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Article

Some New Estimates of Hermite–Hadamard Inequality with Application

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Abstract: This paper establishes several new inequalities of Hermite–Hadamard type for $|f'|^q$ being convex for some fixed $q \in (0, 1]$. As application, some error estimates on special means of real numbers are given.

Keywords: Hermite–Hardamard inequality; convex function; integral inequalities; special means

MSC: 26A51; 26D15

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1. Introduction

For simplicity, in this paper we let $I \subseteq \mathbb{R} = (-\infty, +\infty)$ be an interval.

Definition 1. *A function $f : I \to \mathbb{R}$ is convex if*

$$f[t\alpha + (1-t)\beta] \le tf(\alpha) + (1-t)f(\beta) \tag{1}$$

is true for any $\alpha, \beta \in I$ and $0 \le t \le 1$. The inequality (1) is reversed if f is concave on I.

Suppose that the function $f : I \to \mathbb{R}$ is convex on I, $\alpha, \beta \in I$ with $\alpha < \beta$, then

$$f\left(\frac{\alpha+\beta}{2}\right) \le \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x \le \frac{f(\alpha)+f(\beta)}{2}. \tag{2}$$

It is well known in the literature as the Hermite–Hadamard inequality.

In [1], Dragomir and Agarwal obtained the following inequalities for the right part of (2).

Theorem 1 (Theorem 2.2 in [1])**.** *Suppose that $\alpha, \beta \in I^\circ$ and $\alpha < \beta$, the function $f : I^\circ \to \mathbb{R}$ is differentiable and $|f'|$ is convex on $[\alpha, \beta]$, then*

$$\left|\frac{f(\alpha)+f(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \le \frac{(\beta-\alpha)(|f'(\alpha)|+|f'(\beta)|)}{8}. \tag{3}$$

Theorem 2 (Theorem 2.3 in [1])**.** *Suppose that $\alpha, \beta \in I^\circ$, $\alpha < \beta$, and $p > 1$, the function $f : I^\circ \to \mathbb{R}$ is differentiable and $|f'|^{\frac{p}{p-1}}$ is convex on $[\alpha, \beta]$, then*

$$\left|\frac{f(\alpha)+f(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \le \frac{\beta-\alpha}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(\alpha)|^{\frac{p}{p-1}}+|f'(\beta)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}. \tag{4}$$

In the literature, the extensions of the arithmetic, geometric, identric, logarithmic, and generalized logarithmic mean from two positive real numbers are, respectively, defined by

$$\begin{aligned}
&A(s,t) = \frac{s+t}{2}, && s,t \in \mathbb{R},\\
&G(s,t) = \sqrt{st}, && s,t \in \mathbb{R}, \quad s,t > 0,\\
&I(s,t) = \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}}, && s,t > 0,\\
&L(s,t) = \frac{t-s}{\ln|t| - \ln|s|}, && |s| \neq |t|, \quad st \neq 0,\\
&L_n(s,t) = \left[\frac{t^{n+1} - s^{n+1}}{(n+1)(t-s)}\right]^{\frac{1}{n}}, && n \in \mathbb{Z} \setminus \{-1, 0\}, \quad s,t \in \mathbb{R}, \quad s \neq t.
\end{aligned}$$

It is well known that $G(s,t) < L(s,t) < I(s,t) < A(s,t)$ for $s,t > 0$ with $s \neq t$, for example, see [2].

Dragomir and Agarwal used Theorem 1 and Theorem 2 to establish the following error estimates on special means:

Theorem 3 (Propositions 3.1–3.4 in [1])**.** *Suppose that $s,t \in I^\circ$, $s < t$, $n \in \mathbb{Z}$, then*

$$|A(s^n,t^n) - L_n(s,t)^n| \leq \frac{n(t-s)}{4} A\left(|s|^{n-1}, |t|^{n-1}\right), \qquad n \geq 2, \tag{5}$$

$$|A(s^n,t^n) - L_n(s,t)^n| \leq \frac{n(t-s)}{2(p+1)^{\frac{1}{p}}} \left[A\left(|s|^{\frac{(n-1)p}{p-1}}, |t|^{\frac{(n-1)p}{p-1}}\right)\right]^{\frac{p-1}{p}}, \qquad n \geq 2, \ p > 1, \tag{6}$$

$$|A\left(s^{-1},t^{-1}\right) - L(s,t)^{-1}| \leq \frac{t-s}{4} A\left(|s|^{-2}, |t|^{-2}\right), \qquad 0 \notin [s,t], \tag{7}$$

$$|A\left(s^{-1},t^{-1}\right) - L(s,t)^{-1}| \leq \frac{t-s}{2(p+1)^{\frac{1}{p}}} \left[A\left(|s|^{\frac{-2p}{p-1}}, |t|^{\frac{-2p}{p-1}}\right)\right]^{\frac{p-1}{p}}, \qquad 0 \notin [s,t], \ p > 1. \tag{8}$$

In [3], Pearce and Pečarić obtained a better upper bound for the inequality (4). Moreover, they obtained a similar inequality on the left part of (2).

Theorem 4 (Theorems 1 and 2 in [3])**.** *Suppose that $\alpha, \beta \in I^\circ$, $\alpha < \beta$ and $q \geq 1$, the function $f : I^\circ \to \mathbb{R}$ is differentiable and $|f'|^q$ is convex on $[\alpha, \beta]$, then*

$$\left|\frac{f(\alpha) + f(\beta)}{2} - \frac{1}{\beta - \alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \leq \frac{\beta - \alpha}{4}\left(\frac{|f'(\alpha)|^q + |f'(\beta)|^q}{2}\right)^{\frac{1}{q}} \tag{9}$$

and

$$\left|f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta - \alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \leq \frac{\beta - \alpha}{4}\left(\frac{|f'(\alpha)|^q + |f'(\beta)|^q}{2}\right)^{\frac{1}{q}}. \tag{10}$$

By using Theorem 4, Pearce and Pečarić generalized and improved the error estimates (5)–(8) and obtained the following error estimates on special means:

Theorem 5 (Propositions 1 and 2 in [3])**.** *Suppose that $s,t \in \mathbb{R}$, $s < t$, $0 \notin [s,t]$, $n \in \mathbb{Z}$, $|n| \geq 2$, $q \geq 1$, then*

$$|A(s^n,t^n) - L_n(s,t)^n| \le \frac{|n|(t-s)}{4}\left[A\left(|s|^{(n-1)q},|t|^{(n-1)q}\right)\right]^{\frac{1}{q}}, \tag{11}$$

$$|A(s,t)^n - L_n(s,t)^n| \le \frac{|n|(t-s)}{4}\left[A\left(|s|^{(n-1)q},|t|^{(n-1)q}\right)\right]^{\frac{1}{q}}, \tag{12}$$

$$|A\left(s^{-1},t^{-1}\right) - L(s,t)^{-1}| \le \frac{t-s}{4}\left[A\left(|s|^{-2q},|t|^{-2q}\right)\right]^{\frac{1}{q}}, \tag{13}$$

$$|A(s,t)^{-1} - L(s,t)^{-1}| \le \frac{t-s}{4}\left[A\left(|s|^{-2q},|t|^{-2q}\right)\right]^{\frac{1}{q}}. \tag{14}$$

However, using their method could not obtain the corresponding estimate for $q < 1$. In this paper, supposing that $|f'|^q$ is convex for some fixed $0 < q \le 1$, we obtain some estimates of (2). Moreover, if $q = 1$, our results are the same as (9) and (10), respectively. As application, some error estimates on special means are given, then the inequalities (11)–(14) are improved.

2. Main Results

Theorem 6. *Suppose that $\alpha, \beta \in I^\circ$, $\alpha < \beta$ and $0 < q \le 1$, the function $f : I^\circ \to \mathbb{R}$ is differentiable and $|f'|^q$ is convex on $[\alpha, \beta]$.*

(i) *If $0 < q \le \frac{1}{2}$, then*

$$\left|\frac{f(\alpha)+f(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \le \frac{q(\beta-\alpha)}{2(2q+1)}\left[\frac{q\left(q+2^{\frac{1}{q}}\right)}{2(q+1)}(|f'(\alpha)|+|f'(\beta)|) + (1-q)\sqrt{|f'(\alpha)f'(\beta)|}\right]. \tag{15}$$

(ii) *If $\frac{1}{2} < q \le 1$, then*

$$\left|\frac{f(\alpha)+f(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \le \frac{q(\beta-\alpha)}{2(2q+1)}\left[\frac{1+q\cdot 2^{-\frac{1}{q}}}{q+1}(|f'(\alpha)|+|f'(\beta)|) + \left(1-2^{1-\frac{1}{q}}\right)\sqrt{|f'(\alpha)f'(\beta)|}\right]. \tag{16}$$

Clearly, if $q = 1$, then (16) is the same as (9).

Corollary 1. *Suppose that $\alpha, \beta \in I^\circ$ and $\alpha < \beta$, the function $f : I^\circ \to \mathbb{R}$ is differentiable and $|f'|^{\frac{1}{2}}$ is convex on $[\alpha, \beta]$, then for any $q \ge 1$, we have*

$$\begin{aligned}\left|\frac{f(\alpha)+f(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| &\le \frac{\beta-\alpha}{4}\left[\frac{3}{4}A\left(|f'(\alpha)|,|f'(\beta)|\right) + \frac{1}{4}G\left(|f'(\alpha)|,|f'(\beta)|\right)\right] \\ &\le \frac{\beta-\alpha}{4}A\left(|f'(\alpha)|,|f'(\beta)|\right) \\ &\le \frac{\beta-\alpha}{4}\left[A\left(|f'(\alpha)|^q,|f'(\beta)|^q\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{17}$$

Proof. Let $q = \frac{1}{2}$ in the inequality (15) and we have the first inequality. Note that $\sqrt{|f'(\alpha)f'(\beta)|} \le \frac{|f'(\alpha)|+|f'(\beta)|}{2}$, so the second inequality holds. By power–mean inequality, we can obtain the last inequality. □

Theorem 7. *Suppose that $\alpha, \beta \in I^\circ$, $\alpha < \beta$ and $0 < q \le 1$, the function $f : I^\circ \to \mathbb{R}$ is differentiable, and $|f'|^q$ is convex on $[\alpha, \beta]$.*

(i) *If* $0 < q \le \frac{1}{2}$, *then*

$$\left| f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \le \frac{q(\beta-\alpha)}{2(2q+1)}\left[\frac{q^2\left(2^{\frac{1}{q}+1}-1\right)}{2(q+1)}(|f'(\alpha)|+|f'(\beta)|) + (1-q)\left(\left(\frac{1}{3}+\frac{1}{6q}\right)2^{\frac{1}{q}}-1\right)\sqrt{|f'(\alpha)f'(\beta)|}\right]. \tag{18}$$

(ii) *If* $\frac{1}{2} < q \le 1$, *then*

$$\left| f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \le \frac{q(\beta-\alpha)}{2(2q+1)}\left[\frac{q\left(2-2^{-\frac{1}{q}}\right)}{q+1}(|f'(\alpha)|+|f'(\beta)|) + \left(2^{\frac{1}{q}}-2\right)\left(\frac{1}{3}+\frac{1}{6q}-2^{-\frac{1}{q}}\right)\sqrt{|f'(\alpha)f'(\beta)|}\right]. \tag{19}$$

Clearly, if $q = 1$, then (19) is the same as (10). If we let $q = \frac{1}{2}$ in the inequality (18), then we have the following.

Corollary 2. *Suppose that $\alpha, \beta \in I^\circ$ and $\alpha < \beta$, the function $f : I^\circ \to \mathbb{R}$ is differentiable, and $|f'|^{\frac{1}{2}}$ is convex on $[\alpha, \beta]$, then for any $q \ge 1$, we have*

$$\begin{aligned}\left| f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| &\le \frac{\beta-\alpha}{4}\left[\frac{7}{12}A(|f'(\alpha)|,|f'(\beta)|) + \frac{5}{12}G(|f'(\alpha)|,|f'(\beta)|)\right] \\ &\le \frac{\beta-\alpha}{4}A(|f'(\alpha)|,|f'(\beta)|) \\ &\le \frac{\beta-\alpha}{4}\left[A(|f'(\alpha)|^q,|f'(\beta)|^q)\right]^{\frac{1}{q}}.\end{aligned} \tag{20}$$

3. Lemmas

Lemma 1 (Lemma 2.1 in [1])**.** *Suppose that $\alpha, \beta \in I^\circ$ and $\alpha < \beta$, the function $f : I^\circ \to \mathbb{R}$ is differentiable, and $f' \in L[\alpha, \beta]$, then*

$$\frac{f(\alpha)+f(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x = \frac{\beta-\alpha}{2}\int_0^1 (1-2t)f'[t\alpha+(1-t)\beta]\mathrm{d}t. \tag{21}$$

Lemma 2 (Lemma 2.1 in [4])**.** *Suppose that $\alpha, \beta \in I^\circ$ and $\alpha < \beta$, the function $f : I^\circ \to \mathbb{R}$ is differentiable, and $f' \in L[\alpha, \beta]$, then*

$$f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x = (\beta-\alpha)\int_0^1 M(t)f'[t\alpha+(1-t)\beta]\mathrm{d}t, \tag{22}$$

where

$$M(t) = \begin{cases} -t, & t \in \left[0, \frac{1}{2}\right), \\ 1-t, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The following result can be found in [5]. For the convenience of readers, we provide the proof below.

Lemma 3 (Lemma 2.1 in [5]). *Let $x, y > 0, r \in \mathbb{R} \setminus \{0\}$.*

(i) *If $r \geq 2$ or $0 < r \leq 1$, then*
$$x^r + (2^r - 2)x^{\frac{r}{2}}y^{\frac{r}{2}} + y^r \leq (x+y)^r \leq \frac{2^r}{2r}(x^r + (2r-2)x^{\frac{r}{2}}y^{\frac{r}{2}} + y^r). \tag{23}$$

(ii) *If $1 \leq r \leq 2$ or $r < 0$, then*
$$x^r + (2^r - 2)x^{\frac{r}{2}}y^{\frac{r}{2}} + y^r \geq (x+y)^r \geq \frac{2^r}{2r}(x^r + (2r-2)x^{\frac{r}{2}}y^{\frac{r}{2}} + y^r). \tag{24}$$

Each equality is true if and only if $r = 1, 2$ or $x = y$.

Proof. It is easy to see that every equality in (23) and (24) holds when $r = 1, 2$ or $x = y$, so we suppose that $r \neq 1, 2$, $y = 1$ and $x > 1$ in the following.

First, we prove that the left parts of the inequalities (23) and (24) hold, respectively. Let

$$f(x) = (x+1)^r - x^r - (2^r - 2)x^{\frac{r}{2}} - 1, \qquad x > 1,$$
$$g(t) = (1+t)^{r-1} - 1 - (2^{r-1} - 1)t^{\frac{r}{2}}, \qquad 0 < t < 1.$$

Then
$$f'(x) = rx^{r-1}g\left(\frac{1}{x}\right), \qquad x > 1,$$
$$g'(t) = (r-1)(1+t)^{r-2} - \frac{r}{2}(2^{r-1} - 1)t^{\frac{r}{2}-1}, \qquad 0 < t < 1.$$

The following proof is divided into four cases.

(1) If $r < 0$, then
$$g'(t) = -\left[(1-r)(1+t)^{r-2} - \frac{r}{2}(1 - 2^{r-1})t^{\frac{r}{2}-1}\right] < 0, \quad 0 < t < 1.$$

Note that $g(1) = 0$, so $g(t) > 0$ $(0 < t < 1)$. It follows that $f'(x) < 0$ $(x > 1)$. Since $f(1) = 0$, we have $f(x) < 0$.

(2) If $0 < r < 1$, let
$$h_1(t) = -\left[\ln(1-r) + (r-2)\ln(1+t) - \ln\frac{r}{2} - \ln(1 - 2^{r-1}) - \left(\frac{r}{2} - 1\right)\ln t\right], \quad 0 < t < 1,$$

then $h_1'(t) = -(\frac{r-2}{1+t} - \frac{r-2}{2t}) < 0$. It means that $h_1(t)$ is strictly decreasing on $(0,1)$. Note that
$$h_1(0^+) = \ln\frac{r(1-2^{r-1})}{2(1-r)} + \frac{r-2}{2}\lim_{t\to 0^+}\ln t > 0,$$
$$h_1(1^-) = \ln\frac{r}{1-r} - \ln\frac{2^{r-1}}{1-2^{r-1}} < 0;$$

there exists $\xi_1 \in (0,1)$, such that $h_1(t) > 0$ $(0 < t < \xi_1)$ and $h_1(t) < 0$ $(\xi_1 < t < 1)$. Since $g'(t)$ and $h_1(t)$ have the same sign, we obtain $g'(t) > 0$ $(0 < t < \xi_1)$ and $g'(t) < 0$ $(\xi_1 < t < 1)$. Note that $g(1) = g(0) = 0$, and we have $g(t) > 0$ $(0 < t < 1)$. It follows that $f'(x) > 0$ $(x > 1)$. Because $f(1) = 0$, we have $f(x) > 0$ $(x > 1)$.

(3) If $1 < r < 2$, let
$$h_2(t) = \ln(r-1) + (r-2)\ln(1+t) - \ln\frac{r}{2} - \ln(2^{r-1} - 1) - \left(\frac{r}{2} - 1\right)\ln t,$$

then $h_2'(t) = \frac{r-2}{1+t} - \frac{r-2}{2t} > 0$. It means that $h_2(t)$ is strictly increasing on $(0,1)$. Note that

$$h_2(0^+) = \ln\frac{2(r-1)}{r(2^{r-1}-1)} - \frac{r-2}{2}\lim_{t\to 0^+}\ln t < 0,$$
$$h_2(1^-) = \ln\frac{r-1}{r} - \ln\frac{2^{r-1}-1}{2^{r-1}} > 0;$$

there exists $\xi_2 \in (0,1)$, such that $h_2(t) < 0$ $(0 < t < \xi_2)$ and $h_2(t) > 0$ $(\xi_2 < t < 1)$. Since $g'(t)$ and $h_2(t)$ have the same sign and $g(1) = g(0) = 0$, we have $g(t) < 0$ $(0 < t < 1)$. It follows that $f'(x) < 0$ $(x > 1)$. Note that $f(1) = 0$, and we have $f(x) < 0$ $(x > 1)$.

(4) If $r > 2$, then $h_2'(t) = \frac{r-2}{1+t} - \frac{r-2}{2t} < 0$. It means that $h_2(t)$ is strictly decreasing in $(0,1)$. Note that $h_2(0^+) > 0$, $h_2(1^-) < 0$, and there exists $\xi_3 \in (0,1)$, such that $h_2(t) > 0$ $(0 < t < \xi_3)$ and $h_2(t) < 0$ $(\xi_3 < t < 1)$. Since $g'(t)$ and $h_2(t)$ have the same sign and $g(1) = g(0) = 0$, we have $g(t) > 0$ $(0 < t < 1)$. It follows that $f'(x) > 0$ $(x > 1)$. Then by $f(1) = 0$, we have $f(x) > 0$ $(x > 1)$.

Next, we derive that the right parts of the inequalities (23) and (24) are true, respectively. Let

$$L(x) = (x+1)^r - \frac{2^r}{2r}\left(x^r + (2r-2)x^{\frac{r}{2}} + 1\right), \qquad x > 1,$$
$$l(t) = r(t+1)^{r-1} - 2^{r-1} - 2^{r-1}(r-1)t^{\frac{r}{2}}, \qquad 0 < t < 1.$$

Then

$$L'(x) = x^{r-1}l\left(\frac{1}{x}\right), \qquad x > 1,$$
$$l'(t) = 2^{r-2}r(r-1)t^{\frac{r}{2}-1}\left[\left(\frac{t+1}{2\sqrt{t}}\right)^{r-2} - 1\right], \qquad 0 < t < 1.$$

If $0 < r < 1$ or $r > 2$, then $l'(t) > 0$. Note that $l(1) = 0$, we have $l(t) < 0$ $(0 < t < 1)$, and $L'(x) < 0$ $(x > 1)$. Then by $L(1) = 0$, we have $L(x) < 0$ $(x > 1)$, so the right parts of the inequalities (23) holds.

If $r < 0$ or $1 < r < 2$, then $l'(t) < 0$. Note that $l(1) = 0$, we have $l(t) > 0$ $(0 < t < 1)$ and $L'(x) > 0$ $(x > 1)$. Then by $L(1) = 0$, we have $L(x) > 0$ $(x > 1)$, so the right parts of the inequalities (24) holds.

The proof is complete. □

Lemma 4. *Suppose that* $\alpha, \beta \in I^\circ$, $\alpha < \beta$, $0 < q \le 1$, $0 < t < 1$, *the function* $f : I \to [0, +\infty)$ *is positive, and* f^q *is convex on* $[\alpha, \beta]$.

(i) *If* $0 < q \le \frac{1}{2}$, *then*

$$f[t\alpha + (1-t)\beta] \le q2^{\frac{1}{q}-1}\left[t^{\frac{1}{q}}f(\alpha) + (1-t)^{\frac{1}{q}}f(\beta) + \left(\frac{2}{q} - 2\right)(t(1-t))^{\frac{1}{2q}}\sqrt{f(\alpha)f(\beta)}\right]. \tag{25}$$

(ii) *If* $\frac{1}{2} \le q \le 1$, *then*

$$f[t\alpha + (1-t)\beta] \le t^{\frac{1}{q}}f(\alpha) + (1-t)^{\frac{1}{q}}f(\beta) + (2^{\frac{1}{q}} - 2)(t(1-t))^{\frac{1}{2q}}\sqrt{f(\alpha)f(\beta)}. \tag{26}$$

Proof. Since f^q is convex and $q > 0$, we have

$$f[t\alpha + (1-t)\beta] \le [tf^q(\alpha) + (1-t)f^q(\beta)]^{\frac{1}{q}}.$$

If $0 < q \le \frac{1}{2}$, then $\frac{1}{q} \ge 2$, by the right-hand side of the inequalities (23), then

$$[tf^q(\alpha)+(1-t)f^q(\beta)]^{\frac{1}{q}} \leq q2^{\frac{1}{q}-1}\left[t^{\frac{1}{q}}f(\alpha)+(1-t)^{\frac{1}{q}}f(\beta)+\left(\frac{2}{q}-2\right)(t(1-t))^{\frac{1}{2q}}\sqrt{f(\alpha)f(\beta)}\right].$$

Thus, the inequality (25) is valid.

If $\frac{1}{2} \leq q \leq 1$, then $1 \leq \frac{1}{q} \leq 2$, by the left-hand side of the inequalities (24) we have

$$[tf^q(\alpha)+(1-t)f^q(\beta)]^{\frac{1}{q}} \leq t^{\frac{1}{q}}f(\alpha)+(1-t)^{\frac{1}{q}}f(\beta)+(2^{\frac{1}{q}}-2)(t(1-t))^{\frac{1}{2q}}\sqrt{f(\alpha)f(\beta)}.$$

Hence, the inequality (26) is valid.

□

4. Derivation of Theorem 6 and 7

The derivation of Theorem 6: (i) If $0 < q \leq \frac{1}{2}$, then by the inequalities (21) and (25), we can derive that

$$\begin{aligned}
&\left|\frac{f(\alpha)+f(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \\
\leq &\frac{\beta-\alpha}{2}\int_0^1 |1-2t||f'[t\alpha+(1-t)\beta]|\mathrm{d}t \\
\leq &\frac{(\beta-\alpha)q2^{\frac{1}{q}-1}}{2}\int_0^1 |1-2t|\left[t^{\frac{1}{q}}|f'(\alpha)|+(1-t)^{\frac{1}{q}}|f'(\beta)|+\left(\frac{2}{q}-2\right)(t(1-t))^{\frac{1}{2q}}\sqrt{|f'(\alpha)f'(\beta)|}\right]\mathrm{d}t.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^1 |1-2t|(1-t)^{\frac{1}{q}}\mathrm{d}t \\
=&\int_0^1 |1-2t|t^{\frac{1}{q}}\mathrm{d}t \\
=&\int_0^{\frac{1}{2}}(1-2t)t^{\frac{1}{q}}\mathrm{d}t-\int_{\frac{1}{2}}^1(1-2t)t^{\frac{1}{q}}\mathrm{d}t \\
=&\frac{q(1+q\cdot 2^{\frac{-1}{q}})}{(q+1)(2q+1)},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |1-2t|(t(1-t))^{\frac{1}{2q}}\mathrm{d}t \\
=&\int_0^{\frac{1}{2}}(1-2t)(t(1-t))^{\frac{1}{2q}}\mathrm{d}t-\int_{\frac{1}{2}}^1(1-2t)(t(1-t))^{\frac{1}{2q}}\mathrm{d}t \\
=&\frac{2q}{2q+1}(t(1-t))^{\frac{1}{2q}+1}\,|_0^{\frac{1}{2}}-\frac{2q}{2q+1}(t(1-t))^{\frac{1}{2q}+1}|_{\frac{1}{2}}^1 \\
=&\frac{q\cdot 2^{\frac{-1}{q}}}{2q+1},
\end{aligned}$$

so (15) is valid.

(ii) If $\frac{1}{2} < q \leq 1$, then by (21) and (26), we have

$$\begin{aligned}
&\left|\frac{f(\alpha)+f(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \\
\leq&\frac{\beta-\alpha}{2}\int_0^1 |1-2t||f'[t\alpha+(1-t)\beta]|\mathrm{d}t \\
\leq&\frac{\beta-\alpha}{2}\int_0^1 |1-2t|\left[t^{\frac{1}{q}}|f(\alpha)|+(1-t)^{\frac{1}{q}}|f(\beta)|+(2^{\frac{1}{q}}-2)(t(1-t))^{\frac{1}{2q}}\sqrt{|f(\alpha)f(\beta)|}\right]\mathrm{d}t \\
=&\frac{q(\beta-\alpha)}{2(2q+1)}\left[\frac{1+q\cdot 2^{-\frac{1}{q}}}{q+1}(|f'(\alpha)|+|f'(\beta)|)+\left(1-2^{1-\frac{1}{q}}\right)\sqrt{|f'(\alpha)f'(\beta)|}\right].
\end{aligned}$$

so (16) is valid.

The derivation of Theorem 7: (i) If $0 < q \leq \frac{1}{2}$, then by (22) and (25) we can derive that

$$\begin{aligned}
&\left|f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\mathrm{d}x\right| \\
\leq&(\beta-\alpha)\int_0^1 |M(t)||f'[t\alpha+(1-t)\beta]|\mathrm{d}t \\
\leq&(\beta-\alpha)q2^{\frac{1}{q}-1}\int_0^1 |M(t)|\left[t^{\frac{1}{q}}|f'(\alpha)|+(1-t)^{\frac{1}{q}}|f'(\beta)|+\left(\frac{2}{q}-2\right)(t(1-t))^{\frac{1}{2q}}\sqrt{|f'(\alpha)f'(\beta)|}\right]\mathrm{d}t.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^1 |M(t)|(1-t)^{\frac{1}{q}}\mathrm{d}t \\
=&\int_0^1 |M(t)|t^{\frac{1}{q}}\mathrm{d}t \\
=&\int_0^{\frac{1}{2}} t^{\frac{1}{q}+1}\mathrm{d}t+\int_{\frac{1}{2}}^1 (1-t)t^{\frac{1}{q}}\mathrm{d}t \\
=&\frac{q^2(1-2^{-\frac{1}{q}-1})}{(q+1)(2q+1)},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |M(t)|(t(1-t))^{\frac{1}{2q}}\mathrm{d}t \\
=&\int_0^{\frac{1}{2}} t^{\frac{1}{2q}+1}(1-t)^{\frac{1}{2q}}\mathrm{d}t+\int_{\frac{1}{2}}^1 t^{\frac{1}{2q}}(1-t)^{\frac{1}{2q}+1}\mathrm{d}t \\
=&-\frac{q\cdot 2^{-\frac{1}{q}-1}}{2q+1}+B\left(\frac{1}{2q}+1,\frac{1}{2q}+2\right)
\end{aligned}$$

where the beta function is

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\ \ x>0,\ y>0.$$

Clearly, $B(x,x)$ is decreasing on $(0,+\infty)$ and $\frac{1}{2q} \geq 1$, then

$$B\left(\frac{1}{2q}+1,\frac{1}{2q}+2\right)=\frac{1}{2}B\left(\frac{1}{2q}+1,\frac{1}{2q}+1\right)\leq\frac{1}{2}B(2,2)=\frac{1}{12}.$$

Thus, (18) is valid.

(ii) If $\frac{1}{2} < q \leq 1$, then by (22) and (26), we can induce that

$$\begin{aligned}
&\left|f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x)\mathrm{d}x\right| \\
\leq &(\beta-\alpha)\int_0^1 |M(t)||f'[t\alpha+(1-t)\beta]|\mathrm{d}t \\
\leq &(\beta-\alpha)\int_0^1 |M(t)|\left[t^{\frac{1}{q}}|f(\alpha)| + (1-t)^{\frac{1}{q}}|f(\beta)| + (2^{\frac{1}{q}}-2)(t(1-t))^{\frac{1}{2q}}\sqrt{|f(\alpha)f(\beta)|}\right]\mathrm{d}t \\
\leq &\frac{q(\beta-\alpha)}{2(2q+1)}\left[\frac{q\left(2-2^{-\frac{1}{q}}\right)}{q+1}(|f'(\alpha)|+|f'(\beta)|) + \left(2^{\frac{1}{q}}-2\right)\left(\frac{1}{3}+\frac{1}{6q}-2^{-\frac{1}{q}}\right)\sqrt{|f'(\alpha)f'(\beta)|}\right].
\end{aligned}$$

Thus, (19) is valid.

5. Applications

In this section, we will use Corollary 1 and Corollary 2 to establish some error estimates on special means, then the inequalities (11)–(14) are improved.

Proposition 1. *Suppose that $s, t \in \mathbb{R}$, $s < t$, $0 \notin [s, t]$, $n \in \mathbb{Z}$, $n \geq 3$ or $n \leq -2$, then*

$$|A(s^n, t^n) - L_n(s,t)^n| \leq \frac{|n|(t-s)}{4}\left[\frac{3}{4}A\left(|s|^{n-1}, |t|^{n-1}\right) + \frac{1}{4}G\left(|s|^{n-1}, |t|^{n-1}\right)\right], \tag{27}$$

$$|A(s,t)^n - L_n(s,t)^n| \leq \frac{|n|(t-s)}{4}\left[\frac{7}{12}A\left(|s|^{n-1}, |t|^{n-1}\right) + \frac{5}{12}G\left(|s|^{n-1}, |t|^{n-1}\right)\right]. \tag{28}$$

Proof. Let $f(x) = x^n$, $x \in [s, t]$, $n \in \mathbb{Z}$, $n \geq 3$ or $n \leq -2$, then

$$\left(|f'(x)|^{\frac{1}{2}}\right)'' = \frac{\sqrt{|n|}(n-1)(n-3)}{4}|x|^{\frac{n-5}{2}} \geq 0.$$

Thus, $|f'(x)|^{\frac{1}{2}}$ is convex on $[s, t]$. It follows that (27) and (28) hold by using Corollary 1 and Corollary 2, respectively. □

Remark 1. *For any $q \geq 1$, by the inequalities (17) and (20), we have*

$$\frac{|n|(t-s)}{4}\left[\frac{3}{4}A\left(|s|^{n-1}, |t|^{n-1}\right) + \frac{1}{4}G\left(|s|^{n-1}, |t|^{n-1}\right)\right] \leq \frac{|n|(t-s)}{4}\left[A\left(|s|^{(n-1)q}, |t|^{(n-1)q}\right)\right]^{\frac{1}{q}},$$

and

$$\frac{|n|(t-s)}{4}\left[\frac{7}{12}A\left(|s|^{n-1}, |t|^{n-1}\right) + \frac{5}{12}G\left(|s|^{n-1}, |t|^{n-1}\right)\right] \leq \frac{|n|(t-s)}{4}\left[A\left(|s|^{(n-1)q}, |t|^{(n-1)q}\right)\right]^{\frac{1}{q}}.$$

Thus, for $n \geq 3$ or $n \leq -2$, we obtain an improvement of the inequalities (11) and (12), which is an improvement of the inequalities (5) and (6).

Proposition 2. *Suppose that $s, t \in \mathbb{R}$, $s < t$, $0 \notin [s, t]$, then*

$$|A\left(s^{-1}, t^{-1}\right) - L(s,t)^{-1}| \leq \frac{t-s}{4}\left[\frac{3}{4}A\left(s^{-2}, t^{-2}\right) + \frac{1}{4}G\left(s^{-2}, t^{-2}\right)\right], \tag{29}$$

$$|A(s,t)^{-1} - L(s,t)^{-1}| \leq \frac{t-s}{4}\left[\frac{7}{12}A\left(s^{-2},t^{-2}\right) + \frac{5}{12}G\left(s^{-2},t^{-2}\right)\right]. \tag{30}$$

Proof. Let $f(x) = \frac{1}{x}$, $x \in [s,t]$, then

$$\left(|f'(x)|^{\frac{1}{2}}\right)'' = \frac{2}{|x|^3} \geq 0.$$

Thus, $|f'(x)|^{\frac{1}{2}}$ is convex on $[s,t]$. It follows that (29) and (30) hold by using Corollary 1 and Corollary 2, respectively. □

Remark 2. *For any $q \geq 1$, by the inequalities (17) and (20), we have*

$$\frac{t-s}{4}\left[\frac{3}{4}A\left(s^{-2},t^{-2}\right) + \frac{1}{4}G\left(s^{-2},t^{-2}\right)\right] \leq \frac{t-s}{4}\left[A\left(|s|^{-2q},|t|^{-2q}\right)\right]^{\frac{1}{q}},$$

and

$$\frac{t-s}{4}\left[\frac{7}{12}A\left(s^{-2},t^{-2}\right) + \frac{5}{12}G\left(s^{-2},t^{-2}\right)\right] \leq \frac{t-s}{4}\left[A\left(|s|^{-2q},|t|^{-2q}\right)\right]^{\frac{1}{q}}.$$

Thus, we obtain an improvement of the inequalities (13) and (14), which is an improvement of the inequalities (7) and (8).

Proposition 3. *Suppose that $t > s > 0$, then*

$$\ln I(s,t) - \ln G(s,t) \leq \frac{t-s}{4}\left[\frac{3}{4}A\left(s^{-1},t^{-1}\right) + \frac{1}{4}G\left(s^{-1},t^{-1}\right)\right], \tag{31}$$

$$\ln A(s,t) - \ln I(s,t) \leq \frac{t-s}{4}\left[\frac{7}{12}A\left(s^{-1},t^{-1}\right) + \frac{5}{12}G\left(s^{-1},t^{-1}\right)\right]. \tag{32}$$

Proof. Let $f(x) = \ln x$, $x \in [s,t]$, then

$$\left(|f'(x)|^{\frac{1}{2}}\right)'' = \frac{3}{4}|x|^{-\frac{5}{2}} \geq 0.$$

Thus, $|f'(x)|^{\frac{1}{2}}$ is convex on $[s,t]$. It follows that (31) and (32) hold by using Corollary 1 and Corollary 2, respectively. □

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