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The Zakharov–Shabat Spectral Problem for Complexification and Perturbation of the Korteweg–de Vries Equation

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Abstract: In this paper we consider examples of complex expansion (cKdV) and perturbation (pKdV) of the Korteweg–de Vries equation (KdV) and show that these equations have a representation in the form of the zero-curvature equation. In this case, we use the Lie algebra of 4-dimensional quadratic nilpotent matrices. Moreover, it is shown that the simplest possible matrix representation of this algebra leads to the possibility of constructing a countable number of conservation laws for these equations.

Keywords: complexification of the Korteweg–de Vries equation; perturbations of the Korteweg–de Vries equation; zero-curvature equation; conservation laws

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1. Introduction

The modern understanding of the integrability of nonlinear equations is based on the method of the inverse scattering problem. This method is applicable when the equation under study admits a representation in the form of consistency conditions for auxiliary linear systems. In particular, the integrability of the equations of field theory 1 + 1 and condensed matter physics is based on the possibility of their representation in the form of the zero-curvature equation [1–3]. The most productive interpretation of the zero-curvature equation is achieved if we consider it as a consistency condition for a set of commutating Hamiltonian flows in the dual space with some Lie algebra [3–5]. In this case, the corresponding pair of operators $\widehat{P}$, $\widehat{Q}$ is identified with matrix gradients of mutually commutating Hamiltonians with respect to the Lie–Poisson bracket.

The possibility of using this method was found to be relevant, even for classical problems in mechanics related to the motion of a heavy solid body around a stationary point (the Kovalevskaya top). Almost a hundred years later, since Kovalevskaya reduced the problem to hyperelliptic quadratures, Reyman and Semenov-Tian-Shansky found a "natural" Lax representation with a spectral parameter for the Kovalevskaya vertex [6,7]. The method is based on a general group-theoretic approach to integrable systems and uses the Lie algebra so(4).

In 2006, Clarkson, Joshi, and Mazzocco demonstrated an algorithmic method for obtaining Lax pairs for a modified Korteweg–de Vries hierarchy [8]. Thanks to a special reduction under the action of a similarity group, they obtained a proof of isomonodromy for two identical Penlevé hierarchies.

To obtain new Volterra coupled systems and modified Toda field equations in [9,10], the authors used the special quasi-graduated Lie algebras parameterized by some numerical matrices, which are multiparametric deformations of loop algebras, which also generalize a special elliptic algebra so(3). In [11], a new generalized spectral problem with a 5×5 matrix for the Ablowitz-Kaup-Newell-Segur type models associated with an extended matrix Lie superalgebra was considered, and a superintegrable hierarchy corresponding to it was constructed.

The purpose of this paper is to apply the Zakharov–Shabat method for constructing various perturbations of the KdV equation using the Lie algebra of nilpotent 4×4 matrices and to study the existence of a countable number of conservation laws for the resulting models. In this paper, we continue the study of the kKdV and vKdV equations, for which Lax pairs were constructed, and some exact solutions were found in [12]. The application of the Zakharov–Shabat method allows us not only to integrate such complex equations but also to obtain some additional properties, in particular the presence of a countable number of conservation laws. In addition, the approach used is directly related to the dressing method, which not only allows us to construct new integrable nonlinear equations together with the corresponding auxiliary problems but also to obtain broad classes of exact solutions of the constructed equations, as well as wave functions and variable coefficients of the auxiliary problems.

2. Application of the Zakharov and Shabat Method

The advantage of the Lax method is its elementary character; the method is essentially reduced to the calculation of the conditions of conservation of eigenfunctions. However, after obtaining integrable equations to actually integrate them, it is necessary to develop a technique for solving the inverse spectral problem, which is not always easy. This difficulty was overcome by Zakharov and Shabat [2,13], who demonstrated the possibility of constructing integrable equations together with an explicit indication of the way to calculate their exact solutions by means of the dressing method.

The basic idea of the dressing method is based on the representation of the integral operator on a straight line as a product of two Volterra operators. The method was further developed in [14], where the Riemann–Hilbert problem was introduced into the scheme of the dressing method. The method proved to be applicable both for (1 + 1)-dimensional and (2 + 1)-dimensional (two spatial and one time coordinate) nonlinear equations.

The integration of known non-linear evolution equations (the Korteweg–de Vries equation, the non-linear Schrödinger equation, the Sine–Gordon equation, etc.) [15] is usually based on their representation in the form of a consistency condition for two linear problems

$$V_x = \widehat{P}V, \tag{1}$$

$$V_t = \widehat{Q}V, \tag{2}$$

where $V(x,t)$ is a vector function with values in space $\mathbf{C}^n$, which is commonly called auxiliary; $\widehat{P}$ and $\widehat{Q}$ are matrices in this space, parameterized by functions $u_j(x,t)$ $j = 1, ..., N$ (classical fields) included in the nonlinear equation and by the spectral parameter ξ of $\mathbf{C}^1$, on which they depend meromorphically.

It is well known that relation (1.1) underlies the applicability of the inverse scattering problem to the nonlinear evolution equations, so using the terminology adopted in soliton theory, we will call $\widehat{M}$ the scattering operator. The most important feature of the pair of Lax matrix operators $\widehat{M}$, $\widehat{B}_n$ is that the time derivative is not included in the operator $\widehat{M}$. Thus, we can consider t as a parameter and investigate the spectral properties of this operator, i.e., investigate solutions of the equation on eigenvalues.

This representation of the integrable equations is called geometric, since the matrix functions $\widehat{P}(x,t,\xi)$, $\widehat{Q}(x,t,\xi)$ can be interpreted as local coefficients of connectivity in the trivial decomposition $\mathbf{R}^2 \times \mathbf{C}^n$, where the space-time $\mathbf{R}^2$ plays the role of a base and the auxiliary space $\mathbf{C}^n$ plays the role of a layer. Equations (1) and (2) mean that the vector V is covariantly constant, and the condition

$$\widehat{P}_t - \widehat{Q}_x + \left[\widehat{P}, \widehat{Q}\right] = 0 \tag{3}$$

shows that the connectivity of $\widehat{P}$, $\widehat{Q}$ has zero curvature. Therefore, (3) is called the zero-curvature condition. It replaces the Lax-type representation in the modern formalism of the inverse problem method.

Let us apply the Fourier transform to Equations (1) and (2) using the parameter ξ; then, we represent the operators $\widehat{P}$ and $\widehat{Q}$ in the following form

$$\widehat{P}(x,t,\xi) = a_0\xi^n + a_1\xi^{n-1} + \ldots + a_n + \sum_k \sum_{j=1}^{s_k} \frac{p_{kj}}{(\xi - \xi_k)^j}, \tag{4}$$

$$\widehat{Q}(x,t,\xi) = b_0\xi^m + b_1\xi^{m-1} + \ldots + b_m + \sum_k \sum_{j=1}^{l_k} \frac{q_{kj}}{(\xi - \tilde{\xi}_k)^j}, \tag{5}$$

provided that $j \neq k, \xi_k \neq \xi_j, \widetilde{\xi}_k \neq \widetilde{\xi}_j$ are complex constants, a_k, b_k, p_{kj}, q_{kj} are matrices depending on some functions of the variables x and t and their derivatives with respect to these variables.

The problem is reduced to determining the equations that $\widehat{P}$ and $\widehat{Q}$ must satisfy in order for each solution of system (1) and (2) to satisfy the consistency condition (3), i.e., zero-curvature equation. Substituting (4) and (5) into (3), we obtain the conditions on the matrices a_k, b_k, p_{km}, q_{km}. The existence of such conditions requires the commutativity of the matrices a_0 and b_0. In this case, the equations always exist but may not be uniquely determined. The following conditions are necessary and sufficient for uniqueness:

1. At least one of the numbers $m,\ n$ is not equal to zero;
2. None of ξ_i is equal to $\widetilde{\xi}_i$.

If these conditions are satisfied, no new powers of ξ or new prime fractions not contained in $\widehat{P}$ and $\widehat{Q}$ appear in the commutator $\left[\widehat{P}, \widehat{Q}\right] = \widehat{P}\widehat{Q} - \widehat{Q}\widehat{P}$; therefore, the commutator can also be decomposed into the same prime fractions as $\widehat{P}$ and $\widehat{Q}$.

The operators $\widehat{P}$ and $\widehat{Q}$ will be considered as quadratic matrices 4×4, and $V(x,t)$ is a 4-dimensional vector function. For the convenience of further calculations, let us introduce the basis of four-dimensional Lie algebra $\mathfrak{g}$ matrices in the following form:

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{6}$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Lie algebra is a vector space $\mathfrak{g}$ over some field together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a Lie bracket, satisfying the following axioms: bilinearity, skew-symmetry

$$\forall X, Y \in \mathfrak{g}[X,Y] = -[Y,X] \quad ([X,X] = 0),$$

and Jacobi identity

$$\forall X, Y, Z \in \mathfrak{g}[Z,[X,Y]] + [Y,[Z,X]] + [X,[Y,Z]] = 0.$$

Let us make a table (Table 1) of the results of the commutator $[X, Y]$, where X and Y take values from the set of basis matrices $\{M, N, L, K, T, U, S, H, E, F\}$, without making any assumptions about its closure. Further it is used to construct closures of Equation (3).

Table 1. Table of commutations $[X, Y]$ of the Lie algebra $\mathfrak{g}$.

X \ Y	M	N	K	L	T	U	S	H	E	F
M	0	0	K	$-L$	T	$-U$	$2S$	0	0	$-2F$
N	0	0	K	$-L$	$-T$	U	0	$2H$	$-2E$	0
K	$-K$	$-K$	0	$-M-N$	$2S$	$2H$	0	0	T	U
L	L	L	$M+N$	0	$2E$	$2F$	$-T$	$-U$	0	0
T	$-T$	T	$-2S$	$-2E$	0	$M-N$	0	$-K$	0	$-L$
U	U	$-U$	$-2H$	$-2F$	$N-M$	0	$-K$	0	$-L$	0
S	$-2S$	0	0	T	0	K	0	0	0	M
H	0	$-2H$	0	U	K	0	0	0	N	0
E	0	$2E$	T	0	0	L	0	$-N$	0	0
F	$2F$	0	U	0	L	0	$-M$	0	0	0

2.1. Representation of the Complexification of the Korteweg–de Vries Equation in the Form of the Zero-Curvature Equation

Let us consider a special case of Equation (3), when coefficients of expansion (4) and (5) a_k, b_k, p_{km}, q_{km} are square matrices of the fourth order with elements that depend on functions $u(x,t)$, $\overline{u}(x,t)$ ("–" means complex conjugation) and their partial derivatives with respect to the variable x. To write Equations (1) and (2), we choose a column vector $V(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))^T$ as an eigenfunction, where T stands for transpose.

The structure of the operator $\widehat{P}$ (4) is given in a special way when only two coefficients are non-zero: $a_0 = \text{const}$ and p_1 is a functional matrix. $\widehat{P}$ has the following spectral parameter $i\xi$ expansion

$$\widehat{P} = i\xi a_0 + \frac{p_1}{i\xi} = i\xi \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{i\xi} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\overline{u}(x,t) & 0 & 0 \\ u(x,t) & 0 & 0 & 0 \end{pmatrix},$$

where i is the imaginary unit. Let us write $\widehat{P}$ using basis (6)

$$\widehat{P} = i\xi(L + H + S) + \frac{1}{i\xi}[u(x,t)F + \overline{u}(x,t)E]. \tag{7}$$

The consistency condition (3) for the system of Equations (1) and (2) must be satisfied for all values of ξ, so the form of the operator $\widehat{Q}$ cannot be arbitrary. Let us represent $\widehat{Q}$ in the form

$$\widehat{Q}(x,t,\xi) = b_0(i\xi)^m + b_1(i\xi)^{m-1} + \ldots + b_m + \sum_{j=1}^{s} \frac{q_j}{(i\xi)^j}. \tag{8}$$

Considering b_0 as constant and $b_j, q_j, j = 1, \ldots$ as functional matrices, determine possible values of powers m and s. The artificial closure (3) is the means by which Wahlquist and Estabrook came to the expression for $\widehat{P}$ and $\widehat{Q}$ in the context of the Korteweg–de Vries equation [16], and it is in this direction that subsequent authors have acted [17,18]. In (3), we equate the coefficients at the same powers of $i\xi$:

$$(i\xi)^{m+1}: \quad [a_0, b_0] = 0, \tag{9}$$

$$(i\xi)^{m}: \quad [a_0, b_1] = 0, \tag{10}$$

$$(i\xi)^{m-1}: \quad [a_0, b_2] + [p_1, b_0] - b_{1x} = 0, \tag{11}$$

$$\dots$$

$$(i\xi^0 : \quad [a_0, q_1] + [p_1, b_{m-1}] - b_{mx} = 0, \tag{12}$$

$$(i\xi)^{-1} : \quad [a_0, q_2] + [p_1, b_m] + p_{1t} - q_{1x} = 0, \tag{13}$$

$$(i\xi)^{-2} : \quad [a_0, q_3] + [p_1, b_1] - q_{2x} = 0, \tag{14}$$

$$\dots$$

$$(i\xi)^{-s} : \quad [p_1, q_{s-1}] - q_{sx} = 0, \tag{15}$$

$$(i\xi)^{-s-1} : \ [p_1, q_s] = 0. \tag{16}$$

The only equation that contains a derivative with respect to the variable t is equality (13), so it will determine the equation we are looking for; the other equations should identically turn to zero. The coefficient q_1 defines the smallest degree s at which the system (9)–(16) is consistent, i.e., $s \geq 1$. To make the number of equations as small as possible, consider the limiting case when $s = 1$.

Comparing the relations of the system (9)–(12), we can note that the values of matrices b_k are substituted into the following equations, though b_k are not included in these equations themselves but through their derivatives with respect to the variable x; therefore, it is easy to see that m will determine the degree of the highest derivative included in Equation (13).

Consider the case when $m = 3$. Using the table of commutations of the Lie algebra and the previously defined form (7) $a_0 = L + S + H$, we find, by virtue of Equation (9), a constant matrix b_0 commuting with a_0, and in the simplest form

$$b_0 = \alpha(E + F - K), \tag{17}$$

where α is some parameter, which for obtaining a special form of the final equation, we assume to by $\alpha = 4$. Equation (10) is similar to Equation (9), but the matrix b_1 is a functional matrix and cannot be constant, so we assume $b_1 = 0$.

The coefficient at the first power of $i\xi$ depends only on the functions $u(x,t)$, $\overline{u}(x,t)$ and is expressed from (11) using the table as follows:

$$b_2 = (u + \overline{u})L + (u - 3\overline{u})S - (3u - \overline{u})H. \tag{18}$$

The matrix b_3 can be obtained similarly

$$b_3 = \frac{1}{2}([3\overline{u} - u]_x M + [3u - \overline{u}]_x N). \tag{19}$$

The value q_1 is determined from the overdetermined system of Equations (12) and (16), which gives

$$q_1 = \left[u(u - 3\overline{u}) + \frac{1}{2}(3\overline{u} - u)_{xx}\right]F - \left[\overline{u}(3u - \overline{u}) - \frac{1}{2}(3u - \overline{u})_{xx}\right]E. \tag{20}$$

The specified structure of the matrices $\widehat{P}$ and $\widehat{Q}$ in the found form ensures that all members of the system (9)–(16) vanish, except for member (13) at $(i\xi)^{-1}$, which gives a matrix with two complex conjugate equations. The requirement that the resulting matrix be equal to zero means that $u(x,t)$ satisfies the equation

$$6\overline{u}_x u + 3(\overline{u} - u)u_x + \frac{1}{2}(u - 3\overline{u})_{xxx} = u_t. \tag{21}$$

The obtained Equation (21) coincides with the complex extension of the Korteweg–de Vries equation (cKdV) [12]. The conducted studies led to the following conclusion.

Theorem 1. *The complexification of the KdV Equation (21) has a zero-curvature operator representation (3) with operators $\widehat{P}$ and $\widehat{Q}$ in the form*

$$\widehat{P} = i\xi(L+H+S) + \frac{1}{i\xi}[u(x,t)F + \overline{u}(x,t)E], \tag{22}$$

$$\begin{aligned}\widehat{Q} = 4(i\xi)^3(E+F-K) + i\xi[(u+\overline{u})L + (u-3\overline{u})S - (3u-\overline{u})H] + \tfrac{1}{2}([3\overline{u}-u]_x M + [3u-\overline{u}]_x N) \\ + \tfrac{1}{i\xi}\Big[u(u-3\overline{u}) + \tfrac{1}{2}(3\overline{u}-u)_{xx}\Big]F - \tfrac{1}{i\xi}\Big[\overline{u}(3u-\overline{u}) - \tfrac{1}{2}(3u-\overline{u})_{xx}\Big]E,\end{aligned} \tag{23}$$

where $u(x,t)$ is an unknown complex function and $i\xi$ is an imaginary parameter.

2.2. Representation of the Perturbed Korteweg–de Vries Equation in the Form of the Zero-Curvature Equation

Consider the new problem of the consistency of the system (1), (2) with a given operator $\widehat{P}$ in the form of a polynomial of the first order by the parameter $i\xi$

$$\widehat{P} = (i\xi)^2 a_0 + a_1 = (i\xi)^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\mu} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \mu & 0 \\ -w(x,t) & \frac{1}{\mu}u(x,t) & 0 & 0 \\ u(x,t) & w(x,t) & 0 & 0 \end{pmatrix}.$$

Its expansion, we write using the basis matrices (6) in the form

$$\widehat{P} = (i\xi)^2\left(F + \frac{1}{\mu}E\right) + S - \mu H - w(x,t)L + u(x,t)\left(F - \frac{1}{\mu}E\right), \tag{24}$$

where $u(x,t), w(x,t)$ are some functions and μ is an arbitrary non-zero constant.

Using the consistency condition (3), let us define the matrix $\widehat{Q}$. Let us choose $\widehat{Q}$ in the form

$$\widehat{Q}(x,t,\xi) = b_0(i\xi)^{2m} + b_1(i\xi)^{2(m-1)} + \ldots + b_m + \sum_{j=1}^{s}\frac{q_j}{(i\xi)^{2j}}. \tag{25}$$

The representation in the form of a polynomial is considered for even powers of $(i\xi)^{2m}$. Let us determine possible values of powers m and s, considering b_0 as a constant matrix. To do this, we substitute (24) and (25) into (3) and equate the coefficients at the same powers $(i\xi)$ to zero

$$(i\xi)^{2m+2}:\ [a_0, b_0] = 0, \tag{26}$$

$$(i\xi)^{2m}:\ [a_0, b_1] + [a_1, b_0] = 0, \tag{27}$$

$$(i\xi)^{2m-2}:\ [a_0, b_2] + [a_1, b_1] - b_{1x} = 0, \tag{28}$$

$$\ldots$$

$$(i\xi)^{0}:\ [a_0, q_1] + [a_1, b_m] - b_{mx} + a_{1t} = 0, \tag{29}$$

$$(i\xi)^{-2}:\ [a_0, q_2] + [a_1, q_1] - q_{1x} = 0, \tag{30}$$

$$\ldots$$

$$(i\xi)^{-2s+2}:\ [a_0, q_s] + [a_1, q_{s-1}] - q_{s-1x} = 0, \tag{31}$$

$$(i\xi)^{-2s}: \ [a_1, q_s] - q_{sx} = 0, \tag{32}$$

where b_k, q_k are 4×4 matrices, depending on $u(x,t)$ and $w(x,t)$, with $k \neq 0$.

Some remarks are necessary here:

1. Only one equation contains a derivative with respect to the variable t; this is equality (29), so it will determine the sought relations between the functions $u(x,t)$ and $w(x,t)$. The other equations should identically turn to zero.
2. The coefficient q_1 enters Equations (29) and (30) and determines the smallest power s at which the system (29)–(32) is consistent, i.e., $s \geq 0$. To make the number of equations the smallest, consider the limiting case where s = 0.
3. Comparing relations (26)–(29), we can note that the matrices b_k, $k = 0, 1, \ldots$ defined in the upper equations then do not enter into the subsequent relations themselves but their derivatives with respect to the variable x, so it is easy to see that the degree of the highest derivative included in Equation (28) will determine the value of m. Next, consider the case where m = 2.

Now, let us analyze which basis matrices should be part of the expansion of $\widehat{Q}$ in power of $i\xi$ if the highest power is 2:

$$\widehat{Q}(x,t,\xi) = (i\xi)^4 b_0 + (i\xi)^2 b_1 + b_2, \tag{33}$$

where b_k, $k = 0, 1, \ldots$ are 4×4 functional matrices, depending on $u(x,t)$ and $w(x,t)$. This form of operator transforms the system (26)–(32) into four equalities

$$[a_0, b_0] = 0, [a_0, b_1] = [b_0, a_1], [a_0, b_2] = b_{1x} - [a_1, b_1], [a_1, b_2] - b_{2x} + a_{1t} = 0, \tag{34}$$

where $a_0 = F + \mu^{-1}E$, $a_1 = S - \mu H - w(x,t)L + u(x,t)\left(F - \mu^{-1}E\right)$.

Using the representation of $\widehat{P}$ in the form (25), we will use the basis matrices (6) and Table 1 to correctly select the suitable commutative matrix b_0 in the first equality (34). Suitable elements can be L, E, and F, or a linear combination of T, U. Let us consider each case separately.

(A1) (L): choose, for example,

$$b_0 = \alpha L, \tag{35}$$

α is an arbitrary parameter; then,

(B1) (E, F): if we choose E and F, they can be taken with different constant multipliers α_1, α_2, which gives

$$b_0 = \alpha_1 E + \alpha_2 F. \tag{36}$$

(C1) (T, U): in order for the first commutator of system (34) to become zero with $b_0 = \alpha_3 T + \alpha_4 U$, the condition must be satisfied

$$\left[F + \frac{1}{\mu}E, \alpha_3 T + \alpha_4 U\right] = \left(\alpha_3 + \frac{\alpha_4}{\mu}\right)L = 0.$$

Therefore,

$$b_0 = -\frac{\alpha_3}{\mu}T + \alpha_3 U, \tag{37}$$

where α_3 is a constant coefficient.

Cases A, B, and C will be considered separately, but a linear combination of them can also be used. Let us proceed to the analysis of the second equation of the system (34).

(A2). The right part of $[b_0, a_1]$ has the form

$$\left[\alpha L, S - \mu H - wL + u\left(F - \frac{1}{\mu}E\right)\right] = \alpha(\mu U - T), \tag{38}$$

and the left part of $[a_0, b_1]$ defines the form of b_1. Let us use the table of commutations of the Lie algebra $\mathfrak{g}$ and find the right structure for (38). The matrix b_1 should contain the matrix K; then

$$\left[\left(F+\frac{1}{\mu}E\right), \alpha_5 K\right] = \alpha_5\left(U+\frac{1}{\mu}T\right). \tag{39}$$

Comparing (38) and (39), it is easy to see that the expressions are incomparable, so case (A) will not be used further.

(B2). At b_0 defined by expression (36), we obtain a diagonal form with two parameters

$$\left[\alpha_1 E + \alpha_2 F,\ S - \mu H - wL + u\left(F - \frac{1}{\mu}E\right)\right] = -\alpha_1 M - \alpha_2 N. \tag{40}$$

Consequently, b_1 must contain matrices S and H; then, $[a_0, b_1]$ gives

$$\left[F + \frac{1}{\mu}E, \alpha_6 S + \alpha_7 H\right] = -\alpha_6 M - \frac{\alpha_7}{\mu}N. \tag{41}$$

Comparing (40) and (41), we obtain the refined form of b_0 and b_1:

$$b_0 = \frac{\alpha}{\mu}E + \alpha_2 F,\ b_1 = \alpha_2 S + \alpha\mu H. \tag{42}$$

(C2). The linear combination $T,\ U$ in the commutator $[b_0, a_1]$ has the following elements

$$\left[\alpha_4 U - \frac{\alpha_4}{\mu}T, S - \mu H - wL + u\left(F - \frac{1}{\mu}E\right)\right] = 2\alpha_4(wF - K) + 2\frac{\alpha_4}{\mu}(uL - wE),$$

among which there is a matrix K, which cannot be compensated in $[a_0, b_1]$ (see table), so option (C) is not involved in further study.

In order to compensate the terms of the third equality in system (34) and obtain a meaningful fourth equation, matrices b_1 and b_2 must depend on the functions $u, w, u_x, w_x, u_{xx}, w_{xx}$. Let us complement matrix b_1 so that this complement does not affect the commutation in the third equality of system (34). This is possible since

$$\left[F + \mu^{-1}E, f_1 L\right] = 0,\ \left[F + \mu^{-1}E, f_2(T - \mu U)\right] = 0,\ \left[F + \mu^{-1}E, f_3 F + f_4 E\right] = 0. \tag{43}$$

Let us examine all the cases where

$$b_1 = \alpha_2 S + \alpha\mu H + \Omega_k, k = 1,\ 2,\ 3, \tag{44}$$

$$\Omega_1 = f_1 L,\ \Omega_2 = f_2(T - \mu U),\ \Omega_3 = f_3 F + f_4 E, \tag{45}$$

$f_m = f_m(u, w, u_x, w_x, u_{xx}, w_{xx}), m = 1,\ 2,\ 3,\ 4$, will be further defined below. The cases (45) will be considered separately, but their linear combination can also be used.

Let us write an explicit form of the right-hand side of $[a_0, b_2] = b_{1x} - [a_1, b_1]$, using (44) for the value of b_1 without Ω_k

$$(\alpha_2 S + \alpha\mu H)_x - [a_1, \alpha_2 S + \alpha\mu H] = u(\alpha_2 M - \alpha N) - \alpha_2 wT - \alpha\mu wU. \tag{46}$$

Since the equality $[a_0, b_2] = u(\alpha_2 M - \alpha N) - \alpha_2 wT - \alpha\mu wU$ should be fulfilled, and, as can be seen from Table 1, the initial commutator gives matrices T, U only if b_2 contains an element proportional to K, given that $\left[F + \mu^{-1}E, fK\right] = f\mu^{-1}T + fU$ and the relations between the coefficients of result $f = -\mu\alpha_2 w = -\alpha\mu w$, in (46) we should additionally put $\alpha_2 = \alpha$. As a result, the basis part of b_{20} corresponding to (46) will take the form

$$b_{20} = \mu\alpha uH - \alpha uS - \mu\alpha wK. \tag{47}$$

The additional part $b_{2k}, k = 1,\ 2,\ 3$ depends on the choice of the right-hand side of (45):

$$\text{a) } \Omega_{1x} - [a_1, \Omega_1] = f_{1x}L - f_1 T + \mu f_1 U, \tag{48}$$

$$\text{b) } \Omega_{2x} - [a_1, \Omega_2] = f_{2x}T - \mu f_{2x}U + 2f_2(\mu K - uL) + 2wf_2(E - \mu F), \tag{49}$$

$$\text{c) } \Omega_{3x} - [a_1, \Omega_3] = f_{3x}F + f_{4x}E - f_3 M + \mu f_4 N. \tag{50}$$

Using Table 1, it is easy to see that there is no basis matrix whose commutator with $F + \mu^{-1}E$ would allow us to obtain (48), (49) completely, so cases (a) and (b) are not of interest and will not be used further.

Case (c) gives an additional part b_{23}

$$b_{23} = \frac{1}{2}(f_{3x}M + \mu f_{4x}N) + f_3 S - \mu^2 f_4 H. \tag{51}$$

As a result, (44) and the combination of (47) and (51) give

$$b_1 = \alpha\mu H + \alpha S + f_3 F + f_4 E, \tag{52}$$

$$b_2 = \left(\mu\alpha u - \mu^2 f_4\right)H + (f_3 - \alpha u)S - \mu\alpha w K + \frac{1}{2}(f_{3x}M + \mu f_{4x}N). \tag{53}$$

The fourth equation of the system (34) is not trivial but defines differential relations on the functions $u(x,t)$ and $w(x,t)$:

$$\begin{aligned}
&0 = [a_1, b_2] - b_{2x} + a_{1t} = -(2f_{3x} - \alpha u_x)S + (2\mu^2 f_{4x} - \mu\alpha u_x)H - \mu^2 w f_4 U + wf_3 T \\
&-\left(\tfrac{1}{2}w(f_{3x} + \mu f_{4x}) + w_t\right)L + \left(\mu\alpha w^2 - u(f_3 - \alpha u) - \tfrac{1}{2}f_{3xx}\right)M + (uf_{3x} + u_t)F \\
&+\left(u(\alpha u - \mu f_4) + \mu\alpha w^2 - \tfrac{1}{2}\mu f_{4xx}\right)N - \left(uf_{4x} + \tfrac{1}{\mu}u_t\right)E + \mu\alpha w_x K.
\end{aligned}$$

Ten basis matrices participate in writing the result and give ten equations, three of which contain derivatives with respect to the variable t at $L,\ E,\ F$. The remaining seven equalities should give identically zero. Two arbitrary parameters $f_m,\ m = 3,\ 4$ are introduced in the construction of the matrix b_2; their number is insufficient to resolve the problem. As was shown in Equation (43), the commutator of a_0 with $E + F,\ L,\ T - \mu U$ does not give additional elements, so by adding them to matrix $b_{2,}$ we can nullify the superfluous equations. Let b_2 take the form

$$b_2 = \left(\mu\alpha u - \mu^2 f_4\right)H + (f_3 - \alpha u)S - \mu\alpha w K + \frac{1}{2}(f_{3x}M + \mu f_{4x}N) + f_6 E + f_7 F + f_8 L + f_9(T - \mu U), \tag{54}$$

where $f_k = f_k(u, w, u_x, w_x, u_{xx}, w_{xx}), k = 6,7,8,9$ are unknown functions.

Write out all equations derived from the equality $[a_1, b_2] - b_{2x} + a_{1t} = 0$:

$$\begin{aligned}
&\alpha w_x - 2f_9 = 0,\ \ 2f_{3x} - \alpha u_x = 0,\ \ 2\mu f_{4x} - \alpha u_x = 0, \\
&f_{9x} - f_8 - \mu w f_4 = 0,\ \ wf_3 + f_8 - f_{9x} = 0,\ \ uf_{3x} + u_t - f_{7x} + 2\mu w f_9 = 0, \\
&\tfrac{1}{2}w(f_{3x} + \mu f_{4x}) + w_t + f_{8x} - 2uf_9 = 0, \\
&\mu\alpha w^2 - u(f_3 - \alpha u) - \tfrac{1}{2}f_{3xx} + f_7 = 0, \\
&u(\alpha u - \mu f_4) + \mu\alpha w^2 - \tfrac{1}{2}\mu f_{4xx} - \mu f_6 = 0, \qquad uf_{4x} + \tfrac{1}{\mu}u_t + f_{6x} + 2wf_9 = 0.
\end{aligned} \tag{55}$$

Equation (55) determines the coefficients:

$$f_3 = \tfrac{\alpha}{2}u,\ f_4 = \tfrac{\alpha}{2\mu}u,\ f_9 = \tfrac{\alpha}{2}w_x,\ f_8 = \tfrac{\alpha}{2}w_{xx} - \tfrac{\alpha}{2}uw,$$

$$f_7 = \tfrac{\alpha}{4}u_{xx} - \mu\alpha w^2 - \tfrac{\alpha}{2}u^2, f_6 = \alpha w^2 + \tfrac{\alpha}{2\mu}u^2 - \tfrac{\alpha}{4\mu}u_{xx}.$$

The remaining three equalities give equations, two of which coincide

$$\frac{\alpha}{2}u_x w + w_t - \frac{\alpha}{2}(uw)_x + \frac{\alpha}{2}w_{xxx} - \alpha u w_x = 0,$$

$$\frac{3\alpha}{2\mu}u u_x + \frac{1}{\mu}u_t + 2\alpha w w_x - \frac{\alpha}{4\mu}u_{xxx} + \alpha w w_x = 0.$$

Assuming the parameter $\alpha = 4$, we obtain a system of two differential equations relating the functions $u(x,t)$ and $w(x,t)$:

$$\begin{cases} u_t = 6uu_x - u_{xxx} + 12\mu w w_x, \\ w_t = 2w_{xxx} - 6uw_x. \end{cases} \tag{56}$$

The system (56) coincides with the perturbation of the Korteweg–de Vries equation (pKdV) of [12]; hence, an operator representation in the form of the zero-curvature equation is found for it, and the following theorem is proved.

Theorem 2. *The system of Equation (56) has a zero-curvature operator representation (3) with operators $\widehat{P}$ and $\widehat{Q}$ in the form*

$$\widehat{P} = (i\xi)^2\left(F + \frac{1}{\mu}E\right) + S - \mu H - w(x,t)L + u(x,t)\left(F - \frac{1}{\mu}E\right), \tag{57}$$

$$\begin{aligned} \widehat{Q} = {} & 4(i\xi)^4\left(\tfrac{E}{\mu} + F\right) + 4(i\xi)^2\left(\mu H + S + \tfrac{u}{2}F + \tfrac{u}{2\mu}E\right) + 2u(\mu H - S) + u_x(M+N) \\ & + \left(\tfrac{1}{\mu}u_{xx} - 4w^2 - \tfrac{2}{\mu}u^2\right)(\mu F - E) + 2(w_{xx} - uw)L + 2w_x(T - \mu U) - 4\mu w K, \end{aligned} \tag{58}$$

where $u(x,t)$, $w(x,t)$ are unknown functions, $i\xi$ is imaginary parameter, μ is arbitrary constant, and the matrices $E, F, H, K, L, M, N, S, T, U$ have the form (6).

3. Construction of Conservation Laws

An important property of nonlinear evolution equations is the existence of an infinite sequence of local conservation laws. Each of these laws gives, for a class of solutions asymptotically tending to zero at $x \to \pm\infty$, a conserved value expressed as an integral over the entire space of the corresponding nonlinear polynomial combination of functions and their derivatives.

3.1. Conservation Laws for cKdV

Consider the eigenvalue Equation (1) written for Equation (21), with vector function $V(x,\xi) = (v_1(x,\xi), v_2(x,\xi), v_3(x,\xi), v_4(x,\xi))^T$ and operator $\widehat{P}$ (22) as a system

$$\begin{aligned} & v_{1x} = i\xi v_4, \quad v_{3x} = -i\xi v_1 - \tfrac{1}{i\xi}\overline{u}(x,t)v_2, \\ & v_{2x} = -i\xi v_3, \quad v_{4x} = i\xi v_2 + \tfrac{1}{i\xi}u(x,t)v_1. \end{aligned} \tag{59}$$

Let us reduce the resulting system to a single equation for the function $v_1(x,\xi) = v(x,\xi)$

$$v_{xxxx} - (u + \overline{u})v_{xx} - 2u_x v_x + (\overline{u}u - u_{xx})v = \xi^4 v. \tag{60}$$

Let us show that the complexification of the Korteweg–de Vries Equation (21) has an infinite sequence of conservation laws. Let us construct integrals of motion of the form

$$I_n = \int_{-\infty}^{\infty} \sigma_n dx, \tag{61}$$

where σ_n are polynomials from the function $u(x,t)$ and their derivatives with respect to x associated with problem (60); we will use the method proposed by Zakharov and Shabat [19]. Integrals of the form (61) are called local polynomial integrals.

Suppose

$$\sigma(\xi,x) = \frac{d}{dx}\log\left[1+\frac{q_1(x)}{2i\xi}+\frac{q_2(x)}{(2i\xi)^2}+\ldots+\frac{q_n(x)}{(2i\xi)^n}+\frac{q_{n+1}(\xi,x)}{(2i\xi)^{n+1}}\right], \tag{62}$$

then, the solution $v(\xi,x)$ of Equation (60) will take the following form

$$v(\xi,x) = \exp\left(i\xi x + \int_0^x \sigma(\xi,x)dx\right). \tag{63}$$

Theorem 3. *If $u(x,t)$ changes according to Equation (21) and turns to zero together with its derivatives at $x \to \pm\infty$, then*

(1). The solution of (60) can be represented in the form (63), where

$$\sigma(\xi,x) = \sum_{k=1}^{\infty}\frac{\sigma_k(x)}{(2i\xi)^k}. \tag{64}$$

(2). The functions $\sigma_n(x)$ are defined recurrently by the formulas

$$\begin{aligned}\sigma_{n+3} &= 4\overline{u}_x\sigma_n - 3\sigma_{(n+2)x} - 4\sigma_{(n+1)xx} - 2\sigma_{nxxx} + 2(u+\overline{u})\left(\sum_{k=1}^{n-1}\sigma_k\sigma_{n-k}+\sigma_{nx}+\sigma_{n+1}\right)\\ &-\sum_{k=1}^{n-1}\left[4\sigma_k\sum_{j=1}^{n-k}\sigma_j\sigma_{n+1-k-j}+6(\sigma_{kx}\sigma_{n-k})_x+2\sigma_{kxx}\sigma_{n-k}\right]-3\sum_{k=1}^{n+1}\sigma_k\sigma_{n+2-k}-12\sum_{k=1}^{n}\sigma_{kx}\sigma_{n+1-k}\\ &-2\sum_{k=1}^{n-3}\sigma_k\sum_{j=1}^{n-k-2}\sigma_j\sum_{m=1}^{n-k-j-1}\sigma_m\sigma_{n-k-j-m}-12\sum_{k=1}^{n-2}\sigma_{kx}\sum_{j=1}^{n-k-1}\sigma_j\sigma_{n-k-j}, n=3,4,\ldots\end{aligned} \tag{65}$$

$$\sigma_1 = \frac{1}{2}(u+\overline{u}),\ \sigma_2 = \frac{1}{2}(u-3\overline{u})_x,\ \sigma_3 = -\frac{1}{2}(3u-5\overline{u})_{xx} - 2|u|^2 + \frac{1}{4}(u+\overline{u})^2, \tag{66}$$

$$\sigma_4 = \frac{1}{2}(3u-5\overline{u})_{xxx} - ((u-\overline{u})^2+2|u|^2)_x, \tag{67}$$

$$\begin{aligned}\sigma_5 &= \tfrac{1}{2}(\overline{u}+u)_{xxxx} + 2(u+\overline{u})|u|^2 - \tfrac{1}{4}(u+\overline{u})^3 + \left(\tfrac{5}{4}(u+\overline{u})^2 - 10|u|^2\right)_{xx} + u_x\overline{u}_x\\ &-\tfrac{3}{4}(u-\overline{u})_x^2 + [(u-3\overline{u})_x(u-\overline{u})]_x,\end{aligned} \tag{68}$$

from which follows a polynomial dependence on the functions $u(x,t)$, $\overline{u}(x,t)$ and their partial derivatives with respect to the variable x.

Proof of Theorem 3. Let us write the solution in the form (64) and find the derivatives

$$\begin{aligned}v_x(\xi,x) &= (i\xi+\sigma(\xi,x))v(\xi,x),\ v_{xx}(\xi,x) = \left((i\xi+\sigma(\xi,x))^2+\sigma_x(\xi,x)\right)v(\xi,x),\\ v_{xxx}(\xi,x) &= \left((i\xi+\sigma(\xi,x))^3+3\sigma_x(\xi,x)(i\xi+\sigma(\xi,x))+\sigma_{xx}(\xi,x)\right)v(\xi,x),\\ v_{xxxx}(\xi,x) &= \left((i\xi+\sigma)^4+6\sigma_x(i\xi+\sigma)^2+4\sigma_{xx}(i\xi+\sigma)+3\sigma_x^2+\sigma_{xxx}\right)v(\xi,x).\end{aligned} \tag{69}$$

Substituting (69) into Equation (60), we obtain a new equation for the function $\sigma(\xi, x)$:

$$
\begin{aligned}
&(i\xi+\sigma)^4 + 6\sigma_x(i\xi+\sigma)^2 + 4\sigma_{xx}(i\xi+\sigma) + 3\sigma_x^2 + \sigma_{xxx} - (u+\overline{u})\Big((i\xi+\sigma)^2 + \sigma_x\Big) \\
&-2u_x(i\xi+\sigma) + |u|^2 - u_{xx} = (i\xi)^4.
\end{aligned}
$$

After removal of brackets and collecting terms, write

$$
\begin{aligned}
&4(i\xi)^3\sigma + 6(i\xi)^2\sigma^2 + 4i\xi\sigma^3 + \sigma^4 + 6\sigma_x\big((i\xi)^2 + 2i\xi\sigma + \sigma^2\big) + 4\sigma_{xx}(i\xi+\sigma) + 3\sigma_x^2 + \sigma_{xxx} \\
&-(u+\overline{u})\Big((i\xi)^2 + 2i\xi\sigma + \sigma^2 + \sigma_x\Big) - 2u_x(i\xi+\sigma) + |u|^2 - u_{xx} = 0.
\end{aligned}
$$

Let us group the terms by decreasing powers of the product $2i\xi$:

$$
\begin{aligned}
&(2i\xi)^3\tfrac{1}{2}\sigma + (2i\xi)^2\tfrac{1}{4}\left[6\sigma^2 + 6\sigma_x - (u+\overline{u})\right] + 2i\xi[2\sigma^3 + 6\sigma\sigma_x + 2\sigma_{xx} - (u+\overline{u})\sigma - u_x] \\
&+\sigma^4 + 6\sigma_x\sigma^2 + 4\sigma_{xx}\sigma + 3\sigma_x^2 + \sigma_{xxx} - (u+\overline{u})(\sigma^2+\sigma_x) - 2u_x\sigma + |u|^2 - u_{xx} = 0.
\end{aligned} \tag{70}
$$

Let us introduce the following notations in Formula (62) for brevity

$$
\begin{gathered}
T_n(\xi,x) = 1 + \tfrac{q_1(x)}{2i\xi} + \tfrac{q_2(x)}{(2i\xi)^2} + \ldots + \tfrac{q_n(x)}{(2i\xi)^n} + \tfrac{q_n(x)}{(2i\xi)^n}, \\
E_n(\xi,x) = T_n(\xi,x) + \tfrac{q_{n+1}(\xi,x)}{(2i\xi)^{n+1}}.
\end{gathered} \tag{71}
$$

Then, the function $\sigma(\xi, x)$ after performing the operations in Formula (62) will take the form

$$
\sigma(\xi,x) = \frac{T_n'(x,\xi)}{T_n(x,\xi)} + \frac{q_{n+1}'(\xi,x)T_n(x,\xi) - q_{n+1}(\xi,x)T_n'(x,\xi)}{(2i\xi)^{n+1}T_n(x,\xi)E_n(\xi,x)}.
$$

Let us represent the function $\sigma(\xi, x)$ in the neighborhood of an infinitely distant point of the ξ-plane as a series expansion in powers of $(2i\xi)^{-1}$:

$$
\sigma(\xi,x) = \sum_{k=1}^{\infty}\frac{\sigma_k(x)}{(2i\xi)^k},
$$

where

$$
\frac{T_n'(x,\xi)}{T_n(x,\xi)} = \sum_{k=1}^{n}\frac{\sigma_k(x)}{(2i\xi)^k}, \quad \sum_{k=n+1}^{\infty}\frac{\sigma_k(x)}{(2i\xi)^k} = \frac{q_{n+1}'(\xi,x)T_n(x,\xi) - q_{n+1}(\xi,x)T_n'(x,\xi)}{(2i\xi)^{n+1}T_n(x,\xi)E_n(\xi,x)}.
$$

Substituting the right-hand side of Equality (71) into the equation on $\sigma(\xi, x)$ (69) and taking into account the properties of the function: $\sigma(\xi, x) = o(1)$, $\sigma'(\xi, x) = o(\xi)$, derived from the representation (71), let us write out terms with different powers of $2i\xi$:

$$
\begin{gathered}
(2i\xi)^2\colon \sigma_1 - \tfrac{1}{2}(u+\overline{u}) = 0, \\
(2i\xi)^1\colon\ \sigma_2 + 3\sigma_{1x} - 2u_x = 0, \\
(2i\xi)^0\colon\ \sigma_3 + 3\sigma_1^2 + 3\sigma_{2x} + 4\sigma_{1xx} - 2(u+\overline{u})\sigma_1 + 2|u|^2 - 2u_{xx} = 0, \\
(2i\xi)^{-1}:\ \sigma_4 + 6\sigma_1\sigma_2 + 3\sigma_{3x} + 12\sigma_{1x}\sigma_1 + 4\sigma_{2xx} - 2(u+\overline{u})\sigma_2 + 2\sigma_{1xxx} - 2(u+\overline{u})\sigma_{1x} - 4u_x\sigma_1 = 0, \\
(2i\xi)^{-2}\colon\ \sigma_5 + 6\sigma_1\sigma_3 + 3\sigma_2^2 + 3\sigma_{4x} + 4\sigma_1^3 + 12(\sigma_1\sigma_2)_x + 4\sigma_{3xx} - 2(u+\overline{u})\sigma_3 + 8\sigma_{1xx}\sigma_1 + 6\sigma_{1x}^2 \\
+2\sigma_{2xxx} - 2(u+\overline{u})(\sigma_1^2 + \sigma_{2x}) - 4u_x\sigma_2 = 0, \\
\ldots
\end{gathered}
$$

$$(2i\xi)^{-n}:\ \sigma_{n+3}+4\sigma_{(n+1)xx}+2\sigma_{nxxx}+3\sigma_{(n+2)x}-4u_x\sigma_n-2(u+\overline{u})\left(\sum_{k=1}^{n-1}\sigma_k\sigma_{n-k}+\sigma_{nx}+\sigma_{n+1}\right)$$
$$+\sum_{k=1}^{n-1}\left[4\sigma_k\sum_{j=1}^{n-k}\sigma_j\sigma_{n+1-k-j}+6(\sigma_{kx}\sigma_{n-k})_x+2\sigma_{kxx}\sigma_{n-k}\right]+3\sum_{k=1}^{n+1}\sigma_k\sigma_{n+2-k}+12\sum_{k=1}^{n}\sigma_{kx}\sigma_{n+1-k}$$
$$+2\sum_{k=1}^{n-3}\sigma_k\sum_{j=1}^{n-k-2}\sigma_j\sum_{m=1}^{n-k-j-1}\sigma_m\sigma_{n-k-j-m}+12\sum_{k=1}^{n-2}\sigma_{kx}\sum_{j=1}^{n-k-1}\sigma_j\sigma_{n-k-j}=0.$$

The first five equalities contain the incomplete set of terms included in the general equation with power $(2i\xi)^{-n}$, from which we find $\sigma_1,\sigma_2,\ldots,\sigma_5$ (66), (67), and (68). The remaining functions $\sigma_n(x), n=6,7,\ldots$ are determined recurrently via Formula (65).

The values of $\sigma_n(x)$ contain no integration operations and all $\sigma_n(x)$ are polynomially expressed in terms of the functions $u(x,t)$, $\overline{u}(x,t)$ and their partial derivatives with respect to the variable x. □

Note the interesting properties of the found polynomial functions.

Corollary 1.

1. *All $\sigma_n(x), n=1,2,\ldots$ contain a linear part in the form of higher derivatives of order n–1 from $u(x,t)$ and $\overline{u}(x,t)$; in the recurrence formula it is created by terms $-3\sigma_{(n+2)x}-4\sigma_{(n+1)xx}-2\sigma_{nxxx}$.*
2. *In $\sigma_n(x)$, there are no linear terms with derivatives of order lower than n–1.*
3. *All other terms (except for those highlighted in item (1) of the recurrence Formula (65)) give non-linear terms.*
4. *If we give the functions $u(x,t)$ and $\overline{u}(x,t)$ a weight coefficient of 1, and the derivatives f_{x^k} of the k-th order with respect to the variable x a weight coefficient of $m+\frac{1}{2}k$, where m is the weight coefficient of f (when functions and their derivatives are multiplied their weights are added), then all the terms of the polynomial $\sigma_n(x), n=1,\ 2,\ \ldots$ have the same weight: $\frac{n+1}{2}$.*

Proof of Corollary 1. For the proof, we will use the method of mathematical induction. As the general form of the recurrence Formula (60) shows, the higher terms of the expansion depend on the lower ones, and we need to know at least the first three terms of the expansion (65) in order to use this construction, so in the first step of the method, we perform a formal check for σ_1, σ_2, σ_3.

(I) So, for $\sigma_1=\frac{1}{2}(u+\overline{u})$, obviously, each term $u(x,t)$ and $\overline{u}(x,t)$ corresponds to weight 1; for $\sigma_2=\frac{1}{2}(u-3\overline{u})_x$, both terms represent a first-order derivative and, therefore, the weight of these terms will coincide and be $\frac{3}{2}$; $\sigma_3=-\frac{1}{2}(3u-5\overline{u})_{xx}-2|u|^2+\frac{1}{4}(u+\overline{u})^2$: the first pair as a second-order derivative has weight $1+\frac{1}{2}2=2$, in the terms representing the product $|u|^2=u\overline{u},u^2,\overline{u}^2$ we add the weights and have total weight 2.

(II) Assume that the hypothesis is true for all the lowest σ_n up to and including σ_{n+2}. Let us show that all terms σ_{n+3} in the recurrent Formula (64) have the same weight $\frac{n+3+1}{2}$. Let us find the weight of each term:

$$u_x\sigma_n\ \sim\ \frac{3}{2}+\frac{n+1}{2}=\frac{n+4}{2};\qquad u\sigma_k\sigma_{n-k},\overline{u}\sigma_k\sigma_{n-k}\ \sim\ 1+\frac{k+1}{2}+\frac{n-k+1}{2}=\frac{n+4}{2};$$

$$\sigma_{(n+1)xx}\ \sim\ \frac{1}{2}\cdot 2+\frac{n+2}{2}=\frac{n+4}{2};\qquad u\sigma_{n+1},\overline{u}\sigma_{n+1}\ \sim\ 1+\frac{n+2}{2}=\frac{n+4}{2};$$

$$\sigma_{(n+2)x} \sim \frac{1}{2} + \frac{n+3}{2} = \frac{n+4}{2}; \qquad \sigma_k\sigma_j\sigma_{n+1-k-j} \sim \frac{k+1}{2} + \frac{j+1}{2} + \frac{n-k-j+2}{2} = \frac{n+4}{2};$$

$$\sigma_{nxxx} \sim \frac{1}{2}\cdot 3 + \frac{n+1}{2} = \frac{n+4}{2}; \qquad (\sigma_{kx}\sigma_{n-k})_x, \sigma_{kxx}\sigma_{n-k} \sim \frac{k+1}{2} + \frac{1}{2}\cdot 2 + \frac{n-k+1}{2} = \frac{n+4}{2};$$

$$u\sigma_{nx}, \overline{u}\sigma_{nx} \sim 1 + \frac{1}{2} + \frac{n+1}{2} = \frac{n+4}{2}; \qquad \sigma_{kx}\sigma_{n+1-k} \sim \frac{k+1}{2} + \frac{1}{2} + \frac{n-k+2}{2} = \frac{n+4}{2};$$

$$\sigma_k\sigma_{n+2-k} \sim \frac{k+1}{2} + \frac{n-k+3}{2} = \frac{n+4}{2}; \qquad \sigma_{kx}\sigma_j\sigma_{n-k-j} \sim \frac{k+1}{2} + \frac{1}{2} + \frac{j+1}{2} + \frac{n-k-j+1}{2} = \frac{n+4}{2};$$

$$\sigma_k\sigma_j\sigma_m\sigma_{n-k-j-m} \sim \frac{k+1}{2} + \frac{j+1}{2} + \frac{m+1}{2} + \frac{n-k-j-m+1}{2} = \frac{n+4}{2}.$$

All terms have an assumed weight $\frac{n+4}{2}$. The property is proved. □

Note that in Formulas (65)–(68), there is no explicit dependence on x, t. Obviously, all the quantities σ_n are linearly independent (none of them can be expressed in terms of the others), so the quantities σ_n are each conserved separately. The polynomials σ_2, σ_4 contain full derivatives; they appear at even powers n and are conserved trivially. Dropping some full derivatives in (65), we obtain that a nontrivial set of conserved quantities for Equation (60) will look like

$$\begin{aligned} I_{n+3} = \int_{-\infty}^{\infty} \Bigg\{ & 4u_x\sigma_n + 2(u+\overline{u})\left(\sum_{k=1}^{n-1}\sigma_k\sigma_{n-k} + \sigma_{nx} + \sigma_{n+1}\right) - 2\sum_{k=1}^{n-3}\sigma_k\sum_{j=1}^{n-k-2}\sigma_j\sum_{m=1}^{n-k-j-1}\sigma_m\sigma_{n-k-j-m} \\ & -2\sum_{k=1}^{n-1}\left[2\sigma_k\sum_{j=1}^{n-k}\sigma_j\sigma_{n+1-k-j} + \sigma_{kxx}\sigma_{n-k}\right] - 3\sum_{k=1}^{n+1}\sigma_k\sigma_{n+2-k} - 12\sum_{k=1}^{n}\sigma_{kx}\sigma_{n+1-k} \\ & -12\sum_{k=1}^{n-2}\sigma_{kx}\sum_{j=1}^{n-k-1}\sigma_j\sigma_{n-k-j}\Bigg\}dx, \end{aligned} \tag{72}$$

where the first laws have the form:

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty}(u+\overline{u})dx, \quad I_3 = \int_{-\infty}^{\infty}\left\{\frac{1}{4}(u+\overline{u})^2 - 2|u|^2\right\}dx, \tag{73}$$

$$I_5 = \int_{-\infty}^{\infty}\left\{2(u+\overline{u})|u|^2 - \frac{1}{4}(u+\overline{u})^3 + u_x\overline{u}_x - \frac{3}{4}(u-\overline{u})_x^2\right\}dx. \tag{74}$$

This proves the following corollary.

Corollary 2. *The nonlinear partial differential Equation (21) has a countable set of first integrals (72), where the differential polynomials $\sigma_n(u,\overline{u})$are defined by the recurrent Formula (65). The first integrals of motion are defined by (73) and (74).*

3.2. Conservation Laws for pKdV System

Let us construct a countable number of conserved quantities for the system (56). Stages of the algorithm for construction of conservation laws and their justification fully correspond to the previous section, so the presentation of the material is reduced to a demonstration of only qualitatively different moments of their construction.

Let us write the equation for eigenvalues (1) with operator $\widehat{P}$ (57) and $V(x,t) = (v_1, v_2, v_3, v_4)^T$ as a system

$$\begin{aligned} &v_{1x} = v_4, \quad v_{3x} = -wv_1 + \tfrac{1}{\mu}\left[u - (i\xi)^2\right]v_2, \\ &v_{2x} = \mu v_3, \quad v_{4x} = wv_2 + \left[(i\xi)^2 + u(x,t)\right]v_1. \end{aligned} \tag{75}$$

Let us reduce this system to a single equation

$$\left(w^{-1}\left[v_{xx} - \left(i\xi\right)^2 v - uv\right]\right)_{xx} = -\mu wv + w^{-1}\left(u - \left(i\xi\right)^2\right)\left[v_{xx} - \left(i\xi\right)^2 v - uv\right]. \tag{76}$$

Theorem 4. *If the functions $u(x,t)$ and $w(x,t)$ change according to the system of Equation (56) and tend to zero together with their derivatives at $x \to \pm\infty$, then*

(1). The solution of (76) can be represented as (64).

(2). The functions $\sigma_n(x)$ are determined recurrently by the formulas

$$\begin{aligned} &\sigma_{n+3} = 2w^{-1}w_x\left[\sigma_{n+2} + 3\sigma_{(n-1)x} + 2\sigma_{nxx}\right] - 3\sigma_{(n+2)x} - 4\sum_{k=1}^{n-1}\sigma_k\sum_{j=1}^{n-k}\sigma_j\sigma_{n+1-k-j} - 4\sigma_{(n-1)xx} \\ &-4\left\{w^{-1}w_x u - u_x\right\}\sigma_n + 2\left\{w^{-1}w_{xx} - 2w^{-2}w_x^2 + 2u\right\}\left(\sigma_{nx} + \sum_{k=1}^{n-1}\sigma_k\sigma_{n-k} + \sigma_{n-1}\right) - 2\sigma_{nxxx} \\ &+4\sum_{k=1}^{n-2}\sigma_k\sum_{j=1}^{n-k}\sigma_{n-k-j}\left[3\sigma_{jx} - w^{-1}w_x\sigma_j\right] - \sum_{k=1}^{n-1}\left[8\sigma_{kxx}\sigma_{n-k} + 6\sigma_{kx}\sigma_{(n-k)x} - 12w^{-1}w_x\sigma_{kx}\sigma_{n-k}\right] \\ &-6\sum_{k=1}^{n+1}\sigma_k\sigma_{n+2-k} - 6\sum_{k=1}^{n}\sigma_{n+1-k}\left[2\sigma_{kx} - w^{-1}w_x\sigma_k\right] - 2\sum_{k=1}^{n-3}\sigma_k\sum_{j=1}^{n-k-2}\sigma_j\sum_{m=1}^{n-k-j-1}\sigma_m\sigma_{n-k-j-m} = 0, \end{aligned} \tag{77}$$

$$\sigma_1 = u,\ \sigma_2 = -u_x,\ \sigma_3 = u_{xx} - u^2 - 2\mu w^2,\ \sigma_4 = \left[4\mu w^2 + 2u^2\right]_x, \tag{78}$$

$$\sigma_5 = u_x^2 - \left[6\mu w^2 + 3u^2 + 2u_{xx}\right]_{xx} + 4\mu w_x^2 + 2w^{-1}w_x u_{xxx} + 2u\left[u^2 + 2\mu w^2\right]. \tag{79}$$

Proof of Theorem 4. Let us reproduce the proof in fragmentary form. Let us use the form (63) of writing the solution and, substituting it into Equation (76), we obtain the equation on $\sigma(\xi,x)$:

$$\begin{aligned} &4(i\xi)^3w^2\sigma - 2(i\xi)^2w^2u + (i\xi)^2\left[6w^2\sigma_x - 4ww_x\sigma\right] + 2i\xi\left\{ww_xu - w^2u_x\right\} + 6(i\xi)^2w^2\sigma^2 \\ &+i\xi\left[4w^2\sigma_{xx} - 6ww_x\sigma_x + 2\sigma\left\{2w_x^2 - ww_{xx} - 2w^2u\right\}\right] + 2\left\{ww_xu - w^2u_x\right\}\sigma + w^2\sigma_{xxx} \\ &-2ww_x\sigma_{xx} + \left\{2w_x^2 - ww_{xx} - 2w^2u\right\}\sigma_x + (i\xi)\left[12w^2\sigma_x\sigma - 6ww_x\sigma^2\right] + 4w^2\sigma_{xx}\sigma + 3w^2\sigma_x^2 \\ &+\left\{2w_x^2 - ww_{xx} - 2w^2u\right\}\sigma^2 - 6ww_x\sigma\sigma_x + 4(i\xi)w^2\sigma^3 - 2ww_x\sigma^3 + 6w^2\sigma_x\sigma^2 + w^2\sigma^4 \\ &+\mu w^4 + 2ww_xu_x + u^2w^2 - w^2u_{xx} + \left[ww_{xx} - 2w_x^2\right]u = 0. \end{aligned} \tag{80}$$

Replacing $\sigma(\xi,x)$ with series (64) allows us to group terms with different powers of $2i\xi$ in equality (80):

$$\begin{aligned} &(2i\xi)^2:\ \sigma_1 - u = 0; \\ &(2i\xi)^1:\ \ w^2\sigma_2 + 3w^2\sigma_{1x} - 2ww_x\sigma_1 + 2ww_xu - 2w^2u_x = 0; \\ &(2i\xi)^0:\ \ w^2\sigma_3 + 3w^2\sigma_{2x} - 2ww_x\sigma_2 + 3w^2\sigma_1^2 + 4w^2\sigma_{1xx} - 6ww_x\sigma_{1x} + 2\mu w^4 + 4ww_xu_x \\ &+2\sigma_1\left\{2w_x^2 - ww_{xx} - 2w^2u\right\} + 2u^2w^2 - 2w^2u_{xx} + 2\left[ww_{xx} - 2w_x^2\right]u = 0; \\ &(2i\xi)^{-1}:\ \ w^2\sigma_4 + 3w^2\sigma_{3x} - 2ww_x\sigma_3 + 6w^2\sigma_1\sigma_2 + 4w^2\sigma_{2xx} - 6ww_x\sigma_{2x} - 6ww_x\sigma_1^2 + w^2\sigma_{1xxx} \\ &+2\sigma_2\left\{2w_x^2 - ww_{xx} - 2w^2u\right\} + 4\left\{ww_xu - w^2u_x\right\}\sigma_1 - 4ww_x\sigma_{1xx} + 12w^2\sigma_{1x}\sigma_1 \\ &+2\left\{2w_x^2 - ww_{xx} - 2w^2u\right\}\sigma_{1x} = 0; \\ &\qquad\qquad\ldots \end{aligned}$$

$$(2i\xi)^{-n}: \ \frac{1}{2}w^2\sigma_{n+3} + \frac{1}{2}\left[3w^2\sigma_{(n+2)x} - 2ww_x\sigma_{n+2}\right] + w^2\left(\sum_{k=1}^{n+1} 3\sigma_k\sigma_{n+2-k} + \sigma_{nxxx} + 2\sigma_{(n-1)xx}\right)$$
$$-ww_x\left[3\sigma_{(n-1)x} + 2\sigma_{nxx}\right] + 2\{ww_xu - w^2u_x\}\sigma_n + 3\sum_{k=1}^{n}\sigma_{n+1-k}[2w^2\sigma_{kx} - ww_x\sigma_k]$$
$$-\{2w_x^2 - ww_{xx} - 2w^2u\}\left(\sigma_{nx} + \sigma_{n-1} + \sum_{k=1}^{n-1}\sigma_k\sigma_{n-k}\right) + 2w^2\sum_{k=1}^{n-1}\sigma_k\sum_{j=1}^{n-k}\sigma_j\sigma_{n+1-k-j}$$
$$+\sum_{k=1}^{n-1}\left[4w^2\sigma_{kxx}\sigma_{n-k} + 3w^2\sigma_{kx}\sigma_{(n-k)x} - 6ww_x\sigma_{kx}\sigma_{n-k}\right]$$
$$+\sum_{k=1}^{n-2}\sigma_k\sum_{j=1}^{n-k}\sigma_{n-k-j}[6w^2\sigma_{jx} - 2ww_x\sigma_j] + w^2\sum_{k=1}^{n-3}\sigma_k\sum_{j=1}^{n-k-2}\sigma_j\sum_{m=1}^{n-k-j-1}\sigma_m\sigma_{n-k-j-m} = 0.$$

This system allows us to find the functions $\sigma_n(x)$ recurrently by sequentially applying Equations (78), (79), (77). □

Formulas (77)–(79) are linearly independent (none of them can be expressed through the others). The polynomials σ_2, σ_4,... contain only the full derivatives; they appear at odd powers of $(2i\xi)^k$ of the above system and are trivially conserved. A non-trivial set of conserved quantities for Equation (56) would look like

$$I_1 = \int_{-\infty}^{\infty} u dx, \ I_3 = -\int_{-\infty}^{\infty}\left[u^2 + 2\mu w^2\right]dx, \tag{81}$$

$$I_5 = \int_{-\infty}^{\infty}(u_x^2 + 4\mu w_x^2 + 2w^{-1}w_xu_{xxx} + 2u\left[u^2 + 2\mu w^2\right])dx, \tag{82}$$

$$\begin{aligned} I_{2n+1} = \int_{-\infty}^{\infty}\Bigg(&2(\log w)_x\left[\sigma_{2n} + 3\sigma_{(2n-3)x} + 2\sigma_{(2n-2)xx} - 2u\sigma_{2n-2}\right] - 4\sum_{k=1}^{2n-3}\sigma_k\sum_{j=1}^{2n-2-k}\sigma_j\sigma_{2n-1-k-j} \\ &+4u\left(\sum_{k=1}^{2n-3}\sigma_k\sigma_{2n-2-k} + \sigma_{2n-3}\right) - 2w(w^{-1})_{xx}\left(\sigma_{(2n-2)x} + \sum_{k=1}^{2n-3}\sigma_k\sigma_{2n-2-k} + \sigma_{2n-3}\right) \\ &-4(\log w)_x\sum_{k=1}^{2n-4}\sigma_k\sum_{j=1}^{2n-2-k}\sigma_j\sigma_{2n-2-k-j} - \sum_{k=1}^{2n-3}([2\sigma_{kxx} - 12(\log w)_x\sigma_{kx}]\sigma_{2n-2-k}) - 6\sum_{k=1}^{2n-1}\sigma_k\sigma_{2n-k} \\ &+6(\log w)_x\sum_{k=1}^{2n-2}\sigma_k\sigma_{2n-1-k} - 2\sum_{k=1}^{2n-5}\sigma_k\sum_{j=1}^{2n-k-4}\sigma_j\sum_{m=1}^{2n-k-j-3}\sigma_m\sigma_{2n-2-k-j-m}\Bigg)dx. \end{aligned} \tag{83}$$

This proves the following corollary.

Corollary 3. *The nonlinear system of partial differential Equation (56) describing the KdV perturbation has a countable set of first integrals (81)–(83), where the differential polynomials $\sigma_n(u,w)$ are defined by the recurrence Formula (77).*

The constructed first integrals of the system (56) define the following physical characteristics of the densities of conserved quantities:

I_1 (81)—conservation of mass of the wave process (fully coincides with the first law for the classical soliton in the unperturbed KdV equation);
I_3 (81)—shows the conservation of momentum of interacting functions;
I_5 (82)—conservation of energies $(u_x^2 + 2u^3) + 4\mu(w_x^2 + uw^2)$ (similar terms for the KdV equation are $\beta u_x^2 + \frac{1}{3}u^3)$ with the interaction element of $u(x,t)$ and $w(x,t)$ or their functional connection $2w^{-1}w_xu_{xxx}$.

Obviously, if there is no perturbation in the system (56) $w(x,t) = 0$, the solution is a classical soliton:

$$u(x,t) = \frac{-2k^2}{\cos\mathrm{h}^2(kx - 4k^3t)}. \tag{84}$$

The plot of the function $u(x,t)$ corresponding to Formula (84) is shown in Figure 1.

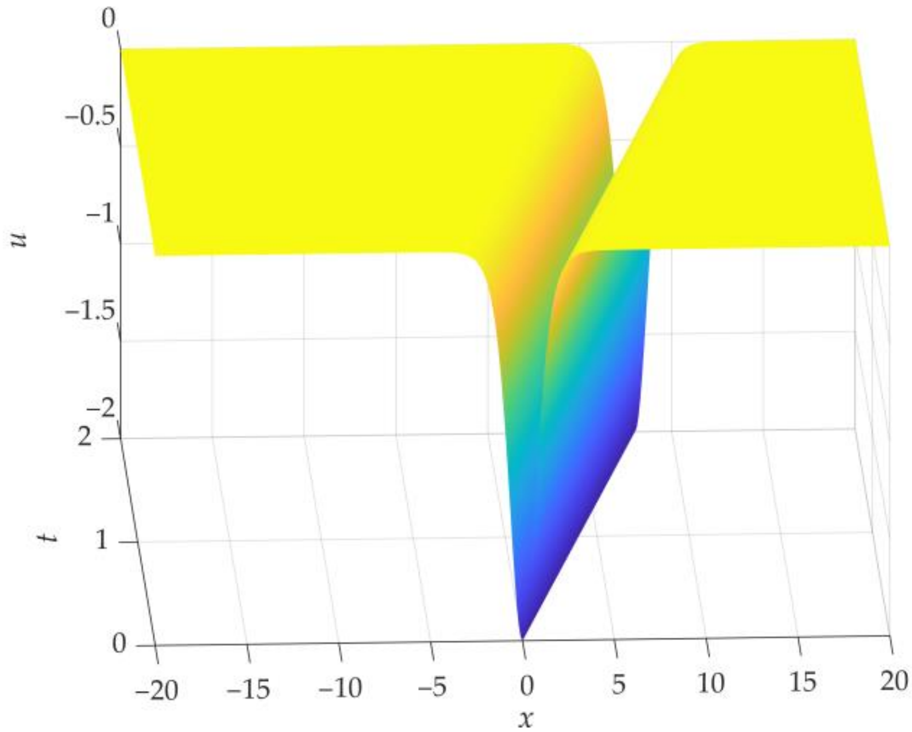

Figure 1. Classical soliton $u(x,t)$ in the case of zero perturbation according to Formula (84).

The question arises as to whether it is possible to preserve the soliton solution in the perturbed system. This is possible if the perturbation function does not create new terms in the first equality of system (56) other than those created by the soliton itself, i.e., those proportional to $\cosh^{-3}(kx+\Omega t)\sinh(kx+\Omega t),\ \cosh^{-5}(kx+\Omega t)\sinh(kx+\Omega t)$.

Corollary 4. *The system of Equation (56) at $\mu > 0$ describes the interaction of the soliton and the kink*

$$u(x,t) = \frac{-2k^2}{\cosh^2(kx+8k^3t)}, \qquad w(x,t) = \pm\frac{2k^2}{\sqrt{\mu}}\tanh\left(kx+8k^3t\right), \tag{85}$$

and at $\mu < 0$

$$u(x,t) = \frac{-2k^2}{\cosh^2(kx+2k^3t)}, \qquad w(x,t) = \pm\frac{\sqrt{2}k^2}{\sqrt{|\mu|}}\operatorname{sech}\left(kx+2k^3t\right). \tag{86}$$

The proof is obtained via simple substitution of these functions.

From Formulas (85) and (86), it follows that when interacting with the kink, the soliton changes the direction of motion and doubles its speed compared to the unperturbed system (84), and when in contact with the hyperbolic secant, it changes the direction of motion and slows down by a factor of two. An illustration of the above is shown in Figure 2, which shows the plots of the corresponding functions according to Formulas (85) and (86).

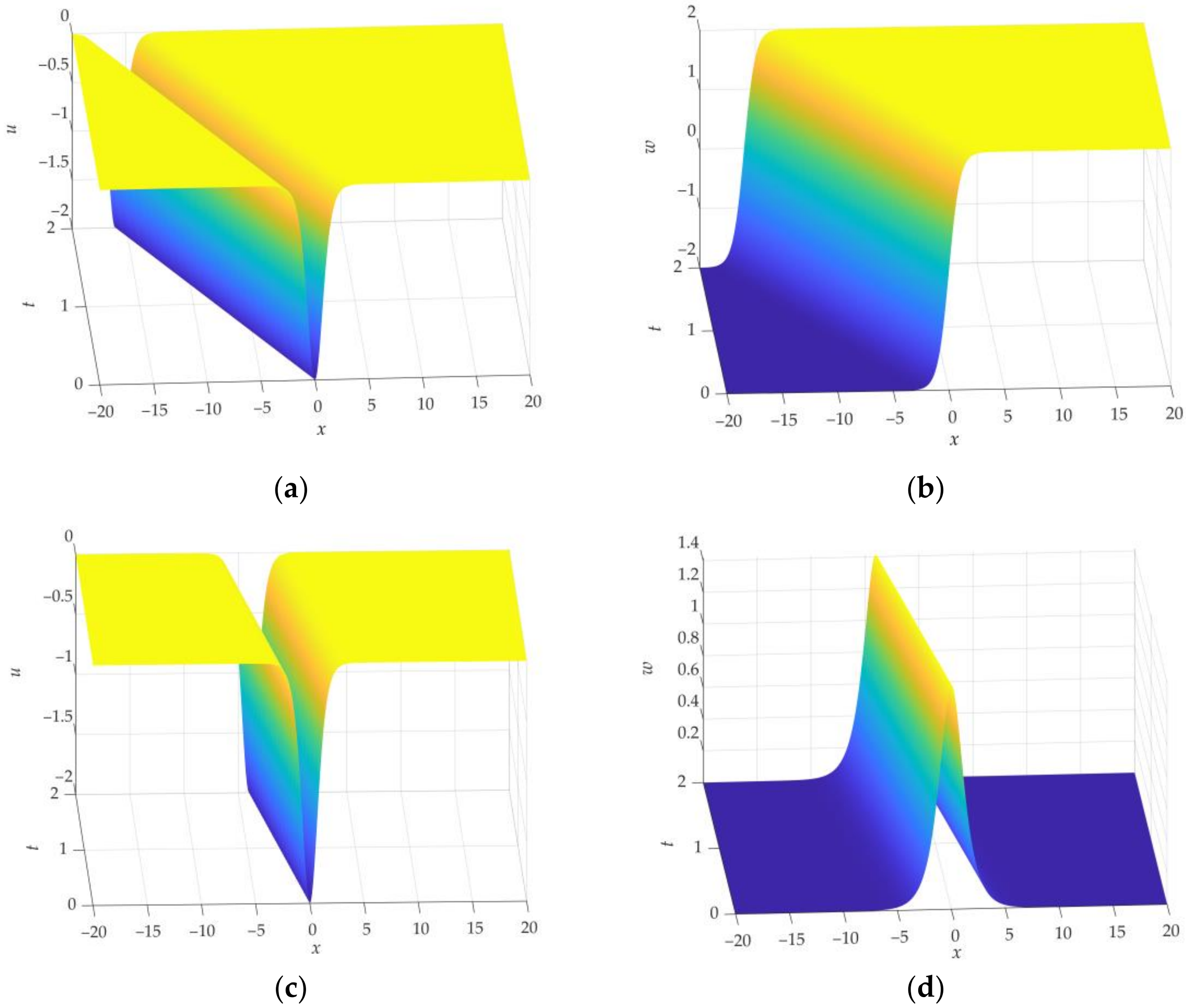

Figure 2. Soliton solution in the perturbed system. (**a**,**b**) Functions $u(x,t)$ and $w(x,t)$ according to Equation (85) at $k=1$, $\mu=1$ and using "+" in corresponding equation. (**c**,**d**) Functions $u(x,t)$ and $w(x,t)$ according to Equation (86) at $k=1$, $\mu=-1$ and using "+" in corresponding equation.

Note that similar interactions were considered in [20]. The dynamics of the kinks of the modified Sine-Gordon equation in a model with localized spatial modulation of the periodic potential were studied numerically. Two qualitatively different scenarios of the dynamic behavior of the kink were demonstrated.

We also conducted numerical studies of the possibility of preserving the soliton solution in the perturbed system. A numerical solution was carried out on the basis of finite-difference approximation of the investigated partial differential equations [21]. A numerical study of the system of Equation (56) on the emergence of stable solutions of the kink and soliton, for initial conditions at $t=0$, corresponding to the form of functions (85) and (86) was performed. The numerical results obtained in this case give a picture of a stable soliton solution and coincide with the analytical solutions shown in Figure 2.

Figure 3 shows numerical solutions of system (56) under other different initial conditions that differ completely or partially from the form of functions (85) and (86). As can be seen from the presented numerical results, a stable soliton solution is not formed in the considered situations.

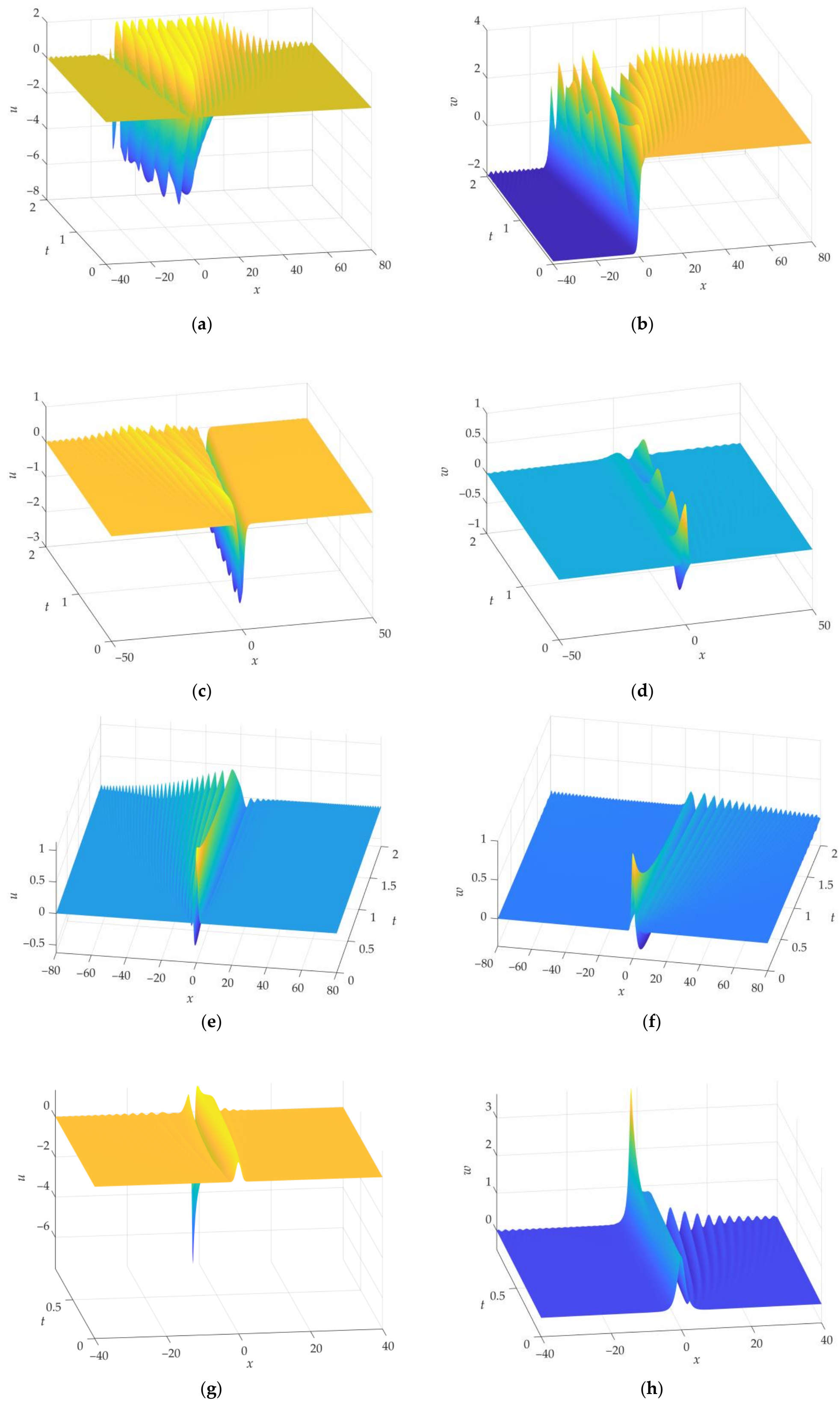

Figure 3. Numerical solution of the system of Equation (56) under different initial conditions. (**a**,**b**) $u(x,0) = \exp(-x^2)$, $w(x,0) = \frac{2}{\sqrt{\mu}}\tanh(x)$ at $\mu = 1$. (**c**,**d**) $u(x,0) = \frac{-2}{\cosh^2(x)}$, $w(x,0) = \exp(-x^2)$ at $\mu = 1$. (**e**,**f**) $u(x,0) = \exp(-x^2)$, $w(x,0) = \exp(-x^2)$ at $\mu = 1$. (**g**,**h**) $u(x,0) = \exp(-x^2)$, $w(x,0) = \sqrt{\frac{2}{|\mu|}}\operatorname{sech}(x)$ at $\mu = -1$.

Thus, numerical simulations confirmed the analytically obtained conclusion that only under the initial conditions corresponding to the analytical solution (85) and (86), the solution splits into a stable soliton and a kink (at $\mu > 0$) and into a soliton and a hyperbolic secant (at $\mu < 0$). In the perturbed state, the propagation velocities in the soliton-kink and soliton-hyperbolic secant pairs coincide but differ from the velocities in the unperturbed state.

4. Conclusions

Using the zero-curvature operator equation with operators represented as polynomials decomposed by the spectral parameter ξ with the Lie algebra of nilpotent 4×4 matrices, perturbations of the KdV equation were constructed, which coincided with the previously obtained cKdV and pKdV equations in [12].

One of the proofs of integrability of nonlinear differential equations is the existence of an infinite number of conserved quantities. The Zakharov–Shabat method allows one, using the equation on eigenvalues, to construct a countable number of conservation laws.

For the cKdV Equation (21), an infinite sequence of polynomial first integrals is found, such that all the terms of the polynomial $\sigma_n(x), n = 1,\ 2,\ \ldots$ have the same weight $\frac{n+1}{2}$.

A countable set of conservation laws was also constructed for the system (56) describing the KdV perturbation. Comparing expressions (81) and (82) with the KdV conservation laws, it is easy to see that the parts of their structure related to the function $u(x,t)$ coincide exactly. This coincidence suggested that system (56) should have a soliton solution. This led to the search for special perturbations in the form of a kink and a hyperbolic secant, which create the conditions for the appearance of a soliton solution of the form (85) and (86), respectively. Numerical simulations confirmed this conclusion.

The system (56) has an applied significance, since it describes the interaction of soliton and kink, soliton and hyperbolic secant, which is widespread in modeling wave phenomena in various real systems (see Refs. [22–24] and the literature therein). Of particular importance in the model is the presence of the parameter μ. By varying this parameter, one can strengthen or weaken the presence of the perturbation $w(x,t)$ and observe the mutual influence of the functions $u(x,t)$ and $w(x,t)$.

Further development of these studies will be the construction of n-soliton solutions.

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