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# Exact and Approximate Solutions for Linear and Nonlinear Partial Differential Equations via Laplace Residual Power Series Method

**Haneen Khresat [1], Ahmad El-Ajou [1], Shrideh Al-Omari [1,*], Sharifah E. Alhazmi [2] and Moa'ath N. Oqielat [1]**

[1] Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan
[2] Mathematics Department, Al-Qunfudah University College, Umm Al-Qura University, Mecca 21955, Saudi Arabia
* Correspondence: shridehalomari@bau.edu.jo; Tel.: +962-772-061-029

**Abstract:** The Laplace residual power series method was introduced as an effective technique for finding exact and approximate series solutions to various kinds of differential equations. In this context, we utilize the Laplace residual power series method to generate analytic solutions to various kinds of partial differential equations. Then, by resorting to the above-mentioned technique, we derive certain solutions to different types of linear and nonlinear partial differential equations, including wave equations, nonhomogeneous space telegraph equations, water wave partial differential equations, Klein–Gordon partial differential equations, Fisher equations, and a few others. Moreover, we numerically examine several results by investing some graphs and tables and comparing our results with the exact solutions of some nominated differential equations to display the new approach's reliability, capability, and efficiency.

**Keywords:** partial differential equation; power series; residual power series; Laplace residual power series

**MSC:** 44A05; 35A22

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## 1. Introduction

Numerous mathematical models have been adapted to aid in realizing some apprised physical phenomena. Although these models produce differential equations (DEs), they involve derivatives of unknown functions. The DEs form a vital branch of mathematics since their inception [1]. Leibniz and the Bernoulli brothers began DEs in the early 1680s, shortly after the Newtonian variable equations in the 1670s. Consequently, various applications in engineering and mechanical topics were developed by many authors, which have latterly strengthened the Leibnizian tradition and expanded its multivariate form [1]. Even though partial differential equations (PDEs) are more general than ordinary differential equations (ODEs), their method of solution is generally different [2–4]. They are challenging and more complex as they involve solutions to multiple independent variables, whilst the topic is enormous and considerable. Indeed, as certain phenomena found some expression through PDEs, solutions of PDEs become of great interest to scientists [5]. Heat and wave equations are two famous forms of linear PDEs, whereas Liouville, Schrödinger, KDV, Poisson, Klein–Gordon, water wave, Fisher, and Dirac are the most popular examples of nonlinear PDEs. Owing to the reason that many types of PDEs do not possess exact solutions, various analytical and numerical methods are described to introduce approximate solutions for linear and nonlinear PDEs involving the homotopy perturbation and analysis method [6–8], Fourier transform technique [9], Laplace transforms (LT) approach [10], operational calculus method [11], Adomian decomposition method [12], operational matrix method [13], variational iteration method [14], recently, the residual power series (RPS) method [15], and many others [16–30].

The RPS method is an analytical method proposed by [15] to determine the coefficients of the power series solutions of a class of DEs. It is based on formulating power series solutions of many linear and nonlinear equations without linearity or perturbation. In addition, the method requires calculating the derivative of the residual function at each stage of finding the coefficients. This paper resorts to a new analytical method called the Laplace residual power series (LRPS) method that introduces solutions to some linear and nonlinear DEs. The method under concern was first presented by Eriqat et al. [31], with advantages arising from reducing the computational efforts required for extracting solutions in the form of a power series of coefficients determined through successive algebraic steps. Although the proposed method does not rely on using a concept of the derivative for determining the coefficients of the series solution as the RPS technique, it uses the limit at infinity concept to reach its primary goal. In fact, the LRPS approach has successfully obtained accurate results for different kinds of linear and nonlinear DEs. The most important feature of the proposed method is its ability to process nonlinear equations, which is missing from the traditional method of solving DEs using LT. Recently, the LRPS method has been adapted to solve some types of fractional DEs, including nonlinear time-fractional dispersive PDEs [32], hyperbolic systems of Caputo-time-fractional PDEs with variable coefficients [33], time-fractional Navier–Stokes equations [34], fuzzy Quadratic Riccati DEs [35], Lane–Emden equations [36], time-fractional nonlinear water wave PDEs [37], nonlinear fractional reaction–diffusion for bacteria growth models [38], Fisher's and logistic system models [39], and a few others, to mention but a few.

The main algorithm of this approach can be summarized as follows. In the first step, we apply the Laplace transform to the entire DE. In the second step, we derive a series solution in the form of a Laurent series expansion in the Laplace space. In the third step, we transfer the attained expansion into a Taylor series form by allowing the inverse Laplace transform to act on the equation.

In the present paper, we construct series solutions for linear and nonlinear PDEs and compare our results with previous results to verify the effectiveness and efficiency of the proposed method. Section 1 discusses PDEs and some previous solution techniques. It also explains the LRPS method for solving linear and nonlinear PDEs. Section 2 reviews some concepts and preliminary results related to the convergence analysis of the proposed method. Section 3 constructs a series solution for a class of linear and nonlinear PDEs using the LRPS method. Section 4 checks the validity and efficiency of the LRPS method by applying the new construction to some examples dealing with wave, space telegraph, water wave, Klein–Gordan, and Fisher equations in some detail.

## 2. Basic Concepts and Auxiliary Results

This section presents some needful concepts in the sequel. It recalls definitions and classifications of PDEs, most of which produce equations containing derivatives of unknown multivariable functions. It also presents the existence and uniqueness of theorems that ensure the existence and uniqueness of the solution of an initial value problem (IVP). In addition, as the construction of the LRPS method needs power and the Laurent series, this paper gives forms, theorems, and properties of the given series. Further, it discusses definitions and allied results associated with the Laplace transform and its inversion as well.

**Definition 1 ([15]).** *An expansion of the form:*

$$\sum_{n=0}^{\infty} h_n(x)(t-t_0)^n = h_0(x) + h_1(x)(t-t_0) + h_2(x)(t-t_0)^2 + \dots, \tag{1}$$

*is called a PS about $t = t_0$, where $(x,t) \in I \times (t_0, \infty)$ and $h_n(x)$, $n = 0, 1, 2, \dots$ are the coefficients of the series.*

**Theorem 1 ([15]).** *Let $u$ have the following power series representation about $t = 0$,*

$$u(x,t) = \sum_{n=0}^{\infty} h_n(x)t^n, x \in I, 0 \le t < R, \tag{2}$$

*and $\partial_t^n u$ be continuous on $I \times (0,R)$, for $n = 0,1,2,\ldots$. Then, the coefficients $h_n$ of Equation (2) are given by*

$$h_n(x) = \frac{\partial_t^n u(x,0)}{n!}, n = 0,1,2,\ldots, \tag{3}$$

*where $\partial_t^n = \frac{\partial^n}{\partial t^n}$ and $R = \min_{c \in I} R_c$, $R_c$ being the radius of convergence of the PS $\sum_{n=0}^{\infty} h_n(x)t^n$.*

The substitution of $h_n(x) = \frac{\partial_t^n u(x,0)}{n!}, n = 0,1,2,\ldots$, in the series representation Equation (2), indeed, leads to an expansion for $u$, about $t = 0$, given by

$$u(x,t) = \sum_{n=0}^{\infty} \frac{\partial_t^n u(x,0)}{n!} t^n, x \in I, 0 \le t < R. \tag{4}$$

The above expansion represents Taylor's series of variable coefficients.

**Definition 2 ([40]).** *Let $u$ be a function defined for $t \ge 0$. Then, the integral*

$$U(x,s) = \mathcal{L}[u(x,t)] := \int_0^{\infty} e^{-st} u(x,t)dt, \tag{5}$$

*is said to be the Laplace transform of $u$, provided the integral converges on an interval of $s$.*

**Definition 3 ([40]).** *The inverse Laplace transform of a function $U$ is the function $u(x,t), t \ge 0$, defined by*

$$u(x,t) = \mathcal{L}^{-1}[U(x,s)] := \int_{\mathrm{a}-\mathrm{i}\infty}^{\mathrm{a}+\mathrm{i}\infty} e^{st} U(x,s)ds, a = Re(s) > a_0, \tag{6}$$

*where $\mathrm{a}_0$ is located in the right half plane of the absolute convergence of the integral.*

**Lemma 1 ([40]).** *Let $\partial_t^{k-1} u$ and $\partial_t^k u$ be piecewise continuous functions of exponential order $\lambda$ defined on $I \times [0,\infty), k = 1,2,\ldots,n$. Then, for $(x,s) \in D =: \left\{(x,s) : \sqrt{x^2+s^2} > \lambda(x)\right\}$, we have*

$$\text{(i) } \mathcal{L}[\partial_t(x,t)] = s\mathcal{L}[u(x,t)] - u(x,0) \tag{7}$$

$$\text{(ii) } \mathcal{L}\left[\partial_t^2(x,t)\right] = s^2\mathcal{L}[u(x,t)] - u_t(x,0) - su(x,0) \tag{8}$$

$$\text{(iii) } \mathcal{L}[\partial_t^n(x,t)] = s^n\mathcal{L}[u(x,t)] - \textstyle\sum_{k=0}^{n-1} s^{n-k-1}\partial_t^k(x,0) \tag{9}$$

**Definition 4 ([41]).** *An expansion that has the following representation*

$$\sum_{n=-\infty}^{\infty} h_n(x)t^n = \sum_{n=1}^{\infty} \frac{h_{-n}(x)}{t^n} + \sum_{n=0}^{\infty} h_n(x)t^n, \tag{10}$$

*is called the Laurent series (LS) about $t = 0$, where $t$ is a variable, and the coefficients of the series $h_n$ are functions of $x$.*

The series $\sum_{n=0}^{\infty} h_n(x)(t-t_0)^n$ is said to be the analytic (regular) part of LS, while $\sum_{n=1}^{\infty} \frac{h_{-n}(x)}{(t-t_0)^n}$ is the singular (principal) part of Laurent's series.

**Lemma 2 ([40]).** *Let* $U(x,s) = \mathcal{L}[u(x,t)]$, $(x,s) \in I \times (0,\infty)$. *Then, we have*

$$\text{(i)}\ \lim_{s\to\infty} U(x,s) = 0. \tag{11}$$

$$\text{(ii)}\ \lim_{s\to\infty} sU(x,s) = u(x,0). \tag{12}$$

**Theorem 2.** *Let the function* $U(x,s) = \mathcal{L}[u(x,t)]$, $(x,s) \in I \times (0,\infty)$ *be expressed by the LS representation:*

$$U(x,s) = \frac{h_0(x)}{s} + \sum_{n=1}^{\infty} \frac{h_n(x)}{s^{n+1}}, s > 0, x \in I. \tag{13}$$

*Then, we have* $h_n(x) = \partial_t^n u(x,0), n = 0,1,2,3,\ldots$

**Proof.** Assume the hypothesis of the theorem is satisfied and that $U$ is represented by the LS expansion of Equation (13). Then, we have

$$sU(x,s) = h_0(x) + \sum_{n=1}^{\infty} \frac{h_n(x)}{s^n}, s > 0. \tag{14}$$

By using Lemma 2 (ii), we derive that $h_0(x) = u(x,0)$. Hence, multiplying Equation (14) by $s$ leads to the expansion:

$$s^2U(x,s) - su(x,0) = h_1(x) + \sum_{n=2}^{\infty} \frac{h_n(x)}{s^n}, s > 0. \tag{15}$$

Therefore, rearranging Equation (15), allowing $s \to \infty$ on both sides of the preceding equation, and using Equation (12) suggests writing

$$\begin{aligned} h_1(x) &= \lim_{s\to\infty}\left(s^2U(x,s) - su(x,0) - \sum_{n=2}^{\infty} \tfrac{h_n(x)}{s^n}\right) \\ &= \lim_{s\to\infty}\left(s^2U(x,s) - su(x,0)\right) \\ &= \lim_{s\to\infty} s(sU(x,s) - u(x,0)) = \lim_{s\to\infty} s(\mathcal{L}[u_t(x,t)]) \\ &= u_t(x,0). \end{aligned}$$

Similarly, multiplying Equation (15) by $s$ and extracting the summations give the expansion form:

$$s\left(s^2U(x,s) - su(x,0) - u_t(x,0)\right) = h_2(x) + \sum_{n=3}^{\infty} \frac{h_n(x)}{s^n}, s > 0. \tag{16}$$

Isolating the term $h_2$ on one side of Equation (16), taking the limit to both sides as $s \to \infty$, and using Equation (12) give rise to

$$\begin{aligned} h_2(x) &= \lim_{s\to\infty}\left(s\left(s^2U(x,s) - su(x,0) - u_t(x,0)\right) - \sum_{n=3}^{\infty} \tfrac{h_n(x)}{s^n}\right) \\ &= \lim_{s\to\infty} s\left(s^2U(x,s) - su(x,0) - u_t(x,0)\right) \\ &= \lim_{s\to\infty} s(\mathcal{L}[u_{tt}(x,t)]) = u_{tt}(x,0). \end{aligned}$$

Now, finding the general formula of the coefficient $h_n$ follows from multiplying Equation (13) by $s^{n+1}$ and taking the limit of the resulting equation as $s \to \infty$. This indeed gives $h_n(x) = \partial_t^n u(x,0), n = 0,1,2,\ldots$. The proof is, therefore, finished. □

By following Theorem (2), we state, without proof, the following remark.

**Remark 1.** *The inversion formula of the Laplace transform Equation (13) can be written in the form:*

$$u(x,t) = \sum_{n=0}^{\infty} \frac{u_n(x,0)}{n!} t^n, t \geq 0, \tag{17}$$

*which is equivalent to Taylor's series Equation (16) of variable coefficients.*

**Theorem 3.** *Let $u$ be a piecewise continuous function on $I \times [0,\infty)$ of order $\delta$, and $U(x,s) = \mathcal{L}[u(x,t)]$ be represented by Equation (13). If $\left| s\mathcal{L}\left[\partial_t^{(n+1)} u(x,t)\right]\right| \leq M(x)$, on $I \times (\delta,\gamma]$, then the reminder $\mathcal{R}_n$ of the Laplace transform satisfies the following inequality:*

$$|\mathcal{R}_n(x,s)| \leq \frac{M(x)}{s^{1+(n+1)}}, x \in I, \delta < s \leq \gamma. \tag{18}$$

**Proof.** First, suppose that $\mathcal{L}\left[\partial_t^k u(x,t)\right]$ is defined on $I \times (\delta,\gamma]$ for $k = 0,1,2,\ldots,n+1$. Assume that

$$\left| s\mathcal{L}\left[\partial_t^{(n+1)} u(x,t)\right]\right| \leq M(x), x \in I, \delta < s \leq \gamma. \tag{19}$$

Then, from the definition of the remainder, $\mathcal{R}_n(x,s) = \mathrm{U}(x,s) - \sum_{k=0}^{n} \frac{\partial_t^k u(x,0)}{s^{1+k}}$, one can obtain

$$\begin{aligned} s^{1+(n+1)}\mathcal{R}_n(x,s) &= s^{1+(n+1)} U(x,s) \\ &- \sum_{k=0}^{n} s^{(n+1-k)} \partial_t^k u(x,0) = s\left( s^{(n+1)} U(x,s) - \sum_{k=0}^{n} s^{(n+1-k)-1} \partial_t^k u(x,0)\right) \\ &= s\mathcal{L}\left[\partial_t^{(n+1)} u(x,t)\right]. \end{aligned}$$

Hence, it follows from Equation (19) that $\left| s^{1+(n+1)}\mathcal{R}_n(x,s)\right| \leq M(x)$. Therefore,

$$-M(x) \leq s^{1+(n+1)}\mathcal{R}_n(x,s) \leq M(x), x \in I, \delta < s \leq \gamma. \tag{20}$$

Thus, our result in Equation (18) can be obtained by reformulating Equation (20). This ends the proof of the theorem. □

The following is the theorem considered the primary tool for the LRPS method.

**Theorem 4.** *Let $U(x,s) = 0$, for all $(x,s) \in I \times (s,\infty)$, have the LS representation:*

$$U(x,s) = \sum_{n=0}^{\infty} \frac{h_n(x)}{s^{n+1}}, s > 0, x \in I. \tag{21}$$

*Then, we have*

$$\lim_{s\to\infty} s^{k+1} U(x,s) = \lim_{s\to\infty} s^{k+1} U_k(x,s) = 0, s > 0, x \in I, k = 0,1,2,\ldots, \tag{22}$$

*where $U_k$ is the kth- truncated series of the expansion of $U$.*

**Proof.** The first part of Equation (22) is trivial. As $U(x,s) = 0$, we have $s^{k+1}U(x,s) = 0$. Thus, we have $\lim_{s\to\infty} s^{k+1} U(x,s) = 0, s > 0, x \in I, k = 0,1,2,\ldots$.

Employing Expansion (21) and the last trivial result, we write

$$\begin{aligned}\lim_{s\to\infty} s^{k+1}U(x,s) &= \lim_{s\to\infty} s^{k+1}\left(\sum_{n=0}^{\infty}\frac{h_n(x)}{s^{n+1}}\right)\\ &= \lim_{s\to\infty} s^{k+1}\left(U_k(x,s)+\sum_{n=k+1}^{\infty}\frac{h_n(x)}{s^{n+1}}\right)\\ &= \lim_{s\to\infty}\left(s^{k+1}U_k(x,s)+\sum_{n=k+1}^{\infty}\frac{h_n(x)}{s^{n-k}}\right)=\lim_{s\to\infty}s^{k+1}U_k(x,s)\\ &= 0.\end{aligned}$$

This ends the proof of our result. □

## 3. The Laplace Residual Power Series Method

In this section, we apply the LRPS method to generate analytical solutions for linear and nonlinear PDEs. Without the loss of generality, we confine ourselves to constructing an LRPS solution for second-order linear PDEs to simplify our construction process.

### *3.1. Laplace Residual Power Series Method for Solving Linear PDEs*

We employ the LRPS method to provide a series solution to second-order linear PDEs in this part. This process can be generalized to any higher-order linear PDE.

Consider the following second-order linear PDEs:

$$\frac{\partial^2 u}{\partial t^2}=A(x,t)\frac{\partial^2 u}{\partial x^2}+B(x,t)\frac{\partial^2 u}{\partial x\partial t}+C(x,t)\frac{\partial u}{\partial t}+D(x,t)\frac{\partial u}{\partial x}+E(x,t)u+F(x,t), \tag{23}$$

subject to the initial conditions:

$$u(x,0)=f(x),u_t(x,0)=g(x), \tag{24}$$

where $A(x,t)$, $B(x,t)$, $C(x,t)$, $D(x,t)$, $E(x,t)$ and $F(x,t)$ are analytic functions, $x\in I\subseteq\mathbb{R}$ and $t\geq 0$.

To derive an analytical series solution for the IVP (23)–(24) by the LRPS method, we transfer Equation (23) into a Laplace space by applying the LT to both sides of Equation (23) and inserting the initial conditions (24) as follows:

$$\begin{aligned}U(x,s)=\frac{f(x)}{s} &+\frac{g(x)}{s^2}+\frac{\mathcal{F}(x,s)}{s^2}+\frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{A}(x,s)\right]\mathcal{L}^{-1}[U_{xx}(x,s)]\right]}{s^2}\\ &+\frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{B}(x,s)\right]\mathcal{L}^{-1}[sU_x(x,s)-f'(x)]\right]}{s^2}\\ &+\frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{C}(x,s)\right]\mathcal{L}^{-1}[sU(x,s)-f(x)]\right]}{s^2}\\ &+\frac{\mathcal{L}\left[\mathcal{L}^{-1}[\mathcal{D}(x,s)]\mathcal{L}^{-1}[U_x(x,s)]\right]}{s^2}+\frac{\mathcal{L}\left[\mathcal{L}^{-1}[\mathcal{E}(x,s)]\mathcal{L}^{-1}[U(x,s)]\right]}{s^2}.\end{aligned} \tag{25}$$

The next step of the procedure is to provide a series solution to Equation (25). Therefore, we assume the exact solution of Equation (25) has the following Laurent expansion:

$$U(x,s)=\sum_{m=0}^{\infty}\frac{h_m(x)}{s^{1+m}},s>0, \tag{26}$$

where $h_m(x)=\partial_t^m u(x,0)$, $m=0,1,2,....$

According to the initial conditions Equation (24), we infer that $h_0(x)=f(x)$ and $h_1(x)=g(x)$. So, Equation (26) can be rewritten as

$$U(x,s)=\frac{f(x)}{s}+\frac{g(x)}{s^2}+\sum_{m=2}^{\infty}\frac{h_m(x)}{s^{1+m}},s>0. \tag{27}$$

We can approximate $U$ by the truncated series Equation (27). So, the $k$th-approximation of $U$ is given by the following $k$th- truncation of the Expansion (27):

$$U_k(x,s) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \sum_{m=2}^{k} \frac{h_m(x)}{s^{1+m}}, s > 0. \tag{28}$$

The major tools of the LRPS method are the Laplace residual function (LRF) Equation (25) and the kth-LRF, which are, respectively, given by

$$\begin{aligned} LRes(x,s) = U(x,s) - \tfrac{f(x)}{s} - \tfrac{g(x)}{s^2} - \tfrac{\mathcal{F}(x,s)}{s^2} \\ - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{A}(x,s)\right]\mathcal{L}^{-1}\left[U_{xx}(x,s)\right]\right]}{s^2} \\ - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{B}(x,s)\right]\mathcal{L}^{-1}\left[sU_x(x,s)-f'(x)\right]\right]}{s^2} \\ - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{C}(x,s)\right]\mathcal{L}^{-1}\left[sU(x,s)-f(x)\right]\right]}{s^2} \\ - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{D}(x,s)\right]\mathcal{L}^{-1}\left[U_x(x,s)\right]\right]}{s^2} \\ - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{E}(x,s)\right]\mathcal{L}^{-1}\left[U(x,s)\right]\right]}{s^2}, s > 0, \end{aligned} \tag{29}$$

and

$$\begin{aligned} LRes_k(x,s) = \quad & U_k(x,s) - \tfrac{f(x)}{s} - \tfrac{g(x)}{s^2} - \tfrac{\mathcal{F}(x,s)}{s^2} - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{A}(x,s)\right]\mathcal{L}^{-1}\left[(U_k)_{xx}(x,s)\right]\right]}{s^2} \\ & - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{B}(x,s)\right]\mathcal{L}^{-1}\left[s(U_k)_x(x,s)-f'(x)\right]\right]}{s^2} \\ & - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{C}(x,s)\right]\mathcal{L}^{-1}\left[sU_k(x,s)-f(x)\right]\right]}{s^2} \\ & - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{D}(x,s)\right]\mathcal{L}^{-1}\left[(U_k)_x(x,s)\right]\right]}{s^2} - \frac{\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{E}(x,s)\right]\mathcal{L}^{-1}\left[U_k(x,s)\right]\right]}{s^2}. \end{aligned} \tag{30}$$

Since $A(x,t)$, $B(x,t)$, $C(x,t)$, $D(x,t)$, $E(x,t)$, and, $F(x,t)$ are analytic functions, they have Taylor's expansion according to Theorem 1. So, the functions $\mathcal{A}(x,s)$, $\mathcal{B}(x,s)$, $\mathcal{C}(x,s)$, $\mathcal{D}(x,s)$, $\mathcal{E}(x,s)$, and $\mathcal{F}(x,s)$ have LS expansions as follows:

$$\begin{aligned} \mathcal{A}(x,s) = \sum_{m=0}^{\infty} \tfrac{\mathbb{A}_m(x)}{s^{m+1}}, \mathcal{B}(x,s) = \sum_{m=0}^{\infty} \tfrac{\mathbb{B}_m(x)}{s^{m+1}}, \\ \mathcal{C}(x,s) = \sum_{m=0}^{\infty} \tfrac{\mathbb{C}_m(x)}{s^{m+1}}, \mathcal{D}(x,s) = \sum_{m=0}^{\infty} \tfrac{\mathbb{D}_m(x)}{s^{m+1}}, \\ \mathcal{E}(x,s) = \sum_{m=0}^{\infty} \tfrac{\mathbb{E}_m(x)}{s^{m+1}}, \mathcal{F}(x,s) = \sum_{m=0}^{\infty} \tfrac{\mathbb{F}_m(x)}{s^{m+1}}, \end{aligned} \tag{31}$$

where $\mathbb{A}_m(x) = \partial_t^m A(x,0)$, $\mathbb{B}_m(x) = \partial_t^m B(x,0)$, $\mathbb{C}_m(x) = \partial_t^m C(x,0)$, $\mathbb{D}_m(x) = \partial_t^m D(x,0)$, $\mathbb{E}_m(x) = \partial_t^m E(x,0)$, and $\mathbb{F}_m(x) = \partial_t^m F(x,0)$.

Substitute the expansions of $\mathcal{A}(x,s)$, $\mathcal{B}(x,s)$, $\mathcal{C}(x,s)$, $\mathcal{D}(x,s)$, $\mathcal{E}(x,s)$, and $\mathcal{F}(x,s)$ into Equation (29) to obtain the following form of the LRF:

$$LRes(x,s) = \sum_{m=2}^{\infty} \frac{h_m(x)}{s^{1+m}} - \sum_{m=0}^{\infty} \frac{\mathbb{F}_m(x) + \mathbb{H}_m(x)}{s^{3+m}}, s > 0, \tag{32}$$

where $\mathbb{H}_m(x)$ is defined as

$$\begin{aligned} \mathbb{H}_m(x) = \sum_{j=0}^{m} \lambda_{1m}\left(\mathbb{A}_{m-j}h_j''\right)(x) + \lambda_{2m}\left(\mathbb{B}_{m-j}h_{j+1}'\right)(x) + \lambda_{3m}\left(\mathbb{C}_{m-j}h_{j+1}\right)(x) \\ + \sum_{j=0}^{m} \lambda_{4m}\left(\mathbb{D}_{m-j}h_j'\right)(x) + \sum_{j=0}^{m} \lambda_{5m}\left(\mathbb{E}_{m-j}h_j\right)(x), \end{aligned} \tag{33}$$

where $\lambda_{jm}$, $j = 1,2,3,4,5$ are known constants.

Similarly, the $k$th-LRF can be demonstrated in the expansion:

$$LRes_k(x,s) = \sum_{m=2}^{k} \frac{h_m(x)}{s^{1+m}} - \sum_{m=0}^{k} \frac{\mathbb{F}_m(x) + \mathbb{H}_m(x)}{s^{3+m}} - \sum_{m=k+1}^{\infty} \frac{\mathbb{F}_m(x) + \mathbb{G}_m(x)}{s^{3+m}}, \tag{34}$$

where $\mathbb{G}_m$ has a meaning in terms of sums as

$$\begin{aligned}\mathbb{G}_m(x) = \sum_{j=0}^{k} \lambda_{1m}\left(\mathbb{A}_{m-j}h_j''\right)(x) + \lambda_{2m}\left(\mathbb{B}_{m-j}h_{j+1}'\right)(x) + \lambda_{3m}\left(\mathbb{C}_{m-j}h_{j+1}\right)(x) \\ + \sum_{j=0}^{k} \lambda_{4m}\left(\mathbb{D}_{m-j}h_j'\right)(x) + \sum_{j=0}^{k} \lambda_{5m}\left(\mathbb{E}_{m-j}h_j\right)(x).\end{aligned} \tag{35}$$

To obtain a form of the coefficient formulas $h_m, m = 2, 3, \ldots, k$ described in Equation (28), we multiply Equation (34) by $s^{k+1}$ to obtain the following formula:

$$\begin{aligned}s^{k+1}LRes_k(x,s) &= \sum_{m=2}^{k} \frac{h_m(x)}{s^{m-k}} - \sum_{m=0}^{k} \frac{\mathbb{F}_m(x)+\mathbb{H}_m(x)}{s^{2+m-k}} \\ &- \sum_{m=k+1}^{\infty} \frac{\mathbb{F}_m(x)+\mathbb{G}_m(x)}{s^{2+m-k}}.\end{aligned} \tag{36}$$

For $= 2$, the fact $\lim_{s\to\infty} s^{k+1}LRes_k(x,s) = 0$ leads to the algebraic equation $h_2(x) - \mathbb{F}_0(x) - \mathbb{H}_0(x) = 0$.

Therefore, the third coefficient in the series (27) has the form:

$$\begin{aligned}h_2(x) = \mathbb{F}_0(x) + \lambda_{10}\left(\mathbb{A}_0 h_0''\right)(x) + \lambda_{20}\left(\mathbb{B}_0 h_1'\right)(x) + \lambda_{30}\left(\mathbb{C}_0 h_1\right)(x) \\ + \lambda_{40}\left(\mathbb{D}_0 h_0'\right)(x) + \lambda_{50}\left(\mathbb{E}_0 h_0\right)(x).\end{aligned} \tag{37}$$

Analogously, for $k = 3$, the fourth coefficient of the series (27) has the following form:

$$h_3(x) = \mathbb{F}_1(x) + \mathbb{H}_1(x). \tag{38}$$

In general, such a procedure can be repeated to an arbitrary order in the coefficients of the bivariate solution of the obtained Equations (23) and (24). Therefore, the coefficients $h_k$ of the series (27) can be given by the recurrence relation:

$$\begin{aligned}h_0(x) = f(x), \\ h_1(x) = g(x),\end{aligned} \tag{39}$$

$$h_k(x) = \mathbb{F}_{k-2}(x) + \mathbb{H}_{k-2}(x), k = 2, 3, \ldots,$$

where $\mathbb{F}_{m-2}(x), \mathbb{H}_{m-2}(x)$ have the significance of Equation (33). Hence, the aimed solution of the Laplace form of Equations (23) and (24) is obtained in a series form as

$$U(x,s) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \sum_{m=2}^{\infty} \frac{\mathbb{F}_{m-2}(x) + \mathbb{H}_{m-2}(x)}{s^{1+m}}. \tag{40}$$

Now, by applying the inverse LT to Equation (42), the solution of Equations (23) and (24) is summarized as follows:

$$u(x,t) = f(x) + \frac{g(x)t}{2} + \sum_{m=2}^{\infty} \frac{(\mathbb{F}_{m-2}(x) + \mathbb{H}_{m-2}(x))t^m}{m!}. \tag{41}$$

This reveals that the $k$th-approximation of the solution of Equations (23) and (24) has the series form:

$$u_k(x,t) = f(x) + \frac{g(x)t}{2} + \sum_{m=2}^{k} \frac{(\mathbb{F}_{m-2}(x) + \mathbb{H}_{m-2}(x))t^m}{m!}. \tag{42}$$

*3.2. Laplace Residual Power Series Method for Solving Non-Linear PDEs*

In this part, we will make use of the LRPS method to construct the LRPS solution to certain nonlinear PDEs. For this end, consider the following operator form of the PDE:

$$\frac{\partial^2 u}{\partial t^2} = N_{x,t}[u(x,t)], x \in I, t \geq 0, \tag{43}$$

subject to the initial conditions:

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned} \tag{44}$$

where $N_{x,t}$ is a nonlinear operator with respect to $x$ and $t$ of order 2. The multivariable function $u(x,t)$ is assumed to be a casual function of time and space, and analytic on $t \geq 0$. In addition, $f(x)$ and $g(x)$ are functions of $x \in I$. To establish the LRPS solution for the IVP (43)–(44), we operate the LT on both sides of Equation (43) to achieve the following equation:

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = \mathcal{L}[N_{x,t}[u(x,t)]]. \tag{45}$$

By using Equation (44) and Lemma 1, we can rewrite Equation (45) as

$$U(x,s) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{N}_{x,s}[U(x,s)], \tag{46}$$

where $U(x,s) = \mathcal{L}[u(x,t)]$, $\mathcal{N}_{x,s}[U(x,s)] = \mathcal{L}[N_{x,t}[u(x,t)]]$ is a nonlinear operator with respect to $x$ and $s$.

We define the LRF of Equation (46) as follows:

$$LRes(x,s) = U(x,s) - \frac{f(x)}{s} - \frac{g(x)}{s^2} - \frac{1}{s^2}\mathcal{N}_{x,s}[U(x,s)], s > 0. \tag{47}$$

Undoubtedly, the $LRes(x,s)$ has an LS expansion. So, it can be expressed as

$$LRes(x,s) = \sum_{m=2}^{\infty} \frac{h_m(x)}{s^{1+m}} + \sum_{m=0}^{\infty} \frac{\mathcal{M}_m[f(x), g(x), h_2(x), h_3(x), \ldots, h_{m+1}(x)]}{s^{3+m}}, \tag{48}$$

where $\mathcal{M}_m$, $m = 0,1,2,\ldots$ are operators relying on the operators $\partial_x$ and $\partial_x^2$.

The $k$th-LRF is not a truncated series of the expansion of $LRes(x,s)$, but it is the series achieved from substituting $U_k(x,s)$ into the LRF (47). Therefore, it takes the form:

$$\begin{aligned} LRes_k(x,s) &= \sum_{m=2}^{k} \frac{h_m(x) + \mathcal{M}_{m-2}[f(x), g(x), h_2(x), h_3(x), \ldots, h_{m-1}(x)]}{s^{1+m}} \\ &+ \sum_{m=k+1}^{r_k} \frac{\mathcal{Z}_m\left[h_j(x)\right]}{s^{1+m}}, \end{aligned} \tag{49}$$

where $j \in \{0,1,2,\ldots,k+1\}$, $\mathcal{Z}_m$, $m = k+1, k+2, \ldots, m_k$ are operators, and $\mathcal{Z}_m\left[h_j(x)\right] \neq 0$.

Now, to determine a form of the unknown coefficient $h_2$ in Equation (28), we substitute the second-truncated series $U_2(x,s) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{h_2(x)}{s^3}$ into the $LRes_2(x,s)$ to yield

$$\begin{aligned} LRes_2(x,s) = &\frac{h_2(x) - \mathcal{M}_0[f(x), g(x)]}{s^3} + \frac{\mathcal{Z}_3[f(x), g(x), h_2(x)]}{s^4} + \ldots \\ &+ \frac{\mathcal{Z}_{r_2}[f(x), g(x), h_2(x)]}{s^{1+r_2}}. \end{aligned} \tag{50}$$

Then, we multiply Equation (49) by $s^3$ to have

$$s^3 LRes_2(x,s) = h_2(x) - \mathcal{M}_0[f(x), g(x)] + \frac{\mathcal{Z}_3[f(x), g(x), h_2(x)]}{s} + \ldots + \frac{\mathcal{Z}_{r_2}[f(x), g(x), h_2(x)]}{s^{r_2-2}}. \tag{51}$$

Taking the limit of Equation (50) as $s \to \infty$ implies that

$$h_2(x) = \mathcal{M}_0[f(x), g(x)]. \tag{52}$$

In general, to determine the $m$th unknown coefficient $h_m(x)$, in Equation (51), we substitute $k = m$ into $LRes_k(x,s)$ to have

$$U_m(x,s) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{M_0[f(x), g(x)]}{s^3} + \frac{M_1[f(x), g(x), h_2(x)]}{s^4} + \ldots + \frac{h_m(x)}{s^m}$$

Therefore, multiplying both sides in the new formula by $s^{k+1}$ produces

$$h_m(x) = \mathcal{M}_{m-2}[f(x), g(x), h_2(x), \ldots, h_{m-1}(x)]. \tag{53}$$

Hence, we repeat the procedure for the required coefficients to represent a solution of Equation (28). Therefore, the $k$th-approximation of the solution of Equation (46) can be expressed in the following finite series:

$$\begin{aligned} U_k(x,s) = \frac{f(x)}{s} \quad & + \frac{g(x)}{s^2} + \frac{\mathcal{M}_0[f(x), g(x)]}{s^3} + \frac{\mathcal{M}_1[f(x), g(x), h_2(x)]}{s^4} \\ & + \frac{\mathcal{M}_2[f(x), g(x), h_2(x), h_3(x)]}{s^5} + \ldots \\ & + \frac{\mathcal{M}_{k-2}[f(x), g(x), h_2(x), \ldots, h_{k-1}(x)]}{s^{k+1}}. \end{aligned} \tag{54}$$

Now, taking the inverse LT for both sides of Equation (54) yields the $k$th-approximation of the solution of the IVP (43)–(44), which takes the expression:

$$\begin{aligned} u_k(x,t) = f(x) \quad & + g(x)\mathrm{t} + \mathcal{M}_0[f(x), g(x)]\frac{t^2}{2} + \mathcal{M}_1[f(x), g(x), h_2(x)]\frac{t^3}{3!} \\ & + \mathcal{M}_2[f(x), g(x), h_2(x), h_3(x)]\frac{t^4}{4!} + \ldots \\ & + \mathcal{M}_{k-2}[f(x), g(x), h_2(x), \ldots, h_{k-1}(x)]\frac{t^k}{k!}. \end{aligned} \tag{55}$$

## 4. Applications

In this section, some applications are presented to illustrate the performance and efficiency of the LRPS method in solving PDEs. During this section, all symbolic and numerical calculations were made using Mathematica 12.

**Problem 1 ([15]).** *Consider the following homogeneous wave equation:*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}, x \in \mathbb{R}, t \geq 0, \tag{56}$$

*subject to the initial condition:*

$$\begin{aligned} u(x,0) &= x, \\ u_t(x,0) &= x^2. \end{aligned} \tag{57}$$

To apply the LRPS method, we transfer Equation (56) into a Laplace space and substitute the initial conditions (57). Therefore, we can rewrite equation (58) as follows:

$$s^2 U(x,s) - sx - x^2 - \frac{1}{2}x^2 U_{xx}(x,s) = 0, s > 0, \tag{58}$$

where $U(x,s) = \mathcal{L}[u(x,t)]$.

The next step in the LRPS method is to find a series solution to Equation (58). For this, we assume the solution of Equation (58) satisfies the initial conditions of Equation (57). Hence, we can conclude that $h_0(x) = x$ and $h_1(x) = x^2$. So, Equation (58) can be expanded to the following LE:

$$U(x,s) = \frac{x}{s} + \frac{x^2}{s^2} + \sum_{m=2}^{\infty} \frac{h_m(x)}{s^{1+m}}, s > 0. \tag{59}$$

Now, we approximate $U(x,s)$ by a truncated series of the series form (59). Therefore, the $k$th approximation of $U(x,s)$ is given by the following truncation:

$$U_k(x,s) = \frac{x}{s} + \frac{x^2}{s^2} + \sum_{m=2}^{k} \frac{h_m(x)}{s^{1+m}}, s > 0. \tag{60}$$

The LRF $LRes(s)$, of Equation (58), is defined as follows:

$$LRes(x,s) = U(x,s) - \frac{x}{s} - \frac{x^2}{s^2} - \frac{x^2}{2}\frac{U_{xx}(x,s)}{s^2}, s > 0, \tag{61}$$

and the $k$th-LRF is defined as

$$LRes_k(x,s) = U_k(x,s) - \frac{x}{s} - \frac{x^2}{s^2} - \frac{x^2}{2}\frac{(U_{xx})_k(x,\mathrm{s})}{s^2}, s > 0. \tag{62}$$

To determine the first unknown coefficients $h_2$ in the Expansion (60), we substitute the second-truncated series $U_2(x,s) = \frac{x}{s} + \frac{x^2}{s^2} + \frac{h_2(x)}{s^3}$ in the second-LRF, $LRes_2(s)$, in Equation (62). This indeed gives

$$LRes_2(x,s) = \frac{h_2(x)}{s^3} - \frac{x^2}{2}\left(\frac{2}{s^4} + \frac{h_2''(x)}{s^5}\right), s > 0. \tag{63}$$

Now, multiple both sides of Equation (63) by $s^3$ to have

$$s^3 LRes_2(x,s) = h_2(x) - \frac{x^2}{2}\left(\frac{2}{s} + \frac{h_2''(x)}{s^2}\right), s > 0. \tag{64}$$

Appealing to the fact that $\lim_{s\to\infty} s^{k+1} LRes_k(x,s) = 0$ for $k = 2$ through Equation (64), we infer that $h_2(x) = 0$.

Now, according to the results obtained by applying the same steps, the fifth-truncated series (60) can be summarized as follows:

$$\begin{aligned} h_0(x) &= x, \\ h_1(x) &= x^2, \\ h_2(x) &= 0, \\ h_3(x) &= x^2, \\ h_4(x) &= 0, \\ h_5(x) &= x^2. \end{aligned} \tag{65}$$

Therefore, the solution of Equation (58) can be expressed in an infinite series form as

$$U(x,s) = \frac{x}{s} + \frac{x^2}{s^2} + \frac{x^2}{s^4} + \frac{x^2}{s^6} + \ldots, s > 0. \tag{66}$$

By taking the inverse LT of Equation (74), we derive the LRPS solution of the IVP (56)–(57) as follows:

$$u(x,t) = x + x^2\left(t + \frac{t^3}{6} + \frac{t^5}{120} + \ldots\right). \tag{67}$$

So, the exact solution of Equations (56) and (57) in the closed form is $u(x,t) = x + x^2 \sinh t$.

Note that the LRPS solution in Equation (67) is alike to the series solution obtained by the RPS method [15], ADM [42], HPM [43], and the VIM [42].

In addition, Figure 1 shows the agreement of the fifth approximate solution of the IVP (56)–(57) with the exact solution.

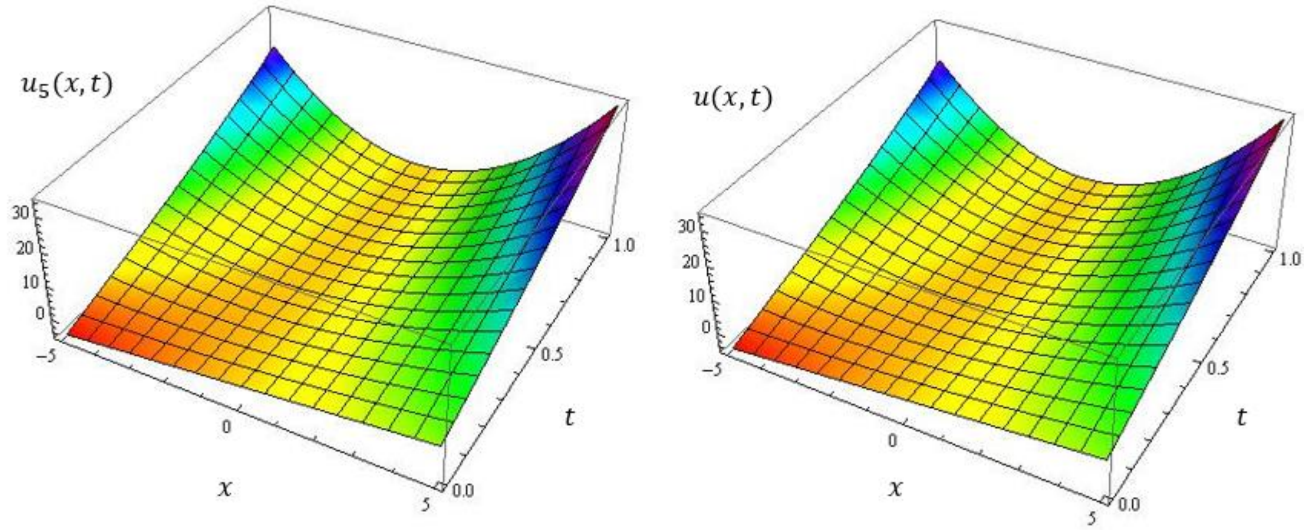


**Figure 1.** The surface graphs of the 5th approximate and exact solutions of the IVP (56)–(57).

**Problem 2 ([15]).** *Consider the following nonhomogeneous space-telegraph equation:*

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) - x^2 - t + 1, x \geq 0, t \geq 0, \tag{68}$$

*subject to the initial conditions:*

$$\begin{aligned} u(0,t) &= t, \\ u_x(0,t) &= 0. \end{aligned} \tag{69}$$

Apply the LT (with respect to $x$) to Equation (68) to obtain the algebraic equation:

$$s^2 U(s,t) - su(0,t) - u_x(0,t) = U_{tt}(s,t) + U_t(s,t) + U(s,t) - \frac{2}{s^3} - \frac{t}{s} + \frac{1}{s}, s > 0, \tag{70}$$

where $U(s,t) = \mathcal{L}[u(x,t)]$.

Use the initial conditions (69) to rewrite Equation (70) as

$$U(s,t) - \frac{t}{s} - \frac{1}{s^2} U_{tt}(s,t) - \frac{1}{s^2} U_t(s,t) - \frac{1}{s^2} U(s,t) + \frac{2}{s^5} + \frac{t}{s^3} - \frac{1}{s^3} = 0, s > 0. \tag{71}$$

The LRF of Equation (71) is therefore defined by

$$LRes(s,t) = U(s,t) - \frac{t}{s} - \frac{U_{tt}(s,t)}{s^2} - \frac{U_t(s,t)}{s^2} - \frac{U(s,t)}{s^2} + \frac{2}{s^5} + \frac{t}{s^3} - \frac{1}{s^3}, s > 0, \tag{72}$$

and the $k$th-LRF is given as

$$LRes_k(s,t) = U_k(s,t) - \frac{t}{s} - \frac{(U_{tt})_k(s,t)}{s^2} - \frac{(U_t)_k(s,t)}{s^2} - \frac{U_k(s,t)}{s^2} + \frac{2}{s^5} + \frac{t}{s^3} - \frac{1}{s^3}. \tag{73}$$

Repeating the previous steps several times informs that $h_k(t) = 0$, for $k = 4, 5, 6, \ldots$.. Therefore, the exact solution of Equation (71) is that

$$U(s,t) = \frac{t}{s} + \frac{2}{s^3}, s > 0. \tag{74}$$

Applying the inverse LT to Equation (74) gives the exact solution of the nonhomogeneous space telegraph Equation (68) subject to the initial conditions (69), which are given as

$$u(x,t) = t + x^2. \tag{75}$$

**Problem 3 ([15]).** *Consider the following nonhomogeneous linear PDE:*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + t\frac{\partial^2 u(x,t)}{\partial t \partial x} + u(x,t) - xt, x \in \mathbb{R}, t \geq 0 \tag{76}$$

*subject to the initial conditions:*

$$\begin{aligned} u(x,0) &= 1, \\ u_t(x,0) &= x. \end{aligned} \tag{77}$$

Run the LT on Equation (76). Consider the initial conditions (77) with some operations and make rearrangements to establish the algebraic equation:

$$s^2 U(x,s) - s - x - U_{xx}(x,s) + s\frac{\partial}{\partial s}U_x(x,s) + U_x(x,s) - U(x,s) + \frac{x}{s^2} = 0, s > 0. \tag{78}$$

As per the initial conditions in Equation (77), we can conclude that $h_0(x) = 1$ and $h_1(x) = x$. So, $U$ has the series form:

$$U(x,s) = \frac{1}{s} + \frac{x}{s^2} + \sum_{m=2}^{\infty} \frac{h_m(x)}{s^{1+m}}, s > 0. \tag{79}$$

The LRF of Equation (78), $LRes(s)$, is defined by

$$LRes(x,s) = U(x,s) - \frac{1}{s} - \frac{x}{s^2} - \frac{U_{xx}(x,s)}{s^2} + \frac{U_{xs}(x,s)}{s} + \frac{U_x(x,s)}{s^2} - \frac{U(x,s)}{s^2} + \frac{x}{s^4}. \tag{80}$$

Thus, according to the results, obtained above, the coefficients of the fifth-truncated series (79) are summarized as follows:

$$\begin{aligned} h_0(x) &= 1, \\ h_1(x) &= x, \\ h_2(x) &= 1, \\ h_3(x) &= -1, \\ h_4(x) &= 3, \\ h_5(x) &= -1. \end{aligned} \tag{81}$$

Therefore, the solution of Equation (78) can be expressed in an infinite series form as

$$U(x,s) = \frac{1}{s} + \frac{x}{s^2} + \frac{1}{s^3} + \frac{-1}{s^4} + \frac{3}{s^5} + \frac{-1}{s^6} + \ldots, s > 0. \tag{82}$$

By taking the inverse of the LT of Equation (82), we obtain the fifth approximate LRPS solution of the IVP (76)–(77) as follows:

$$u_5(x,t) = 1 + tx + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{8} - \frac{t^5}{120}. \tag{83}$$

It is of great importance to mention here that the fifth approximate solution in Equation (83) agrees with the fifth RPS solution that was obtained by El-Ajou et al. [15]. The exact solution of Equations (76) and (77) is $u(x,t) = -t + xt + e^t$. Figure 2 illustrates a comparison between the fifth approximate LRPS solution of the IVP (76)–(77) with the exact solution. Indeed, there is a great deal of agreement between the two surfaces.

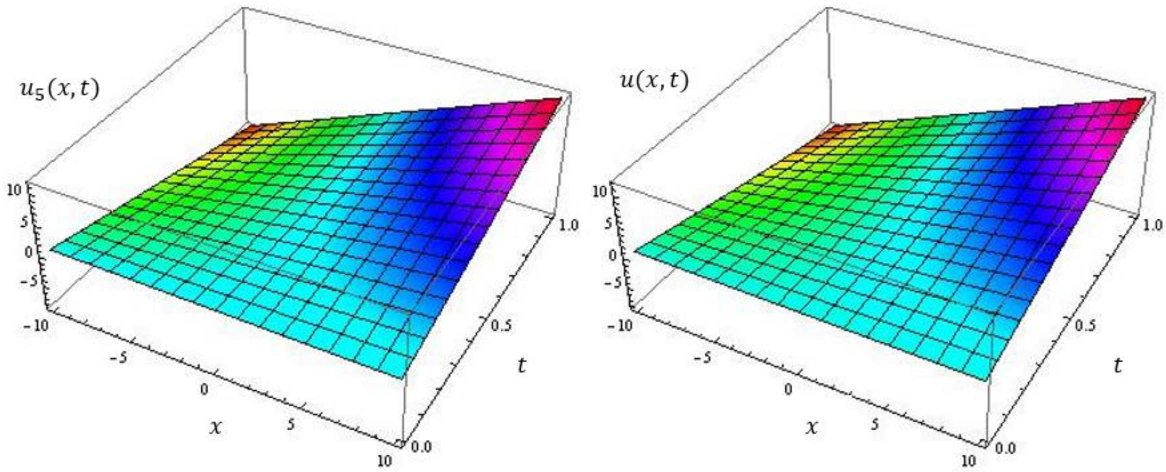


**Figure 2.** The surface graphs of the 5th approximate and exact solutions of the IVP (76)–(77).

The absolute error is defined by Abs. Err.$(x,t) = |u(x,t) - u_5(x,t)|$, and Figure 3 shows the absolute error of the fifth approximate LRPS solution of the IVP (76)–(77) in different regions. It is clear that the absolute error increases whenever the time increases. So, the region of convergence is a strip with a small radius.

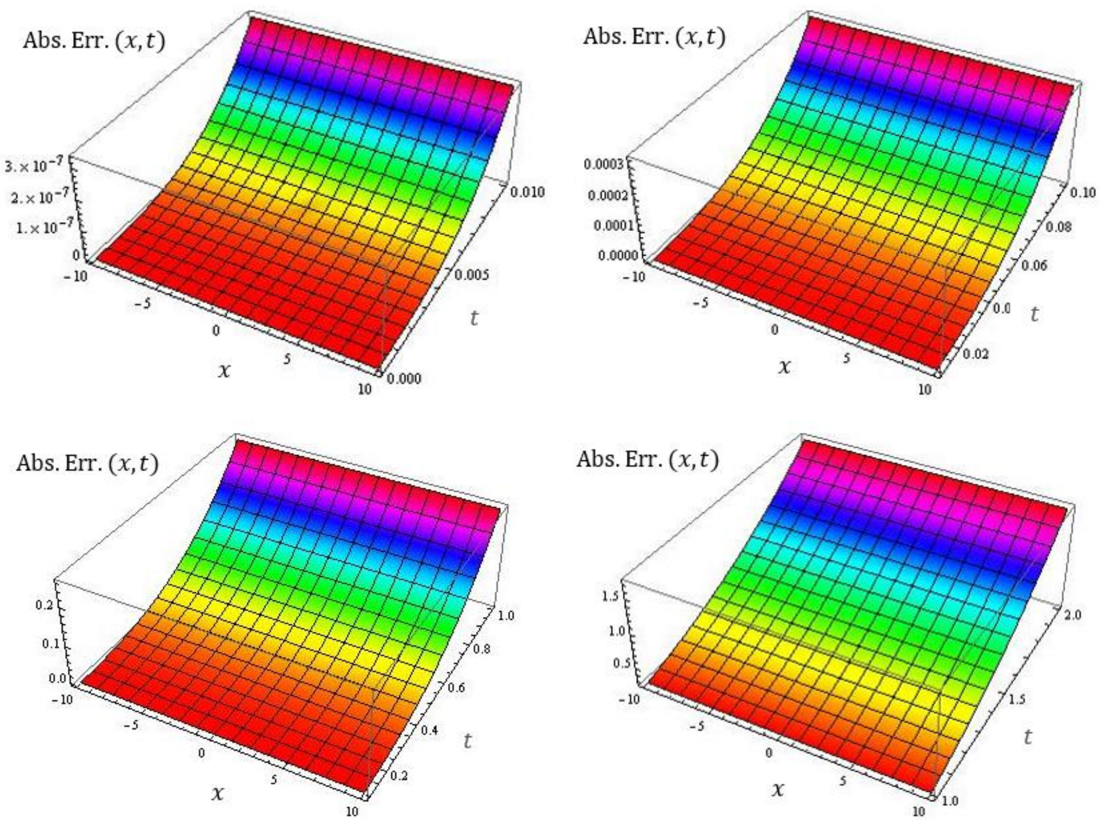


**Figure 3.** The surface graphs of the absolute error of the IVP (76)–(77) in the strip $[-10, 10] \times [0, 2]$.

**Problem 4 ([15]).** *Consider the following nonlinear water wave PDE:*

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} = -\frac{\partial u(x,t)}{\partial x} + u(x,t)\frac{\partial u(x,t)}{\partial x} - 2\frac{\partial^3 u(x,t)}{\partial x^3} + \frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2} \\ -u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} + 2\frac{\partial^5 u(x,t)}{\partial x^5}, x \in \mathbb{R}, t \geq 0,\end{aligned} \tag{84}$$

*subject to the initial condition:*

$$u(x,0) = 48 - \frac{480\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x}+1)^2}. \tag{85}$$

To find the LRPS solution for the IVP (84)–(85), we run the LT on both sides of Equation (84) and use the initial condition (85) as follows:

$$\begin{aligned}U(x,s) - \frac{1}{s}\left(48 - \frac{480\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x}+1)^2}\right) + \frac{1}{s}\frac{\partial U(x,s)}{\partial x} - \frac{1}{s}\mathcal{L}\left[\mathcal{L}^{-1}\left[U(x,s)\right]\mathcal{L}^{-1}\left[\frac{\partial U(x,s)}{\partial x}\right]\right] \\ +\frac{2}{s}\frac{\partial^3 U(x,s)}{\partial x^3} - \frac{1}{s}\mathcal{L}\left[\mathcal{L}^{-1}\left[\frac{\partial U(x,s)}{\partial x}\right]\mathcal{L}^{-1}\left[\frac{\partial^2 U(x,s)}{\partial x^2}\right]\right] \\ +\frac{1}{s}\mathcal{L}\left[\mathcal{L}^{-1}[U(x,s)]\mathcal{L}^{-1}\left[\frac{\partial^3 U(x,s)}{\partial x^3}\right]\right] - \frac{2}{s}\frac{\partial^5 U(x,s)}{\partial x^5} = 0, s > 0,\end{aligned} \tag{86}$$

Assume that the solution of Equation (86) is given by the initial conditions (85). Then, we deduce that $h_0(x) = 48 - \frac{480\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x}+1)^2}$ has the following expansion:

$$U(x,s) = \left(48 - \frac{480\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x}+1)^2}\right)\frac{1}{s} + \sum_{m=1}^{\infty}\frac{h_m(x)}{s^{1+m}}, s > 0. \tag{87}$$

To determine the coefficients of the series (87), we define the LRF as follows:

$$\begin{aligned}LRes(x,s) = \quad & U(x,s) - \left(48 - \frac{480\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x}+1)^2}\right)\frac{1}{s} + \frac{1}{s}\frac{\partial U(x,s)}{\partial x} \\ & -\frac{1}{s}\mathcal{L}\left[\mathcal{L}^{-1}\left[U(x,s)\right]\mathcal{L}^{-1}\left[\frac{\partial U(x,s)}{\partial x}\right]\right] + \frac{2}{s}\frac{\partial^3 U(x,s)}{\partial x^3} \\ & -\frac{1}{s}\mathcal{L}\left[\mathcal{L}^{-1}\left[\frac{\partial U(x,s)}{\partial x}\right]\mathcal{L}^{-1}\left[\frac{\partial^2 U(x,s)}{\partial x^2}\right]\right] \\ & +\frac{1}{s}\mathcal{L}\left[\mathcal{L}^{-1}[U(x,s)]\mathcal{L}^{-1}\left[\frac{\partial^3 U(x,s)}{\partial x^3}\right]\right] - \frac{2}{s}\frac{\partial^5 U(x,s)}{\partial x^5}.\end{aligned} \tag{88}$$

By calculating two additional iterations, our results can be summarized as

$$\begin{aligned}h_0(x) &= 48 - \frac{480\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x}+1)^2}, \\ h_1(x) &= -\frac{116160\mathrm{e}^{2x}(\mathrm{e}^{2x}-1)}{(1+\mathrm{e}^{2x})^3}, \\ h_2(x) &= -\frac{28110720\mathrm{e}^{2x}(1-4\mathrm{e}^{2x}+\mathrm{e}^{4x})}{(1+\mathrm{e}^{2x})^4}, \\ h_3(x) &= -\frac{6802794240\mathrm{e}^{2x}\left(-1+11\mathrm{e}^{2x}-11\mathrm{e}^{4x}+\mathrm{e}^{6x}\right)}{(1+\mathrm{e}^{2x})^5}, \\ h_4(x) &= -\frac{1646276206080\mathrm{e}^{2x}\left(1-26\mathrm{e}^{2x}+66\mathrm{e}^{4x}-26\mathrm{e}^{6x}+\mathrm{e}^{8x}\right)}{(1+\mathrm{e}^{2x})^6}, \\ h_5(x) &= -\frac{398398841871360\mathrm{e}^{2x}\left(-1+57\mathrm{e}^{2x}-302\mathrm{e}^{4x}+302\mathrm{e}^{6x}-57\mathrm{e}^{8x}+\mathrm{e}^{10x}\right)}{(1+\mathrm{e}^{2x})^7}.\end{aligned} \tag{89}$$

So, the solution of Equation (86) has the following approximation:

$$
\begin{aligned}
&U(x,s)\\
&=\left(48-\frac{480e^{-2x}}{(e^{-2x}+1)^2}\right)\frac{1}{s}-\frac{116160e^{2x}(e^{2x}-1)}{(1+e^{2x})^3}\frac{1}{s^2}-\frac{28110720e^{2x}(1-4e^{2x}+e^{4x})}{(1+e^{2x})^4}\frac{1}{s^3}\\
&-\frac{6802794240e^{2x}(-1+11e^{2x}-11e^{4x}+e^{6x})}{(1+e^{2x})^5}\frac{1}{s^4}\\
&-\frac{1646276206080e^{2x}(1-26e^{2x}+66e^{4x}-26e^{6x}+e^{8x})}{(1+e^{2x})^6}\frac{1}{s^5}\\
&-\frac{398398841871360e^{2x}(-1+57e^{2x}-302e^{4x}+302e^{6x}-57e^{8x}+e^{10x})}{(1+e^{2x})^7}\frac{1}{s^6}+\ldots.
\end{aligned}
\tag{90}
$$

The inverse LT of Equation (90) presents the LRPS solution of Equations (85) and (84) in the form:

$$
\begin{aligned}
&u(x,t)\\
&=48-\frac{480e^{-2x}}{(e^{-2x}+1)^2}-\frac{116160e^{2x}(-1+e^{2x})t}{(1+e^{2x})^3}-\frac{14055360e^{2x}(1-4e^{2x}+e^{4x})t^2}{(1+e^{2x})^4}\\
&-\frac{1133799040e^{2x}(-1+11e^{2x}-11e^{4x}+e^{6x})t^3}{(1+e^{2x})^5}\\
&-\frac{68594841920e^{2x}(1-26e^{2x}+66e^{4x}-26e^{6x}+e^{8x})t^4}{(1+e^{2x})^6}\\
&-\frac{3319990348928e^{2x}(-1+57e^{2x}-302e^{4x}+302e^{6x}-57e^{8x}+e^{10x})t^5}{(1+e^{2x})^7}+\ldots
\end{aligned}
\tag{91}
$$

It is worth noting that the exact solution of Equations (84) and (85) in terms of the elementary functions is $u(x,t)=48-120\text{sech}^2(x-121t)$. Again, the approximate solution of Equation (91) is in complete agreement with the approximate solutions obtained by the RPS method [15]. Figure 4 shows the whole agreement between the 5th approximate LRPS and the exact solutions of the IVP in Problem 4.

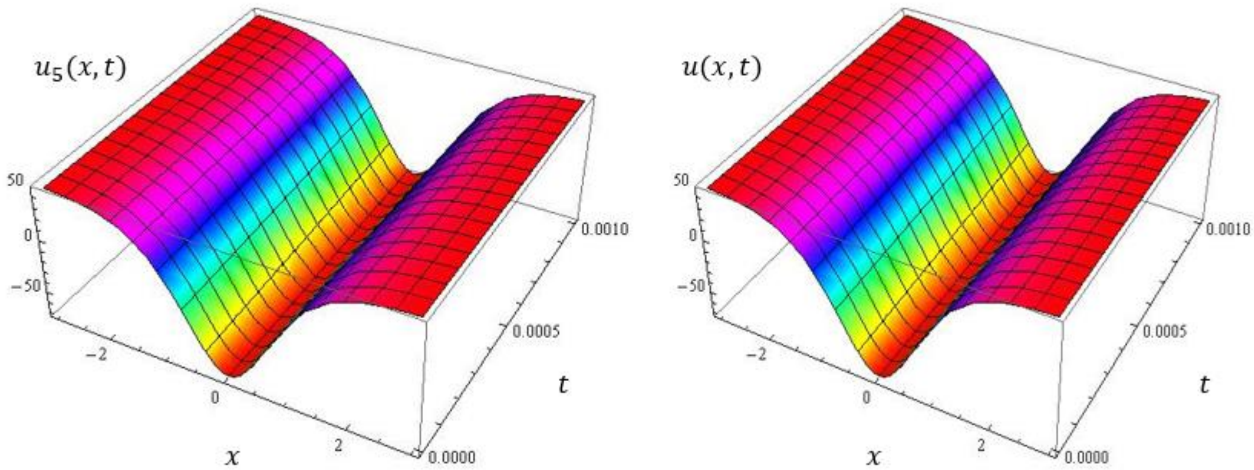


**Figure 4.** The 5th approximate LRPS and the exact solutions of the IVP (84)–(85).

**Problem 5 ([15]).** *Consider the following nonlinear Klein–Gordon PDE:*

$$
\frac{\partial^2 u(x,t)}{\partial t^2}-a\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2+b\left(\frac{\partial^4 u(x,t)}{\partial x^4}\right)^2=0, x\in\mathbb{R}, t\geq 0,
\tag{92}
$$

*subject to the initial conditions:*

$$
\begin{aligned}
u(x,0)&=-4\frac{w^2}{3a}\sinh^2\left(\frac{1}{4}\sqrt{\frac{a}{b}}x\right),\\
u_t(x,0)&=\frac{w^3}{3\sqrt{ab}}\sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right).
\end{aligned}
\tag{93}
$$

To create the LRPS solution to the IVP (92)–(93), we transfer Equation (92) into the Laplace space and use the initial conditions to have

$$U(x,s) + 4\frac{w^2}{3sa}\sinh^2\left(\frac{1}{4}\sqrt{\frac{a}{b}}x\right) - \frac{w^3}{3s^2\sqrt{ab}}\sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right) - \frac{a}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{\partial^2 U(x,s)}{\partial x^2}\right]\right)^2\right] + \frac{b}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{\partial^4 U(x,s)}{\partial x^4}\right]\right)^2\right] = 0. \tag{94}$$

The LRF of Equation (94) is given by

$$LRes(x,s) = U(x,s) + 4\frac{w^2}{3sa}\sinh^2\left(\frac{1}{4}\sqrt{\frac{a}{b}}x\right) - \frac{w^3}{3s^2\sqrt{ab}}\sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right) - \frac{a}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{\partial^2 U(x,s)}{\partial x^2}\right]\right)^2\right] + \frac{b}{s^2}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\frac{\partial^4 U(x,s)}{\partial x^4}\right]\right)^2\right], s > 0. \tag{95}$$

We can summarize the obtained coefficients as

$$\begin{aligned} h_0(x) &= -4\frac{w^2}{3a}\sinh^2\left(\frac{1}{4}\sqrt{\frac{a}{b}}x\right), \\ h_1(x) &= \frac{w^3}{3\sqrt{ab}}\sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right), \\ h_2(x) &= -\frac{w^4}{6b}\cosh(\frac{1}{2}\sqrt{\frac{a}{b}}x), \\ h_3(x) &= \frac{\sqrt{ab}w^5}{12b^2}sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right), \\ h_4(x) &= -\frac{aw^6}{24b^2}\cosh(\frac{1}{2}\sqrt{\frac{a}{b}}x), \\ h_5(x) &= \frac{(ab)^{\frac{3}{2}}w^7\sinh(\frac{1}{2}\sqrt{\frac{a}{b}}x)}{48b^4}. \end{aligned} \tag{96}$$

Therefore, the solution of Equation (94) can be expressed in the following infinite series:

$$U(x,s) = -4\frac{w^2}{3a}\sinh^2\left(\frac{1}{4}\sqrt{\frac{a}{b}}x\right)\frac{1}{s} + \frac{w^3}{3\sqrt{ab}}\sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right)\frac{1}{s^2} - \frac{w^4}{6b}\cosh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right)\frac{1}{s^3} + \frac{\sqrt{ab}w^5}{12b^2}\sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right)\frac{1}{s^4} - \frac{aw^6}{24b^2}\cosh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right)\frac{1}{s^5} + \frac{(ab)^{\frac{3}{2}}w^7\sinh\left(\frac{1}{2}\sqrt{\frac{a}{b}}x\right)}{48b^4}\frac{1}{s^6} + \ldots, s > 0. \tag{97}$$

Hence, the inverse LT of Equation (97) yields the LRPS solution of the IVP (92)–(93), which is written as

$$u(x,t) = -\frac{2w^2}{3a}\left[\cosh\left(\frac{x}{2}\sqrt{\frac{a}{b}}\right)\left(1 + \frac{aw^2}{2^2b}\frac{t^2}{2!} + \frac{a^2w^4}{2^4b^2}\frac{t^4}{4!} + \frac{a^3w^6}{2^6b^3}\frac{t^6}{6!} + \ldots\right) - 1\right] + \frac{2w^2}{3a}\sinh\left(\frac{x}{2}\sqrt{\frac{a}{b}}\right)\left[\frac{w}{2}\sqrt{\frac{a}{b}}\frac{t}{1!} + \frac{w^3}{2^3}\sqrt{\frac{a^3}{b^3}}\frac{t^3}{3!} + \frac{w^5}{2^5}\sqrt{\frac{a^5}{b^5}}\frac{t^5}{5!} + \ldots\right]. \tag{98}$$

So, the exact solution of Equations (92) and (93) in a closed form of elementary function is

$$u(x,t) = -\frac{2w^2}{3a}\left[\cosh\left(\frac{x}{2}\sqrt{\frac{a}{b}}\right)\cosh\left(\frac{w}{2}\sqrt{\frac{a}{b}}t\right) - \sinh\left(\frac{x}{2}\sqrt{\frac{a}{b}}\right)\sinh\left(\frac{w}{2}\sqrt{\frac{a}{b}}t\right) - 1\right]. \tag{99}$$

Figure 5 shows a comparison between the exact solution in Equation (99) and the fifth approximate LRPS solution of the IVP (92)–(93). It appears in the figure that there is significant congruence between the two solutions in a strip. This means that the region of convergence of the series solution is within the thin strip. Therefore, further analysis was carried out by calculating the absolute and the relative errors of the fifth approximate LRPS solution to confirm the appropriateness of the resulting solution, where the relative error is defined by Rel. Err.$(x,t) = \left|\frac{u(x,t)-u_5(x,t)}{u(x,t)}\right|$. Table 1 shows the exact and fifth approximate solutions in addition to the exact and relative errors at different points in the region $[0,10] \times [0,1]$.

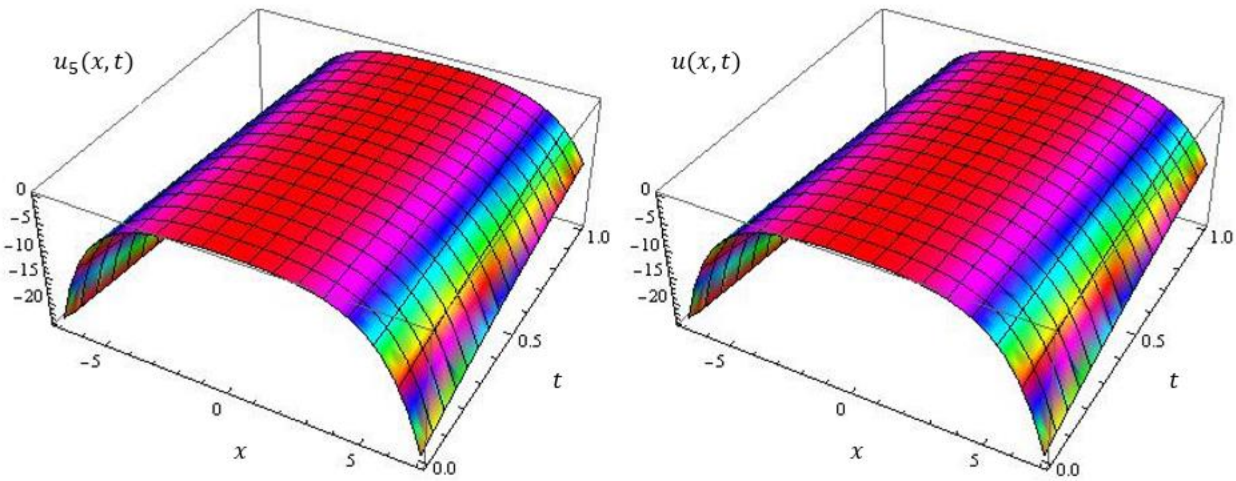


**Figure 5.** The surface graphs of the 5th approximate LRPS and exact solutions of the IVP (92)–(93) at $a = 2, b = w = 1., b = w = 1$.

**Table 1.** Numerical comparisons between the 5th approximate LRPS and the exact solutions of IVP (92)–(93) and the exact and relative errors when $a = 2$, $b = w1$.

| $t$ | $x$ | $u_5(x,t)$ | $u(x,t)$ | Absolute Error | Relative Error |
|---|---|---|---|---|---|
| 0.01 | 1 | −0.0850653 | −0.0850653 | $3.76920 \times 10^{-14}$ | $4.43095 \times 10^{-13}$ |
| | 5 | −5.3501506 | −5.3501506 | $8.43769 \times 10^{-13}$ | $1.57709 \times 10^{-13}$ |
| | 10 | −194.51822 | −194.51822 | $2.87911 \times 10^{-11}$ | $1.48013 \times 10^{-13}$ |
| 0.10 | 1 | −0.0698091 | −0.0698091 | $3.76933 \times 10^{-9}$ | $5.39949 \times 10^{-8}$ |
| | 5 | −5.0003477 | −5.0003477 | $8.41855 \times 10^{-8}$ | $1.68359 \times 10^{-8}$ |
| | 10 | −182.50430 | −182.50429 | $2.81469 \times 10^{-6}$ | $1.58415 \times 10^{-8}$ |
| 0.50 | 1 | −0.0210630 | −0.0210512 | $1.18616 \times 10^{-5}$ | $5.61029 \times 10^{-4}$ |
| | 5 | −3.6895853 | −3.6893215 | $2.63798 \times 10^{-4}$ | $7.15032 \times 10^{-5}$ |
| | 10 | −137.46925 | −137.46019 | $9.05949 \times 10^{-3}$ | $6.59063 \times 10^{-5}$ |
| 1.00 | 1 | −0.00380753 | $3.70074 \times 10^{-17}$ | $3.80752 \times 10^{-4}$ | $1.02885 \times 10^{-3}$ |
| | 5 | −2.50483 | −2.4963224 | $8.50954 \times 10^{-3}$ | $3.40883 \times 10^{-3}$ |
| | 10 | −96.7161 | −96.423845 | 0.292239 | $3.03780 \times 10^{-3}$ |

It is noteworthy that the solution in Equation (99) matches the solution obtained in the variational iteration method [33].

**Problem 6 ([32]).** *Consider the following nonlinear Fisher equation:*

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + 6u(x,t)(1 - u(x,t)), x \in \mathbb{R}, t \geq 0, \tag{100}$$

*with the initial condition:*

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{101}$$

Similar to the previous examples, we transform Equation (100) to the Laplace space with the initial conditions as

$$U(x,s) - \frac{1}{s(1+e^x)^2} - \frac{1}{s}U_{xx}(x,s) - \frac{6}{s}(U(x,s)) - \frac{6}{s}(\mathcal{L}\left[\left(\mathcal{L}^{-1}[U(x,s)]\right)^2\right]) = 0. \tag{102}$$

The LRF of Equation (102) is given by

$$\begin{aligned} LRes(x,s) = U(x,s) - \tfrac{1}{s(1+e^x)^2} - \tfrac{1}{s}U_{xx}(x,s) - \tfrac{6}{s}(U(x,s)) \\ -\tfrac{6}{s}\left(\mathcal{L}\left[\left(\mathcal{L}^{-1}[U(x,s)]\right)^2\right]\right). \end{aligned} \tag{103}$$

Now, we collect the obtained coefficients for Equation (103) in the following box:

$$\begin{aligned} h_0(x) &= \tfrac{1}{(1+e^x)^2}, \\ h_1(x) &= \tfrac{10e^x}{(1+e^x)^3}, \\ h_2(x) &= \tfrac{50e^x(-1+2e^x)}{(1+e^x)^4}, \\ h_3(x) &= \tfrac{250e^x(1-7e^x+4e^{2x})}{(1+e^x)^5}, \\ h_4(x) &= \tfrac{1250e^x\left(-1+18e^x-33e^{2x}+8e^{3x}\right)}{(1+e^x)^6}, \\ h_5(x) &= \tfrac{6250e^x(1-41e^x+171e^{2x}-131e^{3x}+16e^{4x})}{(1+e^x)^7}. \end{aligned} \tag{104}$$

Therefore, the sixth approximate solution of Equation (102) can be expressed in the following form:

$$\begin{aligned} U(x,s) \quad &= \left(\tfrac{1}{(1+e^x)^2}\right)\tfrac{1}{s} + \left(\tfrac{10e^x}{(1+e^x)^3}\right)\tfrac{1}{s^2} + \left(\tfrac{50e^x(-1+2e^x)}{(1+e^x)^4}\right)\tfrac{1}{s^3} \\ &+ \left(\tfrac{250e^x(1-7e^x+4e^{2x})}{(1+e^x)^5}\right)\tfrac{1}{s^4} \\ &+ \left(\tfrac{1250e^x(-1+18e^x-33e^{2x}+8e^{3x})}{(1+e^x)^6}\right)\tfrac{1}{s^5} \\ &+ \left(\tfrac{6250e^x(1-41e^x+171e^{2x}-131e^{3x}+16e^{4x})}{(1+e^x)^7}\right)\tfrac{1}{s^6}, s > 0. \end{aligned} \tag{105}$$

By applying the inverse LT on Equation (105), we obtain the sixth approximate LRPS solution to the IVP (100)–(101):

$$\begin{aligned} u(x,t) = \tfrac{1}{(1+e^x)^2} \quad &+ \tfrac{10e^x t}{(1+e^x)^3} + \tfrac{25e^x(-1+2e^x)t^2}{(1+e^x)^4} + \tfrac{125e^x(1-7e^x+4e^{2x})t^3}{3(1+e^x)^5} \\ &+ \tfrac{625e^x(-1+18e^x-33e^{2x}+8e^{3x})t^4}{12(1+e^x)^6} \\ &+ \tfrac{625e^x(1-41e^x+171e^{2x}-131e^{3x}+16e^{4x})t^5}{12(1+e^x)^7}. \end{aligned} \tag{106}$$

The solution indicated in Equation (106) matches the solution obtained with the homotopy perturbation method in solving the PDEs. Unfortunately, it is not easy to predict the exact solution of Application 4.6. So, we only provide approximate solutions. Figure 6 shows the 6th and 11th approximate solutions to the IVP (100)–(101). The figure shows that by increasing the number of approximate solution terms, the surface becomes smoother at the boundary. To determine the extent of the compatibility and convergence of the two approximate solutions in Figure 6, we illustrate, in Table 2, the approximate exact error and the approximate relative error, which are, respectively, given by

$$\text{App. Exa. Err.}(x,t) = |u_{k+m}(x,t) - u_k(x,t)|,$$
$$\text{App. Rel. Err.}(x,t) = \left|\frac{u_{k+m}(x,t) - u_k(x,t)}{u_{k+m}(x,t)}\right|, m \text{ is sufficiently large.}$$

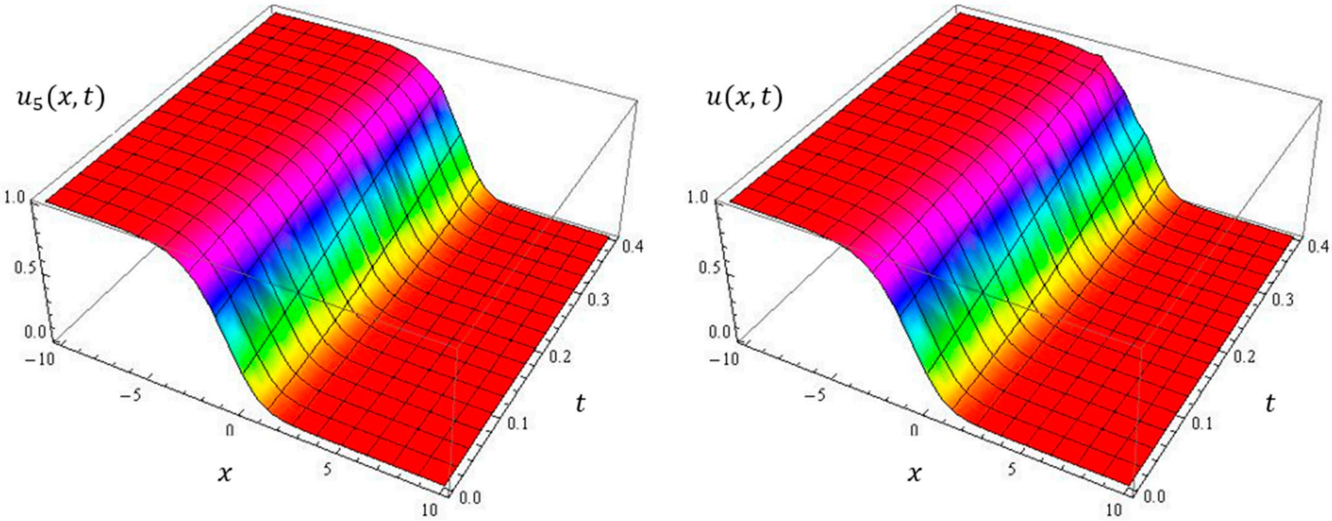


**Figure 6.** The surface graphs of the 6th and 11th approximate LRPS solutions of the IVP (100) and (101).

**Table 2.** Numerical comparisons between the 6th and 11th approximate LRPS solutions of IVP (100)–(101) and the approximate exact and relative errors.

| $t$ | $x$ | $u_6(x,t)$ | $u_{11}(x,t)$ | App. Exa. Err. | App. Rel. Err. |
|---|---|---|---|---|---|
| 0.01 | −10 | 0.99990920 | 0.9999092 | $1.94803 \times 10^{-15}$ | $1.948202 \times 10^{-15}$ |
| | 0 | 0.25 | 0.25 | $2.28855 \times 10^{-11}$ | $8.713212 \times 10^{-11}$ |
| | 10 | $2.060966 \times 10^{-9}$ | $2.060966 \times 10^{-9}$ | $2.90110 \times 10^{-18}$ | $1.273692 \times 10^{-9}$ |
| 0.10 | −10 | 0.99990920 | 0.9999092 | $1.83057 \times 10^{-9}$ | $1.830673 \times 10^{-9}$ |
| | 0 | 0.25 | 0.25 | $2.07248 \times 10^{-5}$ | $5.348967 \times 10^{-5}$ |
| | 10 | $2.060966 \times 10^{-9}$ | $2.060966 \times 10^{-9}$ | $3.32530 \times 10^{-12}$ | $5.935969 \times 10^{-4}$ |
| 0.20 | −10 | 0.99990920 | 0.9999092 | $1.09697 \times 10^{-7}$ | $1.097008 \times 10^{-7}$ |
| | 0 | 0.25 | 0.25 | $1.11782 \times 10^{-3}$ | $2.091544 \times 10^{-3}$ |
| | 10 | $2.060966 \times 10^{-9}$ | $2.060966 \times 10^{-9}$ | $2.51822 \times 10^{-10}$ | $1.653888 \times 10^{-2}$ |
| 0.40 | −10 | 0.99990920 | 0.9999092 | $6.21665 \times 10^{-6}$ | $6.216732 \times 10^{-6}$ |
| | 0 | 0.25 | 0.25 | $5.35626 \times 10^{-2}$ | $6.806822 \times 10^{-2}$ |
| | 10 | $2.060966 \times 10^{-9}$ | $2.060966 \times 10^{-9}$ | $2.38222 \times 10^{-8}$ | $2.124263 \times 10^{-1}$ |

## 5. Conclusions

In this paper, we discuss an analytical method called the LRPS to address the problems in the traditional Laplace transform technique that deals with only some types of linear equations. Using the new LT approach, it is no longer impossible to solve nonlinear DEs. The presented method is the same as the traditional series method, except it provides a smooth and fast technique for determining the series coefficients. The equation is converted into a Laplace space, and the new equation is solved using a series method by the LS, which is the LT for the PS. It is also known that the series solution method requires a recurrence relation to determine the coefficient values for solving DEs. In addition, finding the recurrence relation in the case of non-linear equations is not easy and requires much time and effort during the technique. It is sometimes impossible, but by using the LRPS method it becomes easy to determine the values of the series coefficients for the nonlinear equations. However, using the LRPS method does not need recurrence relations, but these coefficients are determined iteratively using any mathematical program. Finally, we must point out the possibility of applying the proposed method in solving other PDEs, such as nonlinear Schrödinger-type equations, modified Korteweg–de Vries-type equations, Sasa–Satsuma-type equations, and Laplace Equations.

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