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*Article*

# On Sufficiency Conditions for Some Robust Variational Control Problems

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**Abstract:** We study the sufficient optimality conditions for a class of fractional variational control problems involving data uncertainty in the cost functional. Concretely, by using the parametric technique, we prove the sufficiency of the robust necessary optimality conditions by considering convexity, quasi-convexity, strictly quasi-convexity, and/or monotonic quasi-convexity assumptions of the involved functionals.

**Keywords:** fractional variational control problem; sufficient optimality conditions; robust necessary optimality conditions; robust optimal solution; convexity; quasi-convexity; monotonic quasi-convexity

**MSC:** 26B25; 49J20; 90C32



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## 1. Introduction

To study practical, concrete, or real-life problems coming from economics, decision theory, production inventory, data classification, game theory, or portfolio selection, optimization models and control theory are widely used. Since practical problems are often governed by estimation or measurement, some errors may occur. Most of the time, the presence of various errors may contradict the computational results associated with the original problem. To overcome this issue, the use of a robust approach to represent data, the presence of fuzzy numbers, or the use of interval analysis, has become an important research direction in the last decades.

Optimizing the ratio of two objective or cost functions or functionals, means to study a fractional optimization problem. Thus, Dinkelbach [1] and Jagannathan [2] succeeded to transform it into an equivalent non-fractional optimization problem, by considering a parametric approach. In time, many researchers have considered this technique to study and solve various fractional variational problems. We mention the works of Mititelu [3], Antczak and Pitea [4], Mititelu and Treanţă [5], Antczak [6]. For other ideas on this subject, interested readers are directed to Patel [7], Nahak [8], Manesh et al. [9], Kim and Kim [10–12] and references therein. Noor [13] considered and introduced some new concepts of the biconvex functions involving an arbitrary bifunction and function. More precisely, Noor has shown that the optimality conditions for the general biconvex functions can be characterized by a class of bivariational-like inequalities.

Optimization and variational problems with uncertain data arise when we have inadequate information, old sources, a large volume of data, sample disparity, or some other factors leading to data uncertainty. To investigate these cases, the robust technique is intensively used in studying the optimization problem with data uncertainty. This approach reduces the uncertainty associated with the original problem. Several researchers stated and investigated different optimization or variational problems involving data uncertainty, and they tried to establish novel and efficient results (see Jeyakumar et al. [14], Beck and Tal [15], Baranwal et al. [16], Treanţă [17,18], Preeti et al. [19], Jayswal et al. [20]).

Over time, optimal control problems subject to nonlinear equality and/or inequality-type constraints (governed by ordinary differential equations) have been formulated and studied by many researchers. But, since so many phenomena are subject to laws involving partial differential equations or partial differential inequations (PDEs/PDIs), it is generated the need for a consistent analysis of scalar/vector variational control problems with PDE/PDI or isoperimetric-type constraints and multiple/path-independent curvilinear integral cost functionals. Of course, the multiple/curvilinear integrals in the calculus of variations have been considered and studied so far, but, these multiple/curvilinear integrals were not sufficiently analyzed in the context of robust optimal control models. Because of the increasing complexity of the environment, the initial data often suffer from inaccuracy. Therefore, an adequate uncertainty framework is necessary to formulate the model and new methods have to be adapted or developed to provide optimal or efficient solutions in a certain sense. The current paper is situated around the studies of robust/uncertain optimization problems.

Next, we formulate a fractional variational control problem with mixed constraints and data uncertainty in the cost functional (given by path-independent curvilinear-type integral). Further, we state the robust necessary optimality conditions and prove their sufficiency by using the convexity, quasi-convexity, strictly quasi-convexity, and/or monotonic quasi-convexity assumptions of the involved functionals. Moreover, we introduce and describe the *robust Kuhn-Tucker points* associated with the considered optimization problem. The most important and principal credits of the present paper are the following: (i) we introduce, by using the parametric technique, the notions of robust optimal solution and robust Kuhn-Tucker point for the case of curvilinear integral-type functionals, (ii) we state novel proofs for the main results, and (iii) we build a new framework determined by spaces of functions and by curvilinear integral-type functionals.

We continue the paper as follows. Section 2 states the basic concepts, notations, and assumptions used to formulate the principal results. In Section 3, by considering suitable convexity, quasi-convexity, strictly quasi-convexity, and/or monotonic quasi-convexity hypotheses, we establish robust sufficient optimality conditions for the considered problem. In addition, we describe the notion of robust Kuhn-Tucker point. Finally, in Section 4 we present the conclusions and formulate some future research directions for this paper.

## 2. Auxiliary Tools

In the following, we consider the basic concepts, notations, and assumptions used to formulate the principal results. Thus, we start with the classical finite-dimensional Euclidean spaces $R^m$, $R^n$ and $R^l$, with $t = (t^\eta)$, $\eta = \overline{1,m}$ (that is, $t = (t^1, \cdots, t^m)$), $u = (u^\iota)$, $\iota = \overline{1,n}$, and $v = (v^j)$, $j = \overline{1,l}$ as arbitrary points of $R^m$, $R^n$ and $R^l$, respectively. Let $\mathcal{H} = \mathcal{H}_{t_0,t_1} \subset R^m$ be a hyper-parallelepiped, having the diagonally opposite corners $t_0 = (t_0^\eta)$ and $t_1 = (t_1^\eta)$, $\eta = \overline{1,m}$, and let $\mathcal{C} \subset \mathcal{H}$ be a curve (piecewise differentiable), joining the points $t_0 = (t_0^\eta)$ and $t_1 = (t_1^\eta)$ in $R^m$. Define

$$U = \Big\{ u : \mathcal{H} \mapsto R^n \,|\, u = \text{piecewise smooth function} \Big\},$$

$$V = \Big\{ v : \mathcal{H} \mapsto R^l \,|\, v = \text{piecewise continuous function} \Big\}$$

as the space of piecewise smooth functions (*state* variables), and the space of piecewise continuous functions (*control* variables), respectively, and assume the product space $U \times V$ is endowed with the norm induced by the following inner product

$$\langle (u,v),(b,z) \rangle = \int_{\mathcal{C}} \Big[ u(t) \cdot b(t) + v(t) \cdot z(t) \Big] dt^\pi$$

$$= \int_{\mathcal{C}} \Big[ \sum_{\iota=1}^{n} u^{\iota}(t) b^{\iota}(t) + \sum_{j=1}^{l} v^{j}(t) z^{j}(t) \Big] dt^{\pi}, \quad \forall (u,v),(b,z) \in U \times V.$$

Using the above mathematical notations and elements, by denoting $u_{\eta}(t) = \dfrac{\partial u}{\partial t^{\eta}}(t)$, we formulate the following first-order $PDE\&PDI$-constrained fractional optimization problem, with data uncertainty in the objective functional, as follows

$$(\mathcal{P}) \qquad \min_{(u(\cdot),v(\cdot))} \frac{\displaystyle\int_{\mathcal{C}} \Delta_{\pi}(t,u(t),u_{\eta}(t),v(t),f) dt^{\pi}}{\displaystyle\int_{\mathcal{C}} \Theta_{\pi}(t,u(t),u_{\eta}(t),v(t),g) dt^{\pi}}$$

subject to

$$\begin{aligned}
& A_{\beta}(t,u(t),u_{\eta}(t),v(t)) \le 0, \;\; \beta = \overline{1,q}, \; t \in \mathcal{H}, \\
& B_{\eta}^{\iota}(t,u(t),u_{\eta}(t),v(t)) := \frac{\partial u}{\partial t^{\eta}}(t) - Q_{\eta}^{\iota}(t,u(t),v(t)) = 0, \\
& \iota = \overline{1,n}, \; \eta = \overline{1,m}, \; t \in \mathcal{H}, \\
& u(t_0) = u_0 = \text{given}, \; u(t_1) = u_1 = \text{given},
\end{aligned}$$

where $f$ and $g$ are some uncertainty parameters in the convex compact sets $F \subset R$ and $G \subset R$, respectively, and $\Delta = (\Delta_{\pi}) : \mathcal{H} \times U^2 \times V \times F \mapsto \mathbb{R}^m$, $\Theta = (\Theta_{\pi}) : \mathcal{H} \times U^2 \times V \times G \mapsto \mathbb{R}^m \setminus \{0\}$, $A_{\beta} : \mathcal{H} \times U^2 \times V \mapsto R$, $\beta = \overline{1,q}$, $B_{\eta}^{\iota} : \mathcal{H} \times U^2 \times V \mapsto R$, $\iota = \overline{1,n}$, $\eta = \overline{1,m}$, are assumed to be continuously differentiable functionals.

**Definition 1.** *The above functionals*

$$\int_{\mathcal{C}} \Delta_{\pi}(t,u(t),u_{\eta}(t),v(t),f) dt^{\pi}$$

*and*

$$\int_{\mathcal{C}} \Theta_{\pi}(t,u(t),u_{\eta}(t),v(t),g) dt^{\pi}$$

*are named path-independent if* $D_{\eta}\Delta_{\pi} = D_{\pi}\Delta_{\eta}$ *and* $D_{\eta}\Theta_{\pi} = D_{\pi}\Theta_{\eta}$, *for* $\pi \neq \eta$.

**Assumption 1.** *By considering the above functionals*

$$\int_{\mathcal{C}} \Delta_{\pi}(t,u(t),u_{\eta}(t),v(t),f) dt^{\pi}$$

*and*

$$\int_{\mathcal{C}} \Theta_{\pi}(t,u(t),u_{\eta}(t),v(t),g) dt^{\pi}$$

*are path-independent, the following working hypothesis is assumed:*

$$dL := D_{\eta} \left[ \frac{\partial h_{\pi}}{\partial u_{\eta}} (u - u^0) \right] dt^{\pi}$$

*is a total exact differential, with* $L(t_0) = L(t_1)$, $h \in \{\Delta, \Theta\}$.

The robust counterpart for $(\mathcal{P})$, reducing the possible uncertainties in $(\mathcal{P})$, is given as

$$(\mathcal{RP}) \quad \min_{(u(\cdot),v(\cdot))} \frac{\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(t,u(t),u_\eta(t),v(t),f)dt^\pi}{\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(t,u(t),u_\eta(t),v(t),g)dt^\pi}$$

subject to

$$A_\beta(t,u(t),u_\eta(t),v(t)) \le 0, \ \ \beta = \overline{1,q}, \ t\in\mathcal{H},$$
$$B_\eta^\iota(t,u(t),u_\eta(t),v(t)) = 0, \ \ \iota = \overline{1,n}, \ \eta = \overline{1,m}, \ t\in\mathcal{H},$$
$$u(t_0) = u_0 = \text{given}, \ u(t_1) = u_1 = \text{given},$$

where $\Delta = (\Delta_\pi), \Theta = (\Theta_\pi), A = (A_\beta)$ and $B = (B_\eta^\iota)$ are defined as in $(\mathcal{P})$.

The set of all feasible solutions to $(\mathcal{RP})$, which is the same as the set of all feasible solutions to $(\mathcal{P})$, is defined as

$$\mathcal{D} = \{(u,v)\in U\times V \mid A_\beta(t,u(t),u_\eta(t),v(t)) \le 0, \ B_\eta^\iota(t,u(t),u_\eta(t),v(t)) = 0,$$
$$u(t_0) = u_0 = \text{given}, \ u(t_1) = u_1 = \text{given}, \ t\in\mathcal{H}\}.$$

For $(u,v)\in\mathcal{D}$, we assume that $\Delta \ge 0$ and $\Theta > 0$. Further, by considering the positive real number

$$R^0_{f,g} = \min_{(u(\cdot),v(\cdot))} \frac{\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(t,u(t),u_\eta(t),v(t),f)dt^\pi}{\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(t,u(t),u_\eta(t),v(t),g)dt^\pi} = \frac{\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(t,u^0(t),u^0_\eta(t),v^0(t),f)dt^\pi}{\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(t,u^0(t),u^0_\eta(t),v^0(t),g)dt^\pi},$$

on the line of Jagannathan [2], Dinkelbach [1], and following Mititelu and Treanţă [5], we build a non-fractional optimization problem associated with $(\mathcal{P})$, as

$$(\mathcal{NP}) \quad \min_{(u(\cdot),v(\cdot))} \left\{ \int_{\mathcal{C}} \Delta_\pi(t,u(t),u_\eta(t),v(t),f)dt^\pi - R^0_{f,g}\int_{\mathcal{C}} \Theta_\pi(t,u(t),u_\eta(t),v(t),g)dt^\pi \right\}$$

subject to

$$A_\beta(t,u(t),u_\eta(t),v(t)) \le 0, \ \ \beta = \overline{1,q}, \ t\in\mathcal{H},$$
$$B_\eta^\iota(t,u(t),u_\eta(t),v(t)) = 0, \ \ \iota = \overline{1,n}, \ \eta = \overline{1,m}, \ t\in\mathcal{H},$$
$$u(t_0) = u_0 = \text{given}, \ u(t_1) = u_1 = \text{given}.$$

The robust counterpart for $(\mathcal{NP})$ is given by

$$(\mathcal{RNP}) \quad \min_{(u(\cdot),v(\cdot))} \left\{ \int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(t,u(t),u_\eta(t),v(t),f)dt^\pi - R^0_{f,g}\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(t,u(t),u_\eta(t),v(t),g)dt^\pi \right\}$$

subject to

$$A_\beta(t,u(t),u_\eta(t),v(t)) \le 0, \ \ \beta = \overline{1,q}, \ t\in\mathcal{H},$$
$$B_\eta^\iota(t,u(t),u_\eta(t),v(t)) = 0, \ \ \iota = \overline{1,n}, \ \eta = \overline{1,m}, \ t\in\mathcal{H},$$
$$u(t_0) = u_0 = \text{given}, \ u(t_1) = u_1 = \text{given}.$$

Next, for a simple presentation, we will use the following abbreviations throughout the paper: $u = u(t)$, $v = v(t)$, $\overline{u} = \overline{u}(t)$, $\overline{v} = \overline{v}(t)$, $\hat{u} = \hat{u}(t)$, $\hat{v} = \hat{v}(t)$, $\zeta = (t,u(t),u_\eta(t),v(t))$, $\overline{\zeta} = (t,\overline{u}(t),\overline{u}_\eta(t),\overline{v}(t))$, $\hat{\zeta} = (t,\hat{u}(t),\hat{u}_\eta(t),\hat{v}(t))$.

**Definition 2.** *A point $(\overline{u},\overline{v}) \in \mathcal{D}$ is said to be a robust optimal solution to $(\mathcal{P})$, if*

$$\frac{\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(\overline{\zeta},f)dt^\pi}{\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(\overline{\zeta},g)dt^\pi} \leq \frac{\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(\zeta,f)dt^\pi}{\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(\zeta,g)dt^\pi},$$

*for all $(u,v) \in \mathcal{D}$.*

**Definition 3.** *A point $(\overline{u},\overline{v}) \in \mathcal{D}$ is said to be a robust optimal solution to $(\mathcal{NP})$, if*

$$\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(\overline{\zeta},f)dt^\pi - R^-_{f,g}\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(\overline{\zeta},g)dt^\pi$$
$$\leq \int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(\zeta,f)dt^\pi - R^-_{f,g}\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(\zeta,g)dt^\pi,$$

*for all $(u,v) \in \mathcal{D}$.*

**Remark 1.** *We can observe that $\mathcal{D}$ is the set of feasible solutions to $(\mathcal{NP})$ (and, also, for $(\mathcal{RNP})$).*

**Remark 2.** *The robust optimal solutions to $(\mathcal{P})$ (or $(\mathcal{NP})$) are also robust optimal solutions to $(\mathcal{RP})$ (or $(\mathcal{RNP})$).*

Next, in order to prove the principal results of this paper, we present the definition of convex, quasi-convex, strictly quasi-convex, and monotonic quasi-convex curvilinear integral functionals (see, for instance, Treanţă [21]).

**Definition 4.** *A curvilinear integral functional $\int_{\mathcal{C}} \Delta_\pi(\zeta,\overline{f})dt^\pi$ is said to be convex at $(\overline{u},\overline{v}) \in U \times V$ if the following inequality*

$$\int_{\mathcal{C}} \Delta_\pi(\zeta,\overline{f})dt^\pi - \int_{\mathcal{C}} \Delta_\pi(\overline{\zeta},\overline{f})dt^\pi \geq \int_{\mathcal{C}} \Big\{(u-\overline{u})\frac{\partial \Delta_\pi}{\partial u}(\overline{\zeta},\overline{f}) + (v-\overline{v})\frac{\partial \Delta_\pi}{\partial v}(\overline{\zeta},\overline{f})\Big\}dt^\pi$$
$$+\int_{\mathcal{C}} \Big\{(u_\eta-\overline{u}_\eta)\frac{\partial \Delta_\pi}{\partial u_\eta}(\overline{\zeta},\overline{f})\Big\}dt^\pi$$

*holds, for all $(u,v) \in U \times V$.*

**Definition 5.** *A curvilinear integral functional $\int_{\mathcal{C}} \Delta_\pi(\zeta,\overline{f})dt^\pi$ is said to be quasi-convex at $(\overline{u},\overline{v}) \in U \times V$ if the following inequality*

$$\int_{\mathcal{C}} \Delta_\pi(\zeta,\overline{f})dt^\pi \leq \int_{\mathcal{C}} \Delta_\pi(\overline{\zeta},\overline{f})dt^\pi$$

*implies*

$$\int_{\mathcal{C}} \Big\{(u-\overline{u})\frac{\partial \Delta_\pi}{\partial u}(\overline{\zeta},\overline{f}) + (v-\overline{v})\frac{\partial \Delta_\pi}{\partial v}(\overline{\zeta},\overline{f})\Big\}dt^\pi$$
$$+\int_{\mathcal{C}} \Big\{(u_\eta-\overline{u}_\eta)\frac{\partial \Delta_\pi}{\partial u_\eta}(\overline{\zeta},\overline{f})\Big\}dt^\pi \leq 0,$$

*for all $(u,v) \in U \times V$.*

**Definition 6.** *A curvilinear integral functional* $\int_{\mathcal{C}} \Delta_\pi(\zeta, \overline{f})dt^\pi$ *is said to be strictly quasi-convex at* $(\overline{u}, \overline{v}) \in U \times V$ *if the following inequality*

$$\int_{\mathcal{C}} \Delta_\pi(\zeta, \overline{f})dt^\pi \leq \int_{\mathcal{C}} \Delta_\pi(\overline{\zeta}, \overline{f})dt^\pi$$

*implies*

$$\int_{\mathcal{C}} \left\{ (u - \overline{u}) \frac{\partial \Delta_\pi}{\partial u}(\overline{\zeta}, \overline{f}) + (v - \overline{v}) \frac{\partial \Delta_\pi}{\partial v}(\overline{\zeta}, \overline{f}) \right\} dt^\pi + \int_{\mathcal{C}} \left\{ (u_\eta - \overline{u}_\eta) \frac{\partial \Delta_\pi}{\partial u_\eta}(\overline{\zeta}, \overline{f}) \right\} dt^\pi < 0,$$

*for all* $(u, v) \neq (\overline{u}, \overline{v}) \in U \times V$.

**Definition 7.** *A curvilinear integral functional* $\int_{\mathcal{C}} \Delta_\pi(\zeta, \overline{f})dt^\pi$ *is said to be monotonic quasi-convex at* $(\overline{u}, \overline{v}) \in U \times V$ *if the following inequality*

$$\int_{\mathcal{C}} \Delta_\pi(\zeta, \overline{f})dt^\pi = \int_{\mathcal{C}} \Delta_\pi(\overline{\zeta}, \overline{f})dt^\pi$$

*implies*

$$\int_{\mathcal{C}} \left\{ (u - \overline{u}) \frac{\partial \Delta_\pi}{\partial u}(\overline{\zeta}, \overline{f}) + (v - \overline{v}) \frac{\partial \Delta_\pi}{\partial v}(\overline{\zeta}, \overline{f}) \right\} dt^\pi + \int_{\mathcal{C}} \left\{ (u_\eta - \overline{u}_\eta) \frac{\partial \Delta_\pi}{\partial u_\eta}(\overline{\zeta}, \overline{f}) \right\} dt^\pi = 0,$$

*for all* $(u, v) \in U \times V$.

**Remark 3.** *The relationships between the various convexities proposed in this article are discussed and illustrated with suitable examples in previous research works (see, for instance, Mititelu and Treanţă [5] and Jayswal et al. [22]).*

## 3. Robust Sufficient Optimality Conditions

Next, by considering suitable convexity, quasi-convexity, strictly quasi-convexity, and/or monotonic quasi-convexity hypotheses, we establish robust sufficient optimality conditions for the considered problem. In addition, in accordance with Treanţă and Arana-Jiménez [23], we describe the notion of *robust Kuhn-Tucker point*.

The next Proposition provides an auxiliary result to establish the robust sufficient optimality conditions for $(\mathcal{P})$ (see Saeed [24]).

**Proposition 1.** *If* $(\overline{u}, \overline{v}) \in \mathcal{D}$ *is a robust optimal solution to* $(\mathcal{P})$*, then there exists the positive real number* $R^-_{f,g}$ *such that* $(\overline{u}, \overline{v}) \in \mathcal{D}$ *is a robust optimal solution to* $(\mathcal{NP})$*. Moreover, if* $(\overline{u}, \overline{v}) \in \mathcal{D}$ *is a robust optimal solution to* $(\mathcal{NP})$ *and* $R^-_{f,g} = \dfrac{\int_{\mathcal{C}} \max_{f \in F} \Delta_\pi(\overline{\zeta}, f)dt^\pi}{\int_{\mathcal{C}} \min_{g \in G} \Theta_\pi(\overline{\zeta}, g)dt^\pi}$*, then* $(\overline{u}, \overline{v}) \in \mathcal{D}$ *is a robust optimal solution to* $(\mathcal{P})$.

The next result formulates the robust necessary conditions of optimality for $(\mathcal{P})$ (see Saeed [24]).

**Theorem 1.** *Consider* $(\bar{u}, \bar{v}) \in \mathcal{D}$ *is a robust optimal solution for the robust fractional optimization problem* $(\mathcal{P})$ *and* $\max_{f \in F} \Delta_\pi(\zeta, f) = \Delta_\pi(\zeta, \bar{f})$, $\min_{g \in G} \Theta_\pi(\zeta, g) = \Theta_\pi(\zeta, \bar{g})$*. Then, there* *exist $\bar{\theta} \in \mathbb{R}$ and the piecewise differentiable functions $\bar{\mu} = (\bar{\mu}_\beta(t)) \in \mathbb{R}^q_+, \bar{\lambda} = (\bar{\lambda}^\iota_\eta(t)) \in \mathbb{R}^{nm}$, satisfying*

$$\bar{\theta}\Big[\Delta_{\pi,u}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta},\bar{g})\Big] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta}) \tag{1}$$

$$-\frac{\partial}{\partial t^\eta}\Big\{\bar{\theta}\Big[\Delta_{\pi,u_\eta}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u_\eta}(\bar{\zeta},\bar{g})\Big] + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\Big\} = 0,$$

$$\bar{\theta}\Big[\Delta_{\pi,v}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta},\bar{g})\Big] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta}) = 0, \tag{2}$$

$$\bar{\mu}^T A(\bar{\zeta}) = 0,\ \bar{\mu}_\beta \geq 0,\ \beta = \overline{1,q}, \tag{3}$$

$$\bar{\theta} \geq 0, \tag{4}$$

*for $t \in \mathcal{H},\ \pi = \overline{1,m}$, except at points of discontinuity.*

**Remark 4.** *The relations (1)–(4) in Theorem 1 are called robust necessary optimality conditions for the robust fractional optimization problem $(\mathcal{P})$.*

**Definition 8.** *The feasible solution $(\bar{u},\bar{v}) \in \mathcal{D}$ is said to be a normal robust optimal solution to $(\mathcal{P})$ if $\bar{\theta} > 0$ (see Theorem 1).*

Next, in accordance to Treanţă and Arana-Jiménez [23], we introduce and describe the *robust Kuhn-Tucker point* associated with $(\mathcal{P})$.

**Definition 9.** *Let $\max_{f\in F}\Delta_\pi(\zeta,f) = \Delta_\pi(\zeta,\bar{f})$, $\min_{g\in G}\Theta_\pi(\zeta,g) = \Theta_\pi(\zeta,\bar{g})$. The robust feasible solution $(\bar{u},\bar{v})$ is said to be a robust Kuhn-Tucker point of $(\mathcal{P})$ if there exist the piecewise differentiable functions $\bar{\mu} = (\bar{\mu}_\beta(t)) \in \mathbb{R}^q_+, \bar{\lambda} = (\bar{\lambda}^\iota_\eta(t)) \in \mathbb{R}^{nm}$, satisfying*

$$\Delta_{\pi,u}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta},\bar{g}) + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta})$$

$$-\frac{\partial}{\partial t^\eta}\Big\{\Delta_{\pi,u_\eta}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u_\eta}(\bar{\zeta},\bar{g}) + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\Big\} = 0,$$

$$\Delta_{\pi,v}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta},\bar{g}) + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta}) = 0,$$

$$\bar{\mu}^T A(\bar{\zeta}) = 0,\ \bar{\mu}_\beta \geq 0,\ \beta = \overline{1,q},$$

*for $t \in \mathcal{H},\ \pi = \overline{1,m}$, except at points of discontinuity.*

Taking into account the above-mentioned definition, we formulate the following theorem.

**Theorem 2.** *If $(\bar{u},\bar{v}) \in \mathcal{D}$ is a normal robust optimal solution for the robust fractional optimization problem $(\mathcal{P})$, with $\max_{f\in F}\Delta_\pi(\zeta,f) = \Delta_\pi(\zeta,\bar{f})$, $\min_{g\in G}\Theta_\pi(\zeta,g) = \Theta_\pi(\zeta,\bar{g})$, then $(\bar{u},\bar{v}) \in \mathcal{D}$ is a robust Kuhn-Tucker point of $(\mathcal{P})$.*

**Proof.** Let us consider $\max_{f\in F}\Delta_\pi(\zeta,f) = \Delta_\pi(\zeta,\bar{f})$, $\min_{g\in G}\Theta_\pi(\zeta,g) = \Theta_\pi(\zeta,\bar{g})$. Since $(\bar{u},\bar{v}) \in \mathcal{D}$ is a robust optimal solution of $(\mathcal{P})$, by Theorem 1, there exist $\bar{\theta} \in \mathbb{R}$ and the piecewise differentiable functions $\bar{\mu} = (\bar{\mu}_\beta(t)) \in \mathbb{R}^q_+, \bar{\lambda} = (\bar{\lambda}^\iota_\eta(t)) \in \mathbb{R}^{nm}$, satisfying

$$\bar{\theta}\Big[\Delta_{\pi,u}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta},\bar{g})\Big] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta})$$

$$-\frac{\partial}{\partial t^\eta}\Big\{\bar{\theta}\Big[\Delta_{\pi,u_\eta}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u_\eta}(\bar{\zeta},\bar{g})\Big] + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\Big\} = 0,$$

$$\bar{\theta}\Big[\Delta_{\pi,v}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta},\bar{g})\Big] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta}) = 0,$$

$$\bar{\mu}^T A(\bar{\zeta}) = 0,\ \bar{\mu}_\beta \geq 0,\ \beta = \overline{1,q},$$

$$\bar{\theta} \geq 0,$$

for $t \in \mathcal{H}$, $\pi = \overline{1,m}$, except at points of discontinuity. As $(\bar{u}, \bar{v}) \in \mathcal{D}$ is assumed to be a normal robust optimal solution, we can take $\bar{\theta} = 1 > 0$ and this completes the proof. □

Next, under only convexity assumptions of the considered functionals, a result is provided for the sufficiency of the robust necessary optimality conditions established in Theorem 1.

**Theorem 3.** *Let $(\bar{u}, \bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied, $\max_{f\in F} \Delta_\pi(\zeta, f) = \Delta_\pi(\zeta, \overline{f})$, $\min_{g\in G} \Theta_\pi(\zeta, g) = \Theta_\pi(\zeta, \overline{g})$, and consider*

$$\int_{\mathcal{C}} \bar{\theta}\Big[\Delta_\pi(\zeta, \bar{f}) - R^-_{f,g}\Theta_\pi(\zeta, \bar{g})\Big]dt^\pi,\ \int_{\mathcal{C}} \bar{\mu}^T A(\zeta)dt^\pi,\ \int_{\mathcal{C}} \bar{\lambda}^T B(\zeta)dt^\pi$$

*are convex at $(\bar{u}, \bar{v}) \in \mathcal{D}$. Then the pair $(\bar{u}, \bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** By contrary, let us suppose that $(\bar{u}, \bar{v})$ is not a robust optimal solution to $(\mathcal{P})$. Then, there exists $(\hat{u}, \hat{v}) \in \mathcal{D}$ with the property (according to Proposition 1)

$$\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(\hat{\zeta}, f)dt^\pi - R^-_{f,g}\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(\hat{\zeta}, g)dt^\pi$$

$$< \int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(\bar{\zeta}, f)dt^\pi - R^-_{f,g}\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(\bar{\zeta}, g)dt^\pi.$$

By considering $\max_{f\in F} \Delta_\pi(\zeta, f) = \Delta_\pi(\zeta, \overline{f})$, $\min_{g\in G} \Theta_\pi(\zeta, g) = \Theta_\pi(\zeta, \overline{g})$, we get

$$\int_{\mathcal{C}} \Delta_\pi(\hat{\zeta}, \bar{f})dt^\pi - R^-_{f,g}\int_{\mathcal{C}} \Theta_\pi(\hat{\zeta}, \bar{g})dt^\pi$$

$$< \int_{\mathcal{C}} \Delta_\pi(\bar{\zeta}, \bar{f})dt^\pi - R^-_{f,g}\int_{\mathcal{C}} \Theta_\pi(\bar{\zeta}, \bar{g})dt^\pi. \tag{5}$$

By hypothesis, we have considered $(\bar{u}, \bar{v})$ fulfills the conditions (1)–(4). By multiplying Equations (1) and (2) by $(\hat{u} - \bar{u})$ and $(\hat{v} - \bar{v})$, respectively, and integrating them, we get

$$\int_{\mathcal{C}} (\hat{u} - \bar{u})\{\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta}, \bar{f}) - R^-_{f,g}\Theta_{\pi,u}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta}) \tag{6}$$

$$-\frac{\partial}{\partial t^\eta}\Big[\bar{\theta}[\Delta_{\pi,u_\eta}(\bar{\zeta}, \bar{f}) - R^-_{f,g}\Theta_{\pi,u_\eta}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\Big]\}dt^\pi$$

$$+ \int_{\mathcal{C}} (\hat{v} - \bar{v})\{\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta}, \bar{f}) - R^-_{f,g}\Theta_{\pi,v}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta})\}dt^\pi$$

$$= \int_{\mathcal{C}} \Big[(\hat{u} - \bar{u})\{\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta}, \bar{f}) - R^-_{f,g}\Theta_{\pi,u}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta})\}$$

$$+(\hat{u}_\eta - \bar{u}_\eta)\{\bar{\theta}[\Delta_{\pi,u_\eta}(\bar{\zeta}, \bar{f}) - R^-_{f,g}\Theta_{\pi,u_\eta}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\}\Big]dt^\pi$$

$$+ \int_{\mathcal{C}} (\hat{v} - \bar{v})\{\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta}, \bar{f}) - R^-_{f,g}\Theta_{\pi,v}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta})\}dt^\pi = 0,$$

by using the method of integration by parts, the boundary conditions, and the divergence formula.

On the other hand, since $\int_{\mathcal{C}} \bar{\theta}\Big[\Delta_\pi(\zeta, \bar{f}) - R^-_{f,g}\Theta_\pi(\zeta, \bar{g})\Big]dt^\pi$ is convex at $(\bar{u}, \bar{v})$, we have

$$\int_{\mathcal{C}} \Big\{\bar{\theta}[\Delta_\pi(\hat{\zeta}, \bar{f}) - R^-_{f,g}\Theta_\pi(\hat{\zeta}, \bar{g})] - \bar{\theta}[\Delta_\pi(\bar{\zeta}, \bar{f}) - R^-_{f,g}\Theta_\pi(\bar{\zeta}, \bar{g})]\Big\}dt^\pi$$

$$\geq \int_{\mathcal{C}} (\hat{u} - \bar{u})\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta}, \bar{g})]dt^{\pi}$$
$$+ \int_{\mathcal{C}} (\hat{u}_{\eta} - \bar{u}_{\eta})\bar{\theta}[\Delta_{\pi,u_{\eta}}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,u_{\eta}}(\bar{\zeta}, \bar{g})]dt^{\pi}$$
$$+ \int_{\mathcal{C}} (\hat{v} - \bar{v})\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta}, \bar{g})]dt^{\pi},$$

and, by using inequality (5), it follows

$$\int_{\mathcal{C}} (\hat{u} - \bar{u})\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta}, \bar{g})]dt^{\pi} + \int_{\mathcal{C}} (\hat{u}_{\eta} - \bar{u}_{\eta})\bar{\theta}[\Delta_{\pi,u_{\eta}}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,u_{\eta}}(\bar{\zeta}, \bar{g})]dt^{\pi}$$
$$+ \int_{\mathcal{C}} (\hat{v} - \bar{v})\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta}, \bar{g})]dt^{\pi} < 0. \tag{7}$$

Now, by using the convexity property at $(\bar{u}, \bar{v})$ of the functional $\int_{\mathcal{C}} \bar{\mu}^T A(\zeta)dt^{\pi}$, we get

$$\int_{\mathcal{C}} \left\{\bar{\mu}^T A(\hat{\zeta}) - \bar{\mu}^T A(\bar{\zeta})\right\}dt^{\pi} \geq \int_{\mathcal{C}} (\hat{u} - \bar{u})\bar{\mu}^T A_u(\bar{\zeta})dt^{\pi}$$
$$+ \int_{\mathcal{C}} (\hat{u}_{\eta} - \bar{u}_{\eta})\bar{\mu}^T A_{u_{\eta}}(\bar{\zeta})dt^{\pi} + \int_{\mathcal{C}} (\hat{v} - \bar{v})\bar{\mu}^T A_v(\bar{\zeta})dt^{\pi}.$$

The previous inequality along with the robust feasibility of $(\hat{u}, \hat{v})$ to $(\mathcal{P})$ and the optimality condition (3), give

$$\int_{\mathcal{C}} (\hat{u} - \bar{u})\bar{\mu}^T A_u(\bar{\zeta})dt^{\pi} + \int_{\mathcal{C}} (\hat{u}_{\eta} - \bar{u}_{\eta})\bar{\mu}^T A_{u_{\eta}}(\bar{\zeta})dt^{\pi}$$
$$+ \int_{\mathcal{C}} (\hat{v} - \bar{v})\bar{\mu}^T A_v(\bar{\zeta})dt^{\pi} \leq 0. \tag{8}$$

Further, by considering, in the same manner, the convexity property at $(\bar{u}, \bar{v})$ of the integral functional $\int_{\mathcal{C}} \bar{\lambda}^T B(\zeta)dt^{\pi}$, and the robust feasibility of $(\hat{u}, \hat{v})$ to $(\mathcal{P})$, we obtain

$$\int_{\mathcal{C}} (\hat{u} - \bar{u})\bar{\lambda}^T B_u(\bar{\zeta})dt^{\pi} + \int_{\mathcal{C}} (\hat{u}_{\eta} - \bar{u}_{\eta})\bar{\lambda}^T B_{u_{\eta}}(\bar{\zeta})dt^{\pi}$$
$$+ \int_{\mathcal{C}} (\hat{v} - \bar{v})\bar{\lambda}^T B_v(\bar{\zeta})dt^{\pi} \leq 0. \tag{9}$$

Finally, by adding the relations (7), (8) and (9), side by side, we have

$$\int_{\mathcal{C}} \Big[(\hat{u} - \bar{u})\{\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta})\}$$
$$+(\hat{u}_{\eta} - \bar{u}_{\eta})\{\bar{\theta}[\Delta_{\pi,u_{\eta}}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,u_{\eta}}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_{u_{\eta}}(\bar{\zeta}) + \bar{\lambda}^T B_{u_{\eta}}(\bar{\zeta})\}\Big]dt^{\pi}$$
$$+ \int_{\mathcal{C}} (\hat{v} - \bar{v})\{\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta}, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta}, \bar{g})] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta})\}dt^{\pi} < 0,$$

being a contradiction with the relation (6), and this completes the proof. □

The next theorems assert new robust sufficient optimality conditions under (strictly, monotonic) quasi-convexity assumptions.

**Theorem 4.** *Let $(\bar{u}, \bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied,* $\max_{f \in F} \Delta_{\pi}(\zeta, f) = \Delta_{\pi}(\zeta, \overline{f})$, $\min_{g \in G} \Theta_{\pi}(\zeta, g) = \Theta_{\pi}(\zeta, \overline{g})$*, and consider*

$$F(u, v; \bar{f}, \bar{g}) := \int_{\mathcal{C}} \bar{\theta}\Big[\Delta_{\pi}(\zeta, \bar{f}) - R^{-}_{\bar{f},\bar{g}}\Theta_{\pi}(\zeta, \bar{g})\Big]dt^{\pi}, \ V(u, v) := \int_{\mathcal{C}} \bar{\mu}^T A(\zeta)dt^{\pi}$$

*are quasi-convex and strictly quasi-convex at* $(\bar{u}, \bar{v}) \in \mathcal{D}$*, respectively, and* $X(u,v) := \int_{\mathcal{C}} \bar{\lambda}^T B(\zeta) dt^{\pi}$ *is monotonic quasi-convex at* $(\bar{u}, \bar{v}) \in \mathcal{D}$*. Then the pair* $(\bar{u}, \bar{v})$ *is a robust optimal solution to* $(\mathcal{P})$*.*

**Proof.** Let us assume that $(\bar{u}, \bar{v})$ is not a robust optimal solution to $(\mathcal{P})$, and consider the following non-empty set

$$S = \big\{(u,v) \in \mathcal{D} \mid F(u,v;\bar{f},\bar{g}) \le F(\bar{u},\bar{v};\bar{f},\bar{g}),\ X(u,v) = X(\bar{u},\bar{v}),\ V(u,v) \le V(\bar{u},\bar{v})\big\}.$$

By hypothesis, for $(u,v) \in S$, we get

$$F(u,v;\bar{f},\bar{g}) \le F(\bar{u},\bar{v};\bar{f},\bar{g}),$$

then

$$\int_{\mathcal{C}} \Big\{\bar{\theta}\Big[\Delta_{\pi,u}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta},\bar{g})\Big](u-\bar{u}) + \bar{\theta}\Big[\Delta_{\pi,v}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta},\bar{g})\Big](v-\bar{v})\Big\}dt^{\pi}$$
$$+ \int_{\mathcal{C}} \Big\{\bar{\theta}\Big[\Delta_{\pi,u_\eta}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u_\eta}(\bar{\zeta},\bar{g})\Big](u_\eta - \bar{u}_\eta)\Big\}dt^{\pi} \le 0. \tag{10}$$

For $(u,v) \in S$, the equality $X(u,v) = X(\bar{u},\bar{v})$ holds and it follows

$$\int_{\mathcal{C}} \Big\{\bar{\lambda}^T B_u(\bar{\zeta})(u-\bar{u}) + \bar{\lambda}^T B_v(\bar{\zeta})(v-\bar{v})\Big\}dt^{\pi}$$
$$+ \int_{\mathcal{C}} \Big\{\bar{\lambda}^T B_{u_\eta}(\bar{\zeta})(u_\eta - \bar{u}_\eta)\Big\}dt^{\pi} = 0. \tag{11}$$

Also, for $(u,v) \in S$, the inequality $V(u,v) \le V(\bar{u},\bar{v})$ gives

$$\int_{\mathcal{C}} \Big\{\bar{\mu}^T A_u(\bar{\zeta})(u-\bar{u}) + \bar{\mu}^T A_v(\bar{\zeta})(v-\bar{v})\Big\}dt^{\pi}$$
$$+ \int_{\mathcal{C}} \Big\{\bar{\mu}^T A_{u_\eta}(\bar{\zeta})(u_\eta - \bar{u}_\eta)\Big\}dt^{\pi} < 0. \tag{12}$$

By hypothesis, we have considered $(\bar{u}, \bar{v})$ fulfills the conditions (1)–(4). By multiplying Equations (1) and (2) by $(u-\bar{u})$ and $(v-\bar{v})$, respectively, and integrating them, we get

$$\int_{\mathcal{C}} (u-\bar{u})\{\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta})$$
$$-\frac{\partial}{\partial t^\eta}\Big[\bar{\theta}[\Delta_{\pi,u_\eta}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u_\eta}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\Big]\}dt^{\pi}$$
$$+ \int_{\mathcal{C}} (v-\bar{v})\{\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta})\}dt^{\pi}$$
$$= \int_{\mathcal{C}} \Big[(u-\bar{u})\{\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta})\}$$
$$+(u_\eta - \bar{u}_\eta)\{\bar{\theta}[\Delta_{\pi,u_\eta}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u_\eta}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\}\Big]dt^{\pi}$$
$$+ \int_{\mathcal{C}} (v-\bar{v})\{\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta})\}dt^{\pi} = 0, \tag{13}$$

by using the method of integration by parts, the boundary conditions, and the divergence formula. On the other hand, by adding the relations (10), (11) and (12), side by side, we have

$$\int_{\mathcal{C}} \Big[(u-\bar{u})\{\bar{\theta}[\Delta_{\pi,u}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_u(\bar{\zeta}) + \bar{\lambda}^T B_u(\bar{\zeta})\}$$

$$+(u_\eta - \bar{u}_\eta)\{\bar{\theta}[\Delta_{\pi,u_\eta}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,u_\eta}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_{u_\eta}(\bar{\zeta}) + \bar{\lambda}^T B_{u_\eta}(\bar{\zeta})\}\Big]dt^\pi$$

$$+\int_{\mathcal{C}}(v-\bar{v})\{\bar{\theta}[\Delta_{\pi,v}(\bar{\zeta},\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_{\pi,v}(\bar{\zeta},\bar{g})] + \bar{\mu}^T A_v(\bar{\zeta}) + \bar{\lambda}^T B_v(\bar{\zeta})\}dt^\pi < 0,$$

being a contradiction with the relation (13), and this completes the proof. □

Next, some immediate consequences of the previous theorem can be formulated as follows.

**Theorem 5.** *Let $(\bar{u},\bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied, $\max_{f\in F}\Delta_\pi(\zeta,f) = \Delta_\pi(\zeta,\overline{f})$, $\min_{g\in G}\Theta_\pi(\zeta,g) = \Theta_\pi(\zeta,\overline{g})$, and consider*

$$F(u,v;\bar{f},\bar{g}) := \int_{\mathcal{C}}\bar{\theta}\Big[\Delta_\pi(\zeta,\bar{f}) - R^-_{\bar{f},\bar{g}}\Theta_\pi(\zeta,\bar{g})\Big]dt^\pi,\ V(u,v) := \int_{\mathcal{C}}\bar{\mu}^T A(\zeta)dt^\pi$$

*are strictly quasi-convex and quasi-convex at $(\bar{u},\bar{v}) \in \mathcal{D}$, respectively, and $X(u,v) := \int_{\mathcal{C}}\bar{\lambda}^T B(\zeta)dt^\pi$ is monotonic quasi-convex at $(\bar{u},\bar{v}) \in \mathcal{D}$. Then the pair $(\bar{u},\bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** The proof follows in the same manner as in Theorem 4, by replacing the sign "$\leq$" in (10) with "$<$", and the sign "$<$" in (12) with "$\leq$". □

**Theorem 6.** *Let $(\bar{u},\bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied, $\max_{f\in F}\Delta_\pi(\zeta,f) = \Delta_\pi(\zeta,\overline{f})$, $\min_{g\in G}\Theta_\pi(\zeta,g) = \Theta_\pi(\zeta,\overline{g})$, and consider*

$$F(u,v;\bar{f},\bar{g}) := \int_{\mathcal{C}}\bar{\theta}\Big[\tilde{\Theta}_\pi(\overline{\zeta},\bar{f})\Delta_\pi(\zeta,\bar{f}) - \tilde{\Delta}_\pi(\overline{\zeta},\bar{f})\Theta_\pi(\zeta,\bar{g})\Big]dt^\pi,\ V(u,v) := \int_{\mathcal{C}}\bar{\mu}^T A(\zeta)dt^\pi$$

*are quasi-convex and strictly quasi-convex at $(\bar{u},\bar{v}) \in \mathcal{D}$, respectively, and $X(u,v) := \int_{\mathcal{C}}\bar{\lambda}^T B(\zeta)dt^\pi$ is monotonic quasi-convex at $(\bar{u},\bar{v}) \in \mathcal{D}$. Then the pair $(\bar{u},\bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** The proof follows in the same manner as in Theorem 4, by replacing the $R^-_{\bar{f},\bar{g}} = \dfrac{\int_{\mathcal{C}}\max\limits_{f\in F}\Delta_\pi(\overline{\zeta},f)dt^\pi}{\int_{\mathcal{C}}\min\limits_{g\in G}\Theta_\pi(\overline{\zeta},g)dt^\pi} := \dfrac{\tilde{\Delta}_\pi(\overline{\zeta},\bar{f})}{\tilde{\Theta}_\pi(\overline{\zeta},\bar{f})}$. □

**Theorem 7.** *Let $(\bar{u},\bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied, $\max_{f\in F}\Delta_\pi(\zeta,f) = \Delta_\pi(\zeta,\overline{f})$, $\min_{g\in G}\Theta_\pi(\zeta,g) = \Theta_\pi(\zeta,\overline{g})$, and consider*

$$F(u,v;\bar{f},\bar{g}) := \int_{\mathcal{C}}\bar{\theta}\Big[\tilde{\Theta}_\pi(\overline{\zeta},\bar{f})\Delta_\pi(\zeta,\bar{f}) - \tilde{\Delta}_\pi(\overline{\zeta},\bar{f})\Theta_\pi(\zeta,\bar{g})\Big]dt^\pi,\ V(u,v) := \int_{\mathcal{C}}\bar{\mu}^T A(\zeta)dt^\pi$$

*are strictly quasi-convex and quasi-convex at $(\bar{u},\bar{v}) \in \mathcal{D}$, respectively, and $X(u,v) := \int_{\mathcal{C}}\bar{\lambda}^T B(\zeta)dt^\pi$ is monotonic quasi-convex at $(\bar{u},\bar{v}) \in \mathcal{D}$. Then the pair $(\bar{u},\bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** The proof follows in the same manner as in Theorem 4, by replacing the $R^-_{\bar{f},\bar{g}} = \dfrac{\int_{\mathcal{C}}\max\limits_{f\in F}\Delta_\pi(\overline{\zeta},f)dt^\pi}{\int_{\mathcal{C}}\min\limits_{g\in G}\Theta_\pi(\overline{\zeta},g)dt^\pi} := \dfrac{\tilde{\Delta}_\pi(\overline{\zeta},\bar{f})}{\tilde{\Theta}_\pi(\overline{\zeta},\bar{f})}$, the sign "$\leq$" in (10) with "$<$", and the sign "$<$" in (12) with "$\leq$". □

**Theorem 8.** *Let $(\bar{u}, \bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied,* $\max_{f\in F} \Delta_\pi(\zeta, f) = \Delta_\pi(\zeta, \overline{f})$, $\min_{g\in G} \Theta_\pi(\zeta, g) = \Theta_\pi(\zeta, \overline{g})$*, and consider*

$$F(u, v; \bar{f}, \bar{g}) := \int_{\mathcal{C}} \bar{\theta}\Big[\Delta_\pi(\zeta, \bar{f}) - R^{-}_{f,g}\Theta_\pi(\zeta, \bar{g})\Big]dt^\pi,$$

$$\tilde{V}(u, v) := \int_{\mathcal{C}} \Big[\bar{\mu}^T A(\zeta) + \bar{\lambda}^T B(\zeta)\Big]dt^\pi$$

*are quasi-convex and strictly quasi-convex at $(\bar{u}, \bar{v}) \in \mathcal{D}$, respectively. Then the pair $(\bar{u}, \bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** The proof follows in the same manner as in Theorem 4, by considering the sign "$<$" in (11) and (12), then adding them. □

**Theorem 9.** *Let $(\bar{u}, \bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied,* $\max_{f\in F} \Delta_\pi(\zeta, f) = \Delta_\pi(\zeta, \overline{f})$, $\min_{g\in G} \Theta_\pi(\zeta, g) = \Theta_\pi(\zeta, \overline{g})$*, and consider*

$$F(u, v; \bar{f}, \bar{g}) := \int_{\mathcal{C}} \bar{\theta}\Big[\Delta_\pi(\zeta, \bar{f}) - R^{-}_{f,g}\Theta_\pi(\zeta, \bar{g})\Big]dt^\pi,$$

$$\tilde{V}(u, v) := \int_{\mathcal{C}} \Big[\bar{\mu}^T A(\zeta) + \bar{\lambda}^T B(\zeta)\Big]dt^\pi$$

*are strictly quasi-convex and quasi-convex at $(\bar{u}, \bar{v}) \in \mathcal{D}$, respectively. Then the pair $(\bar{u}, \bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** The proof follows in the same manner as in Theorem 4, by considering the sign "$<$" in (10), and the sign "$\leq$" in (11) and (12), then adding them. □

**Theorem 10.** *Let $(\bar{u}, \bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied,* $\max_{f\in F} \Delta_\pi(\zeta, f) = \Delta_\pi(\zeta, \overline{f})$, $\min_{g\in G} \Theta_\pi(\zeta, g) = \Theta_\pi(\zeta, \overline{g})$*, and consider*

$$F(u, v; \bar{f}, \bar{g}) := \int_{\mathcal{C}} \bar{\theta}\Big[\tilde{\Theta}_\pi(\overline{\zeta}, \bar{f})\Delta_\pi(\zeta, \bar{f}) - \tilde{\Delta}_\pi(\overline{\zeta}, \bar{f})\Theta_\pi(\zeta, \bar{g})\Big]dt^\pi,$$

$$\tilde{V}(u, v) := \int_{\mathcal{C}} \Big[\bar{\mu}^T A(\zeta) + \bar{\lambda}^T B(\zeta)\Big]dt^\pi$$

*are quasi-convex and strictly quasi-convex at $(\bar{u}, \bar{v}) \in \mathcal{D}$, respectively. Then the pair $(\bar{u}, \bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** The proof follows in the same manner as in Theorem 4, by replacing the $R^{-}_{f,g} = \dfrac{\int_{\mathcal{C}} \max_{f\in F} \Delta_\pi(\overline{\zeta}, f)dt^\pi}{\int_{\mathcal{C}} \min_{g\in G} \Theta_\pi(\overline{\zeta}, g)dt^\pi} := \dfrac{\tilde{\Delta}_\pi(\overline{\zeta}, \bar{f})}{\tilde{\Theta}_\pi(\overline{\zeta}, \bar{f})}$, and by considering the sign "$<$" in (11) and (12), then adding them. □

**Theorem 11.** *Let $(\bar{u}, \bar{v}) \in \mathcal{D}$ be a feasible solution to $(\mathcal{P})$ such that the robust necessary optimality conditions given in (1)–(4) are satisfied,* $\max_{f\in F} \Delta_\pi(\zeta, f) = \Delta_\pi(\zeta, \overline{f})$, $\min_{g\in G} \Theta_\pi(\zeta, g) = \Theta_\pi(\zeta, \overline{g})$*, and consider*

$$F(u, v; \bar{f}, \bar{g}) := \int_{\mathcal{C}} \bar{\theta}\Big[\tilde{\Theta}_\pi(\overline{\zeta}, \bar{f})\Delta_\pi(\zeta, \bar{f}) - \tilde{\Delta}_\pi(\overline{\zeta}, \bar{f})\Theta_\pi(\zeta, \bar{g})\Big]dt^\pi,$$

$$\tilde{V}(u,v) := \int_{\mathcal{C}} \left[\bar{\mu}^T A(\zeta) + \bar{\lambda}^T B(\zeta)\right] dt^{\pi}$$

*are strictly quasi-convex and quasi-convex at $(\bar{u}, \bar{v}) \in \mathcal{D}$, respectively. Then the pair $(\bar{u}, \bar{v})$ is a robust optimal solution to $(\mathcal{P})$.*

**Proof.** The proof follows in the same manner as in Theorem 4, by replacing the $R^{-}_{f,g} = \dfrac{\int_{\mathcal{C}} \max_{f\in F} \Delta_{\pi}(\overline{\zeta}, f) dt^{\pi}}{\int_{\mathcal{C}} \min_{g\in G} \Theta_{\pi}(\overline{\zeta}, g) dt^{\pi}} := \dfrac{\tilde{\Delta}_{\pi}(\overline{\zeta}, \bar{f})}{\tilde{\Theta}_{\pi}(\overline{\zeta}, \bar{f})}$, the sign "$\leq$" in (10) with "$<$", and by considering the sign "$\leq$" in (11) and (12), then adding them. □

**Remark 5.** *(i) In order to justify the main elements formulated in the paper, some illustrative applications and numerical simulations can be consulted by the reader in the recent research work of Jayswal et al. [22].*

*(ii) Regarding the research limitations associated with this paper, we could mention the study of the case where the second-order partial derivatives are presented, and, also, the situation when the involved functionals are not necessarily (quasi-) convex.*

*(iii) In order to highlight the above-mentioned theorems, a suitable illustrative application (from mechanics) is presented and investigated in Treanţă [25].*

## 4. Conclusions

In this paper, under the convexity, quasi-convexity, strictly quasi-convexity, and/or monotonic quasi-convexity hypotheses of the involved functionals, we have established robust sufficient optimality conditions for the considered problem. Also, a characterization of the associated Kuhn-Tucker points has been stated. To the best of the authors' knowledge, the results presented in this paper are new in the specialized literature. As future research directions of this paper, the authors mention the presence of data uncertainty in the constraints and the associated duality theory.

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