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Article

Representation of Some Ratios of Horn's Hypergeometric Functions H_7 by Continued Fractions

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Abstract: The paper deals with the problem of representation of Horn's hypergeometric functions via continued fractions and branched continued fractions. We construct the formal continued fraction expansions for three ratios of Horn's hypergeometric functions H_7. The method employed is a two-dimensional generalization of the classical method of constructing a Gaussian continued fraction. It is proved that the continued fraction, which is an expansion of each ratio, uniformly converges to a holomorphic function of two variables on every compact subset of some domain of $\mathbb{C}^2$, and that this function is an analytic continuation of such a ratio in this domain. To illustrate this, we provide some numerical experiments at the end.

Keywords: Horn function; continued fraction; holomorphic functions of several complex variables; numerical approximation; convergence

MSC: 33C65; 30B70; 32A10; 33F05; 40A15

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1. Introduction

Families of hypergeometric functions (such as Appell [1,2], Horn [3–5], Lauricella [6], and others) were and remain the object of our research, as they have many different applications both in mathematics and in other fields of science [7–10]. In particular, their properties [11–14], integral representations [15–18], and representations in the form of branched continued fractions [19–26] are studied.

In [27], the expansion of the Horn's hypergeometric function H_4 into a branched continued fraction, which is a continued fraction according to its structure, was obtained. However, this do not provide an opportunity to apply the well-known results from the analytical theory of continued fractions through the so-called 'figure approximants' (see, for example, [28])—that is, different approaches to the determining of approximants. Therefore, the question naturally arises: is there an expansion of Horn's hypergeometric function into a pure continued fraction?

In this paper, we give a partial answer to the above question by converting three expansions of certain ratios of Horn's hypergeometric functions H_7 into continued fractions as functions of two complex variables. To do this, using the technique of establishing recurrence relations [29], it is shown that one three- and two four-term recurrence relations for the function H_7 are valid. It is proved that every continued fraction, which is an expansion of a certain ratio, uniformly converges to a holomorphic function on every compact subset of some domain of $\mathbb{C}^2$, and that this function is an analytic continuation of such a ratio in this domain. At the end of the paper, we undertake some numerical experiments.

2. Expansions

Horn's hypergeometric function H_7 is defined by double power series (see, [3–5])

$$\mathrm{H}_7(a;c_1,c_2;\mathbf{z}) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(c_1)_m(c_2)_n} \frac{z_1^m z_2^n}{m!\,n!}, \quad |z_1| < 1/4, \tag{1}$$

where a, c_1, and c_2 are complex constants, c_1 and c_2 are not equal to a non-positive integer, $(\cdot)_k$ is the Pochhammer symbol defined for any complex number α and non-negative integer n by $(\alpha)_0 = 1$, and $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

Let us prove the three- and four-term recurrence relations for function (1).

Lemma 1. *The following relations hold true:*

$$\begin{aligned}\mathrm{H}_7(a;c_1,c_2;\mathbf{z}) &= \mathrm{H}_7(a+1;c_1+1,c_2;\mathbf{z}) - \frac{(a+1)(2c_1-a)}{c_1(c_1+1)} z_1 \mathrm{H}_7(a+2;c_1+2,c_2;\mathbf{z}) \\ &\quad - \frac{1}{c_2} z_2 \mathrm{H}_7(a+1;c_1+1,c_2+1;\mathbf{z}),\end{aligned} \tag{2}$$

$$\begin{aligned}\mathrm{H}_7(a;c_1,c_2;\mathbf{z}) &= \mathrm{H}_7(a+1;c_1,c_2+1;\mathbf{z}) - \frac{2(a+1)}{c_1} z_1 \mathrm{H}_7(a+2;c_1+1,c_2+1;\mathbf{z}) \\ &\quad - \frac{c_2-a}{c_2(c_2+1)} z_2 \mathrm{H}_7(a+1;c_1,c_2+2;\mathbf{z}),\end{aligned} \tag{3}$$

$$\mathrm{H}_7(a;c_1,c_2;\mathbf{z}) = \mathrm{H}_7(a;c_1,c_2+1;\mathbf{z}) + \frac{a}{c_2(c_2+1)} z_2 \mathrm{H}_7(a+1;c_1,c_2+1;\mathbf{z}). \tag{4}$$

Proof. By definition (1), we get

$$\begin{aligned}
&\mathrm{H}_7(a;c_1,c_2;\mathbf{z}) - \mathrm{H}_7(a+1;c_1+1,c_2;\mathbf{z}) \\
&= \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(c_1)_m(c_2)_n} \frac{z_1^m z_2^n}{m!n!} - \sum_{m,n=0}^{\infty} \frac{(a+1)_{2m+n}}{(c_1+1)_m(c_2)_n} \frac{z_1^m z_2^n}{m!n!} \\
&= \sum_{m,n\geq 0,\, m+n\geq 1} \frac{(a+1)_{2m+n-1}}{(c_2)_n} \frac{z_1^m z_2^n}{m!n!} \left(\frac{a}{(c_1)_m} - \frac{a+2m+n}{(c_1+1)_m} \right) \\
&= \sum_{m\geq 1,\, n=0} \frac{(a+2)_{2(m-1)+n}}{(c_1+2)_{m-1}(c_2)_n} \frac{(a+1)(a-2c_1)}{c_1(c_1+1)} \frac{m z_1^m z_2^n}{m!n!} - \sum_{m=0,\, n\geq 1} \frac{(a+1)_{2m+n-1}}{c_2(c_2+1)_{n-1}} \frac{n z_1^m z_2^n}{m!n!} \\
&\quad + \sum_{m\geq 1,\, n\geq 1} \frac{(a+1)(a+2)_{2(m-1)+n}}{(c_1+1)_{m-1}(c_2)_n} \left(\frac{a}{c_1} - \frac{a+2m+n}{c_1+m} \right) \frac{z_1^m z_2^n}{m!n!} \\
&= -\frac{(a+1)(2c_1-a)}{c_1(c_1+1)} z_1 \sum_{m\geq 1,\, n\geq 0} \frac{(a+2)_{2(m-1)+n}}{(c_1+2)_{m-1}(c_2)_n} \frac{z_1^{m-1} z_2^n}{(m-1)!n!} \\
&\quad - \frac{z_2}{c_2} \sum_{m\geq 0,\, n\geq 1} \frac{(a+1)_{2m+n-1}}{(c_1+1)_m(c_2+1)_{n-1}} \frac{z_1^m z_2^{n-1}}{m!(n-1)!} \\
&= -\frac{(a+1)(2c_1-a)}{c_1(c_1+1)} z_1 \sum_{m\geq 0,\, n\geq 0} \frac{(a+2)_{2m+n}}{(c_1+2)_m(c_2)_n} \frac{z_1^m z_2^n}{m!n!} \\
&\quad - \frac{z_2}{c_2} \sum_{m\geq 0,\, n\geq 0} \frac{(a+1)_{2m+n}}{(c_1+1)_m(c_2+1)_n} \frac{z_1^m z_2^n}{m!n!} \\
&= -\frac{(a+1)(2c_1-a)}{c_1(c_1+1)} z_1 \mathrm{H}_7(a+2;c_1+2,c_2;\mathbf{z}) - \frac{z_2}{c_2} \mathrm{H}_7(a+1;c_1+1,c_2+1;\mathbf{z}),
\end{aligned}$$

and this means that the four-term recurrence relation (2) is correct.

Let us prove the four-term recurrence relation (3). We have

$$\begin{aligned}
&\mathrm{H}_7(a;c_1,c_2;\mathbf{z})-\mathrm{H}_7(a+1;c_1,c_2+1;\mathbf{z})\\
&\quad=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(c_1)_m(c_2)_n}\frac{z_1^m z_2^n}{m!n!}-\sum_{m,n=0}^{\infty}\frac{(a+1)_{2m+n}}{(c_1)_m(c_2+1)_n}\frac{z_1^m z_2^n}{m!n!}\\
&\quad=\sum_{m,n\geq 0,\ m+n\geq 1}\frac{(a+1)_{2m+n-1}}{(c_1)_m}\frac{z_1^m z_2^n}{m!n!}\left(\frac{a}{(c_2)_n}-\frac{a+2m+n}{(c_2+1)_n}\right)\\
&\quad=-\sum_{m\geq 1,\ n=0}\frac{(a+1)(a+2)_{2(m-1)+n}}{(c_1+1)_m}\frac{2mz_1^m z_2^n}{m!n!}\\
&\qquad+\sum_{m=0,\ n\geq 1}\frac{(a+1)_{2m+n-1}}{(c_2+1)_{n-1}}\left(\frac{a}{c_2}-\frac{a+n}{c_2+n}\right)\frac{z_1^m z_2^n}{m!n!}\\
&\qquad+\sum_{m\geq 1,\ n\geq 1}\frac{(a+1)_{2m+n-1}}{(c_1)_m(c_2+1)_{n-1}}\left(\frac{a}{c_2}-\frac{a+2m+n}{c_2+n}\right)\frac{z_1^m z_2^n}{m!n!}\\
&\quad=-\sum_{m\geq 1,\ n\geq 0}\frac{(a+1)(a+2)_{2(m-1)+n}}{c_1(c_1+1)_{m-1}(c_2+1)_n}\frac{2mz_1^m z_2^n}{m!n!}\\
&\qquad+\sum_{m\geq 0,\ n\geq 1}\frac{(a+1)_{2m+n-1}}{(c_1)_m c_2(c_2+1)(c_2+2)_{n-1}}\frac{(a-c_2)nz_1^m z_2^n}{m!n!}\\
&\quad=-2\frac{a+1}{c_1}z_1\sum_{m\geq 1,\ n\geq 0}\frac{(a+2)_{2(m-1)+n}}{(c_1+1)_{m-1}(c_2+1)_n}\frac{z_1^{m-1}z_2^n}{(m-1)!n!}\\
&\qquad-\frac{c_2-a}{c_2(c_2+1)}z_2\sum_{m\geq 0,\ n\geq 1}\frac{(a+1)_{2m+n-1}}{(c_1)_m(c_2+2)_{n-1}}\frac{z_1^m z_2^{n-1}}{m!(n-1)!}\\
&\quad=-2\frac{a+1}{c_1}z_1\sum_{m\geq 0,\ n\geq 0}\frac{(a+2)_{2m+n}}{(c_1+1)_m(c_2+1)_n}\frac{z_1^m z_2^n}{m!n!}\\
&\qquad-\frac{c_2-a}{c_2(c_2+1)}z_2\sum_{m\geq 0,\ n\geq 0}\frac{(a+1)_{2m+n}}{(c_1)_m(c_2+2)_n}\frac{z_1^m z_2^n}{m!n!}\\
&\quad=-2\frac{a+1}{c_1}z_1\,\mathrm{H}_7(a+2;c_1+1,c_2+1;\mathbf{z})-\frac{c_2-a}{c_2(c_2+1)}z_2\,\mathrm{H}_7(a+1;c_1,c_2+2;\mathbf{z}),
\end{aligned}$$

which had to be proved.

Finally,

$$\begin{aligned}
&\mathrm{H}_7(a;c_1,c_2;\mathbf{z})-\mathrm{H}_7(a;c_1,c_2+1;\mathbf{z})\\
&\quad=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(c_1)_m(c_2)_n}\frac{z_1^m z_2^n}{m!n!}-\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(c_1)_m(c_2+1)_n}\frac{z_1^m z_2^n}{m!n!}\\
&\quad=\sum_{m,n\geq 0,\ m+n\geq 1}^{\infty}\frac{a(a+1)_{2m+n-1}}{(c_1)_m}\left(\frac{1}{(c_2)_n}-\frac{1}{(c_2+1)_n}\right)\frac{z_1^m z_2^n}{m!n!}\\
&\quad=\frac{a}{c_2(c_2+1)}z_2\sum_{m\geq 0,\ n\geq 1}^{\infty}\frac{(a+1)_{2m+n-1}}{(c_1)_m(c_2+2)_{n-1}}\frac{z_1^m z_2^{n-1}}{m!(n-1)!}\\
&\quad=\frac{a}{c_2(c_2+1)}z_2\sum_{m\geq 0,\ n\geq 0}^{\infty}\frac{(a+1)_{2m+n}}{(c_1)_m(c_2+2)_n}\frac{z_1^m z_2^n}{m!n!}\\
&\quad=\frac{a}{c_2(c_2+1)}z_2\,\mathrm{H}_7(a+1;c_1,c_2+2;\mathbf{z}),
\end{aligned}$$

which is the desired three-term recurrence relation. □

We set

$$\mathrm{R}_1(a;c_1,c_2;\mathbf{z}) = \frac{\mathrm{H}_7(a;c_1,c_2;\mathbf{z})}{\mathrm{H}_7(a+1;c_1+1,c_2;\mathbf{z})}, \quad \mathrm{R}_2(a;c_1,c_2;\mathbf{z}) = \frac{\mathrm{H}_7(a;c_1,c_2;\mathbf{z})}{\mathrm{H}_7(a+1;c_1,c_2+1;\mathbf{z})},$$

and

$$\mathrm{R}_3(a;c_1,c_2;\mathbf{z}) = \frac{\mathrm{H}_7(a;c_1,c_2;\mathbf{z})}{\mathrm{H}_7(a;c_1,c_2+1;\mathbf{z})}.$$

Then, dividing (2) by $\mathrm{H}_7(a+1;c_1+1,c_2;\mathbf{z})$, (3) by $\mathrm{H}_7(a+1;c_1,c_2+1;\mathbf{z})$, and (4) by $\mathrm{H}_7(a;c_1,c_2+1;\mathbf{z})$, we get

$$\mathrm{R}_1(a;c_1,c_2;\mathbf{z}) = 1 - \frac{\frac{2(a+1)(2c_1-a)}{c_1(c_1+1)}z_2}{\mathrm{R}_1(a+1;c_1+1,c_2;\mathbf{z})} - \frac{\frac{1}{c_2}z_2}{\mathrm{R}_3(a+1;c_1+1,c_2;\mathbf{z})}, \tag{5}$$

$$\mathrm{R}_2(a;c_1,c_2;\mathbf{z}) = 1 - \frac{\frac{2(a+1)}{c_1}z_1}{\mathrm{R}_1(a+1;c_1,c_2+1;\mathbf{z})} - \frac{\frac{c_2-a}{c_2(c_2+1)}z_2}{\mathrm{R}_3(a+1;c_1,c_2+1;\mathbf{z})}, \tag{6}$$

and

$$\mathrm{R}_3(a;c_1,c_2;\mathbf{z}) = 1 + \frac{\frac{a}{c_2(c_2+1)}z_2}{\mathrm{R}_2(a;c_1,c_2+1;\mathbf{z})}. \tag{7}$$

Using the recurrence relations (5)–(7), it is possible to convert the formal expansions of the relations $\mathrm{R}_k(a;c_1,c_2;\mathbf{z})$, $k \in \{1,2,3\}$ into branched continued fractions, as it is for Horn's hypergeometric functions H_3 and H_4 in works [27,30,31]. We will show that for certain values of the parameters, it is possible to convert the formal expansions of these ratios into continued fractions.

The following is true.

Theorem 1. *A ratio*

$$\mathrm{R}_2(a;(a+1)/2,a;\mathbf{z}) \tag{8}$$

has a formal continued fraction of the form

$$1 + \cfrac{a_1 z_1}{1 + \cfrac{a_2 z_2}{1 + \cfrac{a_3 z_2}{1 + \ddots}}}, \tag{9}$$

where

$$a_{3k+1} = -4, \quad a_{3k+2} = -\frac{1}{a+2k+1}, \quad a_{3k+3} = \frac{1}{a+2k+1}, \quad k \geq 0. \tag{10}$$

Proof. We set $c_1 = (a+1)/2$, $c_2 = a$. Then, at Step 1.1 from (6), we obtain

$$\mathrm{R}_2(a;(a+1)/2,a;\mathbf{z}) = 1 - \frac{4z_1}{\mathrm{R}_1(a+1;(a+1)/2,a+1;\mathbf{z})}.$$

At Step 1.2, replacing a, c_2 by $a+1$ and c_2+1, respectively, in (5), we get

$$\mathrm{R}_1(a+1;c_1,c_2+1;\mathbf{z}) = 1 - \frac{\dfrac{2(a+2)(2c_1-a-1)}{c_1(c_1+1)}z_2}{\mathrm{R}_1(a+2;c_1+1,c_2+1;\mathbf{z})} - \frac{\dfrac{1}{c_2+1}z_2}{\mathrm{R}_3(a+2;c_1+1,c_2+1;\mathbf{z})},$$

which gives us

$$\mathrm{R}_2(a;(a+1)/2,a;\mathbf{z}) = 1 - \cfrac{4z_1}{1 - \cfrac{z_2/(a+1)}{\mathrm{R}_3(a+2;(a+3)/2,a+1;\mathbf{z})}}.$$

Since it follows from (7) that

$$\mathrm{R}_3(a+2;c_1+1,c_2+1;\mathbf{z}) = 1 + \frac{\dfrac{a+2}{(c_2+1)(c_2+2)}z_2}{\mathrm{R}_2(a+2;c_1+1,c_2+2;\mathbf{z})},$$

at Step, 1.3 we have

$$\mathrm{R}_2(a;(a+1)/2,a;\mathbf{z}) = 1 - \cfrac{4z_1}{1 - \cfrac{z_2/(a+1)}{1 + \cfrac{z_2/(a+1)}{\mathrm{R}_2(a+2;(a+3)/2,a+2;\mathbf{z})}}}. \quad (11)$$

We will continue with the next construction of a continued fraction using the ideas outlined in Steps 1.1–1.3.

By analogy, it is clear that for all $k \geq 1$, the following relation holds:

$$\mathrm{R}_2(a+2k;(a+1)/2+k,a+2k;\mathbf{z}) = 1 - \cfrac{4z_1}{1 - \cfrac{z_2/(a+2k+1)}{1 + \cfrac{z_2/(a+2k+1)}{\mathrm{R}_2(a+2k+2;(a+1)/2+k+1,a+2k+2;\mathbf{z})}}}. \quad (12)$$

Substituting relation (12) with $k=1$ in (11) in Steps 2.1–2.3 we obtain

$$\mathrm{R}_2(a;(a+1)/2,a;\mathbf{z}) = 1 - \cfrac{4z_1}{1 - \cfrac{z_2/(a+1)}{1 + \cfrac{z_2/(a+1)}{1 - \cfrac{4z_1}{1 - \cfrac{z_2/(a+3)}{1 + \cfrac{z_2/(a+3)}{\mathrm{R}_2(a+4;(a+5)/2,a+4;\mathbf{z})}}}}}}.$$

Next, by recurrence relation (12) after the nth block of Steps n.1–n.3, we get

$$\mathrm{R}_2(a;(a+1)/2,a;\mathbf{z}) = 1 - \cfrac{4z_1}{1 - \cfrac{z_2/(a+1)}{1 + \cfrac{z_2/(a+1)}{1 - \ddots - \cfrac{4z_1}{1 - \cfrac{z_2/(a+2n-1)}{1 + \cfrac{z_2/(a+2n-1)}{\mathrm{R}_2(a+2n;(a+1)/2+n,a+2n;\mathbf{z})}}}}}}.$$

Finally, by (12), one obtains the continued fraction (9) for ratio (8). □

The following two theorems can be proved analogously.

Theorem 2. *A ratio* $\mathrm{R}_1(a; a/2, a; \mathbf{z})$ *has a formal continued fraction of the form* (9)*, where*

$$a_{3k+1} = -\frac{1}{a+2k}, \quad a_{3k+2} = \frac{1}{a+2k}, \quad a_{3k+3} = -4, \quad k \geq 0. \tag{13}$$

Theorem 3. *A ratio* $\mathrm{R}_3(a; (a+1)/2, a-1; \mathbf{z})$ *has a formal continued fraction of the form* (9)*, where*

$$a_{3k+1} = \frac{1}{a+2k-1}, \quad a_{3k+2} = -4, \quad a_{3k+3} = -\frac{1}{a+2k+1}, \quad k \geq 0. \tag{14}$$

3. Convergence of Continued Fraction Expansions

To prove our next result, we recall the following theorem (see, [32], Theorem 4.42).

Theorem 4. *If all elements of a continued fraction*

$$1 + \cfrac{c_1}{1 + \cfrac{c_2}{1 + \cfrac{c_3}{1 + \ddots}}}$$

lie in a parabolic region

$$\Theta_\alpha = \left\{ w : \ |w| - \mathrm{Re}(we^{-2i\alpha}) \leq \frac{1}{2}\cos^2\alpha \right\}, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2},$$

then the continued fraction converges to a finite value if and only if at least one of the series

$$\sum_{n=1}^{\infty} \left| \frac{c_2 c_4 \ldots c_{2n}}{c_3 c_5 \ldots c_{2n+1}} \right|, \quad \sum_{n=1}^{\infty} \left| \frac{c_3 c_5 \ldots c_{2n+1}}{c_4 c_6 \ldots c_{2n+2}} \right|$$

is divergent.

The following is true.

Theorem 5. *Let a be a real constant such that* $a \neq -2k-1$ *for all* $k \geq 0$*, and let*

$$\Omega = \bigcup_{-\pi/2 < \alpha < \pi/2} \Omega_\alpha, \tag{15}$$

where

$$\Omega_\alpha = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| + \mathrm{Re}(z_1 e^{-2i\alpha}) < \frac{1}{8}\cos^2\alpha, \right. \\ \left. |z_2| + \mathrm{Re}(z_2 e^{-2i\alpha}) < \frac{a+1}{2}\cos^2\alpha, \ |z_2| - \mathrm{Re}(z_2 e^{-2i\alpha}) < \frac{a+1}{2}\cos^2\alpha \right\}.$$

Then:

(A) The continued fraction (9)*, whose coefficients are defined by* (10)*, converges uniformly on every compact subset of* (15) *to a function* $f(\mathbf{z})$ *holomorphic in* Ω*;*

(B) The function $f(\mathbf{z})$ *is an analytic continuation of* (8) *in the domain* Ω*.*

Proof. Let α be an arbitrary number from the interval $(-\pi/2, \pi/2)$, and let $\mathbf{z}$ be an arbitrary fixed point from Ω_α. It is clear that the coefficients of (9) satisfy the conditions of Theorem 4.

This yields the uniform convergence of (9) to a holomorphic function on all compact subsets of Ω_α, and, consequently, in whole domain Ω by virtue of arbitrariness α. This proves part (A). Proof of (B) is analogous to the proof of Theorem 3 [27], so it is omitted. □

An analogous two theorems could be proved in a similar way.

Theorem 6. *Let a be a real constant such that $a \neq -2k$ for all $k \geq 0$, and let*

$$\Phi = \bigcup_{-\pi/2<\alpha<\pi/2} \Phi_\alpha, \tag{16}$$

where

$$\Phi_\alpha = \Big\{ \mathbf{z} \in \mathbb{C}^2 : \, |z_1| + \mathrm{Re}(z_1 e^{-2i\alpha}) < \frac{1}{8}\cos^2\alpha, \\ |z_2| + \mathrm{Re}(z_2 e^{-2i\alpha}) < \frac{a}{2}\cos^2\alpha, \; |z_2| - \mathrm{Re}(z_2 e^{-2i\alpha}) < \frac{a}{2}\cos^2\alpha \Big\}.$$

Then:

(A) The continued fraction (9), whose coefficients are defined by (13), converges uniformly on every compact subset of (16) to a function $g(\mathbf{z})$ holomorphic in Φ;
(B) The function $g(\mathbf{z})$ is an analytic continuation of $\mathrm{R}_1(a; a/2, a; \mathbf{z})$ in the domain Φ.

Theorem 7. *Let a be a real constant such that $a \neq -2k+1$ for all $k \geq 0$. Then:*

(A) The continued fraction (9), whose coefficients are defined by (14), converges uniformly on every compact subset of (15) to a function $h(\mathbf{z})$ holomorphic in Ω;
(B) The function $h(\mathbf{z})$ is an analytic continuation of $\mathrm{R}_3(a; (a+1)/2, a-1; \mathbf{z})$ in the domain Ω.

4. Numerical Experiments

By Theorem 5, one obtains

$$\frac{z_2 \exp\left\{\frac{z_2}{1-2\sqrt{z_1}}\right\}\gamma\left(1, \frac{4\sqrt{z_1}z_2}{1-4z_1}\right)}{\gamma\left(2, \frac{-z_2}{1-2\sqrt{z_1}}\right) - \gamma\left(2, \frac{-z_2}{1+2\sqrt{z_1}}\right)} = \frac{\mathrm{H}_7\left(2; \frac{3}{2}, 2; \mathbf{z}\right)}{\mathrm{H}_7\left(3; \frac{3}{2}, 3; \mathbf{z}\right)} \\ = 1 + \cfrac{a_1 z_1}{1 + \cfrac{a_2 z_2}{1 + \cfrac{a_3 z_2}{1 + \ddots}}}, \tag{17}$$

where

$$\gamma(a, z) = \int_0^z e^{-t} t^{a-1} dt$$

is an incomplete gamma function, and $a_{3k+1} = -4$, $a_{3k+2} = -1/(2k+3)$, $a_{3k+3} = 1/(2k+3)$, $k \geq 0$.

The continued fraction in (17) converges and represents a single-valued branch of the function

$$\frac{z_2 \exp\left\{\frac{z_2}{1-2\sqrt{z_1}}\right\}\gamma\left(1, \frac{4\sqrt{z_1}z_2}{1-4z_1}\right)}{\gamma\left(2, \frac{-z_2}{1-2\sqrt{z_1}}\right) - \gamma\left(2, \frac{-z_2}{1+2\sqrt{z_1}}\right)} \tag{18}$$

in the domain (15).

The numerical illustration of series

$$\frac{z_2 \exp\left\{\dfrac{z_2}{1-2\sqrt{z_1}}\right\}\gamma\left(1, \dfrac{4\sqrt{z_1}z_2}{1-4z_1}\right)}{\gamma\left(2, \dfrac{-z_2}{1-2\sqrt{z_1}}\right) - \gamma\left(2, \dfrac{-z_2}{1+2\sqrt{z_1}}\right)} = \frac{\mathrm{H}_7\left(2; \frac{3}{2}, 2; \mathbf{z}\right)}{\mathrm{H}_7\left(3; \frac{3}{2}, 3; \mathbf{z}\right)}$$
$$= 1 - 4z_1 - \frac{4}{3}z_1z_2 + \frac{16}{9}z_1^2z_2^2 + \frac{16}{45}z_1^2z_2^3 + \frac{64}{9}z_1^3z_2^2 + \ldots \quad (19)$$

and the continued fraction (17) is given in the Table 1.

Table 1. Relative error of 10th partial sum and 10th approximant for (18).

$\mathbf{z}$	(18)	(19)	(17)
$(0.02, -0.05)$	0.921335	1.4231×10^{-13}	9.8450×10^{-14}
$(0.9, -0.7)$	-1.9733	$1.1404 \times 10^{+03}$	1.2458×10^{-07}
$(0.5, -1.5)$	-0.559039	$2.2218 \times 10^{+02}$	6.8000×10^{-04}
$(1.1, 2.1)$	-18.7215	$3.1446 \times 10^{+03}$	5.0892×10^{-04}
$(2.5, -2.5)$	-4.7833	$1.7296 \times 10^{+07}$	5.8165×10^{-06}
$(1.5, -2.5)$	-2.61448	$5.7421 \times 10^{+05}$	3.9358×10^{-05}
$(2.1, -2.5)$	-3.91426	$5.3717 \times 10^{+06}$	1.0927×10^{-05}
$(3, -3.5)$	-5.04522	$1.3625 \times 10^{+08}$	2.4661×10^{-05}
$(0.2, 10)$	0.106699	$1.9893 \times 10^{+03}$	6.2873×10^{-01}
$(0.4, -10)$	-0.27212	$2.8133 \times 10^{+04}$	4.2133×10^{-01}

In Figure 1a–d, we can see the plots where the 20th approximant of (17) guarantees certain truncation error bounds for function (18).

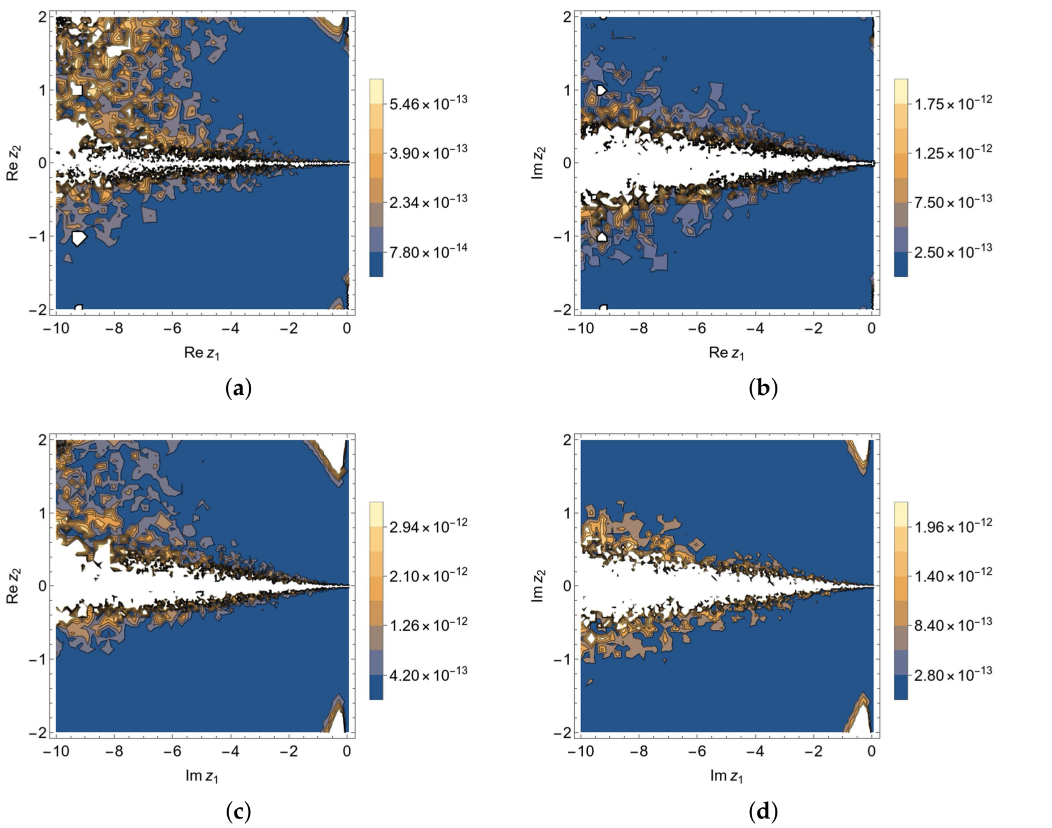

Figure 1. The plots where the approximant $f_{20}(\mathbf{z})$ of (17) guarantees certain truncation error bounds for function (18).

Calculations and plots were performed using Wolfram Mathematica software 13.1.0.0 for Linux.

5. Discussion

In this work, for the first time, expansions of ratios of hypergeometric functions of two complex variables into continued fractions were constructed. This made it possible to apply one of the well-known convergence criteria of continued fractions—the parabolic theorem—to the study of convergence. Numerical experiments showed that the domain of convergence of the constructed expansions is wider; that is, the problem of studying the convergence of such fractions remains open. One should the specific periodicity of the coefficients of the constructed expansions. One should also note that the method of establishing an analytical continuation remains the same as for branched continued fractions. More on branched continued representations of the functions of several variables can be found in the papers [33–40].

Finally, let us point out a rather interesting and promising direction of investigation: representing discrete hypergeometric series (see, [41]) via branched continued fractions.

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