axioms

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N-Widths of Multivariate Sobolev Spaces with Common Smoothness in Probabilistic and Average Settings in the S_q Norm

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Abstract: In this article, we give the sharp bounds of probabilistic Kolmogorov (N, δ)-widths and probabilistic linear (N, δ)-widths of the multivariate Sobolev space W_2^A with common smoothness on a S_q norm equipped with the Gaussian measure μ, where $A \subset \mathbb{R}^d$ is a finite set. And we obtain the sharp bounds of average width from the results of the probabilistic widths. These results develop the theory of approximation of functions and play important roles in the research of related approximation algorithms for Sobolev spaces.

Keywords: probabilistic width; average width; Sobolev space with common smoothness; Gaussian measure; asymptotic order

MSC: 41A46; 42A61; 41A63

Citation: Liu, Y.; Li, X.; Li, H. N-Widths of Multivariate Sobolev Spaces with Common Smoothness in Probabilistic and Average Settings in the S_q Norm. *Axioms* **2023**, *12*, 698. https://doi.org/10.3390/axioms12070698

Academic Editors: Roman Dmytryshyn and Tuncer Acar

Received: 19 June 2023
Revised: 14 July 2023
Accepted: 16 July 2023
Published: 17 July 2023

1. Introduction

The approximation theory of functions is a classical theory of basic mathematics and computational mathematics, and width theory plays a very important role in approximation theory. With the gradual development of modern mathematics and science, the system of width theory has also been improved, which has greatly promoted the research of algorithms and computational complexity. Different types of widths correspond to different calculation methods, and then result in different errors. The different definitions of algorithm errors and costs lead to different computational models. The most common models are the worst case setting, probabilistic setting, and average-case setting. Temlyakov [1] calculated the bounds of approximation of functions with a bounded mixed derivative. Maiorov [2–4] gave the definition of probabilistic Kolmogorov and linear (N, δ)-widths and obtained the sharp bounds of probabilistic Kolmogorov (N, δ)-widths of Sobolev space W_2^r in L_q by using discretization. Fang and Ye [5,6] estimated the exact order of linear N-widths in the probabilistic setting and average-case setting of finite dimensional space. Chen and Fang [7,8] discussed probabilistic Kolmogorov (N, δ)-widths and probabilistic linear (N, δ)-widths of the multivariate Sobolev space $MW_2^r\left(\mathbb{T}^d\right)$ with a mixed derivative, and they obtained the sharp bounds of p-average Kolmogorov and linear N-widths of $MW_2^r\left(\mathbb{T}^d\right)$. Tan et al. [9] gave the definition of probabilistic Gel'fand (N, δ)-width and obtained the sharp bounds of probabilistic Gel'fand (N, δ)-width of Sobolev space $W_2^r(\mathbb{T})$. Liu et al. [10] gave the definition of p-average Gel'fand N-width and obtained the sharp bounds of p-average Gel'fand N-widths of Sobolev space $W_2^r(\mathbb{T})$ and $MW_2^r\left(\mathbb{T}^d\right)$. Dai and Wang [11] obtained the sharp bounds of probabilistic linear (N, δ)-widths and p-average linear N-widths of finite dimensional space with a diagonal matrix. Wang [12,13] estimated the sharp bounds of probabilistic linear (N, δ)-widths and p-average linear N-widths of weighted Sobolev spaces on the ball and Sobolev spaces on compact two-point homogeneous spaces.

Let us recall some definitions of N-widths, which can be found from the book of Pinkus [14].

Let W be a bounded subset of a normed linear space X with norm $\|\cdot\|$, and F_N be a N-dimensional subspace of X. The following quantity is called the deviation of W to F_N:

$$E(W, F_N, X) := \sup_{x \in W} e(x, F_N).$$

where $e(x, F_N) := \inf_{y \in F_N} \|x - y\|$. It shows how well the "worst" elements of W can be approximated by F_N. And the Kolmogorov N-width of W in X is defined as follows:

$$d_N(W, X) := \inf_{F_N} E(W, F_N, X) = \inf_{F_N} \sup_{x \in W} \inf_{y \in F_N} \|x - y\|. \tag{1}$$

where the leftmost infimum is taken over all N-dimensional linear subspaces of X.

Next, let T be a linear operator from X to X. The linear distance of the image TW from the set W is defined as follows:

$$\lambda(W, T, X) = \sup_{x \in W} \|x - Tx\|.$$

and the linear N-width of W in X is defined as follows:

$$\lambda_N(W, X) := \inf_{T_N} \lambda(W, T_N, X). \tag{2}$$

where the infimum is taken over all linear operators T_N whose rank is at most N.

Now we give the definition of probabilistic (N, δ)-widths and p-average N-widths from the article of Maiorov [2–4].

Definition 1. *Let W be a bounded subset of normed linear space $(X, \|\cdot\|)$. Assume that W contains a Borel field B consisting of open subsets of W and is equipped with a probability measure μ, i.e., μ is a σ-additive nonnegative function on B , and satisfies the condition that $\mu(W) = 1$. For any $\delta \in (0, 1]$, the probabilistic Kolmogorov (N, δ)-width and probabilistic linear (N, δ)-width of W in X are defined as follows:*

$$d_{N,\delta}(W, \mu, X) := \inf_{G_\delta} d_N(W \backslash G_\delta, X). \tag{3}$$

$$\lambda_{N,\delta}(W, \mu, X) := \inf_{G_\delta} \lambda_N(W \backslash G_\delta, X). \tag{4}$$

where G_δ runs through all possible subsets in B, which satisfies the condition that $\mu(G_\delta) \le \delta$.

Definition 2. *Let W, X and μ be the same to Definition 1. Given $0 < p < \infty$, the p-average Kolmogorov N-width and p-average linear N-width are defined, respectively, by*

$$d_N^{(a)}(W, \mu, X)_p := \inf_{F_N} \left(\int_W e(x, F_N)^p \mathrm{d}\mu \right)^{1/p}, \tag{5}$$

$$\lambda_N^{(a)}(W, \mu, X)_p := \inf_{T_N} \left(\int_W \|x - T_N x\|_X^p \mathrm{d}\mu \right)^{1/p}. \tag{6}$$

It can be seen from the definition that N-widths are defined by the errors generated by the "worst" elements of the functions class during the approximation process in the worst case setting. For example, the classical Kolmogorov N-widths of functional classes are defined by the optimal errors generated by the approximation of the "worst" element in the set by a finite dimensional subspace. To satisfy the demands of practical applications and theoretical analysis, the concepts of N-widths in the probabilistic and average-case setting are introduced. The sharp bounds of those widths are often used to solve the optimal solution of numerical problems. Like classical N-widths, probabilistic (N,δ)-widths reflect the best approximation of functional classes. From the definitions, we know that it needs to delete some functions with the "worst" properties before defining N-widths of functional classes in the probabilistic setting, and these widths are still defined by the "worst" elements of the remaining functions. Therefore, although the probabilistic (N,δ)-widths can allow the algorithm to generate "errors" within a given range, it does not reflect the overall optimal approximation situation. The N-widths in the average-case setting are defined by the integral of the errors under a given measure, which give the average approximation degree of a function class under a given probability measure. They reflect the optimal approximation degree of most elements in spaces, and more profoundly reflect the essential characteristics of the structure of the functional classes.

Next, we will provide two asymptotic relationships. Let $a(x)$ and $b(x)$ be two positive functions of x. If there is a positive constant $c>0$, such that $a(x)\leq cb(x)$ for all x from the domain of the functions a and b, then we write $a(x)\ll b(x)$ or $b(x)\gg a(x)$. If $a(x)\ll b(x)$ and $a(x)\gg b(x)$, then we write $a(x)\asymp b(x)$.

2. Main Results

In this article, we will discuss probabilistic Kolmogorov and linear (N,δ)-widths. Then, we will estimate the sharp bounds of p-average Kolmogorov and linear N-widths by using the results of probabilistic Kolmogorov and linear (N,δ)-widths. First, we introduce the concept of multivariate Sobolev space $W_2^A\left(\mathbb{T}^d\right)$, where $\mathbb{T}=[0,2\pi)$.

Let $s\in\mathbb{R}$, $y=(y_1,\ldots,y_d)\in\mathbb{R}^d$, $t=(t_1,\ldots,t_d)\in\mathbb{R}^d$, $k=(k_1,\ldots,k_d)\in\mathbb{Z}^d\subset\mathbb{R}^d$. We write $(y,t)=\sum\limits_{i=1}^{d}y_it_i$, $|y|^s=\prod\limits_{i=1}^{d}|y_i|^s$, $y+s=(y_1+s,\ldots,y_d+s)$.

Assume $L_2\left(\mathbb{T}^d\right)$ is a classical Lebesgue square integrable space. For any $x,y\in L_2\left(\mathbb{T}^d\right)$, this space is a Hilbert space with the inner product

$$\langle x,y\rangle=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}x(t)\overline{y(t)}\mathrm{d}t.$$

For $x\in L_2\left(\mathbb{T}^d\right)$, the Fourier series of x is defined as follows:

$$c_k=\hat{x}(k)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}x(t)e_k(-t)\mathrm{d}t,$$

where $e_k(t):=\exp\mathrm{i}(k,t)$.

For any $\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{R}^d$, we define the Wyel α-derivative for $x\in L_2\left(\mathbb{T}^d\right)$ as follows:

$$x^{(\alpha)}(t):=(D^\alpha x)(t)=\sum_{k\in\mathbb{Z}_0^d}(\mathrm{i}k)^\alpha c_ke_k(t),$$

where $\mathbb{Z}_0^d=\left\{k=(k_1,\ldots,k_d)\in\mathbb{Z}^d:k_i\neq0,i=1,\ldots,d\right\}$, $(\mathrm{i}k)^\alpha=\prod\limits_{j=1}^{d}|k_j|^{\alpha_j}\exp\left(\mathrm{i}\frac{\pi}{2}sgn\alpha_j\right)$.

Given the finite subset A of $\mathbb{R}^d$, the multivariate Sobolev $W_2^A\left(\mathbb{T}^d\right)$ with common smoothness is defined by

$$W_2^A\left(\mathbb{T}^d\right):=\left\{x\in L_2\left(\mathbb{T}^d\right):x^{(\alpha)}(t)\in L_2\left(\mathbb{T}^d\right),\alpha\in A,\int_0^{2\pi}x(t)\mathrm{d}t_j=0,j=1,\ldots,d\right\}.\quad(7)$$

From Equation (7), we need to give the definition of the common Weyl-derivative as follows:

$$x^{(A)}(t) := \left(D^A x\right)(t) := \sum_{k\in\mathbb{Z}_0^d} (\mathrm{i}k)^A c_k e_k(t), \tag{8}$$

where $(\mathrm{i}k)^A = \sum_{\alpha\in A} (\mathrm{i}k)^\alpha$. We can know that the Sobolev space $W_2^A\left(\mathbb{T}^d\right)$ is a Hilbert space with the inner $\langle x,y\rangle_A := \left\langle x^{(A)}, y^{(A)}\right\rangle$ and with the norm $\|x\|_{W_2^A} = \left\langle x^{(A)}, x^{(A)}\right\rangle^{\frac{1}{2}}$.

Our results of the Sobolev space $W_2^A\left(\mathbb{T}^d\right)$ with common smoothness can be a generalization of the sharp bounds of N-widths in the probabilistic and average setting of Sobolev spaces with smoothness. For example, if $A = \{\alpha\}$, then $W_2^A\left(\mathbb{T}^d\right) = MW_2^\alpha\left(\mathbb{T}^d\right)$. Space $MW_2^\alpha\left(\mathbb{T}^d\right)$ is a Sobolev space with a mixed derivative, and the related conclusions can be found in papers [7,8].

We denote by A and B any two subsets of $\mathbb{R}^d$, and we denote that

$$A+B := \{x+y : x\in A, y\in B\}, A+\eta := \{x+\eta : x\in A, \eta\in\mathbb{R}\}.$$

Let $co(A)$ be the convex hull of a set A, $N(A) := co(A) - \mathbb{R}_+^d$, and $IN(A)$ be the set of interior points of $N(A)$. We write $A_+^0 := \left\{x\in\mathbb{R}_+^d : (\alpha,x)\le 1, \alpha\in A\right\}, r = \left(\sup\{(s,1) : s\in A_+^0\}\right)^{-1}$, $v := \dim\left\{x\in A_+^d : (x,1) = r^{-1}\right\}$, $A' = \frac{1}{r+\frac{\rho}{2}}\left(A+\frac{\rho}{2}\right)$. In the research process of this article, we always assume that $0\in IN(A)$ and $r > 1/2$.

Now, we give the definition of the space $S_q\left(\mathbb{T}^d\right)$:

$$S_q\left(\mathbb{T}^d\right) := \left\{x\in L_1\left(\mathbb{T}^d\right) : \{\hat{x}(k)\}_{k\in\mathbb{Z}^d}\in l_q\right\}$$

where l_q is the infinite vector space with the norm for any $a = \{a_n\}_{n=-\infty}^{\infty}$:

$$\|a\|_{l_q} = \begin{cases} \left(\sum\limits_{j=-\infty}^{\infty} |a_j|^q\right)^{1/q}, 1\le q<\infty \\ \sup\limits_{j\in\mathbb{Z}} |a_j|, q=\infty \end{cases}$$

For any $x\in S_q\left\{\mathbb{T}^d\right\}$, let $\|x\|_{q,S} := \left\|\{\hat{x}(k)\}_{k\in\mathbb{Z}_0^d}\right\|_{l_q}$ be the norm of $S_q\left(\mathbb{T}^d\right)$.

Next, we equip a Gaussian measure for $W_2^A\left(\mathbb{T}^d\right)$. Let μ be a Gaussian measure whose mean value is 0 and whose correlation operator is C_μ which has eigenfunctions $e_k(t)$ and eigenvalues $\lambda_k = |k|^{-\rho} (\rho > 1)$, that is,

$$C_\mu e_k = \lambda_k e_k, k\in\mathbb{Z}_0^d \tag{9}$$

Let $y_1,\dots,y_n$ be any orthogonal system of functions in $L_2\left(\mathbb{T}^d\right)$, $\sigma_j = \langle C_\mu y_j, y_j\rangle$, $j = 1,\dots,n$, and $\mathcal{D}$ be an arbitrary Borel subset of $\mathbb{R}^n$. Then, the Gaussian measure μ on the cylindrical subsets in the space $W_2^A\left(\mathbb{T}^d\right)$:

$$G = \left\{x\in W_2^A\left(\mathbb{T}^d\right) : \left(\left\langle x, y_1^{(-r)}\right\rangle_r, \ \cdots, \left\langle x, y_n^{(-r)}\right\rangle_r\right)\in\mathcal{D}\right\}$$

is given by

$$\mu(G)=\prod_{j=1}^{n}\left(2\pi\sigma_j\right)^{-\frac{1}{2}}\int_{\mathcal{D}}\exp\left(-\sum_{j=1}^{n}\frac{\left|u_j\right|^2}{2\sigma_j}\right)\mathrm{d}u_1\cdots\mathrm{d}u_n. \tag{10}$$

More results and research of Gaussian measures can be found in paper [15–17].

The aim of this paper is to determine the asymptotic order of probabilistic Kolmogorov and linear (N,δ)-widths as well as p-average Kolmogorov and linear N-widths of the multivariate Sobolev space $W_2^A\left(\mathbb{T}^d\right)$ with common smoothness. The main results are as follows:

Theorem 1. *Assume that* $r>\frac{1}{2}$, $1\le q<\infty$, $\rho>1$, $\delta\in\left(0,\frac{1}{2}\right]$, $\mathbb{T}=[0,2\pi)$. *Let* A *be a finite subset of* $\mathbb{R}^d$ *and* $0\in IN(A)$. *Note* $d_{N,\delta}:=d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)$. *Then,*

$$d_{N,\delta}\asymp\left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\sqrt{1+\frac{1}{N}\ln\left(\frac{1}{\delta}\right)},\ 1\le q<\infty. \tag{11}$$

Theorem 2. *Assume that* $r>\frac{1}{2}$, $1\le q<\infty$, $\rho>1$, $\delta\in\left(0,\frac{1}{2}\right]$, $\mathbb{T}=[0,2\pi)$. *Let* A *be a finite subset of* $\mathbb{R}^d$ *and* $0\in IN(A)$. *Note* $\lambda_{N,\delta}:=\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)$. *Then,*

$$\lambda_{N,\delta}\asymp\left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\sqrt{1+\frac{1}{N}\ln\left(\frac{1}{\delta}\right)},\ 1\le q<2; \tag{12}$$

$$\lambda_{N,\delta}\asymp\left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\left(1+N^{-\frac{1}{q}}\sqrt{\ln\left(\frac{1}{\delta}\right)}\right),\ 2\le q<\infty. \tag{13}$$

Theorem 3. *Assume that* $r>\frac{1}{2}$, $1\le q<\infty$, $\rho>1$, $\delta\in\left(0,\frac{1}{2}\right]$, $\mathbb{T}=[0,2\pi)$. *Let* A *be a finite subset of* $\mathbb{R}^d$ *and* $0\in IN(A)$, $0<p<\infty$. *Then,*

$$d_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)_p\asymp\left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right),1\le q<\infty. \tag{14}$$

Theorem 4. *Assume that* $r>\frac{1}{2}$, $1\le q<\infty$, $\rho>1$, $\delta\in\left(0,\frac{1}{2}\right]$, $\mathbb{T}=[0,2\pi)$. *Let* A *be a finite subset of* $\mathbb{R}^d$ *and* $0\in IN(A)$, $0<p<\infty$. *Then,*

$$\lambda_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)_p\asymp\left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right),\ 1\le q<\infty. \tag{15}$$

3. Discretization

In order to prove Theorems 1 and 2, we use the discretization method, which is based on the reduction of the calculation of the probabilistic widths of a given class to the computation of the widths of a finite-dimensional set equipped with the standard Gaussian measure. Before we use the discretization, we need the definitions, and cite some results on the probabilistic widths of finite-dimensional spaces. Let l_p^m be the m-dimensional normed space of vectors $x=(x_1,\cdots,x_m)\in\mathbb{R}^m$, with norm

$$\|x\|_{l_p^m}=\begin{cases}\left(\sum\limits_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}}, & 1\le p<\infty.\\ \max\limits_{1\le i\le m}|x_i|, & p=\infty.\end{cases}$$

Consider in $\mathbb{R}^m$ the standard Gaussian measure, which is defined as

$$v(G) = (2\pi)^{-\frac{m}{2}} \int_G \exp\left(-\frac{1}{2}\|x\|_2^2\right) \mathrm{d}x. \tag{16}$$

where G is any Borel subset in $\mathbb{R}^m$. Obviously, $v(\mathbb{R}^m) = 1$.

First, we introduce some results of probabilistic (N, δ)-widths of finite space. These results can be found from papers of Maiorov, Chen, Fang, and Ye [2–8].

Lemma 1 (Maiorov, Chen, and Fang [3,4,7])**.** *Let $m > N$, $1 \leq q \leq \infty$ and $\delta \in \left(0, \frac{1}{2}\right]$. Then,*

$$d_{N,\delta}\left(\mathbb{R}^m, v, l_q^m\right) \asymp m^{1/q-1/2}\sqrt{m + \ln(1/\delta)}, m \geq 2N, 1 \leq q < 2; \tag{17}$$

$$m^{-1/2} \ll \frac{d_{N,\delta}\left(\mathbb{R}^m, v, l_q^m\right)}{m^{1/q}\sqrt{m + \ln(1/\delta)}} \ll N^{-1/2}, 2 \leq q < \infty. \tag{18}$$

Lemma 2 (Maiorov, Fang, and Ye [2,5,6])**.** *Let $m > N$, $1 \leq q \leq \infty$ and $\delta \in \left(0, \frac{1}{2}\right]$. Then,*

$$\lambda_{N,\delta}\left(\mathbb{R}^m, v, l_q^m\right) \asymp \begin{cases} m^{1/q-1/2}\sqrt{m + \ln(1/\delta)}, \ 1 \leq q < 2; \\ m^{1/q} + \sqrt{\ln(1/\delta)}, 2 \leq q < \infty; \\ \sqrt{\ln((m-n)/\delta)}, q = \infty. \end{cases} \tag{19}$$

Lemma 3 (Maiorov [3])**.** *For $\forall \delta \in \left(0, \frac{1}{2}\right]$, there is a positive c_0, such that*

$$v\left(\left\{x \in \mathbb{R}^m : \|x\|_2 \geq c_0\left(\sqrt{m} + \sqrt{\ln\frac{1}{\delta}}\right)\right\}\right) \leq \delta. \tag{20}$$

For $2 \leq q < \infty$ and any $\delta \in \left(0, \frac{1}{2}\right]$, there exists a positive constant c_q, which depends only on the q, such that

$$v\left(\left\{x \in \mathbb{R}^m : \|x\|_{l_q^m} \geq c_q\left(m^{\frac{1}{q}} + \sqrt{\ln\frac{1}{\delta}}\right)\right\}\right) \leq \delta. \tag{21}$$

To establish the discretization theorem, we introduce some notations and lemmas. It is convenient in many cases to split the Fourier series of a function into the sum of diadic blocks. We associate every vector $s = (s_1, \cdots, s_d) \in \mathbb{N}^d$ whose coordinates are natural numbers with the set

$$\square_s := \left\{n = (n_1, \cdots, n_d) \in \mathbb{Z}_0^d : \ 2^{s_j - 1} \leq |n_j| < 2^{s_j}, \ j = 1, \ldots, d\right\}.$$

And we let $x_s(t)$ be the "block" of the Fourier series for $x(t)$, denoted by

$$\delta_s(x_t) := x_s(t) := \sum_{n \in \mathcal{K}_s} c_n \exp(\mathrm{i}(n, t)). \tag{22}$$

After introducing these necessary concepts, we have

Lemma 4 (Galeev [18])**.** *Let $s \in \mathbb{N}^d$. Then, the trigonometric polynomial space $span\{e_n(t) : n \in \square_s\}$ and $\mathbb{R}^{2^{(s,1)}}$ are isomorphic under the following mapping:*

$$x(t) \mapsto \{x_{s,m}(t_j)\}_{m,j}, x_{s,m}(t_j) = \sum_{n \in \square_s, sgn n = sgn m} c_n e_n(t),$$

where $m = (\pm 1, \ldots, \pm 1) \in \mathbb{R}^d$, $t_j = \left(\pi 2^{2-s_1} j_1, \ldots, \pi 2^{2-s_d} j_d\right) \in \mathbb{R}^d$, $j_i = 1, \ldots, 2^{s_i - 1}, i = 1, \ldots, d$.

For natural numbers l and k, we define

$$\begin{aligned} S_{l,k} &:= \left\{ s \in \mathbb{N}^d : l-1 \le S_{A'}(s) < l, (s,1) = k \right\}; \\ F_{l,k} &:= span\{e_n(t) : n \in \square_s, s \in S_{l,k}\}, \end{aligned} \tag{23}$$

where $S_A(s) = \sup\{(s,\alpha) : \alpha \in A\}$. We can know $k \ge d$, and $S_{l,k} = \emptyset$ for $k \ge l$.

Let $\|S_{l,k}\| = \sum\limits_{s \in S_{l,k}} |\square_s|$. We can obtain that $\|S_{l,k}\| = 2^k |S_{l,k}|$. And we define $\Delta_{l,k} x := \sum_{s \in S_{l,k}} \delta_s x$.

From ([7]), for any $\alpha \in A$, $n^\alpha \asymp 2^{(s,\alpha)}$. So,

$$\left|n^A\right| := \left|\sum_{\alpha \in A} n^\alpha\right| \asymp \left|\sum_{\alpha \in A} 2^{(s,\alpha)}\right| \asymp 2^{S_A(\alpha)}.$$

From the definition of A', we know

$$S_A(s) = S_{(r+\frac{\rho}{2})A' - \frac{\rho}{2}}(s) = \left(r + \frac{\rho}{2}\right) S_{A'}(s) - \frac{\rho(s,1)}{2} = \left(r + \frac{\rho}{2}\right) S_{A'}(s) - \frac{k\rho}{2}. \tag{24}$$

Therefore, for any $x \in F_{l,k}$, we have

$$\begin{aligned} \left\|D^A x\right\|_{q,S} &= \left(\sum_{n \in \square_s, s \in S_{l,k}} \left|n^A\right|^q |c_n|^q\right)^{1/q} \\ &\asymp \left(\sum_{n \in \square_s, s \in S_{l,k}} 2^{S_A(s) q} |c_n|^q\right)^{1/q} \\ &= \left(\sum_{n \in \square_s, s \in S_{l,k}} 2^{\left((r+\frac{\rho}{2}) S_{A'}(s) - \frac{k\rho}{2}\right) q} |c_n|^q\right)^{1/q} \\ &\asymp 2^{(r+\frac{\rho}{2}) l - \frac{k\rho}{2}} \left(\sum_{n \in \square_s, s \in S_{l,k}} |c_n|^q\right)^{1/q} \\ &= 2^{(r+\frac{\rho}{2}) l - \frac{k\rho}{2}} \|x\|_{q,S}. \end{aligned}$$

That is,

$$\left\|D^A x\right\|_{q,S} \asymp 2^{(r+\frac{\rho}{2}) l - \frac{k\rho}{2}} \|x\|_{q,S}. \tag{25}$$

We consider a mapping:

$$I_{l,k} : F_{l,k} \to l_q^{\|S_{l,k}\|}, x \mapsto \left\{ \left\langle x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle \right\}_{n \in \square_s, s \in S_{l,k}}.$$

It is not difficult to see that $I_{l,k}$ is a isomorphic mapping. From Equation (9), we know that $\sigma_n := \left\langle C_\mu \frac{e_n(t)}{\sqrt{\lambda_n}}, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle = 1$.

By $|n| \asymp 2^{(s,1)} = 2^k$ ([7]), we obtain

$$\|I_{l,k}\|_{l_q^{\|S_{l,k}\|}} = \left\| \left\{ \left\langle x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle \right\}_{n\in\square_s, s\in S_{l,k}} \right\|_{l_q^{\|S_{l,k}\|}}$$
$$= \left\| \left\{ \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |n|^{\rho/2} x(t) e_n(-t) dt \right\}_{n\in\square_s, s\in S_{l,k}} \right\|_{l_q^{\|S_{l,k}\|}}$$
$$\asymp \left\| 2^{k\rho/2} \hat{x}(n) \right\|_{l_q^{\|S_{l,k}\|}}$$
$$= 2^{k\rho/2} \|x\|_{q,S}.$$

Therefore, from Equation (25), we have

$$\left\| I_{l,k}\left(D^A x\right) \right\|_{l_q^{\|S_{l,k}\|}} \asymp 2^{k\rho/2} \left\| D^A x \right\|_{q,S} = 2^{(r+\rho/2)l} \|x\|_{q,S}. \tag{26}$$

Let $\Delta_{l,k} = \sum_{s\in S_{l,k}} \delta_s x$. Therefore,

$$\left\|\Delta_{l,k} x\right\|_{q,S} \ll \|x\|_{q,S} \ll \left\| \left\{ D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\}_{n\in\square_s, s\in S_{l,k}} \right\|_{l_q^{\|S_{l,k}\|}}. \tag{27}$$

Based on the above description, we establish the discretization theorem. The following theorems reflect the upper bounds of Theorems 1 and 2.

Theorem 5. *Let $r > 1/2$, $1 \le q < \infty$, $\rho > 1$, $\delta \in \left(0, \frac{1}{2}\right]$, $N \in \mathbb{N}$, A satisfy the condition of Theorem 1. Assume that the sequences of numbers $\{N_{l,k}\}$ and $\{\delta_{l,k}\}$ satisfy the condition $0 \le N_{l,k} \le \|S_{l,k}\|$, $\sum_{l,k} N_{l,k} \le N$ and $\sum_{l,k} \delta_{l,k} \le \delta$. Then*

$$d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) \ll \sum_{l,k} 2^{-(r+\rho/2)l} d_{N_{l,k},\delta_{l,k}}\left(\mathbb{R}^{\|S_{l,k}\|}, v, l_q^{\|S_{l,k}\|}\right). \tag{28}$$

Proof. From Definition 1, there would be a subspace $L_{l,k}$ of $l_q^{\|S_{l,k}\|}$ such that $\dim L_{l,k} \le N_{l,k}$ and

$$v\left(\left\{ y \in l_q^{\|S_{l,k}\|} : e\left(y, L_{l,k}, l_q^{\|S_{l,k}\|}\right) > d_{N_{l,k},\delta_{l,k}} \right\}\right) \le \delta_{l,k}. \tag{29}$$

where $d_{N_{l,k},\delta_{l,k}} := d_{N_{l,k},\delta_{l,k}}\left(\mathbb{R}^{\|S_{l,k}\|}, v, l_q^{\|S_{l,k}\|}\right)$.

From Equation (27), there is a constant $c_1 > 0$ independent of l and k, such that

$$e\left(\Delta_{l,k} x, D^{-A} I_{l,k}^{-1} L_{l,k}, S_q\left(\mathbb{T}^d\right)\right) \le c_1 2^{-(r+\rho/2)l} e\left(\left\{ \left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle \right\}_{n\in\square_s, s\in S_{l,k}}, L_{l,k}, l_q^{\|S_{l,k}\|}\right) \tag{30}$$

Consider the set

$$G_{l,k} = \left\{ x \in W_2^A\left(\mathbb{T}^d\right) : e\left(\Delta_{l,k} x, D^{-A} I_{l,k}^{-1} L_{l,k}, S_q\left(\mathbb{T}^d\right)\right) > c_1 2^{-(r+\rho/2)l} d_{N_{l,k},\delta_{l,k}} \right\}.$$

From Equations (29) and (30), the definition of μ and v,

$$\mu(G_{l,k}) \le \mu\left(\left\{x \in W_2^A\left(\mathbb{T}^d\right) : e\left(\left\{\left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}}\right\rangle\right\}_{n\in\square_s, s\in S_{l,k}}, L_{l,k}, l_q^{\|S_{l,k}\|}\right) > d_{N_{l,k},\delta_{l,k}}\right\}\right)$$
$$= v\left(\left\{y \in l_q^{\|S_{l,k}\|} : e\left(y, L_{l,k}, l_q^{\|S_{l,k}\|}\right) > d_{N_{l,k},\delta_{l,k}}\right\}\right)$$
$$\le \delta_{l,k}.$$

Let $G = \bigcup_{l,k} G_{l,k}$, $F_N = \sum_{l,k} D^{-A} I_{l,k}^{-1} L_{l,k}$, where F_N is the direct sum of $D^{-A} I_{l,k}^{-1} L_{l,k}$. Therefore,

$$\mu(G) = \mu\left(\bigcup_{l,k} G_{l,k}\right) \le \sum_{l,k} \mu(G_{l,k}) \le \sum_{l,k} \delta_{l,k} \le \delta,$$

and

$$\dim F_N = \dim \sum_{l,k} D^{-A} I_{l,k}^{-1} L_{l,k} \le \sum_{l,k} \dim D^{-A} I_{l,k}^{-1} L_{l,k} \le \sum_{l,k} N_{l,k} \le N.$$

Consequently, by Definition 1, we have

$$d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) \le \sup_{x\in W_2^A\left(\mathbb{T}^d\right)\backslash G} e\left(x, F_N, S_q\left(\mathbb{T}^d\right)\right)$$
$$\le \sup_{x\in W_2^A\left(\mathbb{T}^d\right)\backslash G} \sum_{l,k} e\left(\Delta_{l,k} x, D^{-A} I_{l,k}^{-1} L_{l,k}, S_q\left(\mathbb{T}^d\right)\right)$$
$$\le \sum_{l,k} \sup_{x\in W_2^A\left(\mathbb{T}^d\right)\backslash G} e\left(\Delta_{l,k} x, D^{-A} I_{l,k}^{-1} L_{l,k}, S_q\left(\mathbb{T}^d\right)\right)$$
$$\ll \sum_{l,k} 2^{-(r+\rho/2)l} d_{N_{l,k},\delta_{l,k}},$$

which completes the proof of Theorem 5. □

To estimate the upper bound of Theorem 1, we need the following lemmas.

Lemma 5 (Romanyuk [19]). *Assume that the set A satisfies the condition of Theorem 1, then*

$$\sum_{s\in Q(u)} 2^{(s,1)} \asymp 2^u u^v, Q(u) = \left\{s \in \mathbb{N}^d : S_{A'}(s) \le u\right\},$$

where $u \in \mathbb{R}_+$.

From Lemma 5, we have

Lemma 6. *For any* $N \in \mathbb{N}$, $\beta > 0$, $N \asymp 2^u u^v$, *let*

$$N_{l,k} = \begin{cases} \|S_{l,k}\|, d \le k \le l, l \le u, \\ \left\lfloor c|S_{l,k}|2^{u+\beta u-2\beta l+\beta k}\right\rfloor, d \le k \le l, l > u, \\ 0, otherwise, \end{cases} \tag{31}$$

where $\lfloor a \rfloor$ *is the integer part of a. Then, we can choose c, such that* $N_{l,k} \le \|S_{l,k}\|$, $\sum_{l,k} N_{l,k} \le N$.

We assume that in Lemma 6, the constant $\beta > 0$ satisfies $0 < \beta < \min\{2r + \rho - 2, 1/2\}$.

To establish the discretization of the lower bound of Theorem 1, we also need the following concepts. Let

$$N \asymp 2^{\frac{k}{r}} k^{v}, S = \left\{ s \in \mathbb{N}^d : (s, 1) > \frac{k}{r}, S_A(s) \le k + c_0 \right\},$$

where the constants c_0 and k are pending. Then,

$$S = \left\{ s \in \mathbb{N}^d : \frac{k}{r} < (s, 1) < \frac{k + c_0}{r}, k \le S_A(s) \le k + c_0 \right\}.$$

Therefore, $|S| \asymp k^v$, $\|S\| := \sum_{s \in S} |\square_s| = \sum_{s \in S} 2^{(s,1)} > |S| 2^{\frac{k}{2}} \gg k^v 2^{\frac{k}{2}} \ge 2N$.

Let $F_S = span\{e_n(t) : n \in \square_s, s \in S\}$. Consider the mapping:

$$I_S : F_S \to l_q^{\|S\|}, x \mapsto \left\{ \left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle \right\}_{n \in \square_s, s \in S}.$$

Then for any $x \in F_S$, by using the method of the proof of Equation (26), we can obtain

$$\left\| \left\{ \left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle \right\}_{n \in \square_s, s \in S} \right\|_{l_q^{\|S\|}} \asymp 2^{(r+\rho/2)k} \|x\|_{q,s}. \tag{32}$$

Theorem 6. *Let $r > 1/2$, $1 \le q < \infty$, $\rho > 1$, $\delta \in \left(0, \frac{1}{2}\right]$, $N \in \mathbb{N}$, A satisfy the condition of Theorem 1. Then*

$$d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) \gg 2^{-(r+\rho/2)k} d_{N,\delta}\left(\mathbb{R}^{\|S\|}, v, l_q^{\|S\|}\right). \tag{33}$$

Proof. From Definition 1, there is a subspace F_1, such that $\dim F_1 \le N$ and

$$\mu\left(\left\{ x \in W_2^A\left(\mathbb{T}^d\right) \bigcap F_S : e\left(x, F_1, S_q\left(\mathbb{T}^d\right)\right) > d_{N,\delta} \right\}\right) \le \delta, \tag{34}$$

where $d_{N,\delta} := d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)$.

Let $G = \left\{ y \in \mathbb{R}^{\|S\|} : e\left(y, I_S D^A F_1, l_q^{\|S\|}\right) > c_2^{-1} 2^{(r+\rho/2)k} d_{N,\delta} \right\}$, where $c_3 > 0$, such that

$$e\left(x, F_1, S_q\left(\mathbb{T}^d\right)\right) = c_2 2^{-(r+\rho/2)k} e\left(\left\{ \left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle \right\}_{n \in \square_s, s \in S}, I_S D^A F_1, l_q^{\|S\|}\right). \tag{35}$$

Equation (35) can be obtained by Equation (32); therefore,

$$\begin{aligned} & v(G) \\ & = \mu\left(\left\{ x \in W_2^A\left(\mathbb{T}^d\right) \bigcap F_S : e\left(\left\{ \left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}} \right\rangle \right\}_{n \in \square_s, s \in S}, I_S D^A F_1, l_q^{\|s\|}\right) > c_2^{-1} 2^{(r+\rho/2)k} d_{N,\delta} \right\}\right) \\ & \le \mu\left(\left\{ x \in W_2^A\left(\mathbb{T}^d\right) \bigcap F_S : e\left(x, F_1, S_q\left(\mathbb{T}^d\right)\right) > d_{N,\delta} \right\}\right) \le \delta. \end{aligned}$$

Due to $\dim I_S D^A F_1 = \dim F_1 = N$ and Definition 1, we have

$$\begin{aligned} d_{N,\delta}\left(\mathbb{R}^{\|S\|}, v, l_q^{\|S\|}\right) & \le E\left(\mathbb{R}^{\|S\|} \backslash G, I_S D^A F_1, l_q^{\|S\|}\right) \\ & = \sup_{y \in \mathbb{R}^{\|S\|} \backslash G} e\left(y, I_S D^A F_1, l_q^{\|S\|}\right) \\ & \ll 2^{(r+\rho/2)k} d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right). \end{aligned}$$

That is, $d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) \gg 2^{-(r+\rho/2)k} d_{N,\delta}\left(\mathbb{R}^{\|S\|}, v, l_q^{\|S\|}\right)$. $\square$

Theorem 7. *Let $r > 1/2$, $1 \le q < \infty$, $\rho > 1$, $\delta \in \left(0, \frac{1}{2}\right]$, $N \in \mathbb{N}$, A satisfy the condition of Theorem 1. Assume that the sequences of numbers $\{N_{l,k}\}$ and $\{\delta_{l,k}\}$ satisfy the condition $0 \le N_{l,k} \le \|S_{l,k}\|$, $\sum_{l,k} N_{l,k} \le N$ and $\sum_{l,k} \delta_{l,k} \le \delta$. Then,*

$$\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) \ll \sum_{l,k} 2^{-(r+\rho/2)l} \lambda_{N_{l,k},\delta_{l,k}}\left(\mathbb{R}^{\|S_{l,k}\|}, v, l_q^{\|S_{l,k}\|}\right). \tag{36}$$

Proof. From Definition 1, there would be a linear operator $T_{l,k}$ of $l_q^{\|S_{l,k}\|}$ into itself, such that $rankT_{l,k} \le N_{l,k}$ and

$$v\left(\left\{y \in l_q^{\|S_{l,k}\|} : \|y - T_{l,k}y\|_{l_q^{\|S_{l,k}\|}} > \lambda_{N_{l,k},\delta_{l,k}}\right\}\right) \le \delta_{l,k}, \tag{37}$$

where $\lambda_{N_{l,k},\delta_{l,k}} := \lambda_{N_{l,k},\delta_{l,k}}\left(\mathbb{R}^{\|S_{l,k}\|}, v, l_q^{\|S_{l,k}\|}\right)$.

From Equation (27), there is a constant $c_3 > 0$ independent of l and k, such that

$$\begin{aligned} &\left\|\Delta_{l,k}x - D^{-A} I_{l,k}^{-1} T_{l,k} \Delta_{l,k} x\right\|_{q,S} \\ &\le c_3 2^{-(r+\rho/2)l} \left\| \left\{\left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}}\right\rangle\right\}_{n\in\square_s, s\in S_{l,k}} - T_{l,k}\left\{\left\langle D^A x, \frac{e_n(t)}{\sqrt{\lambda_n}}\right\rangle\right\}_{n\in\square_s, s\in S_{l,k}} \right\|_{l_q^{\|S_{l,k}\|}}. \end{aligned} \tag{38}$$

Consider the set

$$P_{l,k} = \left\{x \in W_2^A\left(\mathbb{T}^d\right) : \left\|\Delta_{l,k}x - D^{-A} I_{l,k}^{-1} T_{l,k} \Delta_{l,k} x\right\|_{q,s} > c_3 2^{-(r+\rho/2)l} \lambda_{N_{l,k},\delta_{l,k}}\right\}.$$

From Equations (37) and (38), the definition of μ and v,

$$\begin{aligned} \mu(P_{l,k}) &\le \mu\left(\left\{x \in W_2^A\left(\mathbb{T}^d\right) : \left\|I_{l,k}D^A x - T_{l,k} I_{l,k} D^A x\right\|_{l_q^{\|S_{l,k}\|}} > \lambda_{N_{l,k},\delta_{l,k}}\right\}\right) \\ &= v\left(\left\{y \in l_q^{\|S_{l,k}\|} : \|y - T_{l,k}y\|_{l_q^{\|S_{l,k}\|}} > \lambda_{N_{l,k},\delta_{l,k}}\right\}\right) \\ &\le \delta_{l,k}. \end{aligned}$$

Let $P = \bigcup_{l,k} P_{l,k}$, $T_N = \sum_{l,k} D^{-A} I_{l,k}^{-1} T_{l,k}$, where T_N is the direct sum of $D^{-A} I_{l,k}^{-1} T_{l,k}$. Therefore,

$$\mu(P) = \mu\left(\bigcup_{l,k} P_{l,k}\right) \le \sum_{l,k} \mu(P_{l,k}) \le \sum_{l,k} \delta_{l,k} \le \delta$$

and

$$rankT_N = rank \sum_{l,k} D^{-A} I_{l,k}^{-1} T_{l,k} \le \sum_{l,k} rankD^{-A} I_{l,k}^{-1} T_{l,k} \le \sum_{l,k} N_{l,k} \le N.$$

Consequently, by Definition 1, we have

$$\begin{aligned}\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right) &\le \sup_{x\in W_2^A\left(\mathbb{T}^d\right)\setminus G}\|x-T_Nx\|_{q,s} \\ &\le \sup_{x\in W_2^A\left(\mathbb{T}^d\right)\setminus G}\sum_{l,k}\left\|\Delta_{l,k}x-D^{-A}I_{l,k}^{-1}T_{l,k}\Delta_{l,k}x\right\|_{q,s} \\ &\le \sum_{l,k}\sup_{x\in W_2^A\left(\mathbb{T}^d\right)\setminus G}\left\|\Delta_{l,k}x-D^{-A}I_{l,k}^{-1}T_{l,k}\Delta_{l,k}x\right\|_{q,s} \\ &\ll \sum_{l,k}2^{-(r+\rho/2)l}\lambda_{N_{l,k},\delta_{l,k}},\end{aligned}$$

which completes the proof of Theorem 7. □

Theorem 8. *Let* $r>1/2$, $1\le q<\infty$, $\rho>1$, $\delta\in\left(0,\frac{1}{2}\right]$, $N\in\mathbb{N}$, *A satisfy the condition of Theorem 1. Then,*

$$\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)\gg 2^{-(r+\rho/2)k}\lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},v,l_q^{\|S\|}\right). \tag{39}$$

Proof. From Definition 1, there is a linear operator T_1, such that $rankT_1\le N$ and

$$\mu\left(\left\{x\in W_2^A\left(\mathbb{T}^d\right)\bigcap F_S:\|x-T_1x\|_{q,s}>\lambda_{N,\delta}\right\}\right)\le\delta, \tag{40}$$

where $\lambda_{N,\delta}:=\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)$.

Let $P=\left\{y\in\mathbb{R}^{\|S\|}:\left\|y-I_ST_1D^AI_S^{-1}y\right\|_{l_q^{\|S\|}}>c_4^{-1}2^{(r+\rho/2)k}\lambda_{N,\delta}\right\}$, where $c_4>0$, such that

$$\begin{aligned}&\|x-T_1x\|_{q,s}\\ &=c_42^{-(r+\rho/2)k}\left\|\left\{\left\langle D^Ax,\frac{e_n(t)}{\sqrt{\lambda_n}}\right\rangle\right\}_{n\in\square_s,s\in S}-I_ST_1D^AI_S^{-1}\left\{\left\langle D^Ax,\frac{e_n(t)}{\sqrt{\lambda_n}}\right\rangle\right\}_{n\in\square_s,s\in S}\right\|_{l_q^{\|S\|}}.\end{aligned} \tag{41}$$

Equation (41) can be obtained by Equation (32). Let

$$M_x:=\left\|\left\{\left\langle D^Ax,\frac{e_n(t)}{\sqrt{\lambda_n}}\right\rangle\right\}_{n\in\square_s,s\in S}-I_ST_1D^AI_S^{-1}\left\{\left\langle D^Ax,\frac{e_n(t)}{\sqrt{\lambda_n}}\right\rangle\right\}_{n\in\square_s,s\in S}\right\|_{l_q^{\|S\|}}.$$

Therefore,

$$\begin{aligned}&v(P)\\ &=\mu\left(\left\{x\in W_2^A\left(\mathbb{T}^d\right)\bigcap F_S:M_x>c_4^{-1}2^{(r+\rho/2)k}\lambda_{N,\delta}\right\}\right)\\ &\le\mu\left(\left\{x\in W_2^A\left(\mathbb{T}^d\right)\bigcap F_S:\|x-T_1x\|_{q,s}>d_{N,\delta}\right\}\right)\le\delta.\end{aligned}$$

Due to $rankI_SD^AT_1=rankT_1=N$ and Definition 1, we have

$$\begin{aligned}\lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},v,l_q^{\|S\|}\right)&\le\lambda\left(\mathbb{R}^{\|S\|}\setminus G,I_ST_1D^AI_S^{-1},l_q^{\|S\|}\right)\\ &=\sup_{y\in\mathbb{R}^{\|S\|}\setminus G}\left\|y-I_ST_1D^AI_S^{-1}y\right\|_{l_q^{\|S\|}}\\ &\ll 2^{(r+\rho/2)k}\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right).\end{aligned}$$

That is, $\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right) \gg 2^{-(r+\rho/2)k}\lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},v,l_q^{\|S\|}\right)$. □

4. Proof of Main Results

Now we prove Theorem 1 by using Theorems 5 and 6 and Lemma 1, and prove Theorem 2 by using Theorems 7 and 8 and Lemma 2. And then, we prove Theorems 3 and 4 by using results of Theorems 1 and 2. Assume that $N_{l,k}$ satisfies the condition of Lemma 5 and assume that $N \in \mathbb{N}$ satisfies the condition $N \asymp 2^u u^v$. Let

$$\delta_{l,k} = \begin{cases} \delta N_{l,k}/N, d \le k \le l, l > u \\ 0, otherwise \end{cases}$$

Therefore, $\sum\limits_{l,k}\delta_{l,k} \le \delta$.

Proof of Theorem 1. From Theorem 5, Lemma 1, for $1 \le q < 2$, we have

$$\begin{aligned} d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right) &\ll \sum_{l,k}2^{-(r+\rho/2)l}d_{N_{l,k},\delta_{l,k}}\left(\mathbb{R}^{\|S_{l,k}\|},v,l_q^{\|S_{l,k}\|}\right) \\ &\ll \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q-1/2}\sqrt{\|S_{l,k}\|+\ln(1/\delta_{l,k})} \\ &\ll \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q-1/2}\left(\|S_{l,k}\|^{1/2}+\ln^{1/2}(1/\delta_{l,k})\right) \\ &\ll \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q-1/2}\left(\|S_{l,k}\|^{1/2}+(N/N_{l,k})^{1/2}+\ln^{1/2}(1/\delta)\right) \\ &= \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q} \\ &+ \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q-1/2}N^{1/2}N_{l,k}^{-1/2} \\ &+ \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q-1/2}\sqrt{\ln(1/\delta)} \\ &:= I_1+I_2+I_3. \end{aligned}$$

First, we calculate I_1:

$$I_1 = \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q} \ll \sum_{l>u}2^{-(r+\rho/2)l}\sum_{d\le k\le l}|S_{l,k}|^{1/q}2^{k/q}.$$

Split term for $\sum\limits_{d\le k\le l}|S_{l,k}|^{1/q}2^{k/q}$:

$$\sum_{d\le k\le l}|S_{l,k}|^{1/q}2^{k/q} = \sum_{d\le k\le l}{}'|S_{l,k}|^{1/q}2^{k/q} + \sum_{d\le k\le l}{}''|S_{l,k}|^{1/q}2^{k/q},$$

where $\sum'$ is carried out over k for $|S_{l,k}| \le l^v$, and $\sum''$ is carried out over k for $|S_{l,k}| > l^v$. Therefore,

$$\sum_{d\le k\le l}{}'|S_{l,k}|^{1/q}2^{k/q} \le l^{v/q}\sum_{d\le k\le l}{}'2^{k/q} \ll l^{v/q}2^{l/q};$$

and

$$\sum_{d\le k\le l}{}''|S_{l,k}|^{1/q}2^{k/q} = \sum_{d\le k\le l}{}''|S_{l,k}|^{1/q-1}|S_{l,k}|2^{k/q} \le l^{v/q-v}\sum_{S_{A'}(s)\le l}{}''2^{(s,1)/q} \ll l^{v/q}2^{l/q}.$$

Therefore,

$$\sum_{d\le k\le l} \left|S_{l,k}\right|^{1/q} 2^{k/q} \ll l^{v/q} 2^{l/q}. \tag{42}$$

So, we obtain

$$I_1 = \sum_{l>u} 2^{-(r+\rho/2-1/q)l} l^{v/q}. \tag{43}$$

Due to $0 < \beta < \min\{2r+\rho-2, 1/2\}$, we have

$$\begin{aligned} I_1 &= \sum_{l>u} 2^{-(r+\rho/2-1/q)l} l^{v/q} \\ &= 2^{-(r+\rho/2-1/q)u} u^{v/q} \\ &\ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right). \end{aligned}$$

Secondly, we calculate I_2:

$$\begin{aligned} I_2 &= \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \left\|S_{l,k}\right\|^{1/q-1/2} N^{1/2} N_{l,k}^{-1/2} \\ &\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \left|S_{l,k}\right|^{1/q-1/2} 2^{k/q-k/2} N^{1/2} \left|S_{l,k}\right|^{-1/2} 2^{-(u+\beta u-2\beta l+\beta k)/2} \\ &= N^{1/2} 2^{-u/2-\beta u/2} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} \sum_{d\le k\le l} \left|S_{l,k}\right|^{1/q-1} 2^{(1/q-1/2-\beta/2)k}. \end{aligned}$$

By using the method of the proof of Equation (42), we can obtain

$$\sum_{d\le k\le l} \left|S_{l,k}\right|^{1/q-1} 2^{k(1/q+1/2-\beta/2)} \ll l^{v/q-v} 2^{l(1/q+1/2-\beta/2)}.$$

Therefore,

$$I_2 = N^{1/2} 2^{-u/2-\beta u/2} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} l^{v/q-v} 2^{l(1/q+1/2-\beta/2)}.$$

Due to $0 < \beta < \min\{2r+\rho-2, 1/2\}$, we have

$$\begin{aligned} I_2 &\ll N^{1/2} 2^{-u/2-\beta u/2-(r+\rho/2)u+\beta u+u(1/q+1/2-\beta/2)} u^{v/q-v} \\ &\ll 2^{u/2} u^{v/2} 2^{-(r+\rho/2-1/q)u-u} u^{v/q-v} \\ &\ll 2^{-u/2} u^{-v/2} 2^{-(r+\rho/2-1/q)u} u^{v/q} \\ &\ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right). \end{aligned}$$

Finally, we calculate I_3:

$$\begin{aligned} I_3 &= \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \left\|S_{l,k}\right\|^{1/q-1/2} \sqrt{\ln(1/\delta)} \\ &\ll \sqrt{\ln(1/\delta)} \sum_{l>u} 2^{-(r+\rho/2)l} \sum_{d\le k\le l} \left|S_{l,k}\right|^{1/q-1/2} 2^{k/q-k/2}. \end{aligned}$$

By using the method of the proof of Equation (42), we can obtain

$$\sum_{d\le k\le l} \left|S_{l,k}\right|^{1/q-1/2} 2^{k/q-k/2} \ll l^{v/q-v/2} 2^{l(1/q-1/2)}.$$

Therefore,

$$\begin{aligned} I_3 &\ll \sqrt{\ln(1/\delta)} \sum_{l>u} 2^{-(r+\rho/2-1/q)l-l/2} l^{v/q-v/2} \\ &\ll 2^{-(r+\rho/2-1/q)u-u/2} u^{v/q-v/2} \sqrt{\ln(1/\delta)} \\ &\ll \left(N^{-1}\ln^{v} N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) \sqrt{1/N \ln(1/\delta)}. \end{aligned}$$

Summarily, if $1 \le q < 2$,

$$\begin{aligned} d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) &\ll I_1 + I_2 + I_3 \\ &\ll \left(N^{-1}\ln^{v} N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) \sqrt{1 + 1/N \ln(1/\delta)}. \end{aligned}$$

If $2 \le q < \infty$, from Theorem 5, Lemma 1, and the definition of $N_{l,k}$, we have

$$\begin{aligned} d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) &\ll \sum_{l,k} 2^{-(r+\rho/2)l} d_{N_{l,k},\delta_{l,k}}\left(\mathbb{R}^{\|S_{l,k}\|}, v, l_q^{\|S_{l,k}\|}\right) \\ &\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q} N_{l,k}^{-1/2} \sqrt{\|S_{l,k}\| + \ln(1/\delta_{l,k})} \\ &\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q} N_{l,k}^{-1/2} \left(\|S_{l,k}\|^{1/2} + \ln^{1/2}(1/\delta_{l,k})\right) \\ &\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q} N_{l,k}^{-1/2} \left(\|S_{l,k}\|^{1/2} + (N/N_{l,k})^{1/2} + \ln^{1/2}(1/\delta)\right) \\ &= \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q+1/2} N_{l,k}^{-1/2} \\ &+ \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q} N^{1/2} N_{l,k}^{-1} \\ &+ \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q} N_{l,k}^{-1/2} \sqrt{\ln(1/\delta)} \\ &:= I_1{}' + I_2{}' + I_3{}'. \end{aligned}$$

First, we calculate $I_1{}'$:

$$\begin{aligned} I_1{}' &= \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q+1/2} N_{l,k}^{-1/2} \\ &\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l} |S_{l,k}|^{1/q+1/2} 2^{k(1/q+1/2)} |S_{l,k}|^{-1/2} 2^{-(u+\beta u-2\beta l+\beta k)/2} \\ &= 2^{-u/2-\beta u/2} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} \sum_{d\le k\le l} |S_{l,k}|^{1/q} 2^{k(1/q+1/2-\beta/2)}. \end{aligned}$$

By using the method of the proof of Equation (42), we can obtain

$$\sum_{d\le k\le l} |S_{l,k}|^{1/q} 2^{k(1/q+1/2-\beta/2)} \ll l^{v/q} 2^{l(1/q+1/2-\beta/2)}.$$

Therefore,

$$I_1{}' = 2^{-u/2-\beta u/2} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} l^{v/q} 2^{l(1/q+1/2-\beta/2)}.$$

Due to $0 < \beta < \min\{2r + \rho - 2, 1/2\}$, we obtain

$$\begin{aligned} {I_1}' &\ll 2^{-u/2-\beta u/2} 2^{-(r+\rho/2)u+\beta u} u^{v/q} 2^{u(1/q+1/2-\beta/2)} \\ &= 2^{-(r+\rho/2-1/q)u} u^{v/q} \\ &\ll \left(N^{-1} \ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right). \end{aligned}$$

Secondly, we calculate ${I_2}'$:

$$\begin{aligned} {I_2}' &= \sum_{l>u} \sum_{d \le k \le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q} N^{1/2} N_{l,k}^{-1} \\ &\ll N^{1/2} \sum_{l>u} \sum_{d \le k \le l} 2^{-(r+\rho/2)l} |S_{l,k}|^{1/q} 2^{k/q} |S_{l,k}|^{-1} 2^{-(u+\beta u-2\beta l+\beta k)} \\ &= 2^{-u-\beta u} N^{1/2} \sum_{l>u} 2^{-(r+\rho/2)l+2\beta l} \sum_{d \le k \le l} |S_{l,k}|^{1/q-1} 2^{k(1/q-\beta)}. \end{aligned}$$

By using the method of the proof of Equation (42), we can obtain

$$\sum_{d \le k \le l} |S_{l,k}|^{1/q-1} 2^{k(1/q-\beta)} \ll l^{v/q-v} 2^{l(1/q-\beta)}.$$

Therefore,

$${I_2}' = 2^{-u-\beta u} N^{1/2} \sum_{l>u} 2^{-(r+\rho/2)l+2\beta l} l^{v/q-v} 2^{l(1/q-\beta)}.$$

Due to $0 < \beta < \min\{2r + \rho - 2, 1/2\}$, we obtain

$$\begin{aligned} {I_2}' &\ll 2^{-u-\beta u} 2^{-(r+\rho/2)u+2\beta u} u^{v/q-v} 2^{u(1/q-\beta)} N^{1/2} \\ &\ll 2^{-(r+\rho/2-1/q)u-u} u^{v/q-v} 2^{u/2} u^{v/2} \\ &= \left(N^{-1} \ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) 2^{-u/2} u^{-v/2} \\ &\ll \left(N^{-1} \ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) N^{-1/2} \\ &\ll \left(N^{-1} \ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right). \end{aligned}$$

Finally, we calculate ${I_3}'$:

$$\begin{aligned} {I_3}' &= \sum_{l>u} \sum_{d \le k \le l} 2^{-(r+\rho/2)l} \|S_{l,k}\|^{1/q} N_{l,k}^{-1/2} \sqrt{\ln(1/\delta)} \\ &\ll \sqrt{\ln(1/\delta)} \sum_{l>u} \sum_{d \le k \le l} 2^{-(r+\rho/2)l} |S_{l,k}|^{1/q} 2^{k/q} |S_{l,k}|^{-1/2} 2^{-(u+\beta u-2\beta l+\beta k)/2} \\ &= 2^{-u/2-\beta u/2} \sqrt{\ln(1/\delta)} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} \sum_{d \le k \le l} |S_{l,k}|^{1/q-1/2} 2^{k(1/q-\beta/2)}. \end{aligned}$$

By using the method of the proof of Equation (42), we can obtain

$$\sum_{d \le k \le l} |S_{l,k}|^{1/q-1/2} 2^{k(1/q-\beta/2)} \ll l^{v/q-v/2} 2^{l(1/q-\beta/2)}.$$

Therefore,

$${I_3}' = 2^{-u/2-\beta u/2} \sqrt{\ln(1/\delta)} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} l^{v/q-v/2} 2^{l(1/q-\beta/2)}.$$

Due to $0 < \beta < \min\{2r + \rho - 2, 1/2\}$, we obtain

$$\begin{aligned} I_3{}' &\ll 2^{-u/2-\beta u/2} 2^{-(r+\rho/2)u+\beta u} l^{v/q-v/2} 2^{u(1/q-\beta/2)} \sqrt{\ln(1/\delta)} \\ &\ll 2^{-(r+\rho/2-1/q)u-u/2} u^{v/q-v/2} \sqrt{\ln(1/\delta)} \\ &= \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) 2^{-u/2} u^{-v/2} \sqrt{\ln(1/\delta)} \\ &\ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) \sqrt{1/N \ln(1/\delta)}. \end{aligned}$$

Summarily, if $2 \le q < \infty$,

$$\begin{aligned} d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) &\ll I_1{}' + I_2{}' + I_3{}' \\ &\ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) \sqrt{1 + 1/N \ln(1/\delta)}. \end{aligned}$$

That is, if $1 \le q < \infty$

$$d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) \ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) \sqrt{1 + 1/N \ln(1/\delta)}.$$

Now we begin to prove the lower bound of Theorem 1. If $1 \le q < \infty$, from Theorem 6 and Lemma 1, we have

$$\begin{aligned} d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) &\gg 2^{-(r+\rho/2)k} d_{N,\delta}\left(\mathbb{R}^{\|S\|}, v, l_q^{\|S\|}\right) \\ &\gg 2^{-(r+\rho/2)k} \|S\|^{1/q-1/2} \sqrt{\|S\| + \ln(1/\delta)} \\ &\gg 2^{-(r+\rho/2)k} \|S\|^{1/q-1/2} \left(\|S\|^{1/2} + \sqrt{\ln(1/\delta)}\right) \\ &\gg 2^{-(r+\rho/2)k} |S|^{1/q} 2^{k/q} + 2^{-(r+\rho/2)k} |S|^{1/q-1/2} 2^{k/q-k/2} \sqrt{\ln(1/\delta)} \\ &\gg 2^{-(r+\rho/2)k} |S|^{1/q} 2^{k/q} \left(1 + |S|^{-1/2} 2^{-k/2} \sqrt{\ln(1/\delta)}\right) \\ &\gg 2^{-(r+\rho/2-1/q)k} k^{v/q} \left(1 + k^{-v/2} 2^{-k/2} \sqrt{\ln(1/\delta)}\right) \\ &\gg \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) \sqrt{1/N \ln(1/\delta)}. \end{aligned}$$

That is,

$$d_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) \asymp \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q} \left(\ln^{v/q} N\right) \sqrt{1 + 1/N \ln(1/\delta)}, 1 \le q < \infty,$$

which completes the proof of Theorem 1. □

Proof of Theorem 2. First, we prove the upper bound of Theorem 2. From Lemma 2, if $1 \le q < 2$, $\lambda_{N,\delta}\left(\mathbb{R}^m, v, l_q^m\right)$ and $d_{N,\delta}\left(\mathbb{R}^m, v, l_q^m\right)$ have the same sharp bounds. So, we only need to prove the upper bound if $2 \le q < \infty$. From Theorem 7 and Lemma 2, we obtain

$$\begin{aligned}
\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right) &\ll \sum_{l,k} 2^{-(r+\rho/2)l} \lambda_{N_{l,k},\delta_{l,k}}\left(\mathbb{R}^{\|S_{l,k}\|}, v, l_q^{\|S_{l,k}\|}\right) \\
&\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l}\left(\|S_{l,k}\|^{1/q} + \sqrt{\ln(1/\delta_{l,k})}\right) \\
&\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l}\left(\|S_{l,k}\|^{1/q} + (N/N_{l,k})^{1/2} + \sqrt{\ln(1/\delta)}\right) \\
&\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l}\|S_{l,k}\|^{1/q} \\
&+ \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l}(N/N_{l,k})^{1/2} \\
&+ \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l}\sqrt{\ln(1/\delta)} \\
&:= I_1'' + I_2'' + I_3''.
\end{aligned}$$

It is obvious to see that $I_1'' = I_1$. Therefore,

$$I_1'' \ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q} N\right).$$

Now, we calculate I_2'':

$$\begin{aligned}
I_2'' &\ll \sum_{l>u} \sum_{d\le k\le l} 2^{-(r+\rho/2)l}(N/N_{l,k})^{1/2} \\
&\ll N^{1/2} \sum_{l>u} 2^{-(r+\rho/2)l} \sum_{d\le k\le l} |S_{l,k}|^{-1/2} 2^{-(u+\beta u-2\beta l+\beta k)/2} \\
&\ll N^{1/2} 2^{-u/2-\beta u/2} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} \sum_{d\le k\le l} |S_{l,k}|^{-1/2} 2^{-\beta k/2}.
\end{aligned}$$

By using the method of the proof of Equation (42), we can obtain

$$\sum_{d\le k\le l} |S_{l,k}|^{-1/2} 2^{-\beta k/2} \ll l^{-v/2} 2^{-\beta l/2}.$$

Therefore,

$$\begin{aligned}
I_2'' &\ll N^{1/2} 2^{-u/2-\beta u/2} \sum_{l>u} 2^{-(r+\rho/2)l+\beta l} l^{-v/2} 2^{-\beta l/2} \\
&\ll N^{1/2} 2^{-u/2-\beta u/2} 2^{-(r+\rho/2)u+\beta u} u^{-v/2} 2^{-\beta u/2} \\
&\ll 2^{u/2} u^{v/2} 2^{-u/2} 2^{-(r+\rho/2)u} u^{-v/2} \\
&= 2^{-(r+\rho/2-1/q)u} 2^{-u/q} u^{-v/q} u^{v/q} \\
&\ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q} N\right) N^{-1/q}.
\end{aligned}$$

Next, we calculate I_3'':

$$I_3'' = \sum_{l>u}\sum_{d\le k\le l} 2^{-(r+\rho/2)l}\sqrt{\ln(1/\delta)}$$
$$\ll 2^{-(r+\rho/2)u}\sqrt{\ln(1/\delta)}$$
$$\ll \left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)N^{-1/q}\sqrt{\ln(1/\delta)}.$$

Summarily, if $1 \le q < \infty$,

$$\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right) \ll I_1''+I_2''+I_3''$$
$$\ll \left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\left(1+N^{-1/q}\sqrt{\ln(1/\delta)}\right).$$

Finally, we prove the lower bound of Theorem 2. From Lemma 2, we only need to prove the lower bound of Theorem 2 if $2 \le q < \infty$. From Theorem 8 and Lemma 2, we have

$$\lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right) \gg 2^{-(r+\rho/2)k}\lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},v,l_q^{\|S\|}\right)$$
$$\gg 2^{-(r+\rho/2)k}\left(\|S\|^{1/q}+\sqrt{\ln(1/\delta)}\right)$$
$$\gg 2^{-(r+\rho/2)k}|S|^{1/q}2^{k/q}+2^{-(r+\rho/2)k}\sqrt{\ln(1/\delta)}$$
$$\gg 2^{-(r+\rho/2-1/q)k}k^{v/q}\left(1+2^{-k/q}k^{-v/q}\sqrt{\ln(1/\delta)}\right)$$
$$\gg \left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\left(1+N^{-1/q}\sqrt{\ln(1/\delta)}\right).$$

That is, if we note $\lambda_{N,\delta} := \lambda_{N,\delta}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)$, then

$$\lambda_{N,\delta} \asymp \left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\sqrt{1+\frac{1}{N}\ln\left(\frac{1}{\delta}\right)},\ 1<q<2;$$
$$\lambda_{N,\delta} \asymp \left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\left(1+N^{-\frac{1}{q}}\sqrt{\ln\left(\frac{1}{\delta}\right)}\right),\ 2\le q<\infty,$$

which completes the proof of Theorem 2. □

Proof of Theorem 3. We consider the decreasing sequence of sets $\{G_{2^{-k}}\}_{k=0}^{\infty}$, such that $\mu(G_{2^{-k}}) \le 2^{-k}$ for each k and $G_1 = W_2^A\left(\mathbb{T}^d\right)$. Then, $W_2^A\left(\mathbb{T}^d\right) = \bigcup_{k=0}^{\infty}\left(G_{2^{-k}}\backslash G_{2^{-k-1}}\right)$. From Theorem 1, there would be a subspace F_N, such that $\dim F_N \le N$ and

$$e\left(W_2^A\left(\mathbb{T}^d\right)\backslash G_{2^{-k-1}},F_N,S_q\left(\mathbb{T}^d\right)\right) \ll d_{N,2^{-k-1}}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)$$
$$\ll \left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\sqrt{1+1/N\ln\left(2^{k+1}\right)}$$
$$\ll \left(N^{-1}\ln^{v}N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\sqrt{k+2}.$$

Therefore, from Definition 2 :

$$
\begin{aligned}
\left(d_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)_p\right)^p &\le \int_{W_2^A(\mathbb{T}^d)} e\left(x,F_N,S_q\left(\mathbb{T}^d\right)\right)^p \mathrm{d}\mu \\
&\le \sum_{k=0}^{\infty}\int_{G_{2^{-k}}\setminus G_{2^{-k-1}}} e\left(x,F_N,S_q\left(\mathbb{T}^d\right)\right)^p \mathrm{d}\mu \\
&\le \sum_{k=0}^{\infty}\int_{G_{2^{-k}}\setminus G_{2^{-k-1}}} e\left(W_2^A\left(\mathbb{T}^d\right)\setminus G_{2^{-k-1}},F_N,S_q\left(\mathbb{T}^d\right)\right)^p \mathrm{d}\mu \\
&\ll \sum_{k=0}^{\infty}\left(\left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right)\sqrt{k+2}\right)^p \mu(G_{2^{-k}}) \\
&\ll \left(N^{-1}\ln^v N\right)^{(r+\rho/2-1/q)p}\left(\ln^{vp/q}N\right)\sum_{k=0}^{\infty}\left(\sqrt{k+2}\right)^p 2^{-k} \\
&\ll \left(N^{-1}\ln^v N\right)^{(r+\rho/2-1/q)p}\left(\ln^{vp/q}N\right).
\end{aligned}
$$

Due to the astringency of $\sum\limits_{k=0}^{\infty}\sqrt{k+2}2^{-k}$, we have

$$
d_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)_p \ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right),\ 1\le q<\infty.
$$

Next, we prove the lower bound of Theorem 3. We consider the set

$$
G=\left\{x\in W_2^A\left(\mathbb{T}^d\right): e\left(x,F_N,S_q\left(\mathbb{T}^d\right)\right)>\frac{1}{2}d_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)\right\}.
$$

Then, $\mu(G)>\frac{1}{2}$. If not, we have

$$
d_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)_p \le \sup_{x\in W_2^A(\mathbb{T}^d)\setminus G} e\left(x,F_N,S_q\left(\mathbb{T}^d\right)\right)\le\frac{1}{2}d_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right).
$$

So, we obtain contradictions. Therefore,

$$
\begin{aligned}
\int_{W_2^A(\mathbb{T}^d)} e\left(x,F_N,S_q\left(\mathbb{T}^d\right)\right)^p \mathrm{d}\mu &\gg \int_G e\left(x,F_N,S_q\left(\mathbb{T}^d\right)\right)^p \mathrm{d}\mu \\
&\gg \int_G \left(\frac{1}{2}d_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)\right)^p d\mu \\
&\gg 2^{-p}\left(N^{-1}\ln^v N\right)^{(r+\rho/2-1/q)p}\left(\ln^{vp/q}N\right)(1+1/N\ln 2)^{p/2}.
\end{aligned}
$$

That is, $d_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)_p \gg (N^{-1}\ln^v N)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right),\ 1\le q<\infty$.

Finally, we obtain

$$
d_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right),\mu,S_q\left(\mathbb{T}^d\right)\right)_p \asymp \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q}N\right),\ 1\le q<\infty.
$$

□

Proof of Theorem 4. We consider the decreasing sequence of sets $\{G_{2^{-k}}\}_{k=0}^{\infty}$, such that $\mu(G_{2^{-k}})\le 2^{-k}$ for each k and $G_1=W_2^A\left(\mathbb{T}^d\right)$. Then, $W_2^A\left(\mathbb{T}^d\right)=\bigcup\limits_{k=0}^{\infty}(G_{2^{-k}}\setminus G_{2^{-k-1}})$. From Theorem 2, there would be a linear operator T_N from $S_q\left(\mathbb{T}^d\right)$ into itself, such that $rankT_N\le N$ and

$$\lambda\left(W_2^A\left(\mathbb{T}^d\right)\backslash G_{2^{-k-1}}, T_N, S_q\left(\mathbb{T}^d\right)\right) \ll \lambda_{N,2^{-k-1}}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)$$
$$\ll \begin{cases} \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q} N\right)\sqrt{1+1/N\ln(2^{k+1})}, 1<q<2; \\ \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q} N\right)\left(1+N^{-1/q}\sqrt{k+1}\right), 2\le q<\infty. \end{cases}$$

Therefore, from Definition 2,

$$\left(\lambda_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)_p\right)^p \le \int_{W_2^A\left(\mathbb{T}^d\right)} \|x-T_N x\|_{q,S}^p \mathrm{d}\mu$$
$$\le \sum_{k=0}^{\infty}\int_{G_{2^{-k}}\backslash G_{2^{-k-1}}} \lambda\left(W_2^A\left(\mathbb{T}^d\right)\backslash G_{2^{-k-1}}, T_N, S_q\left(\mathbb{T}^d\right)\right)^p \mathrm{d}\mu$$
$$\ll \sum_{k=0}^{\infty} \lambda_{N,2^{-k-1}}\left(W_2^A\left(\mathbb{T}^d\right), T_N, S_q\left(\mathbb{T}^d\right)\right)^p \mu(G_{2^{-k}})$$
$$\ll \begin{cases} \left(N^{-1}\ln^v N\right)^{(r+\rho/2-1/q)p}\left(\ln^{vp/q} N\right)\left(\sqrt{k+2}\right)^p 2^{-k}, 1<q<2; \\ \left(N^{-1}\ln^v N\right)^{(r+\rho/2-1/q)p}\left(\ln^{vp/q} N\right)\left(1+\sqrt{k+1}\right)^p 2^{-k}, 2\le q<\infty. \end{cases}$$

Due to the astringency of $\sum\limits_{k=0}^{\infty} 2^{-k}\left(\sqrt{k+2}\right)^p$ and $\sum\limits_{k=0}^{\infty} 2^{-k}\left(1+\sqrt{k+1}\right)^p$, we have

$$\lambda_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)_p \ll \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q} N\right),\ 1\le q<\infty.$$

Next, we prove the lower bound of Theorem 4. We consider the set

$$G=\left\{x\in W_2^A\left(\mathbb{T}^d\right) : \|x-T_N x\|_{q,S} > \frac{1}{2}\lambda_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)\right\}.$$

Then, $\mu(G) > \frac{1}{2}$. If not, we have

$$\lambda_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)_p \le \sup_{x\in W_2^A\left(\mathbb{T}^d\right)\backslash G} \|x-T_N x\|_{q,S} \le \frac{1}{2}\lambda_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right).$$

So, we obtain contradictions. Therefore,

$$\int_{W_2^A\left(\mathbb{T}^d\right)} \|x-T_N x\|_{q,S}^p d\mu \gg \int_G \|x-T_N x\|_{q,s}^p \mathrm{d}\mu$$
$$\gg \int_G \left(1/2\lambda_{N,1/2}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)\right)^p \mathrm{d}\mu$$
$$\gg \begin{cases} 2^{-p}\left(N^{-1}\ln^v N\right)^{(r+\rho/2-1/q)p}\left(\ln^{vp/q} N\right)(1+1/N\ln 2)^{p/2}, 1\le q<2 \\ 2^{-p}\left(N^{-1}\ln^v N\right)^{(r+\rho/2-1/q)p}\left(\ln^{vp/q} N\right)\left(1+N^{-1/q}\sqrt{\ln 2}\right)^p, 2\le q<\infty. \end{cases}$$

That is, $\lambda_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)_p \gg (N^{-1}\ln^v N)^{r+\rho/2-1/q}\left(\ln^{v/q} N\right),\ 1\le q<\infty.$

Finally, we obtain

$$\lambda_N^{(a)}\left(W_2^A\left(\mathbb{T}^d\right), \mu, S_q\left(\mathbb{T}^d\right)\right)_p \asymp \left(N^{-1}\ln^v N\right)^{r+\rho/2-1/q}\left(\ln^{v/q} N\right),\ 1\le q<\infty.$$

□

In summary, the proof of main results are completed.

5. Summary

In this article, we have obtained the sharp bounds of Kolmogorov and linear N-widths in the probabilistic and average setting of the Sobolev space $W_2^A\left(\mathbb{T}^d\right)$ in the S_q-norm. In the process of calculating, we use discretization. Discretization means that we can transform function space into finite-dimensional space. It can reduce the calculation of the probabilistic (N,δ)-widths. The sharp bounds of the p-average N-widths should be obtained by the sharp bounds of the probabilistic (N,δ)-widths. These results can be used to the research of algorithms and computational complexity. And these results may play important roles of the research of approximation theory of Sobolev spaces.

On the other hand, other related theories have not yet been studied. For example, we can study the sharp bounds of probabilistic Gel'fand (N,δ)-widths and p-average Gel'fand N-widths of $W_2^A\left(\mathbb{T}^d\right)$ in the S_q-norm and L_q-norm. The above issues can be studied later.

Author Contributions: Resources, H.L.; Writing—original draft, Y.L.; Writing—review and editing, X.L. All authors have read and agreed to the published version of the manuscript.

Funding: Capacity building for scientific and technological innovation services – basic scientific research business fees: 110052971921/102.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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