$$\tilde{\varpi}_\lambda(x^*,x^*) = \lim_{\mathfrak{z}\to\infty}\tilde{\varpi}_\lambda(x^*,x_\mathfrak{z}) = \lim_{\mathfrak{z},m\to\infty}\tilde{\varpi}_\lambda(x_\mathfrak{z},x_m) = 0.$$

Moreover,

$$\begin{aligned}\tilde{\varpi}_{2\lambda}(x_\mathfrak{z},\mathcal{H}(x^*,\iota)) &= \tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z}),\mathcal{H}(x^*,\iota))\\ &\leq \mathfrak{s}\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota_\mathfrak{z}),\mathcal{H}(x_\mathfrak{z},\iota)) + \mathfrak{s}\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota),\mathcal{H}(x^*,\iota)).\end{aligned} \tag{20}$$

Similarly, we have

$$\begin{aligned}\psi(\mathfrak{O}(\mathfrak{s}\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota),\mathcal{H}(x^*,\iota)))) &\leq \psi\big(\mathfrak{O}\big(\mathfrak{s}^{p+1}\tilde{\varpi}_\lambda(\mathcal{H}(x_\mathfrak{z},\iota),\mathcal{H}(x^*,\iota))\big)\big)\\ &\leq [\psi(\mathfrak{O}(C(x_\mathfrak{z},x^*)))]^k < \psi(\mathfrak{O}(C(x_\mathfrak{z},x^*))),\end{aligned} \tag{21}$$

where

$$C(x_{\mathfrak{z}}, x^*) = \alpha\tilde{\omega}_\lambda(x_{\mathfrak{z}}, x^*) + \alpha|\tilde{\omega}_\lambda(x_{\mathfrak{z}}, \mathcal{H}(x_{\mathfrak{z}}, \iota)) - \tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota))| + (\mathfrak{s}^p - 2\alpha)\varphi\Big(\tfrac{1}{\mathfrak{s}}\tilde{\omega}_{2\lambda}(x_{\mathfrak{z}}, \mathcal{H}(x^*, \iota))\Big),$$

$$\leq \alpha\tilde{\omega}_\lambda(x_{\mathfrak{z}}, x^*) + \alpha|\tilde{\omega}_\lambda(x_{\mathfrak{z}}, \mathcal{H}(x_{\mathfrak{z}}, \iota)) - \tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota))| + (\mathfrak{s}^p - 2\alpha)\varphi\Big(\tilde{\omega}_\lambda(x_{\mathfrak{z}}, x^*) + \tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota)) - \tfrac{1}{\mathfrak{s}}\tilde{\omega}_\lambda(x^*, x^*)\Big).$$

Consequently, keeping the properties of ψ and $\mathfrak{D}$ in mind, from (21), we conclude that

$$\mathfrak{s}\tilde{\omega}_\lambda(\mathcal{H}(x_{\mathfrak{z}}, \iota), \mathcal{H}(x^*, \iota)) \leq (\mathfrak{s}^p - \alpha)\tilde{\omega}_\lambda(x_{\mathfrak{z}}, x^*) + \alpha|\tilde{\omega}_\lambda(x_{\mathfrak{z}}, \mathcal{H}(x_{\mathfrak{z}}, \iota)) - \tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota))| + (\mathfrak{s}^p - 2\alpha)\tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota)),$$

and thereupon, by using (c), the expression (20) becomes

$$\tilde{\omega}_\lambda(x_{\mathfrak{z}}, \mathcal{H}(x^*, \iota)) \leq |\psi(\iota_{\mathfrak{z}}) - \psi(\iota)| + (\mathfrak{s}^p - \alpha)\tilde{\omega}_\lambda(x_{\mathfrak{z}}, x^*) + \alpha|\tilde{\omega}_\lambda(x_{\mathfrak{z}}, \mathcal{H}(x_{\mathfrak{z}}, \iota)) - \tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota))| + (\mathfrak{s}^p - 2\alpha)\tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota)).$$

Letting $\mathfrak{z} \to \infty$ in the above, we obtain $\lim_{\mathfrak{z}\to\infty} \tilde{\omega}_\lambda(x_{\mathfrak{z}}, \mathcal{H}(x^*, \iota)) = 0$ and hence

$$\tilde{\omega}_\lambda(x^*, \mathcal{H}(x^*, \iota)) = \lim_{\mathfrak{z}\to\infty} \tilde{\omega}_\lambda(x_{\mathfrak{z}}, \mathcal{H}(x_{\mathfrak{z}}, \iota)) = 0,$$

which entails that $x^* = \mathcal{H}(x^*, \iota)$. Since (a) is provided, we gain $x^* \in \Upsilon$. Thus $\iota \in \mathfrak{X}$ and $\mathfrak{X}$ is closed in $[0,1]$.

To obtain the openness of $\mathfrak{X}$ in $[0,1]$, regard that $\iota_0 \in \mathfrak{X}$. Thence, $x_0 \in \Upsilon$ with $x_0 = \mathcal{H}(x_0, \iota_0)$ exists. Due to the openness of Υ, $r > 0$ exists such that $B_{\tilde{\omega}_\lambda}(x_0, r) \subseteq \Upsilon$ in $\mathcal{M}$. Considering $\varepsilon = \frac{\mathfrak{s}(\alpha+1-\mathfrak{s}^p)}{\alpha+s}(\tilde{\omega}_\lambda(x_0, x_0) + r) > 0$ with $\alpha \in (0,1)$ and $\mathfrak{s} \geq 1$ provided that $\alpha + 1 > \mathfrak{s}^p$, there exists $\vartheta(\varepsilon) > 0$ such that $|\psi(\iota) - \psi(\iota_0)| < \varepsilon$ for all $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$ because ψ is continuous on ι_0. Let $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$, for

$$p \in \overline{B_{\tilde{\omega}_\lambda}(x_0, r)} = \{x \in \mathcal{M} : \tilde{\omega}_\lambda(x, x_0) \leq \tilde{\omega}_\lambda(x_0, x_0) + r\},$$

we obtain

$$\tilde{\omega}_{2\lambda}(\mathcal{H}(x, \iota), x_0) = \tilde{\omega}_{2\lambda}(\mathcal{H}(x, \iota), \mathcal{H}(x_0, \iota_0)) \leq \mathfrak{s}\tilde{\omega}_\lambda(\mathcal{H}(x, \iota), \mathcal{H}(x, \iota_0)) + \mathfrak{s}\tilde{\omega}_\lambda(\mathcal{H}(x, \iota_0), \mathcal{H}(x_0, \iota_0)). \tag{22}$$

Also, using (b), we have

$$\psi(\mathfrak{D}(\mathfrak{s}\tilde{\omega}_\lambda(\mathcal{H}(x, \iota_0), \mathcal{H}(x_0, \iota_0)))) \leq \psi\big(\mathfrak{D}\big(\mathfrak{s}^{p+1}\tilde{\omega}_\lambda(\mathcal{H}(x, \iota_0), \mathcal{H}(x_0, \iota_0))\big)\big) \leq [\psi(\mathfrak{D}(C(x, x_0)))]^k < \psi(\mathfrak{D}(C(x, x_0))), \tag{23}$$

where

$$C(x, x_0) = \alpha\tilde{\omega}_\lambda(x, x_0) + \alpha|\tilde{\omega}_\lambda(x, \mathcal{H}(x, \iota_0)) - \tilde{\omega}_\lambda(x_0, \mathcal{H}(x_0, \iota_0))| + (\mathfrak{s}^p - 2\alpha)\varphi\Big(\tfrac{1}{\mathfrak{s}}\tilde{\omega}_{2\lambda}(x, \mathcal{H}(x_0, \iota_0))\Big),$$

$$\leq (\mathfrak{s}^p - \alpha)\tilde{\omega}_\lambda(x_{\mathfrak{z}}, x^*) + \alpha\tilde{\omega}_\lambda(\mathcal{H}(x, \iota), \mathcal{H}(x, \iota_0)).$$

So, in a similar way, the inequality (23) becomes

$$\mathfrak{s}\tilde{\omega}_\lambda(\mathcal{H}(\chi,\iota_0),\mathcal{H}(\varkappa_0,\iota_0)) \leq (\mathfrak{s}^p-\alpha)\tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi^*)+\alpha\tilde{\omega}_\lambda(\mathcal{H}(\chi,\iota),\mathcal{H}(\chi,\iota_0))$$

and subsequently, considering (22) and the inequality (c), we achieve

$$\begin{aligned}\tilde{\omega}_\lambda(\mathcal{H}(\chi,\iota),\varkappa_0) &\leq |\psi(\iota)-\psi(\iota_0)|+(\mathfrak{s}^p-\alpha)\tilde{\omega}_\lambda(\chi_\mathfrak{z},\chi^*)+\tfrac{\alpha}{\mathfrak{s}}|\psi(\iota)-\psi(\iota_0)| \\ &\leq (1+\tfrac{\alpha}{\mathfrak{s}})|\psi(\iota)-\psi(\iota_0)|+(\mathfrak{s}^p-\alpha)(\tilde{\omega}_\lambda(\varkappa_0,\varkappa_0)+r) \\ &\leq (1+\tfrac{\alpha}{\mathfrak{s}})\varepsilon+(\mathfrak{s}^p-\alpha)(\tilde{\omega}_\lambda(\varkappa_0,\varkappa_0)+r) \\ &\leq \tilde{\omega}_\lambda(\varkappa_0,\varkappa_0)+r\end{aligned}$$

and $\mathcal{H}(\chi,\iota)\in\overline{B_{\tilde{\omega}_\lambda}(\varkappa_0,r)}$. Therefore,

$$\mathcal{H}(\cdot,\iota):\overline{B_{\tilde{\omega}_\lambda}(\varkappa_0,r)}\to\overline{B_{\tilde{\omega}_\lambda}(\varkappa_0,r)}$$

for every fixed $\iota\in(\iota_0-\vartheta(\varepsilon),\iota_0+\vartheta(\varepsilon))$. Now, Corollary 5 can be applied to derive that $\mathcal{H}(\cdot,\iota)$ enjoys a fixed point in Λ. Owing to (a), this fixed point must belong to Y. Therefore,

$$(\iota_0-\vartheta(\varepsilon),\iota_0+\vartheta(\varepsilon))\subseteq\mathfrak{X},$$

and thus we deduce that $\mathfrak{X}$ is open in $[0,1]$. □

5. Conclusions

In conclusion, according to the attractiveness of $b-$metric, partial metric, and modular metric spaces, we derive a new generalized metric space structure referred to as partial modular $b-$metric space, which improves the results of the work of Das et al. [11] and Hosseinzadeh and Parvaneh [10]. Furthermore, we describe certain key topological properties and provide instances to back them up.

On the other hand, in the context of this space, we establish a fixed point theorem based on the concept of interpolative type contraction, which was created by Karapınar [29] and has been a valuable source for researchers working on establishing a more general contraction mapping. In addition, we look at a family of simulation functions that have $\mathcal{E}-$contraction and almost contraction mappings. There are still vacancies in the sense of $\mathfrak{P}^{\mathfrak{partial}}_{\mathfrak{m}_\flat\mathfrak{ms}}$ for fixed point outcomes. We demonstrate a basic application of homotopy theory. It should be highlighted that the findings of this study can be advanced in various ways.

Author Contributions: Conceptualization, A.B. and M.Ö.; Methodology, A.B. and M.Ö.; Formal analysis, D.K., A.B. and M.Ö.; Investigation, D.K., A.B. and M.Ö.; Data curation, A.B.; Writing—original draft, D.K., A.B. and M.Ö.; Writing—review and editing, D.K., A.B. and M.Ö.; Supervision, M.Ö. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Banach, S. Sur les operations dans les emsembles abstraits et leurs applications aux equations integrales. *Fund. Math.* **1922**, *1*, 133–181. [CrossRef]
2. Bakhtin, I.A. The contraction mapping principle in quasi-metric spaces. *Funct. Anal. Unianowsk Gos. Ped. Inst.* **1989**, *30*, 26–37.
3. Czerwik, S. Contraction mappings in $b-$metric spaces. *Acta. Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
4. Czerwik, S. Nonlinear set-valued contraction mappings in $b-$metric spaces. *Atti Semin. Mat. Fis. Univ. Modena* **1998**, *46*, 263–276.
5. Matthews, S.G. *Partial Metric Topology*; Research Report 212; Dept. of Computer Science, The University of Warwick: Coventry, UK, 1992.

6. Mustafa, Z.; Roshan, J.R.; Parvaneh, V.; Kadelburg, Z. Some common fixed point results in ordered partial $b-$metric spaces. *J. Inequal. Appl.* **2013**, *562*, 1–26. [CrossRef]
7. Shukla, S. Partial $b-$metric spaces and fixed point theorems. *Mediterr. J. Math.* **2014**, *11*, 703–711. [CrossRef]
8. Chistyakov, V.V. Modular metric spaces, I: Basic concepts. *Nonlinear Anal.* **2010**, *72*, 1–14. [CrossRef]
9. Ege, M.E.; Alaca, C. Some results for modular $b-$metric spaces and an application to system of linear equations. *Azerbaijan J. Math.* **2018**, *8*, 3–14.
10. Hosseinzadeh, H.; Parvaneh, V. Meir-Keeler type contractive mappings in modular and partial modular metric spaces. *Asian-Eur. J. Math.* **2020**, *13*, 1–18. [CrossRef]
11. Das, D.; Narzary, S.; Singh, Y.M.; Khan, M.S.; Sessa, S. Fixed point results on partial modular metric space. *Axioms* **2022**, *11*, 62. [CrossRef]
12. Chistyakov, V.V. Modular metric spaces, II: Application to superposition operators. *Nonlinear Anal.* **2010**, *72*, 15–30. [CrossRef]
13. Chistyakov, V.V. Fixed points of modular contractive maps. *Dokl. Math.* **2012**, *86*, 515–518. [CrossRef]
14. Öztürk, M.; Büyükkaya, A. Fixed point results for Suzuki-type $\Sigma-$contractions via simulation functions in modular $b-$metric spaces. *Math. Meth. Appl. Sci.* **2022**, *45*, 12167–12183. [CrossRef]
15. Büyükkaya, A.; Fulga, A.; Öztürk, M. On generalized Suzuki-Proinov type $\left(\alpha, \mathcal{Z}_E^*\right)-$contractions in modular $b-$metric spaces. *Filomat* **2023**, *37*, 1207–1222. [CrossRef]
16. Khojasteh, F.; Shukla, S.; Radenovic, S. A new approach to the study of fixed point theorems for simulation functions. *Filomat* **2015**, *29*, 1189–1194. [CrossRef]
17. Argoubi, H.; Samet, B.; Vetro, C. Nonlinear contractions involving simulation functions in a metric space with a partial order. *J. Nonlinear Sci. Appl.* **2015**, *8*, 1082–1094. [CrossRef]
18. Roldan-Lopez-de-Hierro, A.F.; Karapınar, E.; Roldan-Lopez-de-Hierro, C.; Martínez-Moreno, J. Coincidence point theorems on metric spaces via simulation functions. *J. Comput. Appl. Math.* **2015**, *275*, 345–355. [CrossRef]
19. Joonaghany, G.H.; Farajzadeh, A.; Azhini, M.; Khojasteh, F. A new common fixed point theorem for Suzuki type contractions via generalized $\Psi-$simulation functions. *Sahand Commun. Math. Anal.* **2019**, *16*, 129–148.
20. Zoto, K.; Mlaiki, N.; Aydi, H. Related fixed point theorems via general approach of simulation functions. *J. Math.* **2020**, *2020*, 4820191. [CrossRef]
21. Fulga, A.; Karapınar, E. Some results on $S-$contractions of type E. *Mathematics* **2018**, *6*, 195. [CrossRef]
22. Aydi, H.; Karapınar, E.; Rakocevic, V. Nonunique fixed point theorems on $b-$metric spaces via simulation functions. *Jordan J. Math. Stat.* **2019**, *12*, 265–288.
23. Agarwal, R.P.; Karapınar, E. Interpolative Rus-Reich-Ciric type contractions via simulation functions. *An. St. Univ. Ovidius Constanta* **2019**, *27*, 137–152.
24. Öztürk, M.; Golkarmanesh, F.; Büyükkaya, A.; Parvaneh, V. Generalized almost simulative $\hat{Z}_{\Psi^*}^{\Theta}-$contraction mappings in modular $b-$metric spaces. *J. Math. Ext.* **2023**, *17*, 1–37.
25. Cho, S.H. Fixed point theorem for $Z-$contraction generalized metric spaces. *Abstr. Appl. Anal.* **2018**, *2018*, 1327691. [CrossRef]
26. Cho, S.H. Fixed point theorems for $\mathcal{L}_\varphi$ contractions in Branciari distance spaces. *Axioms* **2022**, *11*, 479. [CrossRef]
27. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequal. Appl.* **2014**, *38*, 2014. [CrossRef]
28. Liu, X.D.; Chang, S.S.; Xiao Y.; Zhao, L.C. Some fixed point theorems concerning $(\psi, \phi)-$type contraction in complete metric spaces. *J. Nonlinear Sci. Appl.* **2016**, *9*, 4127–4136. [CrossRef]
29. Karapınar, E. Revisiting the Kannan type contractions via interpolation. *Adv. Theory Nonlinear Anal. Appl.* **2018**, *2*, 85–87. [CrossRef]
30. Karapınar, E.; Agarwal, R.; Aydi, H. Interpolative Reich–Rus–Ćirić type contractions on partial metric spaces. *Mathematics* **2018**, *6*, 256. [CrossRef]
31. Dung, N.V.; Hang, V.T.L. Remarks on partial $b-$metric spaces and fixed point. *Mat. Vesn.* **2017**, *69*, 231–240.
32. Karapınar, E.; Chen, C.M.; Alghamdi, M.; Fulga, A. Advances on the fixed point results via simulation function involving rational terms. *Adv. Differ. Equ.* **2021**, *409*, 1–20. [CrossRef]
33. Fulga, A.; Proca, A.M. Fixed point for Geraghty ϑ_E-contractions. *J. Nonlinear Sci. Appl.* **2017**, *10*, 5125–5131. [CrossRef]