axioms

Article

Sampling Plan for the Kavya–Manoharan Generalized Inverted Kumaraswamy Distribution with Statistical Inference and Applications

Najwan Alsadat [1], Amal S. Hassan [2,*], Mohammed Elgarhy [3], Christophe Chesneau [4,*] and Ahmed R. El-Saeed [5]

1 Department of Quantitative Analysis, College of Business Administration, King Saud University, P.O. Box 71115, Riyadh 11587, Saudi Arabia
2 Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt
3 Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef 62521, Egypt
4 Department of Mathematics, Université de Caen Normandie, Campus II, Science 3, 14032 Caen, France
5 Department of Basic Sciences, Obour High Institute for Management & Informatics, Obour 11848, Egypt
* Correspondence: amal52_soliman@cu.edu.eg (A.S.H.); christophe.chesneau@unicaen.fr (C.C.)

Abstract: In this article, we introduce the Kavya–Manoharan generalized inverse Kumaraswamy (KM-GIKw) distribution, which can be presented as an improved version of the generalized inverse Kumaraswamy distribution with three parameters. It contains numerous referenced lifetime distributions of the literature and a large panel of new ones. Among the essential features and attributes covered in our research are quantiles, moments, and information measures. In particular, various entropy measures (Rényi, Tsallis, etc.) are derived and discussed numerically. The adaptability of the KM-GIKw distribution in terms of the shapes of the probability density and hazard rate functions demonstrates how well it is able to fit different types of data. Based on it, an acceptance sampling plan is created when the life test is truncated at a predefined time. More precisely, the truncation time is intended to represent the median of the KM-GIKw distribution with preset factors. In a separate part, the focus is put on the inference of the KM-GIKw distribution. The related parameters are estimated using the Bayesian, maximum likelihood, and maximum product of spacings methods. For the Bayesian method, both symmetric and asymmetric loss functions are employed. To examine the behaviors of various estimates based on criterion measurements, a Monte Carlo simulation research is carried out. Finally, with the aim of demonstrating the applicability of our findings, three real datasets are used. The results show that the KM-GIKw distribution offers superior fits when compared to other well-known distributions.

Keywords: Kavya–Manoharan generated family; generalized inverse Kumaraswamy distribution; entropy; maximum product of spacing; Bayesian estimation

MSC: 60E05; 62F10; 62C10; 62D05

check for updates

Citation: Alsadat, N.; Hassan, A.S.; Elgarhy, M.; Chesneau, C.; El-Saeed, A.R. Sampling Plan for the Kavya–Manoharan Generalized Inverted Kumaraswamy Distribution with Statistical Inference and Applications. *Axioms* **2023**, *12*, 739. https://doi.org/10.3390/axioms12080739

Academic Editors: Yolanda Gómez and Inmaculada Barranco-Chamorro

Received: 3 June 2023
Revised: 21 July 2023
Accepted: 24 July 2023
Published: 27 July 2023

1. Introduction

Nowadays, in order to communicate and use data, developing effective statistical models remains a challenge. The sciences of the environment, biology, economics, and engineering are particularly demanding on such models. The desired characteristics of the (probability) distributions that form the bases of these models have, thus, been the focus of numerous generations of statisticians. In particular, various types of extensions or generalization techniques have been elaborated to enhance the properties of existing distributions. In parallel, modern informatics advancements have stimulated the practical application of complex mathematical changes that have emerged in this area. A traditional approach involves adding scale or shape parameters to make an existing distribution more sensitive to certain key modeling elements (mean, variance, tails of the distributions, skewness, kurtosis, etc.). Consequently, several novel families of continuous distributions were put forth, some of which were created in the references listed in Refs. [1–12]. In particular, Ref. [13] suggested the Dinesh–Umesh–Sanjay (DUS) transformation to obtain original lifetime distributions. Its primary advantage involves the capacity of the created distributions to keep their parameter-parsimonious nature and possesses new functionalities. The cumulative distribution function (CDF) and probability density function (PDF) of the random variable (RV) V based on the DUS transformation are given by

$$F(v) = \frac{1}{e-1}\left[e^{G(v)} - 1\right], \quad v \in \mathbb{R},$$

and

$$f(v) = \frac{1}{e-1} g(v) e^{G(v)}, \quad v \in \mathbb{R},$$

respectively, where $G(.)$ and $g(.)$ are the CDF and the PDF of a baseline continuous distribution, respectively. Ref. [14] suggested an alternative approach that created new lifetime distributions using the sine function. Based on this, Ref. [15] developed a generalized DUS transformation for generating several relevant lifetime distributions. Still in the spirit of the DUS transformation, Ref. [16] proposed a new class of parsimonious distributions, known as the Kavya–Manoharan (KM) transformation, with the following CDF and PDF:

$$F(v) = \frac{e}{e-1}\left(1 - e^{-G(v)}\right), \quad v \in \mathbb{R}, \tag{1}$$

and

$$f(v) = \frac{e}{e-1} g(v) e^{-G(v)}, \quad v \in \mathbb{R}, \tag{2}$$

respectively. Using the exponential and Weibull distributions as the baseline distributions for this modification, Ref. [16] introduced two new distributions. Some analytical properties, parameter estimates, and data analysis were presented. Based on the KM transformation, some recently modified distributions were established. The one-parameter distribution called the KM inverse length biased exponential distribution was suggested in Ref. [17]. Ref. [18] proposed an enhanced version of the Burr X (BX) distribution based on the KM transformation and used ranked set sampling for the estimation of the parameters involved. In regard to biomedical data, Ref. [19] presented an extended version of the log-logistic distribution using this transformation. A new expanded form of the Kumaraswamy (Kw) distribution, still based on the KM distribution, was presented in Ref. [20]. Further, Ref. [21] provided a new three-parameter KM exponentiated Weibull distribution.

On the other hand, the distribution of the inverse of an RV is also a famous transformation technique known as the inverse distribution. Based on an RV, say U, it consists of considering the distribution of the inverse RV of U, i.e., $V = 1/U$. Here are a few applications where the inverse distribution arises: In finance, the inverse distribution is used to model the distribution of returns on investments. More precisely, the returns on investments are often modeled by log-normal or other distributions, and the inverse of these distributions is used to model the distribution of the time to reach a certain investment goal. In actuarial science, the inverse distribution is used to model the time until a claim is made. In queuing theory, the distribution of the time between two arrivals in a queue is often modeled by an inverse distribution. The inverse distribution is also used to model the service times of customers in a queue. Overall, many studies involving inverse distributions have been treated in the literature by different researchers (see, for instance, Refs. [22–29]). In particular, the inverse Kw (IKw) distribution was created in Ref. [29] from the original Kw distribution. The corresponding CDF is indicated as follows:

$$G(v) = \left[1 - \left(1 + v^{\vartheta_1}\right)\right]^{-\vartheta_2}, \quad v > 0, \tag{3}$$

where $\vartheta_1 > 0$ and $\vartheta_2 > 0$ are shape parameters, and $G(v) = 0$ for $v \leq 0$. Ref. [30] has published the generalized version of Equation (3), known as the generalized IKw (GIKw) distribution, along with a new shape parameter. The CDF and PDF of the GIKw distribution are as follows:

$$G(v) = \left[1 - \left(1 + v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}, \quad v > 0, \tag{4}$$

and $G(v) = 0$ for $v \leq 0$, and

$$g(v) = \vartheta_1\vartheta_2\vartheta_3 v^{\vartheta_1 - 1}\left(1 + v^{\vartheta_1}\right)^{-\vartheta_2 - 1}\left[1 - \left(1 + v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3 - 1}, \quad v > 0, \tag{5}$$

and $g(v) = 0$ for $v \leq 0$, respectively, where $\vartheta_1 > 0$, $\vartheta_2 > 0$, and $\vartheta_3 > 0$ are shape parameters.

In light of the above, this article provides a contribution to the topic by introducing the Kavya–Manoharan-GIKw (KM-GIKw) distribution as a new three-parameter distribution based on the KM transformation. The following points provide sufficient justification for studying it:

- The KM-GIKw distribution has a PDF that possesses both symmetric and asymmetric forms (unimodal, inverse J-shaped, and right-skewed).
- The KM-GIKw distribution provides a great deal of versatility and contains a plethora of novel and published sub-distributions.
- The hazard function (HF) forms of the MK-GIKw distribution include decreasing and upside-down shapes.
- The KM-GIKw distribution has a closed-form quantile function (QF); it is easy to compute numerous properties and generate random numbers using it.
- In the setting of the MK-GIKw distribution, an accurate acceptance sampling plan (ASP) based on the truncated life test can be constructed.
- The parameters of the MK-GIKw distribution can be estimated quite efficiently using the Bayesian, maximum likelihood (ML), and maximum product of spacings (MPS) methods.
- In terms of data fitting, thanks to its high level of flexibility, the superiority of the KM-GIKw distribution in comparison to other well-known and comparable distributions is quite possible (this will be shown using three actual datasets; model selection results demonstrated that the suggested distribution is the most suitable choice for them).

All these items are developed through the article, with a maximum amount of information and details.

The rest is divided into the following sections: The KM-GIKw distribution is described in Section 2. Section 3 discusses some of its structural aspects. For the truncated life tests based on the KM-GIKw distribution, the ASP and related numerical work are provided in Section 4. The parameter estimation and study using a Monte Carlo simulation are presented in Section 5. Data analyses for actual world data are given in Section 6. Section 7 summarizes the article and provides its conclusion.

2. The New KM-GIKw Distribution

The CDF of the KM-GIKw distribution is obtained by replacing the CDF of the GIKw distribution in Equation (4) into Equation (1); that is:

$$F(v) = \frac{e}{e-1}\left\{1 - e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}}\right\}, \quad v > 0, \tag{6}$$

and $F(v) = 0$ for $v \leq 0$, where $\vartheta_1 > 0$, $\vartheta_2 > 0$ and $\vartheta_3 > 0$ are shape parameters.

In order to mention the parameters, the KM-GIKw distribution can be expressed by KM-GIKw $(\vartheta_1, \vartheta_2, \vartheta_3)$. The PDF associated with Equation (6) is determined by

$$f(v) = \frac{e\ \vartheta_1\vartheta_2\vartheta_3 v^{\vartheta_1-1}}{e-1}\left(1+v^{\vartheta_1}\right)^{-\vartheta_2-1}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3-1} e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}}, \quad v>0, \tag{7}$$

and $f(v) = 0$ for $v \leq 0$.

The survival function (SF) and HF of the KM-GIKw distribution are defined as follows:

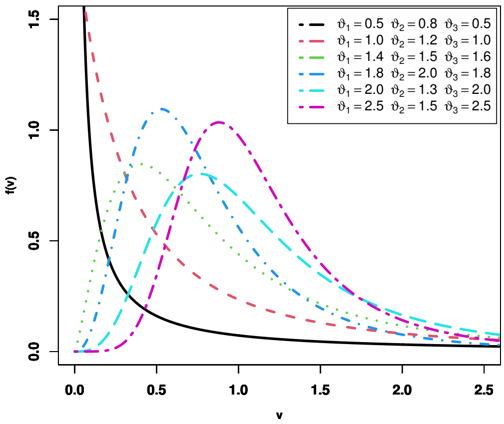
$$\bar{F}(v) = 1 - \frac{e}{e-1}\left\{1 - e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}}\right\}, \quad v>0,$$

and $\bar{F}(v) = 1$ for $v \leq 0$, and

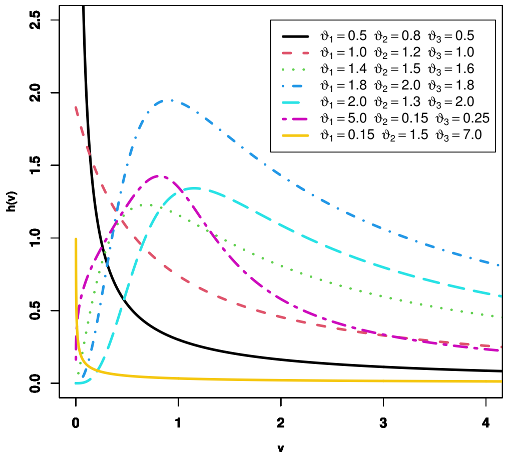
$$h(v) = \frac{e\ \vartheta_1\vartheta_2\vartheta_3 v^{\vartheta_1-1}\left(1+v^{\vartheta_1}\right)^{-\vartheta_2-1}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3-1} e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}}}{e-1-e\left(1-e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}}\right)}, \quad v>0,$$

and $h(v) = 0$ for $v \leq 0$, respectively.

Figure 1 displays the PDF and HF of the KM-GIKw distribution for certain parameter values.



(**a**)



(**b**)

Figure 1. Selected plots of (**a**) $f(v)$ and (**b**) $h(v)$.

This figure shows that the PDF can have symmetric and asymmetric forms. On this side, the HF can be decreasing and upside-down-shaped.

By introducing two RVs Z and V, the following new or existing parsimonious distributions are offered, mainly based on the CDF in Equation (6).

1. For $\vartheta_1 = 1$, the CDF in Equation (6) provides the KM-IKw distribution as a new sub-distribution.
2. Using the transformation $Z = \delta V^{\vartheta_1}$, where V has the CDF in Equation (6), then Z has the KM-exponentiated Lomax distribution with parameters $(\delta, \vartheta_2, \vartheta_3)$ as the new distribution. Hence, the CDF of Z obtains the following form:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-\left(1+\frac{z}{\delta}\right)^{-\vartheta_2}\right]^{\vartheta_3}}\right\}, \quad z>0,$$

and $F(z) = 0$ for $z \leq 0$.

3. Using the transformation $Z = \log(1 + V^{\vartheta_1})$, where V has the CDF in Equation (6), then Z has the KM-exponentiated exponential distribution with parameters $(\vartheta_2, \vartheta_3)$ (see Ref. [31]). Hence, the CDF of Z obtains the following form:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-e^{-\vartheta_2 z}\right]^{\vartheta_3}}\right\}, \quad z > 0.$$

and $F(z) = 0$ for $z \leq 0$. For $\vartheta_3 = 1$, Z has the KM-exponential distribution with parameter (ϑ_2) (see Ref. [16]).

4. Using the transformation $Z = \frac{1}{\vartheta_1}[\log(1 + V^{\vartheta_1})]^{1/\delta}$, where V has the CDF in Equation (6) and let $\vartheta_2 = 1$, then Z has the KM-exponentiated Weibull distribution with parameters $(\vartheta_3, \vartheta_1, \delta)$ (see Ref. [21]). Hence, the CDF of Z obtains the following structure:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-e^{-(\vartheta_1 z)^{\delta}}\right]^{\vartheta_3}}\right\}, \quad z > 0.$$

and $F(z) = 0$ for $z \leq 0$. For $\vartheta_3 = 1$, Z has the KM–Weibull (KM-W) distribution with parameters (ϑ_1, δ) (see Ref. [16]).

5. Using the transformation $Z = [\log(1 + V^{\vartheta_1})]^{1/2}$, where V has the CDF in Equation (6) and let $\vartheta_3 = 1$, then Z has the KM–Rayleigh distribution with parameter ϑ_2 (see Ref. [16]). Hence, the CDF of Z has the following structure:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-e^{-\vartheta_2 z^2}\right]}\right\}, \quad z > 0,$$

and $F(z) = 0$ for $z \leq 0$.

6. Using the transformation $Z = \frac{1}{\vartheta_1}[\log(1 + V^{\vartheta_1})]^{1/2}$, where V has the CDF in Equation (6) and let $\vartheta_2 = 1$, then Z has the KM–Burr X (KM-BX) distribution with parameters $(\vartheta_1, \vartheta_3)$ (see Ref. [18]). Hence, the CDF of Z has the following structure:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-e^{-(\vartheta_1 z)^2}\right]^{\vartheta_3}}\right\}, \quad z > 0,$$

and $F(z) = 0$ for $z \leq 0$.

7. Using the transformation $Z = \delta V$, where V has the CDF in Equation (6) and let $\vartheta_2 = \vartheta_3 = 1$, then Z has the KM–Log logistic distribution with parameters (δ, ϑ_1) (see Ref. [19]). Hence, the CDF of Z has the following structure:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-\left(1+\left(\frac{z}{\delta}\right)^{\vartheta_1}\right)^{-1}\right]}\right\}, \quad z > 0,$$

and $F(z) = 0$ for $z \leq 0$.

8. Using the transformation $Z = V^{-(\vartheta_1/\delta)}$, where V has the CDF in Equation (6) and let $\vartheta_3 = 1$, then Z has the KM–Burr III distribution with parameters (δ, ϑ_2) as the new distribution. Hence, the CDF of Z has the following structure:

$$F(z) = 1 - \frac{e}{e-1}\left\{1 - e^{-\left[1-(1+z^{-\delta})^{-\vartheta_2}\right]}\right\}, \quad z > 0,$$

and $F(z) = 0$ for $z \leq 0$.

9. Using the transformation $Z = V^{-(\vartheta_1/\delta)}$, where V has the CDF in Equation (6), then Z has the KM–Kumaraswamy Burr III distribution with parameters $(\delta, \vartheta_2, \vartheta_3)$ as the new distribution. Hence, the CDF of Z has the following structure:

$$F(z) = 1 - \frac{e}{e-1}\left\{1 - e^{-\left[1-(1+z^{-\delta})^{-\vartheta_2}\right]^{\vartheta_3}}\right\}, \quad z > 0,$$

and $F(z) = 0$ for $z \le 0$.

10. Using the transformation $Z = V^{(\vartheta_1/\delta)}$, where V has the CDF in Equation (6), then Z has the KM-exponentiated Burr XII distribution with parameters $(\delta, \vartheta_2, \vartheta_3)$ as the new distribution. Hence, the CDF of Z has the following structure:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-\left(1+z^{\delta}\right)^{-\vartheta_2}\right]^{\vartheta_3}}\right\}, \quad z > 0,$$

and $F(z) = 0$ for $z \le 0$.

11. Using the transformation $Z = V^{(\vartheta_1/\delta)}$, where V has the CDF in Equation (6), then Z has the KM–Burr XII (KM-XBII) distribution with parameters (δ, ϑ_2) as the new distribution. Hence, the CDF of Z has the following structure:

$$F(z) = \frac{e}{e-1}\left\{1 - e^{-\left[1-\left(1+z^{\delta}\right)^{-\vartheta_2}\right]}\right\}, \quad z > 0,$$

and $F(z) = 0$ for $z \le 0$.

3. Statistical Properties

In this section, we examine the statistical features of the KM-GIKw distribution, such as the QF, moments, and entropy measures.

3.1. Quantile Function

Theoretical considerations, statistical applications, and Monte Carlo techniques all involve the QF. The QF of the KM-GIKw distribution is represented by

$$v_u = Q(u) = \left(\left(1 - \left\{-\log\left[1 - u\left(1 - e^{-1}\right)\right]\right\}^{1/\vartheta_3}\right)^{-(1/\vartheta_2)} - 1\right)^{1/\vartheta_1}, \tag{8}$$

where $u \in (0,1)$. For $u = 0.5$, we obtain the median of the KM-GIKw distribution.

3.2. Moment Measures

The moments with various orders play a key role in defining the characteristics of the variability of a distribution. Here, we determine the main moment measures for the KM-GIKw distribution. To this aim, here and after, we consider an RV V having the KM-GIKw distribution. For any integer q, the qth moment of V is obtained as $\mu'_q = E(V^q)$, so that

$$\mu'_q = \frac{e\ \vartheta_1\vartheta_2\vartheta_3}{e-1}\int_0^{\infty} v^{q+\vartheta_1-1}\left(1+v^{\vartheta_1}\right)^{-\vartheta_2-1}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3-1} e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}} dv. \tag{9}$$

Let us now investigate its existence in the functions of the involved parameters. When $v \to 0$, we have

$$v^{q+\vartheta_1-1}\left(1+v^{\vartheta_1}\right)^{-\vartheta_2-1}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3-1} e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}} \sim \vartheta_2^{\vartheta_3-1} v^{q+\vartheta_1\vartheta_3-1},$$

and, since $q \geq 0$, $\vartheta_1 > 0$ and $\vartheta_3 > 0$, we always have $q + \vartheta_1\vartheta_3 - 1 > -1$. On the other hand, when $v \to \infty$, we obtain

$$v^{q+\vartheta_1-1}\left(1+v^{\vartheta_1}\right)^{-\vartheta_2-1}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3-1} e^{-\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}} \sim e^{-1}v^{q-\vartheta_1\vartheta_2-1}$$

and we have $-q + \vartheta_1\vartheta_2 + 1 > 1$ if $q < \vartheta_1\vartheta_2$. Hence, according to the Riemann integral convergence rules, the qth moment of V exists if and only if $q < \vartheta_1\vartheta_2$.

This integral is complicated to simplify, but two complementary options are possible: (i) numerical computation for the fixed values of the parameters and (ii) series expansions that incorporate as much moment information as possible into discrete coefficients. The numerical approach is taken into account throughout the rest of the study.

Furthermore, based on these moments, the qth central moment of V is defined by

$$\mu_q = E[(V-\mu_1')^q] = \sum_{l=0}^{q}(-1)^l \binom{q}{l} (\mu_1')^l \mu_{q-l}'. \tag{10}$$

Diverse moment measures can be obtained based on Equation (10). The main ones are provided in Table 1, considering the first few moments, i.e., variance (var), skewness (SK), kurtosis (KU), coefficient of variation (CV), and index of dispersion (ID) of V. We conclude from this table that

- As the value of ϑ_3 increases, at fixed values of ϑ_1 and ϑ_3, the values of μ_1', μ_2', μ_3', μ_4', SK, and KU are rising, whereas those of var, CV, and ID are decreasing.
- The values of μ_1', μ_2', μ_3', μ_4', var, SK, KU, CV, and ID are decreasing when the values of ϑ_2 increase at a fixed value of ϑ_1 and ϑ_3 in the majority of situations.
- The KM-GIKw distribution is positively skewed, as indicated by the values of SK.
- The KM-GIKw distribution is platykurtic and leptokurtic based on the values of KU.

Table 1. Numerical values of certain moments associated with the KM-GIKw distribution.

ϑ_1	ϑ_2	ϑ_3	μ_1'	μ_2'	μ_3'	μ_4'	var	SK	KU	CV	ID
3.0	2.0	2.0	0.915	0.969	1.222	1.955	0.139	1.938	14.148	0.398	0.145
		3.0	1.036	1.211	1.645	2.778	0.137	2.101	15.927	0.357	0.132
		4.0	1.123	1.402	2.011	3.539	0.132	2.211	17.162	0.336	0.126
		5.0	1.190	1.563	2.337	4.252	0.127	2.289	18.081	0.322	0.123
	3.0	2.0	0.759	0.644	0.613	0.660	0.068	1.172	6.412	0.344	0.090
		3.0	0.851	0.790	0.804	0.911	0.065	1.279	7.035	0.300	0.077
		4.0	0.915	0.901	0.963	1.132	0.064	1.358	7.495	0.276	0.070
		5.0	0.963	0.991	1.099	1.331	0.063	1.418	7.852	0.261	0.066
	4.0	2.0	0.673	0.500	0.409	0.368	0.047	0.897	4.896	0.322	0.070
		3.0	0.751	0.608	0.530	0.501	0.043	0.982	5.279	0.277	0.058
		4.0	0.804	0.688	0.629	0.616	0.041	1.047	5.566	0.252	0.051
		5.0	0.845	0.753	0.712	0.718	0.040	1.099	5.794	0.236	0.047

Table 1. *Cont.*

ϑ_1	ϑ_2	ϑ_3	μ_1'	μ_2'	μ_3'	μ_4'	var	SK	KU	CV	ID
5.0	2.0	2.0	0.932	0.914	0.946	1.039	0.045	1.017	6.036	0.228	0.048
		3.0	1.008	1.058	1.161	1.341	0.042	1.198	6.844	0.203	0.042
		4.0	1.059	1.163	1.328	1.589	0.040	1.313	7.409	0.190	0.038
		5.0	1.098	1.246	1.467	1.803	0.040	1.394	7.832	0.181	0.036
	3.0	2.0	0.836	0.728	0.660	0.623	0.029	0.602	4.199	0.203	0.034
		3.0	0.899	0.832	0.796	0.786	0.025	0.750	4.606	0.175	0.028
		4.0	0.940	0.906	0.898	0.915	0.023	0.850	4.911	0.161	0.024
		5.0	0.970	0.963	0.980	1.023	0.022	0.923	5.150	0.152	0.022
	4.0	2.0	0.779	0.629	0.527	0.456	0.022	0.423	3.680	0.192	0.029
		3.0	0.835	0.716	0.630	0.570	0.019	0.554	3.952	0.164	0.022
		4.0	0.871	0.776	0.707	0.658	0.017	0.645	4.159	0.148	0.019
		5.0	0.898	0.822	0.767	0.731	0.015	0.712	4.327	0.139	0.017

Figures 2–4 provide further trends by displaying the 3D plots of the mean, variance, skewness, and kurtosis of V for various values of ϑ_1, ϑ_2, and ϑ_3. Various non-monotonic shapes are observed, illustrating the versatility of these measures.

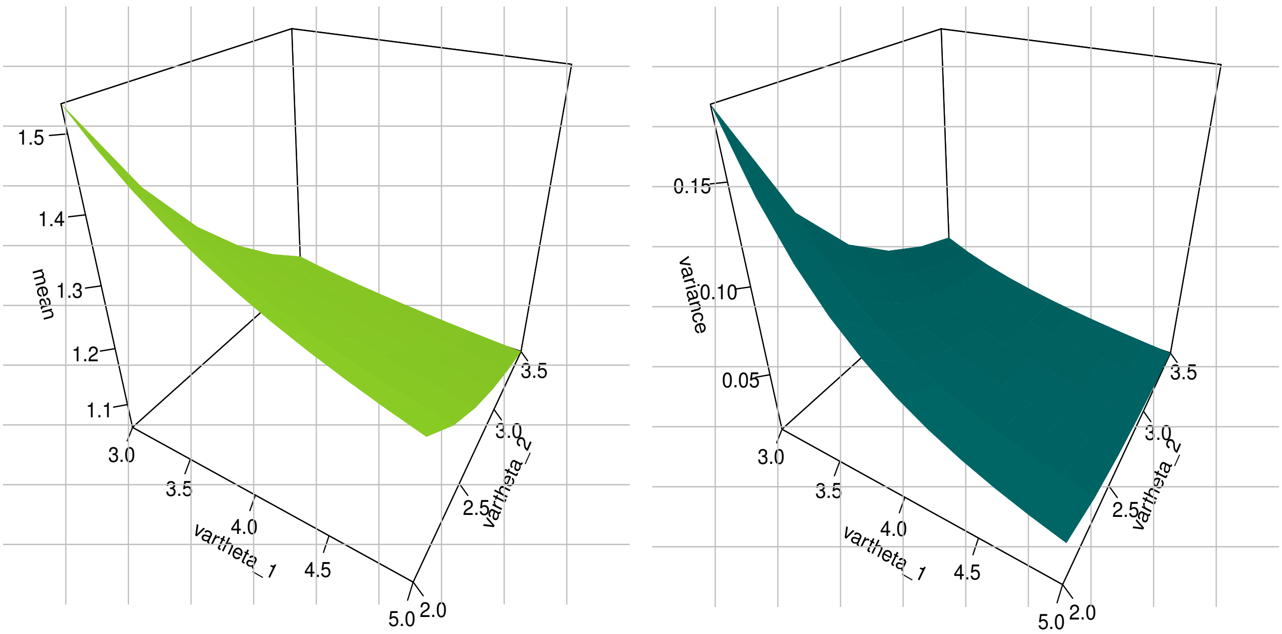


Figure 2. *Cont.*

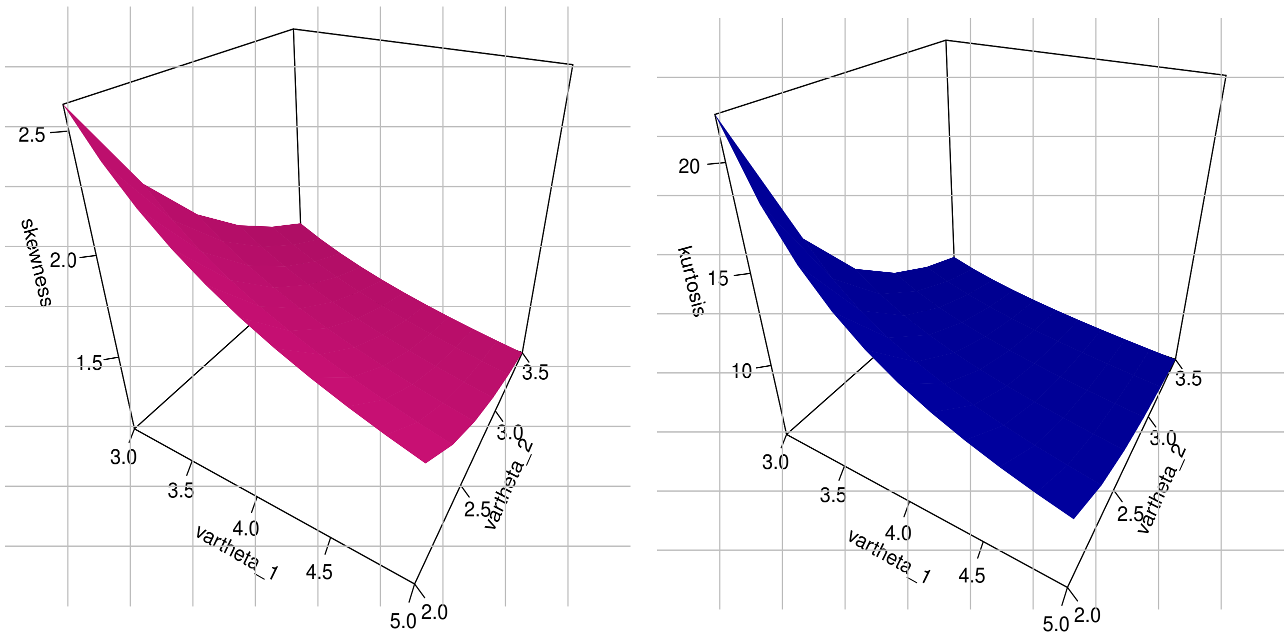


Figure 2. The 3D plots of the mean (light green), variance (dark green), skewness (dark pink), and kurtosis (dark blue) associated with the KM-GIKw distribution at $\vartheta_3 = 15.0$.

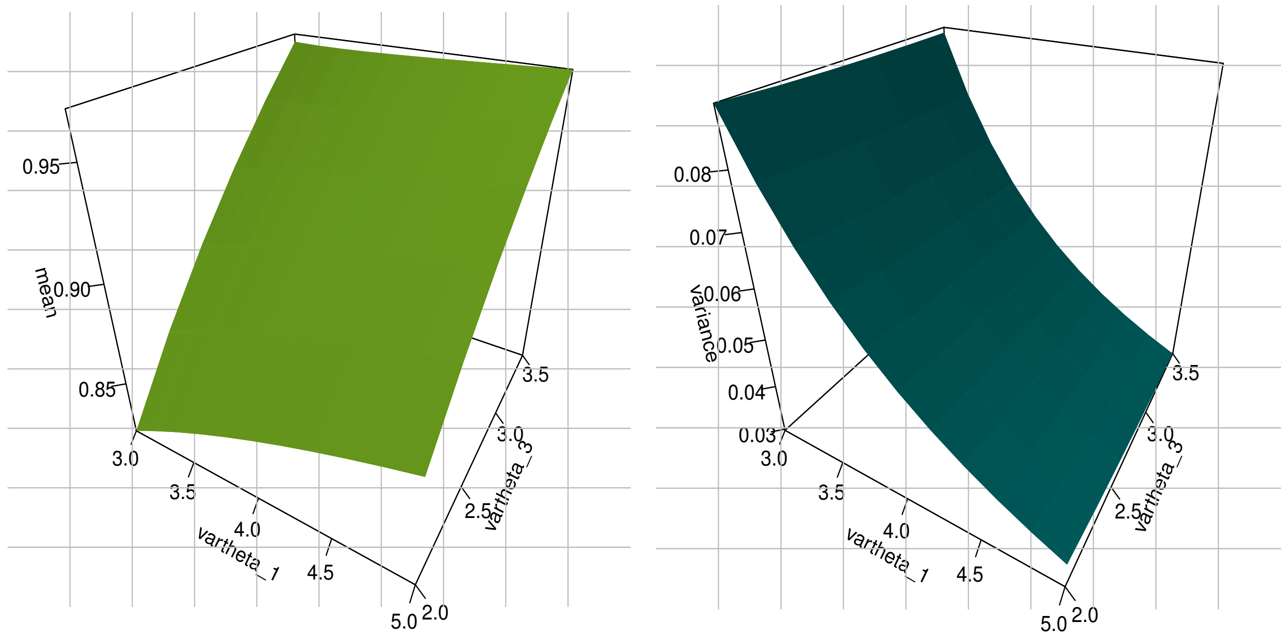


Figure 3. *Cont.*

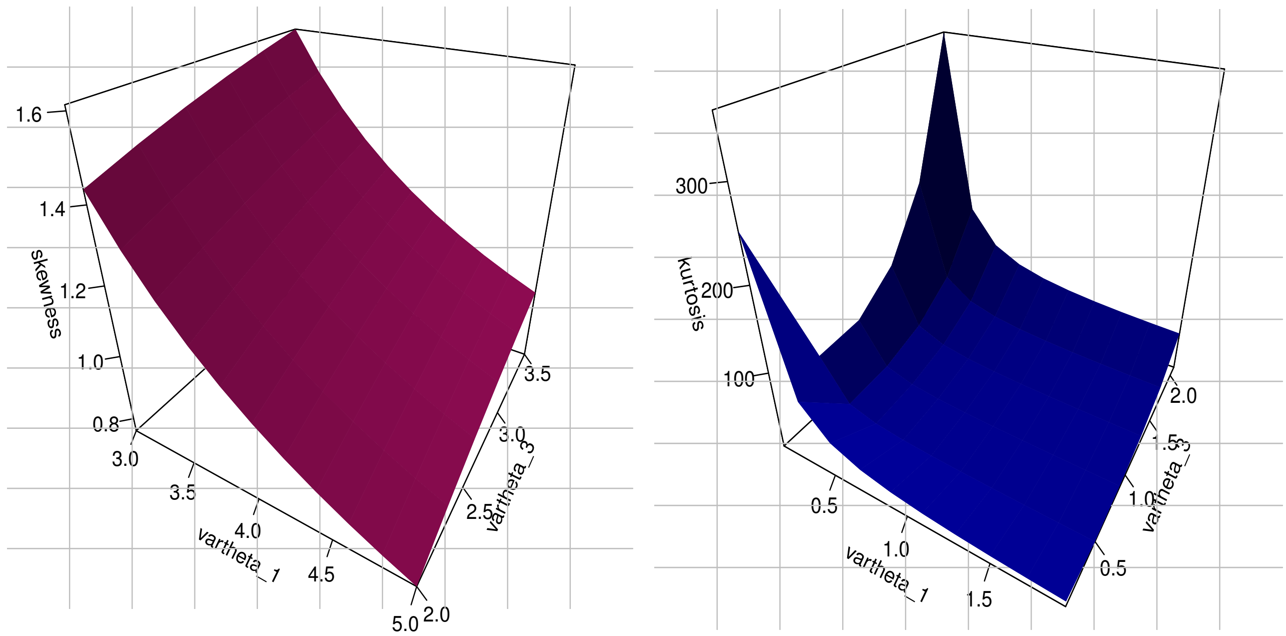


Figure 3. The 3D plots of the mean (light green), variance (dark green), skewness (dark pink), and kurtosis (dark blue) associated with the KM-GIKw distribution at $\vartheta_2 = 2.5$.

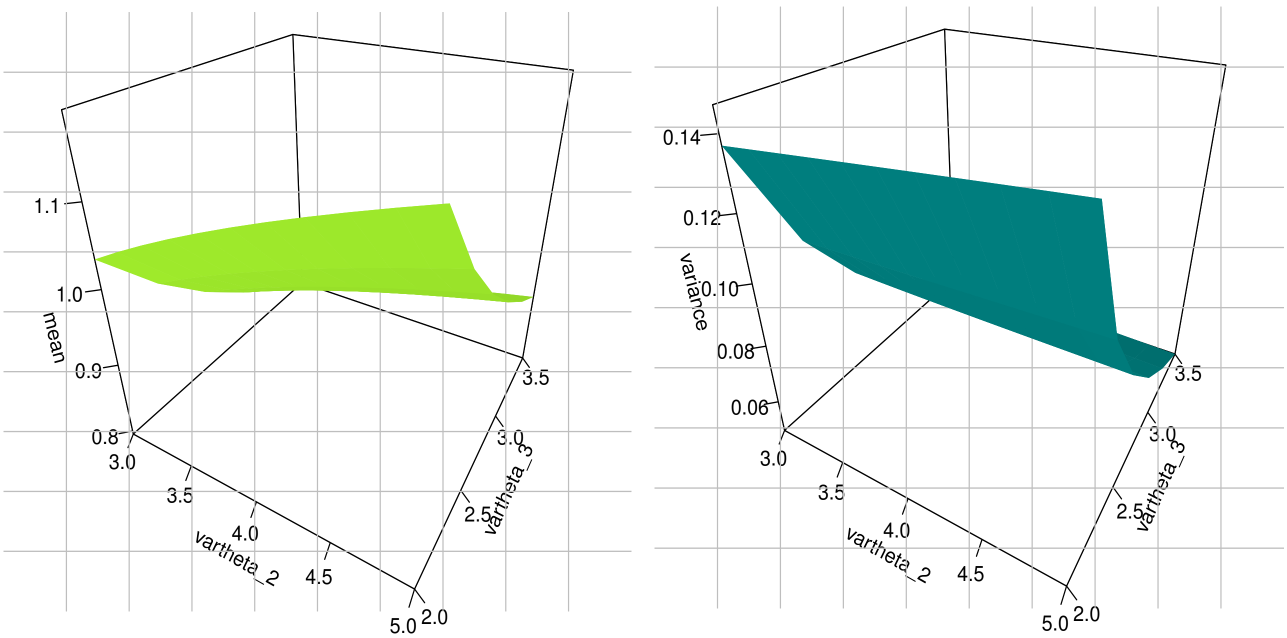


Figure 4. *Cont.*

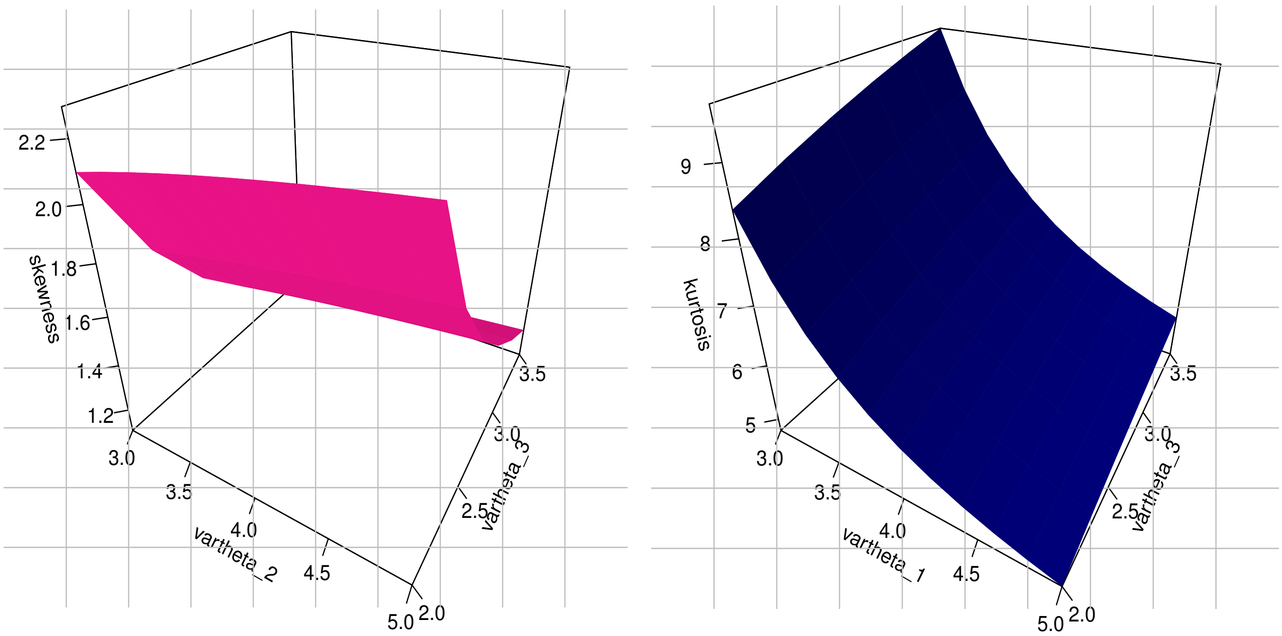


Figure 4. The 3D plots of the mean (light green), variance (dark green), skewness (dark pink), and kurtosis (dark blue) associated with the KM-GIKw distribution at ϑ_1 = 3.0.

3.3. Entropy Measures

In several disciplines, including physics, engineering, and economics, the entropy of an RV is a measure of variation in uncertainty. As evidenced by Rényi in Ref. [32], the Rényi (Ri) entropy is defined as follows:

$$I^{\dagger\dagger}(\tau) = \frac{1}{1-\tau} \log\left[\int_{-\infty}^{\infty} [f(v)]^{\tau} dv\right], \tag{11}$$

with $\tau \neq 1$, $\tau > 0$, and where $f(v)$ denotes the PDF of the considered RV. In the precise context of the KM-GIKw distribution, for $v > 0$, we have

$$[f(v)]^{\tau} = \left(\frac{e\ \vartheta_1\vartheta_2\vartheta_3}{e-1}\right)^{\tau} v^{\tau(\vartheta_1-1)}\left(1+v^{\vartheta_1}\right)^{-\tau(\vartheta_2+1)}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\tau(\vartheta_3-1)} e^{-\tau\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}}.$$

For $v \leq 0$, we obviously have $[f(v)]^{\tau} = 0$. Let us now investigate the existence of $\int_0^{\infty} [f(v)]^{\tau} dv$, and the Ri entropy as well, in the function of the involved parameters. When $v \to 0$, we have

$$v^{\tau(\vartheta_1-1)}\left(1+v^{\vartheta_1}\right)^{-\tau(\vartheta_2+1)}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\tau(\vartheta_3-1)} e^{-\tau\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}} \sim \vartheta_2^{\tau(\vartheta_3-1)} v^{\tau(\vartheta_1\vartheta_3-1)}.$$

Hence, according to the Riemann integral convergence rules, we must have $\tau(\vartheta_1\vartheta_3 - 1) > -1$. On the other hand, when $v \to \infty$, we obtain

$$v^{\tau(\vartheta_1-1)}\left(1+v^{\vartheta_1}\right)^{-\tau(\vartheta_2+1)}\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\tau(\vartheta_3-1)} e^{-\tau\left[1-\left(1+v^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}} \sim e^{-\tau} v^{-\tau(\vartheta_1\vartheta_2+1)}.$$

Hence, according to the Riemann integral convergence rules, we must have $\tau(\vartheta_1\vartheta_2 + 1) > 1$. To summarize, the Ri entropy makes mathematical sense if and only if $\tau(\vartheta_1\vartheta_3 - 1) > -1$ and $\tau(\vartheta_1\vartheta_2 + 1) > 1$.

Under these conditions, it is challenging to simplify this integral, and there are two complementary approaches that can be used: (i) numerical computation of the integral for fixed values of the parameters, and (ii) series expansions that incorporate as much moment information as possible into discrete coefficients. The first point will be investigated in this study.

Based on the Ri entropy, the Shannon entropy can be derived as

$$S^{\dagger\dagger} = \lim_{\tau \to 1} I^{\dagger\dagger}(\tau) = -\int_{-\infty}^{\infty} f(v) \log [f(v)] dv.$$

See [33]. By applying the previous findings, it exists if and only if $\vartheta_1\vartheta_3 > 0$ and $\vartheta_1\vartheta_2 > 0$, which is always the case.

On the other hand, the following formula is used to compute the Tsallis entropy of V:

$$T^{\dagger\dagger}(\tau) = \frac{1}{\tau - 1}\left[1 - \int_{-\infty}^{\infty} [f(v)]^{\tau} dv\right],$$

where $\tau \neq 1$, $\tau > 0$.

The Arimoto entropy is specified by

$$A^{\dagger\dagger}(\tau) = \frac{\tau}{1-\tau}\left[\left(\int_{-\infty}^{\infty} [f(v)]^{\tau} dv\right)^{\frac{1}{\tau}} - 1\right],$$

where $\tau \neq 1$ and $\tau > 0$.

Also, another famous entropy measure, the Havrda–Charvat entropy, is specified by

$$HC^{\dagger\dagger}(\tau) = \frac{1}{2^{1-\tau} - 1}\left[\left(\int_{-\infty}^{\infty} [f(v)]^{\tau} dv\right)^{\frac{1}{\tau}} - 1\right],$$

where $\tau \neq 1$ and $\tau > 0$.

The three above entropy measures make sense in the setting of the KM-GIKw distribution if and only if $\tau(\vartheta_1\vartheta_3 - 1) > -1$ and $\tau(\vartheta_1\vartheta_2 + 1) > 1$.

Table 2 displays some numerical measures of the introduced entropy measures. We conclude from this table that:

- All the entropy measures decrease when the value of τ increases, giving us additional information.
- As the value of ϑ_3 increases, at fixed values of ϑ_1 and ϑ_2, except at $(\vartheta_1, \vartheta_2) = (3, 2)$, we observe that all the entropy measures decrease.
- As the value of ϑ_2 increases, at fixed values of ϑ_1 and ϑ_3, we observe that all the entropy measures decrease.

Table 2. Some numerical values of the considered entropy measures.

ϑ_1	ϑ_2	ϑ_3	$\tau = 0.5$				$\tau = 0.8$				$\tau = 1.2$			
			$I^{\dagger\dagger}$	$T^{\dagger\dagger}$	$A^{\dagger\dagger}$	$HC^{\dagger\dagger}$	$I^{\dagger\dagger}$	$T^{\dagger\dagger}$	$A^{\dagger\dagger}$	$HC^{\dagger\dagger}$	$I^{\dagger\dagger}$	$T^{\dagger\dagger}$	$A^{\dagger\dagger}$	$HC^{\dagger\dagger}$
3.0	2.0	2.0	0.265	0.714	0.842	2.033	0.153	0.366	0.369	0.621	0.083	0.187	0.188	0.241
		3.0	0.271	0.733	0.868	2.095	0.155	0.369	0.372	0.626	0.081	0.184	0.184	0.237
		4.0	0.277	0.751	0.891	2.152	0.157	0.375	0.379	0.637	0.082	0.186	0.187	0.240
		5.0	0.282	0.767	0.914	2.207	0.160	0.384	0.387	0.651	0.085	0.191	0.192	0.247
	3.0	2.0	0.122	0.302	0.325	0.785	0.033	0.077	0.077	0.130	−0.027	−0.063	−0.063	−0.081
		3.0	0.114	0.281	0.301	0.726	0.021	0.048	0.048	0.080	−0.042	−0.098	−0.098	−0.126
		4.0	0.109	0.267	0.285	0.688	0.013	0.030	0.030	0.051	−0.051	−0.120	−0.119	−0.154
		5.0	0.106	0.259	0.276	0.666	0.008	0.019	0.019	0.031	−0.057	−0.133	−0.133	−0.171
	4.0	2.0	0.040	0.093	0.096	0.231	−0.039	−0.090	−0.090	−0.151	−0.095	−0.224	−0.223	−0.287
		3.0	0.024	0.056	0.057	0.138	−0.059	−0.134	−0.134	−0.225	−0.117	−0.277	−0.276	−0.355
		4.0	0.013	0.031	0.031	0.076	−0.072	−0.163	−0.163	−0.274	−0.132	−0.312	−0.311	−0.400
		5.0	0.006	0.013	0.013	0.032	−0.081	−0.184	−0.183	−0.308	−0.142	−0.337	−0.335	−0.432
5.0	2.0	2.0	0.045	0.106	0.108	0.262	−0.046	−0.106	−0.105	−0.177	−0.108	−0.256	−0.255	−0.328
		3.0	0.025	0.059	0.060	0.145	−0.070	−0.158	−0.158	−0.265	−0.134	−0.318	−0.316	−0.407
		4.0	0.015	0.035	0.035	0.084	−0.083	−0.186	−0.186	−0.312	−0.148	−0.352	−0.350	−0.451
		5.0	0.009	0.020	0.020	0.049	−0.090	−0.203	−0.202	−0.340	−0.156	−0.373	−0.371	−0.478
	3.0	2.0	−0.056	−0.125	−0.121	−0.292	−0.134	−0.299	−0.297	−0.499	−0.189	−0.455	−0.452	−0.582
		3.0	−0.088	−0.193	−0.183	−0.442	−0.169	−0.375	−0.372	−0.625	−0.227	−0.550	−0.545	−0.702
		4.0	−0.108	−0.233	−0.219	−0.530	−0.191	−0.421	−0.416	−0.700	−0.249	−0.608	−0.602	−0.776
		5.0	−0.121	−0.260	−0.243	−0.587	−0.206	−0.452	−0.447	−0.751	−0.265	−0.648	−0.642	−0.826
	4.0	2.0	−0.114	−0.245	−0.230	−0.556	−0.186	−0.410	−0.406	−0.682	−0.238	−0.579	−0.574	−0.739
		3.0	−0.152	−0.322	−0.296	−0.714	−0.227	−0.497	−0.491	−0.825	−0.281	−0.692	−0.684	−0.881
		4.0	−0.177	−0.368	−0.334	−0.807	−0.254	−0.551	−0.543	−0.914	−0.309	−0.764	−0.755	−0.972
		5.0	−0.194	−0.400	−0.360	−0.870	−0.272	−0.589	−0.580	−0.975	−0.328	−0.815	−0.804	−1.036

4. Acceptance Sampling Plans

This section is devoted to the construction of an ASP in the context of the KM-GIKw distribution. Hence, we assume that the lifetime of a product can be modeled as an RV with the KM-GIKw $(\vartheta_1, \vartheta_2, \vartheta_3)$ distribution given by the CDF in Equation (6), and the producer-assumed specified median lifetime of the units is M_0. The main objective is to determine if the suggested lot should be approved or disapproved based on the criterion that the unit's actual median lifetime, M, is longer than the indicated lifetime, M_0. The test is typically terminated after a certain period t and the failure number is recorded. In order to detect the median lifetime, the experiment is conducted for $t = aM_0$ units of time, a multiple of the expected median lifetime multiplied by any positive constant a.

The ASP has been the subject of several studies. In particular, a single ASP for the three-parameter inverse Topp–Leone (ITL) and power ITL distributions was found in Refs. [34,35] using the median lifetime of the provided distribution and the truncated life test.

The following is how the idea is described in Ref. [36], regarding acceptance of the offered lot, based on proof that $M \geq M_0$, given the probability of at least p^* (consumer's risk) and the ASP:

1. Pick a sample of n units at random from the indicated lot.
2. Execute the test below for t units of time:
   If the acceptance number, denoted by c, or fewer units malfunction during the test, accept the entire lot; elsewhere, reject the entire lot.

According to the proposed ASP, the probability of accepting a lot is provided by taking into account lots that are large enough to allow for the implementation of the binomial distribution. It is given by

$$L(\delta) = \sum_{i=0}^{c} \binom{n}{i} \delta^i (1-\delta)^{n-i}, \quad i = 1, 2, \ldots, n, \tag{12}$$

where $\delta = F(t) = F(t; \vartheta_1, \vartheta_2, \vartheta_3)$, as defined by the CDF in Equation (6). The function $L(\delta)$ represents the sampling plan's operational characteristic or the acceptance probability of the lot as a function of the failure probability. Additionally, by using the formula $t = aM_0$, δ_0 can be expressed as follows:

$$\delta_0 = F(aM_0; \vartheta_1, \vartheta_2, \vartheta_3) = \frac{e}{e-1} \left\{ 1 - e^{-\left[1-\left(1+(aM_0)^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}} \right\}. \tag{13}$$

The current problem is to find the lowest positive integer n for given values of p^*, aM_0, and c. Thus, the operating characteristic function can be rewritten as follows:

$$L(\delta_0) = \sum_{i=0}^{c} \binom{n}{i} \delta_0^i (1-\delta_0)^{n-i} \le 1 - p^*, \tag{14}$$

where δ_0 is provided in Equation (13).

The low values of n satisfying the inequality in Equation (14) and its corresponding operating characteristic probability are computed and mentioned in Tables 3–10, for the supposed parameters listed below:

1. The consumer risk is assumed as follows: $p^* = 0.1, 0.25, 0.5, 0.75$, and 0.99.
2. The acceptance number of each proposed lot is assumed as follows: $c = 0, 2, 4, 10$, and 20.
3. The factor median lifetime is assumed as $a = 0.15, 0.30, 0.60, 0.90$, and 1. If $a = 1$, then $t = M_0 = 0.5$ for all values of ϑ_1, ϑ_2, and ϑ_3.
4. Eight cases for parameters of the KM-GIKw distribution are considered, where the following values are assumed: $0.25, 0.75, 1.25, 1.5$, and 2.

The following observations may be drawn based on the information shown in the tables:

- For the parameters of the ASP, when p^* and c rise, the necessary sample size n rises as well, whereas $L(\delta_0)$ reduces. While a rises, the necessary n falls, while $L(\delta_0)$ rises.
- For the parameters of the KM-GIKw distribution: With increasing any parameters of $\vartheta_1, \vartheta_2, \vartheta_3$, and keeping the other parameters constant, the required n rises, but $L(\delta_0)$ decreases.

Lastly, we verify each of our outcomes: $L(\delta_0) \le 1 - p^*$. Also, when $a = 1$, we have $\delta_0 = 0.5$ as $t = M_0$ and, hence, all numerical results $(n, L(\delta_0))$ for any vector of parameters $(\vartheta_1, \vartheta_2, \lambda)$ are the same.

Table 3. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters: $(\vartheta_1, \vartheta_2, \vartheta_3) = (0.25, 0.25, 0.25)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	0.15		0.30		0.60		0.90		1	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		3	1.0000	3	1.0000	3	1.0000	3	1.0000	3	1.0000
	4		7	0.9181	7	0.9048	6	0.9685	6	0.9652	6	0.9643
	10		17	0.9351	17	0.9164	16	0.9399	16	0.9302	16	0.9274
	20		36	0.9191	35	0.9174	34	0.9171	33	0.9298	33	0.9258
0.25	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		4	0.8977	4	0.8865	4	0.8743	4	0.8667	4	0.8647
	4		8	0.8230	8	0.7986	8	0.7720	8	0.7553	8	0.7508
	10		20	0.7712	19	0.8000	19	0.7573	18	0.8104	18	0.8044
	20		40	0.7662	39	0.7526	37	0.7944	37	0.7586	36	0.8036
0.50	0		2	0.5323	2	0.5159	1	1.0000	1	1.0000	1	1.0000
	2		6	0.5604	6	0.5297	5	0.6862	5	0.6711	5	0.6671
	4		10	0.5791	10	0.5390	9	0.6348	9	0.6128	9	0.6070
	10		23	0.5377	22	0.5585	21	0.5849	21	0.5495	21	0.5402
	20		44	0.5486	43	0.5211	41	0.5582	41	0.5079	40	0.5599
0.75	0		3	0.2834	3	0.2661	2	0.4991	2	0.4892	2	0.4866
	2		8	0.2829	8	0.2534	7	0.3421	7	0.3237	7	0.3189
	4		13	0.2623	12	0.3113	12	0.2724	12	0.2505	11	0.3444
	10		27	0.2583	26	0.2614	25	0.2677	24	0.3013	24	0.2926
	20		49	0.2875	48	0.2558	46	0.2717	45	0.2755	45	0.2641
0.99	0		8	0.0121	7	0.0188	7	0.0155	7	0.0137	7	0.0133
	2		15	0.0123	14	0.0150	14	0.0110	13	0.0160	13	0.0153
	4		21	0.0128	20	0.0137	19	0.0152	19	0.0122	19	0.0115
	10		38	0.0113	36	0.0136	35	0.0118	34	0.0128	34	0.0119
	20		64	0.0111	61	0.0130	59	0.0119	58	0.0109	57	0.0134

Table 4. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3) = (0.25, 0.25, 0.75)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	0.15		0.30		0.60		0.90		1	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		4	0.9481	4	0.9278	4	0.9005	3	1.0000	3	1.0000
	4		8	0.9276	7	0.9509	7	0.9213	6	0.9711	6	0.9687
	10		21	0.9183	19	0.9235	17	0.9393	17	0.9057	16	0.9407
	20		44	0.9189	40	0.9159	36	0.9265	35	0.9017	34	0.9186
0.25	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		5	0.8505	5	0.8013	4	0.9005	4	0.8806	4	0.8750
	4		10	0.7873	9	0.7998	8	0.8290	8	0.7858	8	0.7734
	10		24	0.7978	22	0.7824	20	0.7825	19	0.7796	19	0.7596
	20		49	0.7823	44	0.7895	40	0.7823	38	0.7734	37	0.7974
0.50	0		2	0.6270	2	0.5835	2	0.5366	2	0.5076	1	1.0000
	2		7	0.6003	7	0.5102	6	0.5683	6	0.5143	5	0.6874
	4		13	0.5163	11	0.5913	10	0.5894	10	0.5188	9	0.6366
	10		29	0.5156	26	0.5189	23	0.5537	22	0.5282	21	0.5880
	20		56	0.5034	50	0.5141	45	0.5147	42	0.5392	41	0.5625
0.75	0		3	0.3932	3	0.3405	3	0.2879	3	0.2577	2	0.5000
	2		10	0.2861	9	0.2820	8	0.2908	7	0.3582	7	0.3437
	4		16	0.2857	14	0.3088	13	0.2721	12	0.2919	12	0.2743
	10		34	0.2609	30	0.2790	27	0.2727	25	0.2962	25	0.2705
	20		62	0.2783	56	0.2569	50	0.2645	47	0.2635	46	0.2756
0.99	0		10	0.0150	9	0.0134	8	0.0128	7	0.0171	7	0.0156
	2		20	0.0103	17	0.0133	15	0.0133	14	0.0129	14	0.0112
	4		27	0.0137	24	0.0129	21	0.0141	20	0.0114	19	0.0154
	10		49	0.0113	43	0.0125	38	0.0129	36	0.0106	35	0.0121
	20		82	0.0112	73	0.0105	65	0.0101	60	0.0125	59	0.0124

Table 5. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3) = (0.25, 0.75, 0.25)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	**0.15**		**0.30**		**0.60**		**0.90**		**1**	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		4	0.9215	4	0.9059	3	1.0000	3	1.0000	3	1.0000
	4		7	0.9445	7	0.9275	7	0.9074	6	0.9699	6	0.9687
	10		19	0.9076	18	0.9111	17	0.9202	17	0.9005	16	0.9408
	20		39	0.9166	37	0.9161	35	0.9227	34	0.9251	34	0.9186
0.25	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		5	0.7868	5	0.7520	4	0.8887	4	0.8779	4	0.8750
	4		9	0.7797	8	0.8407	8	0.8033	8	0.7798	8	0.7734
	10		21	0.8097	20	0.8043	19	0.8073	19	0.7699	19	0.7597
	20		43	0.7844	41	0.7686	39	0.7646	38	0.7593	37	0.7975
0.50	0		2	0.5718	2	0.5452	2	0.5190	2	0.5039	1	1.0000
	2		6	0.6328	6	0.5843	6	0.5355	6	0.5073	5	0.6875
	4		11	0.5611	10	0.6100	10	0.5466	10	0.5095	9	0.6367
	10		25	0.5406	24	0.5091	22	0.5698	22	0.5144	21	0.5881
	20		48	0.5469	46	0.5058	43	0.5371	42	0.5199	41	0.5627
0.75	0		3	0.3270	3	0.2972	3	0.2693	3	0.2539	2	0.5000
	2		9	0.2594	8	0.3071	8	0.2588	7	0.3511	7	0.3437
	4		14	0.2791	13	0.2926	12	0.3187	12	0.2832	12	0.2744
	10		29	0.2874	27	0.3028	26	0.2716	25	0.2835	25	0.2706
	20		54	0.2730	51	0.2633	48	0.2696	46	0.2935	46	0.2757
0.99	0		9	0.0114	8	0.0143	8	0.0101	7	0.0164	7	0.0156
	2		17	0.0105	15	0.0156	14	0.0159	14	0.0121	14	0.0112
	4		23	0.0141	22	0.0113	20	0.0147	20	0.0105	19	0.0154
	10		42	0.0113	39	0.0122	37	0.0105	35	0.0136	35	0.0122
	20		71	0.0100	66	0.0111	62	0.0111	60	0.0107	59	0.0124

Table 6. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3) = (0.25, 0.75, 0.75)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	**0.15**		**0.30**		**0.60**		**0.90**		**1**	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		5	0.9280	4	0.9527	4	0.9135	3	1.0000	3	1.0000
	4		10	0.9197	8	0.9362	7	0.9359	7	0.9012	6	0.9688
	10		27	0.9094	22	0.9047	18	0.9269	17	0.9111	16	0.9408
	20		57	0.9113	46	0.9037	38	0.9141	35	0.9096	34	0.9186
0.25	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		7	0.7735	5	0.8621	5	0.7688	4	0.8836	4	0.8750
	4		13	0.7649	10	0.8080	9	0.7543	8	0.7922	8	0.7734
	10		32	0.7554	25	0.7824	21	0.7727	19	0.7898	19	0.7597
	20		64	0.7702	51	0.7634	42	0.7724	38	0.7881	37	0.7975
0.50	0		3	0.5128	2	0.6383	2	0.5578	2	0.5117	2	0.5000
	2		10	0.5063	8	0.5056	6	0.6074	6	0.5219	6	0.5000
	4		17	0.5071	13	0.5492	11	0.5246	10	0.5287	10	0.5000
	10		38	0.5099	30	0.5090	24	0.5579	22	0.5432	22	0.5000
	20		73	0.5136	57	0.5324	47	0.5209	42	0.5599	42	0.5000
0.75	0		5	0.2629	4	0.2601	3	0.3111	3	0.2618	3	0.2500
	2		13	0.2933	10	0.3107	8	0.3317	7	0.3659	7	0.3438
	4		21	0.2885	17	0.2568	13	0.3239	12	0.3014	12	0.2744
	10		45	0.2573	35	0.2642	28	0.2906	25	0.3102	25	0.2706
	20		83	0.2511	64	0.2773	52	0.2827	47	0.2819	46	0.2757
0.99	0		14	0.0130	11	0.0112	8	0.0168	7	0.0179	7	0.0156
	2		27	0.0106	20	0.0132	16	0.0124	14	0.0139	14	0.0112
	4		37	0.0121	28	0.0135	23	0.0101	20	0.0125	19	0.0154
	10		66	0.0110	51	0.0104	40	0.0132	36	0.0120	35	0.0122
	20		110	0.0109	85	0.0107	68	0.0114	61	0.0110	59	0.0124

Table 7. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3) = (0.75, 0.75, 0.75)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	0.15		0.30		0.60		0.90		1	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		6	0.9124	5	0.9012	4	0.9261	3	1.0000	3	1.0000
	4		12	0.9090	9	0.9259	7	0.9492	7	0.9056	6	0.9688
	10		32	0.9112	24	0.9185	19	0.9194	17	0.9175	16	0.9408
	20		69	0.9021	51	0.9141	40	0.9093	35	0.9190	34	0.9186
0.25	0		2	0.7657	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		8	0.7883	6	0.8108	5	0.7974	4	0.8872	4	0.8750
	4		15	0.7861	11	0.8152	9	0.7944	8	0.8000	8	0.7734
	10		38	0.7666	28	0.7846	22	0.7734	19	0.8021	19	0.7597
	20		78	0.7503	57	0.7770	44	0.7768	39	0.7561	37	0.7975
0.50	0		3	0.5863	2	0.6809	2	0.5803	2	0.5168	2	0.5000
	2		12	0.5048	9	0.5037	7	0.5036	6	0.5314	6	0.5000
	4		20	0.5302	15	0.5218	11	0.5831	10	0.5412	10	0.5000
	10		46	0.5067	34	0.5049	26	0.5058	22	0.5618	22	0.5000
	20		88	0.5209	65	0.5151	49	0.5445	43	0.5258	42	0.5000
0.75	0		6	0.2632	4	0.3157	3	0.3368	3	0.2671	3	0.2500
	2		16	0.2813	12	0.2661	9	0.2757	8	0.2550	7	0.3438
	4		26	0.2698	19	0.2719	14	0.3005	12	0.3135	12	0.2744
	10		54	0.2733	40	0.2563	30	0.2673	26	0.2644	25	0.2706
	20		100	0.2656	73	0.2693	55	0.2774	48	0.2598	46	0.2757
0.99	0		18	0.0107	12	0.0146	9	0.0129	7	0.0190	7	0.0156
	2		33	0.0112	23	0.0132	17	0.0125	14	0.0153	14	0.0112
	4		46	0.0110	33	0.0108	24	0.0119	20	0.0140	19	0.0154
	10		81	0.0110	58	0.0115	43	0.0112	36	0.0140	35	0.0122
	20		135	0.0105	97	0.0112	72	0.0115	62	0.0102	59	0.0124

Table 8. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3) = (1.25, 1.25, 1.25)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	0.15		0.30		0.60		0.90		1	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		2	0.9441	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2		21	0.9022	9	0.9041	4	0.9670	4	0.9017	3	1.0000
	4		45	0.9025	18	0.9135	9	0.9243	7	0.9227	6	0.9687
	10		128	0.9007	51	0.9025	24	0.9157	18	0.9017	16	0.9408
	20		278	0.9017	110	0.9002	51	0.9099	37	0.9029	34	0.9186
0.25	0		6	0.7501	2	0.8557	1	1.0000	1	1.0000	1	1.0000
	2		32	0.7509	13	0.7552	6	0.8084	4	0.9017	4	0.8750
	4		61	0.7565	24	0.7696	11	0.8119	8	0.8317	8	0.7734
	10		155	0.7562	61	0.7582	28	0.7788	20	0.7875	19	0.7596
	20		319	0.7540	125	0.7527	57	0.7685	40	0.7894	37	0.7975
0.50	0		13	0.5016	5	0.5361	2	0.6792	2	0.5385	1	1.0000
	2		48	0.5074	19	0.5074	8	0.5996	6	0.5719	5	0.6875
	4		84	0.5027	33	0.5009	15	0.5162	10	0.5940	9	0.6367
	10		191	0.5042	74	0.5114	33	0.5445	23	0.5609	21	0.5881
	20		370	0.5027	143	0.5122	65	0.5033	45	0.5249	41	0.5626
0.75	0		25	0.2516	9	0.2874	4	0.3134	3	0.2900	2	0.5000
	2		70	0.2517	27	0.2544	12	0.2621	8	0.2944	7	0.3437
	4		112	0.2510	43	0.2560	19	0.2668	13	0.2767	12	0.2744
	10		232	0.2523	89	0.2590	39	0.2836	27	0.2793	25	0.2706
	20		426	0.2517	164	0.2550	73	0.2591	50	0.2734	46	0.2757
0.99	0		81	0.0101	30	0.0109	12	0.0142	8	0.0131	7	0.0156
	2		148	0.0101	55	0.0112	23	0.0127	15	0.0138	14	0.0112
	4		204	0.0103	77	0.0106	33	0.0103	21	0.0147	19	0.0154
	10		356	0.0101	135	0.0103	58	0.0107	38	0.0137	35	0.0121
	20		586	0.0102	223	0.0104	97	0.0102	65	0.0109	59	0.0124

Table 9. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3) = (1.5, 1.5, 1.5)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	0.15		0.30		0.60		0.90		1	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		6	0.9067	2	0.9196	1	1.0000	1	1.0000	1	1.0000
	2		58	0.9010	15	0.9031	5	0.9395	4	0.9099	3	1.0000
	4		127	0.9004	32	0.9008	11	0.9013	7	0.9319	6	0.9687
	10		364	0.9004	89	0.9052	29	0.9004	18	0.9195	16	0.9408
	20		795	0.9009	194	0.9029	61	0.9043	38	0.9017	34	0.9186
0.25	0		15	0.7601	4	0.7777	1	1.0000	1	1.0000	1	1.0000
	2		90	0.7508	22	0.7641	7	0.8039	5	0.7607	4	0.8750
	4		174	0.7533	43	0.7537	14	0.7512	8	0.8492	8	0.7734
	10		445	0.7519	108	0.7589	34	0.7550	21	0.7555	19	0.7597
	20		917	0.7501	223	0.7503	68	0.7715	41	0.7931	37	0.7975
0.50	0		36	0.5036	9	0.5115	3	0.5383	2	0.5517	1	1.0000
	2		138	0.5022	33	0.5194	10	0.5550	6	0.5962	5	0.6875
	4		241	0.5013	58	0.5121	18	0.5116	11	0.5087	9	0.6367
	10		550	0.5010	133	0.5040	40	0.5277	24	0.5342	21	0.5881
	20		1065	0.5011	257	0.5058	78	0.5074	46	0.5407	41	0.5627
0.75	0		71	0.2537	17	0.2617	5	0.2897	3	0.3043	2	0.5000
	2		202	0.2502	48	0.2604	14	0.2855	8	0.3197	7	0.3437
	4		323	0.2504	77	0.2594	23	0.2639	13	0.3085	12	0.2744
	10		670	0.2506	161	0.2532	48	0.2577	28	0.2691	25	0.2706
	20		1229	0.2509	296	0.2508	88	0.2623	52	0.2538	46	0.2757
0.99	0		235	0.0102	55	0.0108	15	0.0131	8	0.0155	7	0.0156
	2		430	0.0101	102	0.0102	29	0.0105	16	0.0110	14	0.0112
	4		595	0.0100	141	0.0103	40	0.0112	22	0.0131	19	0.0154
	10		1033	0.0101	246	0.0102	71	0.0104	40	0.0109	35	0.0122
	20		1700	0.0100	406	0.0101	118	0.0104	67	0.0113	59	0.0124

Table 10. The ASP $(n, L(\delta_0))$ for the KM-GIKw distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3) = (2, 2, 2)$ for selected values of p^*, c, and a.

p^*	c	$a \rightarrow$	0.15		0.30		0.60		0.90		1	
			n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$	n	$L(\delta_0)$
0.10	0		91	0.9008	7	0.9056	1	1.0000	1	1.0000	1	1.0000
	2		951	0.9000	68	0.9023	8	0.9097	4	0.9268	3	1.0000
	4		2098	0.9000	150	0.9004	16	0.9174	7	0.9499	6	0.9688
	10		6052	0.9001	430	0.9009	45	0.9090	19	0.9211	16	0.9408
	20		13259	0.9000	941	0.9005	97	0.9079	40	0.9120	34	0.9186
0.25	0		248	0.7507	18	0.7550	2	0.8373	1	1.0000	1	1.0000
	2		1489	0.7502	106	0.7522	11	0.7861	5	0.7990	4	0.8750
	4		2904	0.7501	206	0.7524	22	0.7516	9	0.7966	8	0.7734
	10		7430	0.7500	527	0.7509	54	0.7641	22	0.7771	19	0.7597
	20		15302	0.7501	1085	0.7502	111	0.7541	44	0.7820	37	0.7975
0.50	0		597	0.5006	42	0.5078	4	0.5869	2	0.5816	2	0.5000
	2		2305	0.5000	163	0.5032	17	0.5041	7	0.5063	6	0.5000
	4		4025	0.5002	285	0.5018	29	0.5121	11	0.5864	10	0.5000
	10		9194	0.5001	651	0.5009	66	0.5049	26	0.5111	22	0.5000
	20		17811	0.5001	1261	0.5005	127	0.5105	50	0.5032	42	0.5000
0.75	0		1194	0.2503	84	0.2536	8	0.2884	3	0.3383	3	0.2500
	2		3378	0.2501	239	0.2504	24	0.2528	9	0.2782	7	0.3438
	4		5407	0.2500	382	0.2512	38	0.2584	14	0.3039	12	0.2744
	10		11219	0.2500	793	0.2509	79	0.2568	30	0.2720	25	0.2706
	20		20580	0.2501	1455	0.2509	145	0.2586	55	0.2839	46	0.2757
0.99	0		3967	0.0100	279	0.0101	26	0.0118	9	0.0131	7	0.0156
	2		7241	0.0100	510	0.0101	49	0.0105	17	0.0128	14	0.0112
	4		9997	0.0100	705	0.0100	68	0.0105	24	0.0123	19	0.0154
	10		17356	0.0100	1224	0.0101	119	0.0105	43	0.0117	35	0.0122
	20		28521	0.0100	2013	0.0100	197	0.0105	72	0.0121	59	0.0124

5. Methods of Estimation for the Unknown Parameters

In this section, we investigate the estimation of the parameters of the KM-GIKw distribution based on the ML, MPS, and Bayesian methods.

5.1. Maximum Likelihood Estimates

In this part, the ML estimates (MLEs) of the parameters of the KM-GIKw distribution are determined on the basis of complete samples. Thus, let $v_1, \ldots, v_n$ be the observed values of an RV with the KM-GIKw distribution. We denote the parameter vector as $\Psi \equiv (\vartheta_1, \vartheta_2, \vartheta_3)^T$. Let us set $f(v) = f(v;\Psi)$ and $F(v) = F(v;\Psi)$. The associated log-likelihood function is provided by

$$\begin{aligned}
\log L(\Psi) &= \sum_{i=1}^{n} \log[f(v_i;\Psi)] \\
&= n - n\log(e-1) + n\log\vartheta_1 + n\log\vartheta_2 + n\log\vartheta_3 + (\vartheta_1 - 1)\sum_{i=1}^{n}\log v_i - (\vartheta_2+1)\sum_{i=1}^{n}\log\left(1+v_i^{\vartheta_1}\right) \\
&\quad +(\vartheta_3-1)\sum_{i=1}^{n}\log\left[1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\right] - \sum_{i=1}^{n}\left[1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}.
\end{aligned}$$

The MLEs $\hat{\vartheta}_1$, $\hat{\vartheta}_2$, and $\hat{\vartheta}_3$ of ϑ_1, ϑ_2, and ϑ_3, respectively, are obtained by maximizing $\log L(\Psi)$ with respect to Ψ. They can be obtained numerically based on the first partial derivatives of $\log L(\Psi)$ with respect to Ψ. These derivatives are given by

$$\begin{aligned}
\frac{\partial \log L(\Psi)}{\partial\vartheta_1} &= \frac{n}{\vartheta_1} + \sum_{i=1}^{n}\log v_i - (\vartheta_2+1)\sum_{i=1}^{n}\frac{\log(v_i)}{1+v_i^{-\vartheta_1}} + \vartheta_2(\vartheta_3-1)\sum_{i=1}^{n}\frac{v_i^{\vartheta_1}\log(v_i)\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2-1}}{1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}} \\
&\quad -\vartheta_2\vartheta_3\sum_{i=1}^{n}v_i^{\vartheta_1}\log(v_i)\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2-1}\left[1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3-1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L(\Psi)}{\partial\vartheta_2} &= \frac{n}{\vartheta_2} - \sum_{i=1}^{n}\log\left(1+v_i^{\vartheta_1}\right) + (\vartheta_3-1)\sum_{i=1}^{n}\frac{\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\log\left(1+v_i^{\vartheta_1}\right)}{1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}} \\
&\quad -\vartheta_3\sum_{i=1}^{n}\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\log\left(1+v_i^{\vartheta_1}\right)\left[1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3-1},
\end{aligned}$$

and

$$\frac{\partial \log L(\Psi)}{\partial\vartheta_3} = \frac{n}{\vartheta_3} + \sum_{i=1}^{n}\log\left[1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\right] - \sum_{i=1}^{n}\left[1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}\log\left[1-\left(1+v_i^{\vartheta_1}\right)^{-\vartheta_2}\right].$$

Setting the system of nonlinear equations $\frac{\partial \log L(\Psi)}{\partial\vartheta_1} = \frac{\partial \log L(\Psi)}{\partial\vartheta_2} = \frac{\partial \log L(\Psi)}{\partial\vartheta_3} = 0$ and solving them simultaneously yields the MLEs. It is often more feasible to use nonlinear optimization techniques, such as the quasi-Newton algorithm, to numerically maximize $\log L(\Psi)$.

5.2. Maximum Product Spacing Estimates

A good alternative to the ML method is the MPS method, which consists of approximating the Kullback–Leibler information measure. In this method, an ordered sample of values of size n is taken from an RV with the KM-GIKw distribution, say $v_{(1)}, \ldots, v_{(n)}$. Then, the uniform spacings can be computed as follows:

$$\zeta^*(\Psi) = \frac{1}{n+1}\log\left\{\prod_{i=1}^{n+1}\Theta_{(i)}\right\},$$

where

$$\begin{cases} \Theta_{(1)} = F(v_{(1)};\Psi) \\ \Theta_{(i)} = F(v_{(i)};\Psi) - F(v_{(i-1)};\Psi) \quad i = 2,\ldots,n \\ \Theta_{(n+1)} = 1 - F(v_{(n)};\Psi). \end{cases}$$

with $F\left(v_{(0)};\Psi\right) = 0$ and $F\left(v_{(n+1)};\Psi\right) = 1$.

The logarithm of the geometric mean of the spacings is defined as follows:

$$\zeta^*(\Psi) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\{\frac{e}{e-1}\left\{e^{-\left[1-\left(1+v_{(i-1)}{}^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}} - e^{-\left[1-\left(1+v_{(i)}{}^{\vartheta_1}\right)^{-\vartheta_2}\right]^{\vartheta_3}}\right\}\right\}.$$

The MPS estimates (MPSEs) $\tilde{\vartheta}_1$, $\tilde{\vartheta}_2$ and $\tilde{\vartheta}_3$ of ϑ_1, ϑ_2 and ϑ_3, respectively, are obtained by maximizing $\zeta^*(\Psi)$ with respect to Ψ. However, the obtained estimates cannot be expressed analytically. As a result, a numerical technique using nonlinear optimization algorithms can be used.

5.3. Bayesian Estimates

This section examines the estimation of the unknown parameters of the KM-GIKw distribution using the Bayesian method. The squared error (SE) loss function (LF) and the linear exponential LF (LELF) are two different types of LFs that may be considered for the Bayesian estimates (BEs). We suggest employing separate gamma priors of parameters ϑ_1, ϑ_2, and ϑ_3 with the following PDFs:

$$\begin{aligned} \pi_1(\vartheta_1) &\propto \vartheta_1^{a_1-1}e^{-b_1\vartheta_1} \quad \vartheta_1 > 0, a_1 > 0, b_1 > 0, \\ \pi_2(\vartheta_2) &\propto \vartheta_2^{a_2-1}e^{-b_2\vartheta_2} \quad \vartheta_2 > 0, a_2 > 0, b_2 > 0, \\ \pi_3(\vartheta_3) &\propto \vartheta_3^{a_3-1}e^{-b_3\vartheta_3} \quad \vartheta_3 > 0, a_3 > 0, b_3 > 0, \end{aligned} \tag{15}$$

where the hyperparameters a_s and b_s, with $s = 1,2,3$, are picked to represent the prior information of the unknown parameters. The joint prior distribution of $\Psi = (\vartheta_1, \vartheta_2, \vartheta_3)$ is given as follows:

$$\pi(\Psi) = \pi_1(\vartheta_1)\pi_2(\vartheta_2)\pi_3(\vartheta_3), \tag{16}$$

that is

$$\pi(\Psi) \propto \vartheta_1^{a_1-1}\vartheta_2^{a_2-1}\vartheta_3^{a_3-1}e^{-b_1\vartheta_1-b_2\vartheta_2-b_3\vartheta_3}. \tag{17}$$

Given the observed data $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, the posterior density is provided via the following equation:

$$\pi(\Psi \mid \mathbf{v}) = \frac{\pi(\Psi)\mathrm{L}(\Psi)}{\int_{(0,\infty)^3}\pi(\Psi)\mathrm{L}(\Psi)d\Psi}.$$

Thus, it is expressed as follows:

$$\pi(\Psi \mid \mathbf{v}) \propto \frac{\vartheta_1^{n+a_1-1}\vartheta_2^{n+a_2-1}\vartheta_3^{n+a_3-1}}{e^{-n+b_1\vartheta_1+b_2\vartheta_2+b_3\vartheta_3}(e-1)^n}\prod_{i=1}^{n}v_i^{\vartheta_1-1}\Omega_i^{-\vartheta_2-1}\left[1-\Omega_i{}^{-\vartheta_2}\right]^{\vartheta_3-1}e^{-\left[1-\Omega_i{}^{-\vartheta_2}\right]^{\vartheta_3}}, \tag{18}$$

where $\Omega_i = \left(1 + v_i^{\vartheta_1}\right)$. The BE of $L(\Psi)$ under the SE loss function, denoted by BE-SELF, is provided via

$$\hat{\vartheta}_{BE-SELF} = E[L(\Psi) \mid \mathbf{v}] = \int_{(0,\infty)^3} L(\Psi)\pi(\Psi \mid \mathbf{v})d\Psi. \tag{19}$$

On the other hand, the LELF is an asymmetric LF that equally emphasizes under- and overestimations. Underestimation can be less beneficial than overestimation in a number of real-world scenarios, and vice versa. In these circumstances, a LELF can be proposed as an alternative to the SELF, which is offered through the followin equation:

$$\left[L(\Psi), \hat{L}(\Psi)\right] = e^{\hat{L}(\Psi)-L(\Psi)} - \tau^*\left[\hat{L}(\Psi) - L(\Psi)\right] - 1,$$

where $\tau^* \neq 0$. Here, $\tau^* > 0$ demonstrates that an overestimation is more serious than an underestimation, and $\tau^* < 0$ indicates the opposite. As τ^* moves closer to zero, it replicates the BE-SELF itself. For further information on this subject, see Refs. [37,38]. The BE of $L(\Psi)$ under this loss can be calculated as follows:

$$\hat{\Psi}_{BE-LELF} = E\left[e^{-\tau^* L(\Psi)} \mid \mathbf{v}\right] = -\frac{1}{\tau^*} \log\left[\int_{(0,\infty)^3} e^{-\tau^* L(\Psi)} \pi(\Psi \mid \mathbf{v}) d\Psi\right]. \tag{20}$$

As can be seen, there is no way to convert the estimates given in Equations (19) and (20) into closed-form expressions. Thus, we use the Metropolis–Hasting (MH) algorithm to create posterior samples and the Markov chain Monte Carlo (MCMC) method to generate the appropriate BEs.

5.4. Markov Chain Monte Carlo

The MCMC technique is a generic simulation technique employed for computing and sampling from posterior values of concern. In fact, the posterior uncertainty regarding the parameters Ψ as well as a kernel estimate of the posterior distribution may both be fully summarized by the MCMC samples (see Ref. [39]).

A discrete-time Markov chain (MC) serves as the foundation for the MCMC technique. An MC is a stochastic process: $\Psi^{(0)}, \Psi^{(1)}, \Psi^{(2)}, \ldots$. There are numerous methods to generate proposals in the MCMC technique, like the MH algorithm.

5.5. MH Algorithm

A recommended distribution and beginning values for the unknown parameters Ψ must be specified in order to implement the MH algorithm for the KM-GIKw distribution. To this end, a multivariate normal distribution is considered; that is, $N_3(\Psi, S_\Psi)$, where S_Ψ comprises the variance–covariance matrix (V-CM) for the recommended distribution. In fact, it is possible to gather unfavorable observations. The MLEs for Ψ are taken into account for the starting values, i.e., $\Psi^{(0)} = \hat{\Psi}^{MLE}$. As the asymptotic V-CM, S_Ψ is selected. $I^{-1}(\hat{\Psi}^{MLE})$, where $I(.)$ is the Fisher information matrix. The selection of S_Ψ is shown to be a key factor in the MH algorithm, where the acceptance rate depends on it. When taking all of this into account, the MH algorithm's steps for selecting a sample from the designated posterior density in Equation (18) are as follows:

I. Put the value of Ψ's initial parameter to $\Psi^{(0)} = \left(\hat{\vartheta}_1{}^{MLE}, \hat{\vartheta}_2{}^{MLE}, \hat{\vartheta}_3{}^{MLE}\right)$.

II. Perform the following operations for $i = 1, 2, \ldots, M$:

- II.1: Put $\Psi = \Psi^{(i-1)}$.
- II.2: Using $N_3(\log \Psi, S_\Psi)$, create a new candidate parameter value Ψ.
- II.3: Specify $\xi' = e^{\Psi}$.
- II.4: Determine $\beta = \frac{\pi(\xi' | \mathbf{v})}{\pi(\Psi | \mathbf{v})}$, where $\pi(\cdot \mid \mathbf{v})$ is given in Equation (18).
- II.5: Create a sample U from the uniform distribution over (0, 1).
- II.6: Take the new candidate or leave it out ξ'

$$\begin{cases} \text{If} \quad U \leq \beta \quad \text{set} \quad \Psi^{(i)} = \xi' \\ \text{otherwise} \quad \text{set} \quad \Psi^{(i)} = \Psi. \end{cases}$$

Finally, it is possible to reject a portion of the size M of random samples taken from the posterior density (burn-in), and then use the remaining samples to obtain the BEs. More precisely, the BEs of $\Psi^{(i)} = \left(\vartheta_1^{(i)}, \vartheta_2^{(i)}, \vartheta_3^{(i)}\right)$ utilizing the MCMC technique under the SELF and LELF can be estimated as follows:

$$\hat{\Psi}_{BE-SELF} = \frac{1}{M-l_B} \sum_{i=l_B}^{M} \Psi^{(i)}, \tag{21}$$

and

$$\hat{\Psi}_{BE-LELF} = -\frac{1}{\tau^*} \log\left[\frac{1}{M-l_B} \sum_{i=l_B}^{M} e^{-\tau^* \Psi^{(i)}}\right], \tag{22}$$

where l_B represents the number of burn-in samples.

5.6. Simulation Study

The aim of this section is to examine the behaviors of the MLE, MPSE, and BE, which were covered in the previous sub-sections. To evaluate the effectiveness of the suggested estimating methods, an MC analysis is employed. The computation is done using the statistical programming language R. Additionally, the *bbmle* and *BMT* packages in R are considered to compute the MPSEs and MLEs, respectively.

Utilizing a number of recommended estimation techniques, the MC simulation is run. The KM-GIKw distribution may be used to create one thousand elements of random data under the following assumptions:

1. The target sample sizes for the KM-GIKw distribution are $n = 20, 30, 50, 100$, and 200.
2. The parameters of the KM-GIKw distribution are assumed to be

   Case 1: $\vartheta_1 = 1.25, \vartheta_2 = 1.25, \vartheta_3 = 1.25$.
   Case 2: $\vartheta_1 = 1.50, \vartheta_2 = 1.75, \vartheta_3 = 1.50$.
   Case 3: $\vartheta_1 = 2.00, \vartheta_2 = 1.75, \vartheta_3 = 2.50$.
   Case 4: $\vartheta_1 = 2.50, \vartheta_2 = 2.50, \vartheta_3 = 2.50$.
   Case 5: $\vartheta_1 = 2.50, \vartheta_2 = 2.50, \vartheta_3 = 3.50$.

Monte Carlo steps:

Step 1: Generate random data from the KM-GIKw distribution from Equation (8) with ϑ_1, ϑ_2, and ϑ_3, given the sample size n.

Step 2: Compute the MLEs of ϑ_1, ϑ_2, and ϑ_3 using the true value of these parameters as the initial values for solving the normal equations.

Step 3: Compute the MPSEs of ϑ_1, ϑ_2, and ϑ_3.

Step 4: The MH method and MCMC technique are used with an informative prior (IP) to determine the BEs. For the IP, assume that

$$a_1 = 0.5,\ b_1 = 1.5,\ a_2 = 0.75,\ b_2 = 1.75,\ a_3 = 0.65,\ b_3 = 1.65.$$

After that, the estimated values are calculated using these values. When utilizing the MH algorithm, the MLEs take into account the initial guess values. Out of the 10,000 samples created from the posterior density and subsequently derived BEs under two distinct LFs, the SELF and LELF (at $\tau^* = -0.5$, and $\tau^* = 0.5$), 2000 burn-in samples are ultimately deleted.

Step 5: Repeat Step 1 to Step 4, the number of times: 1000, saving all estimates.

Step 6: Compute the following statistical measures of the performances for the point estimates, i.e., the average estimated bias (ABias) and root mean square errors (RMSE). These measures are computed as follows:

$$ABias(\Psi) = \frac{1}{1000} \sum_{l=1}^{1000} \hat{\Psi}_l - \Psi, \quad RMSE(\Psi) = \sqrt{\frac{1}{1000} \sum_{l=1}^{1000} \left(\hat{\Psi}_l - \Psi\right)^2},$$

where Ψ refers to the parameter vector, $\hat{\Psi}$ refers to the estimated value of the given parameter, and l indicates the number of the considered sample.

Hence, MLEs and MPSEs are computed for two methods.

All the results of the Monte Carlo simulation for each case for the given parameters $(\vartheta_1, \vartheta_2, \vartheta_3)$ are reported in Tables 11–15, respectively. From these tabulated values, one can indicate the following:

1. As n increases, the ABiases tend to zero and the RMSEs decrease.
2. The ABiases and RMSEs of parameters ϑ_1 and ϑ_2 for the MLEs have larger magnitudes than those of the MPSEs. But, for ϑ_3, they have larger magnitudes for the MPSEs than for the MLEs.
3. For the BEs, one can order the RMSEs as follows: RMSE (BE-LELF: $\tau^* = -0.5$) < RMSE (BE-SELF) < RMSE (BE-LELF: $\tau^* = -0.5$).

Table 11. ABiases and RMSEs of different estimation methods for the KM-GIKw distribution with $(\vartheta_1, \vartheta_2, \vartheta_3) = (1.25, 1.25, 1.25)$ at different sample sizes n.

n		MLE		MPSE		BE-SELF		BE-LELF: $\tau^* = -0.5$		BE-LELF: $\tau^* = 0.5$	
		ABias	RMSE	ABias	RMSE	ABias	RMSE	ABias	RMSE	ABias	RMSE
20	ϑ_1	0.7316	1.7584	−0.0425	1.3785	−0.5916	0.6724	−0.5532	0.7331	−0.6225	0.6705
	ϑ_2	1.8898	4.7333	0.8544	4.0574	−0.5588	0.5863	−0.5327	0.5676	−0.5808	0.6038
	ϑ_3	0.3424	1.5112	1.0926	2.2619	0.8328	0.9379	0.9602	1.0804	0.7827	1.2488
30	ϑ_1	0.6693	1.6406	−0.0303	1.3035	−0.5440	0.6488	−0.5281	0.5777	−0.5823	0.6159
	ϑ_2	1.9725	4.7845	0.6817	3.3038	−0.5182	0.5532	−0.4917	0.5356	−0.5407	0.5700
	ϑ_3	0.2547	1.2547	0.8893	1.8100	0.9111	1.4813	0.8869	0.9990	0.7977	1.3782
50	ϑ_1	0.4162	1.2621	−0.0479	1.1619	−0.4874	0.5438	−0.4558	0.5424	−0.5127	0.5614
	ϑ_2	1.3037	3.6018	0.5193	3.0619	−0.4504	0.5064	−0.4218	0.4925	−0.4748	0.5212
	ϑ_3	0.1706	0.8818	0.6128	1.2320	0.8205	1.6151	0.8234	1.1918	0.7281	1.2419
100	ϑ_1	0.4184	1.0113	0.0026	0.8663	−0.3717	0.4531	−0.3459	0.4475	−0.3952	0.4653
	ϑ_2	1.0295	2.5959	0.3959	2.2452	−0.3434	0.4337	−0.3149	0.4260	−0.3684	0.4437
	ϑ_3	−0.0175	0.5828	0.2960	0.7670	0.5166	0.6365	0.5651	0.6879	0.4915	0.7394
200	ϑ_1	0.2137	0.7564	−0.0919	0.6796	−0.3350	0.4342	−0.3137	0.4240	−0.3552	0.4451
	ϑ_2	0.5632	1.8495	0.1247	1.4341	−0.2966	0.4204	−0.2686	0.4202	−0.3211	0.4253
	ϑ_3	0.0361	0.5049	0.2805	0.6553	0.4592	0.5961	0.4996	0.6502	0.4240	0.5587

Table 12. ABiases and RMSEs of different estimation methods for the KM-GIKw distribution with $(\vartheta_1, \vartheta_2, \vartheta_3) = (1.50, 1.75, 1.50)$ at different sample sizes n.

n		MLE		MPSE		BE-SELF		BE-LELF: $\tau^* = -0.5$		BE-LELF: $\tau^* = 0.5$	
		ABias	RMSE	ABias	RMSE	ABias	RMSE	ABias	RMSE	ABias	RMSE
20	ϑ_1	0.5246	1.6695	−0.3838	1.3168	−0.7705	0.7938	−0.7391	0.7666	−0.7991	0.8192
	ϑ_2	2.3495	6.3725	0.4480	4.6099	−0.8761	0.8986	−0.8374	0.8665	−0.9085	0.9268
	ϑ_3	0.4631	1.5513	1.5144	2.5522	1.0225	1.1076	1.1902	1.2888	0.8873	0.9657
30	ϑ_1	0.5510	1.6154	−0.1520	1.4451	−0.7220	0.7483	−0.6932	0.7237	−0.7487	0.7716
	ϑ_2	2.6210	6.6031	1.1645	5.7568	−0.8251	0.8504	−0.7863	0.8193	−0.8575	0.8782
	ϑ_3	0.3648	1.5363	1.1933	2.2814	1.0141	1.1110	1.1632	1.2751	0.8924	0.9809
50	ϑ_1	0.4939	1.4495	−0.0869	1.2708	−0.6301	0.6669	−0.6020	0.6438	−0.6561	0.6890
	ϑ_2	2.3578	6.1063	0.9671	4.7559	−0.7399	0.7775	−0.6984	0.7468	−0.7748	0.8057
	ϑ_3	0.2193	1.1584	0.7910	1.6130	0.9391	1.4851	0.9978	1.1121	0.8195	1.0244
100	ϑ_1	0.3337	1.0898	−0.1169	0.9897	−0.5622	0.6114	−0.5377	0.5925	−0.5853	0.6299
	ϑ_2	1.3954	3.9828	0.4382	3.0738	−0.6601	0.7196	−0.6202	0.6962	−0.6944	0.7431
	ϑ_3	0.0970	0.8204	0.5065	1.0929	0.7672	0.8745	0.8401	0.9510	0.7015	0.8066
200	ϑ_1	0.2369	0.8830	−0.1593	0.8164	−0.4943	0.5620	−0.4720	0.5471	−0.5158	0.5771
	ϑ_2	0.9288	2.8265	0.1694	2.1749	−0.5844	0.6790	−0.5437	0.6690	−0.6194	0.6943
	ϑ_3	0.0641	0.6343	0.4225	0.8734	0.6613	0.7745	0.7171	0.8335	0.6108	0.7222

Table 13. ABiases and RMSEs of different estimation methods for the KM-GIKw distribution with $(\vartheta_1, \vartheta_2, \vartheta_3) = (2.00, 1.75, 2.50)$ at different sample sizes n.

n		MLE		MPSE		BE-SELF		BE-LELF: $\tau^* = -0.5$		BE-LELF: $\tau^* = 0.5$	
		ABias	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**
20	ϑ_1	0.5611	1.7686	−0.1133	1.7178	−.8147	0.8634	−0.7595	0.8172	-0.8653	0.9073
	ϑ_2	2.7515	7.3533	1.8432	6.6930	−0.7407	0.7662	−0.6887	0.7238	−0.7834	0.8033
	ϑ_3	0.9808	3.7088	2.4913	5.5586	0.9436	1.0320	1.2018	1.3005	0.8197	1.8058
30	ϑ_1	0.4477	1.6176	−0.1606	1.5830	−0.7126	0.7578	−0.6616	0.7147	−0.7599	0.7991
	ϑ_2	2.1123	6.0099	1.0897	5.1813	−0.6580	0.6906	−0.6014	0.6479	−0.7041	0.7292
	ϑ_3	0.6810	2.8021	2.0666	4.6228	0.9620	1.0637	1.1915	1.3043	0.7798	0.8808
50	ϑ_1	0.3479	1.2937	−0.1061	1.2867	−0.5821	0.6500	−0.5380	0.6158	−0.6240	0.6839
	ϑ_2	1.5401	4.5769	0.8561	4.0290	−0.5681	0.6308	−0.5122	0.5986	−0.6144	0.6630
	ϑ_3	0.3670	2.0177	1.2711	3.2743	0.9065	1.0394	1.0907	1.2393	0.7583	0.8870
100	ϑ_1	0.1935	0.9246	−0.1235	0.9579	−0.4646	0.5504	−0.4277	0.5240	−0.5002	0.5773
	ϑ_2	0.8289	2.7821	0.3507	2.3355	−0.4700	0.5689	−0.4178	0.5480	−0.5150	0.5933
	ϑ_3	0.1523	1.0608	0.6896	1.8525	0.7481	0.8986	0.8733	1.0322	0.6417	0.7910
200	ϑ_1	0.1686	0.7778	−0.0629	0.7797	−.3791	0.5035	−0.3462	0.4838	−0.4111	0.5243
	ϑ_2	0.5959	1.7857	0.2476	1.5078	−0.3603	0.5352	−0.3060	0.5352	−0.4072	0.5463
	ϑ_3	0.0954	1.0296	0.4097	1.4428	0.6088	0.8107	0.7064	0.9182	0.5245	0.7242

Table 14. ABiases and RMSEs of different estimation methods for the KM-GIKw distribution with $(\vartheta_1, \vartheta_2, \vartheta_3) = (2.50, 2.50, 2.50)$ at different sample sizes n.

n		MLE		MPSE		BE-SELF		BE-LELF: $\tau^* = -0.5$		BE-LELF: $\tau^* = 0.5$	
		ABias	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**
20	ϑ_1	0.4491	1.9251	−0.3490	1.9073	−1.1529	1.2137	−1.0588	1.3444	−1.2202	1.2509
	ϑ_2	4.2616	11.3458	2.1601	8.7924	−1.3623	1.3793	−1.3010	1.3237	−1.4125	1.4263
	ϑ_3	1.2525	4.0293	3.0214	6.2343	1.2724	1.3530	1.5482	1.6413	1.2070	2.4814
30	ϑ_1	0.3770	1.6916	−0.1718	1.7480	−1.0427	1.0855	−0.9886	1.0367	−1.0931	1.1317
	ϑ_2	3.5462	9.6190	1.9969	7.6894	−1.2911	1.3116	−1.2258	1.2535	−1.3438	1.3604
	ϑ_3	0.9190	3.2612	2.2280	5.3408	1.3217	1.4105	1.5663	1.6703	1.1262	1.2085
50	ϑ_1	0.4073	1.4816	−0.0103	1.5210	−0.8714	0.9164	−0.8239	0.8745	−0.9168	0.9573
	ϑ_2	2.7688	7.6290	1.7529	6.7930	−1.1285	1.1640	−1.0582	1.1073	−1.1869	1.2144
	ϑ_3	0.4457	2.3392	1.2457	3.5957	1.1896	1.2912	1.3756	1.4911	1.0390	1.1345
100	ϑ_1	0.1672	0.9408	−0.1319	0.9760	−0.7570	0.8105	−0.7170	0.7753	−0.7959	0.8453
	ϑ_2	1.1330	4.2257	0.5191	3.5660	−1.0104	1.0638	−0.9429	1.0121	−1.0679	1.1111
	ϑ_3	0.1391	0.9752	0.5157	1.6269	1.0352	1.1518	1.1717	1.3014	0.9207	1.0307
200	ϑ_1	0.1454	0.8496	−0.0860	0.8652	−0.6253	0.7003	−0.5902	0.6713	−0.6598	0.7296
	ϑ_2	0.8290	3.0290	0.4561	3.3005	−0.8457	0.9369	−0.7745	0.8957	−0.9067	0.9798
	ϑ_3	0.1127	1.0067	0.3814	1.4589	0.8284	0.9687	0.9288	1.0842	0.7429	0.8758

Table 15. ABiases and RMSEs of different estimation methods for the KM-GIKw distribution with $(\vartheta_1, \vartheta_2, \vartheta_3) = (2.50, 2.50, 3.50)$ at different sample sizes n.

n		MLE		MPSE		BE-SELF		BE-LELF: $\tau^* = -0.5$		BE-LELF: $\tau^* = 0.5$	
		ABias	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**	**ABias**	**RMSE**
20	ϑ_1	0.4693	1.8287	−0.2074	1.8708	−0.9543	0.9991	−0.8896	0.9418	−1.0147	1.0538
	ϑ_2	3.4209	9.1172	2.3421	8.9577	−1.1298	1.1524	−1.0441	1.0763	−1.1980	1.2156
	ϑ_3	1.4743	4.7706	3.5656	7.9989	1.0206	1.1420	1.3566	1.4808	0.8484	2.1064
30	ϑ_1	0.4582	1.6699	−0.1520	1.6820	−0.8169	0.8657	−0.7596	0.8158	−0.8711	0.9140
	ϑ_2	2.9969	8.1020	1.7638	7.3588	−1.0134	1.0451	−0.9206	0.9677	−1.0868	1.1106
	ϑ_3	0.8390	3.6453	2.5670	6.5515	1.1261	1.2277	1.4255	1.5356	0.8874	0.9900
50	ϑ_1	0.3047	1.4393	−0.1342	1.5184	−0.6835	0.7323	−0.6352	0.6895	-0.7302	0.7744
	ϑ_2	2.1835	6.9038	1.4187	6.5544	−0.8928	0.9375	−0.8033	0.8663	−0.9667	1.0008
	ϑ_3	0.5551	2.4754	1.8370	4.7715	1.0950	1.2145	1.3467	1.4806	0.8930	1.0108
100	ϑ_1	0.1940	0.9748	−0.1029	1.0387	−0.5449	0.6194	−0.5035	0.5858	−0.5851	0.6532
	ϑ_2	1.1916	3.7559	0.5906	3.3470	−0.7221	0.8177	−0.6244	0.7661	−0.8012	0.8731
	ϑ_3	0.1698	1.3761	0.7391	2.3793	0.9181	1.0719	1.0980	1.2616	0.7669	0.9200
200	ϑ_1	0.1452	0.8401	−0.0928	0.8728	−0.4836	0.5750	−0.4498	0.5495	−0.5168	0.6011
	ϑ_2	0.8788	3.1260	0.4142	2.7938	−0.6414	0.7807	−0.5578	0.7475	−0.7124	0.8211
	ϑ_3	0.1242	1.2396	0.5395	2.0225	0.8120	0.9940	0.9476	1.1376	0.6945	0.8747

6. Real Data Analysis

In this section, we present applications to two datasets to demonstrate the usefulness and adaptability of the KM-GIKw distribution. More precisely, adopting the modeling viewpoint, the KM-GIKw model is fitted to these two datasets and compared with the three-parameter GIKw model as well as two-parameter models, including the KM–Lomax (KM-L), Lomax (L), KM–Burr XII (KM-BXII), BXII, KM-BX, BX, KM-W, and W models. For the decision about the best-fitting competing model, we compute two important criteria, e.g., the Kolmogorov–Smirnov statistic (KS) and the p-value (PVKS). The model that has the lowest KS value and the largest PVKS value is the best model. Table 16 displays several statistical measures for the two datasets.

Table 16. Descriptive analysis for both datasets.

Datasets	Minimum	Var	Median	Mean	Standard Deviation	Maximum	SK	KU
Dataset 1	1.00	4995.173	22.00	59.60	70.677	261.00	1.784	2.569
Dataset 2	1.10	0.471	1.70	1.90	0.686	4.10	1.862	0.686

6.1. The First Dataset

The first dataset consists of a sample of 30 failure times of an airplane's air-conditioning system. It was provided in Ref. [40]. The data are reported in Table 17.

Table 17. The failure time dataset.

246	21	120	23	261	87	7	14	62	47	225	71	42	20	5
12	71	11	14	120	11	3	16	52	95	14	11	16	90	1

6.2. The Second Dataset

The second dataset consists of the relief times of twenty patients receiving an analgesic introduced in Ref. [41]. The data are described in Table 18.

Table 18. The values of the relief-time data.

1.5	1.2	1.4	1.9	1.8	1.6	2.2	1.1	1.4	1.3	1.7	1.7	2.7	4.1	1.8
2.3	1.6	2	3	1.7										

Tables 19 and 20 show the numerical results of the MLEs, standard errors (SEs), KS, and PVKS for both datasets. From these tables, we can note that the KM-GIKw model has the lowest value of KS and the largest value of PVKS for both datasets. Then, we can conclude that the KM-GIKw model gives the best fit. Figures 5–10 display the plots of the empirical CDF (ECDF), empirical PDF (EPDF), and probability–probability (PP) plots for both datasets. They support the numerical values in Tables 19 and 20; visually, the KM-GIKw model provides the best fitting results. Tables 21 and 22 show the estimates for the KM-GIKw model by using different methods of estimation.

Table 19. Numerical results of the MLE, SE, KS, and PVKS for the first dataset.

Measures	KM-GIKw	GIKw	KM-L	L	KM-BXII	BXII	KM-BX	BX	KM-W	W
$\hat{\vartheta}_1$	0.331	0.352					0.006	0.007	0.013	0.018
$\hat{\vartheta}_2$	2.803	2.958	7.533	3.296	0.019	0.029				
$\hat{\vartheta}_3$	43.765	45.251					0.353	0.290		
$\hat{\delta}$			511.530	141.265	10.513	10.198			0.938	0.854
SE($\hat{\vartheta}_1$)	0.250	0.265					0.001	0.001	0.003	0.004
SE($\hat{\vartheta}_2$)	2.727	2.844	18.296	3.075	0.083	0.142				
SE($\hat{\vartheta}_3$)	91.054	97.513					0.063	0.059		
SE($\hat{\delta}$)			1365.278	168.443	45.141	49.620			0.128	0.119
KS	0.131	0.135	0.145	0.142	0.372	0.377	0.181	0.196	0.146	0.153
PVKS	0.680	0.641	0.557	0.585	$<$0.001	$<$0.001	0.282	0.201	0.546	0.481

Table 20. Numerical results of the MLE, SE, KS, and PVKS for the second dataset.

Measures	KM-GIKw	GIKw	KM-L	L	KM-BXII	BXII	KM-BX	BX	KM-W	W
$\hat{\vartheta}_1$	3.628	5.111					0.655	0.691	0.422	0.469
$\hat{\vartheta}_2$	1.093	0.809	1670.930	1405.725	0.022	0.032				
$\hat{\vartheta}_3$	9.050	6.470					3.563	3.246		
$\hat{\delta}$			4716.450	2669.973	53.623	52.862			3.114	2.787
SE($\hat{\vartheta}_1$)	6.075	6.631					0.085	0.086	0.033	0.040
SE($\hat{\vartheta}_2$)	2.418	1.316	19,573.048	12,641.637	0.053	0.077				
SE($\hat{\vartheta}_3$)	19.360	7.799					1.338	1.321		
SE($\hat{\delta}$)			55,248.332	24,014.725	129.668	125.750			0.458	0.427
KS	0.094	0.096	0.436	0.440	0.291	0.285	0.172	0.190	0.186	0.185
PVKS	0.995	0.993	0.001	0.001	0.068	0.078	0.595	0.465	0.492	0.501

Table 21. Different methods of estimation of the KM-GIKw distribution for the first dataset.

	MLE	MPSE	BE-LELF: $\tau^* = -0.5$	BE-LELF: $\tau^* = 0.5$	BE-SELF
$\hat{\vartheta}_1$	0.3310	0.0117	0.0090	0.0090	0.0090
$\hat{\vartheta}_2$	2.8030	4.8698	2.7485	2.7346	2.7417
$\hat{\vartheta}_3$	43.7650	45.7956	59.8282	35.2123	43.2794

Table 22. Different methods of estimation of the KM-GIKw distribution for the second dataset.

	MLE	MPSE	BE-LELF: $\tau^* = -0.5$	BE-LELF: $\tau^* = 0.5$	BE-SELF
$\hat{\vartheta}_1$	3.6280	0.8197	1.0706	1.0687	1.0696
$\hat{\vartheta}_2$	1.0930	5.6366	10.1222	9.6864	9.9097
$\hat{\vartheta}_3$	9.0500	3.9271	3.8744	3.7049	3.7908

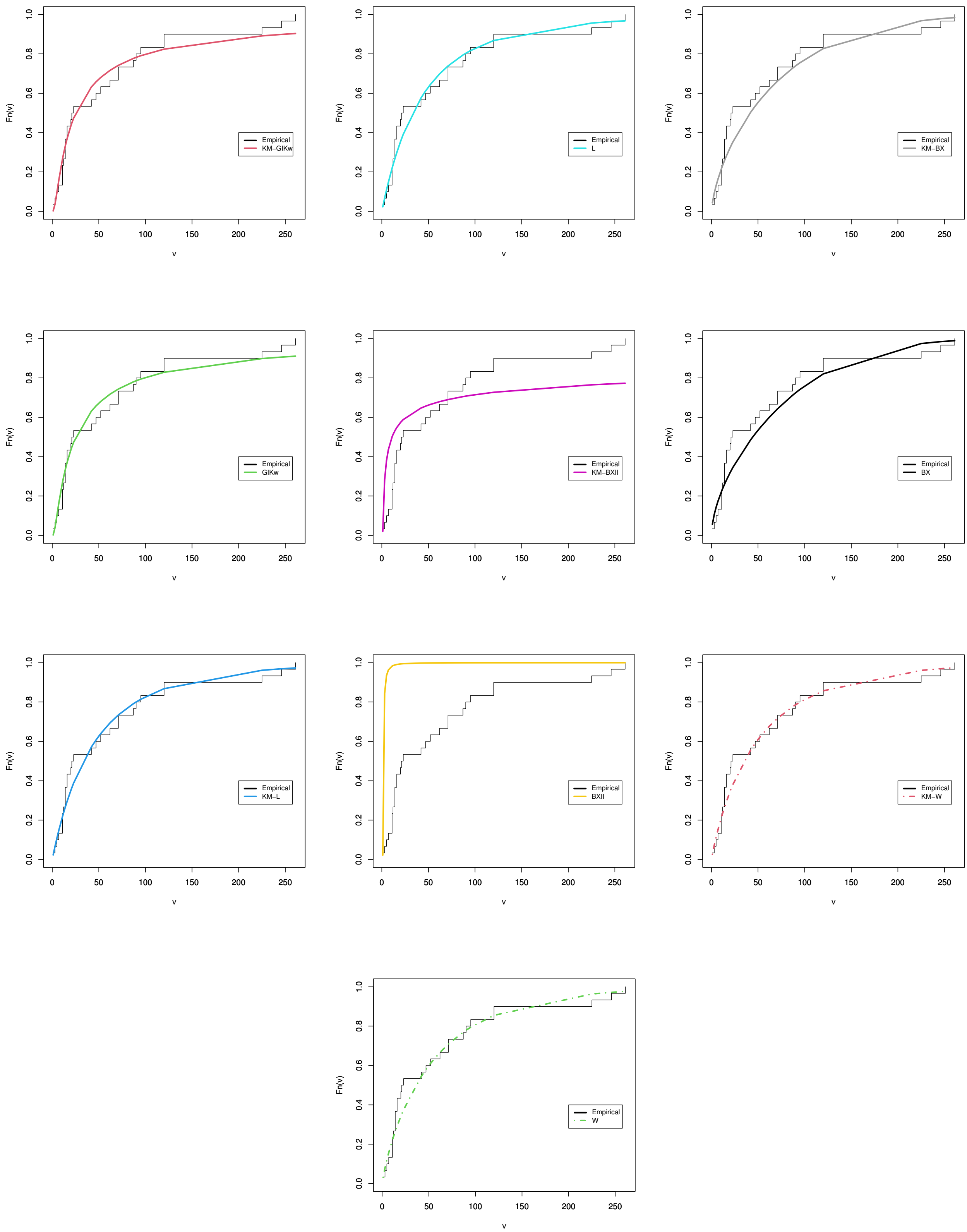


Figure 5. ECDF plots of all the competitive models for the first dataset.

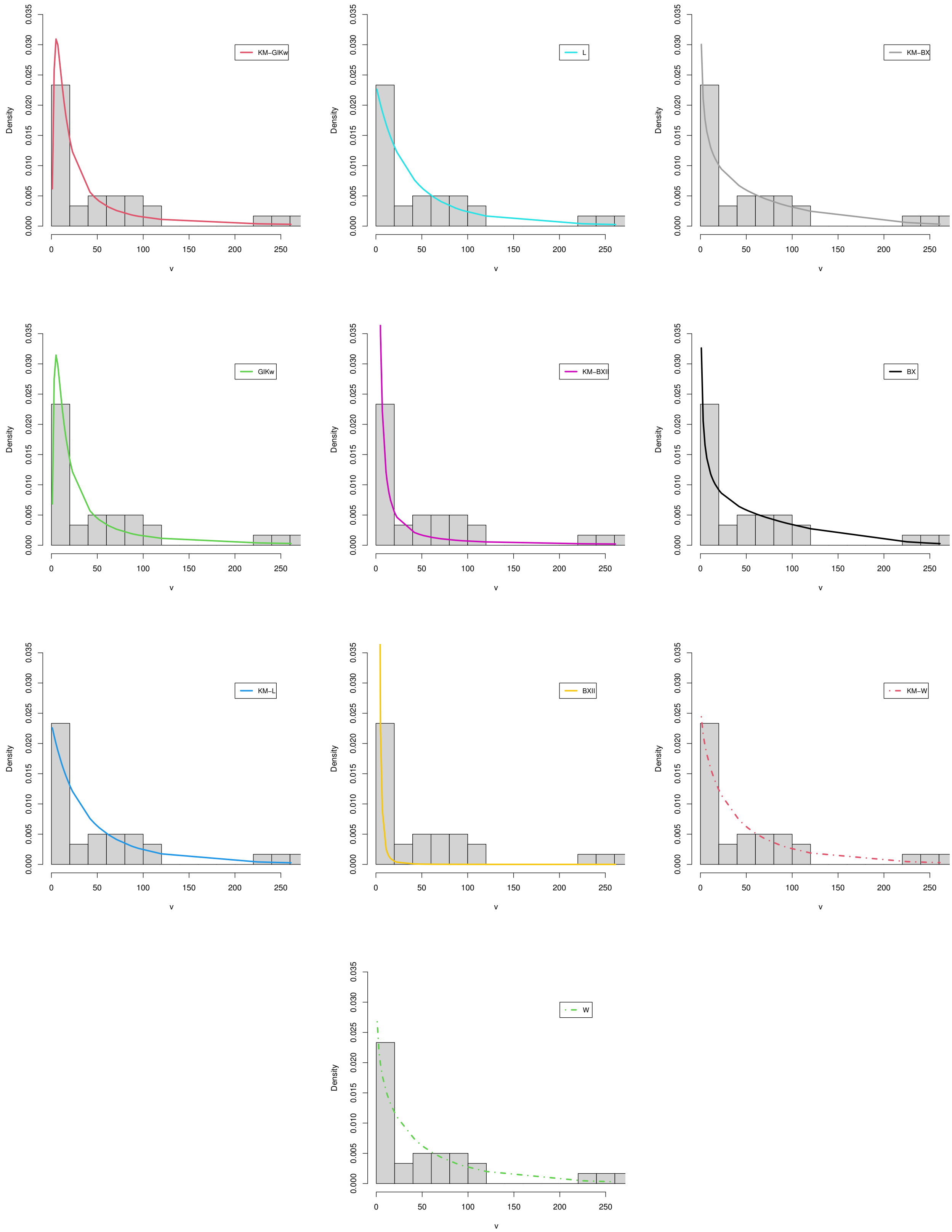


Figure 6. EPDF plots of all the competitive models for the first dataset.

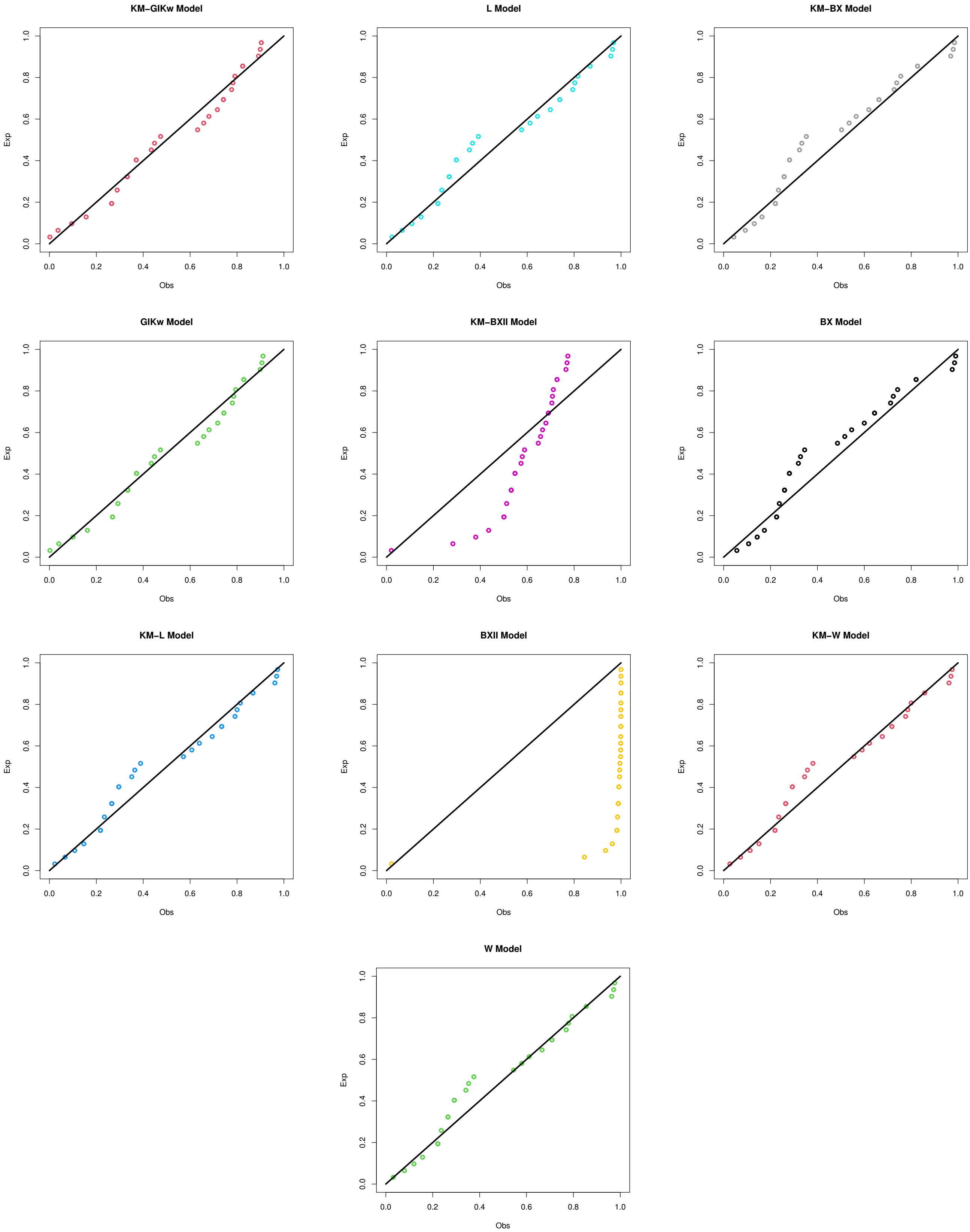


Figure 7. PP plots of all the competitive models for the first dataset.

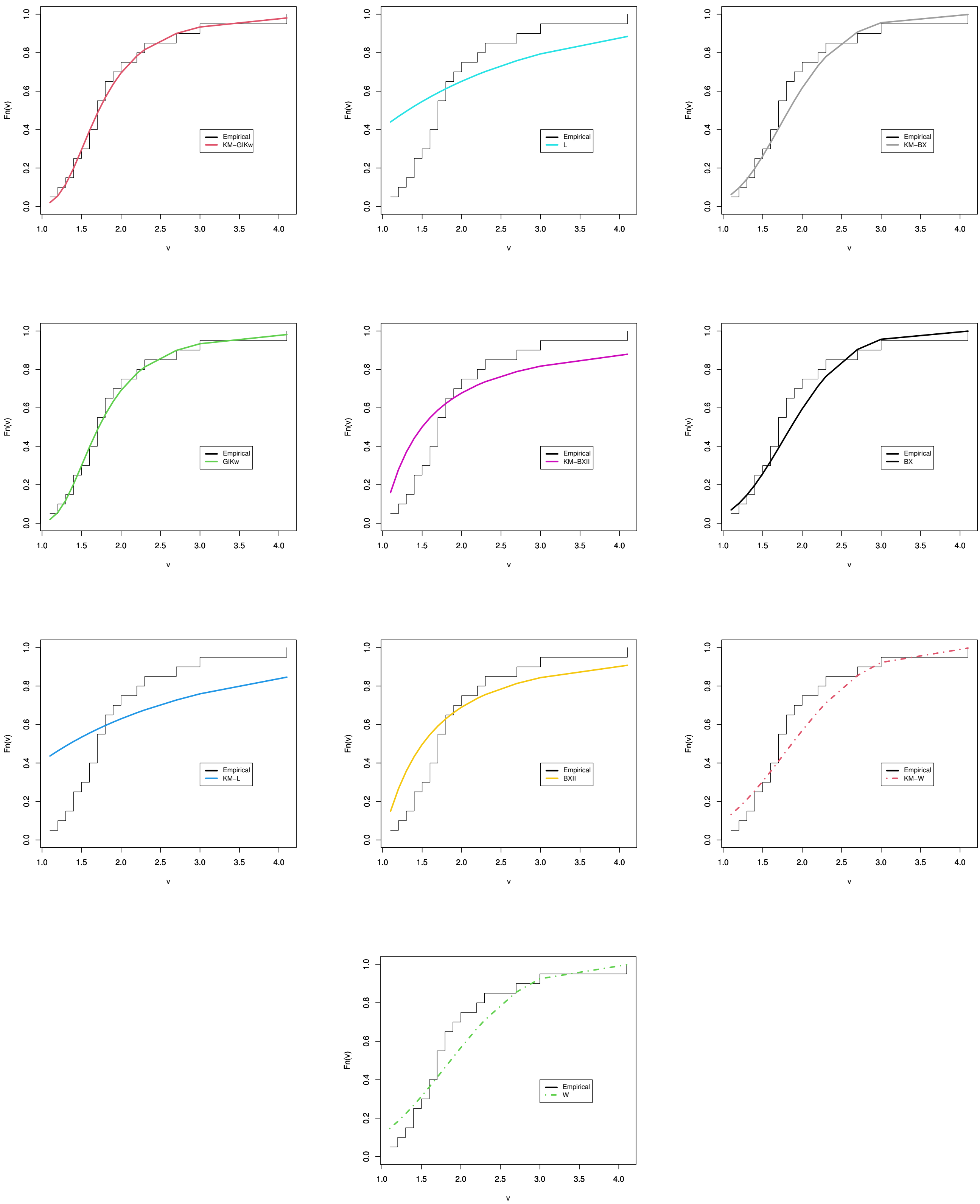


Figure 8. ECDF plots of all the competitive models for the second dataset.

Figure 9. EPDF plots of all the competitive models for the second dataset.

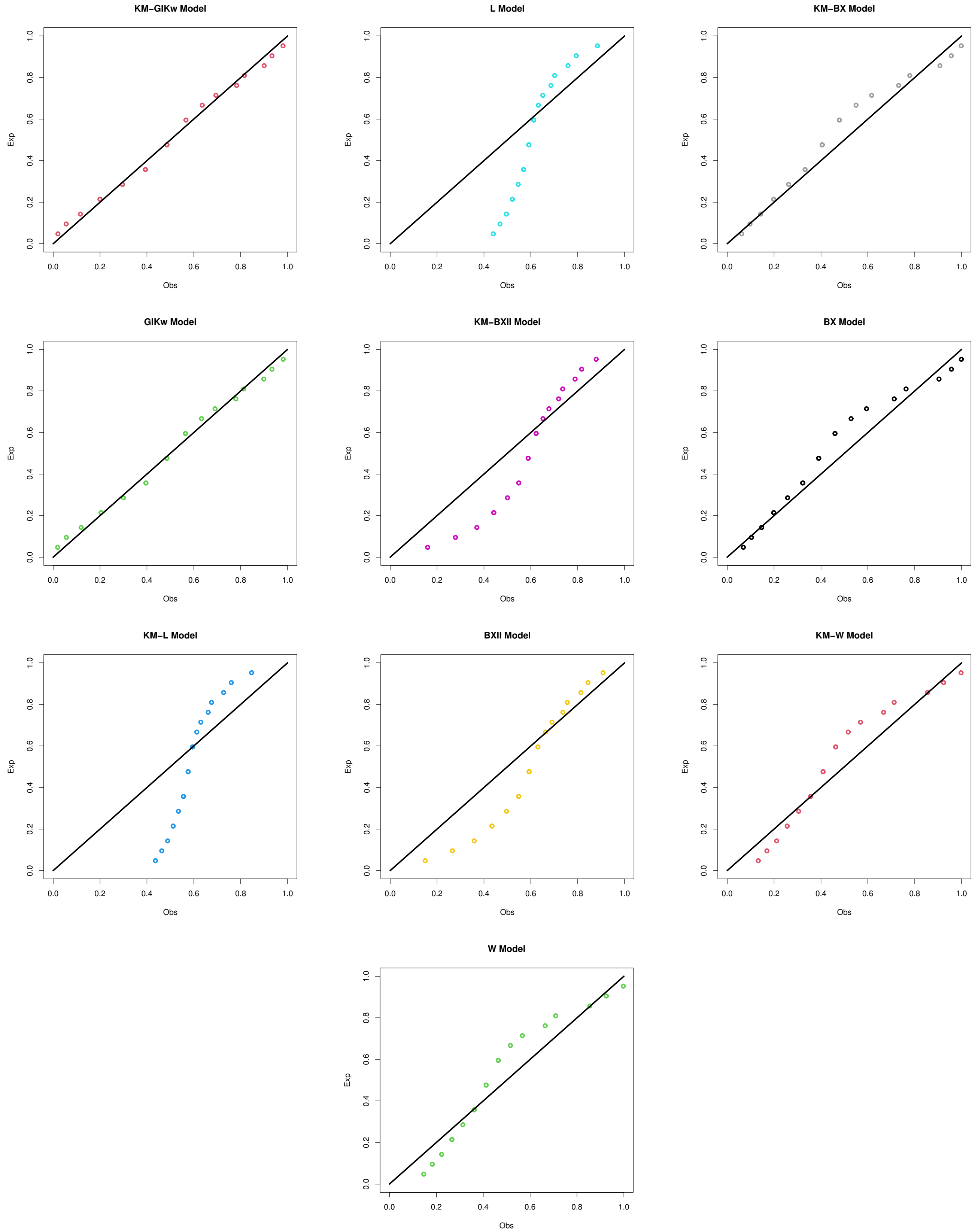


Figure 10. PP plots of all the competitive models for the second dataset.

7. Discussion and Conclusions

This study combines the generalized inverse Kumaraswamy distribution with the Kavya–Manoharan-G family to produce a new three-parameter distribution known as the Kavya–Manoharan generalized inverse Kumaraswamy distribution, abbreviated as KM-GIKw. The corresponding PDF includes a wide range of forms, which increases its versatility in estimating a variety of data types. This was shown by precise graphics. The corresponding HF is a decreasing or upside-down function. Some fundamental mathematical characteristics of the KM-GIKw distribution were derived, including the Rényi, Tsallis, Arimoto, and Havrda and Charvat entropy measures. On the other hand, an ASP was constructed using the KM-GIKw distribution when the life test was terminated at the median lifetime of the suggested distribution. The required sample size was determined using a variety of truncation periods and different characteristics of the proposed distribution and degrees of consumer risk. Additionally, it was determined that the probability of acceptance at the obtained sample sizes must be less than or equal to the complement of the consumer's risk. In the statistical part, the model parameters were estimated using the ML, MPS, and Bayesian estimation methods. Based on these different methods, the simulation study examined the performance of the model parameters. The adaptability and possibilities of the KM-GIKw model were then illustrated by looking at real-world data applications. It was shown that it can provide a better fit than previous competing lifetime models.

Author Contributions: Conceptualization, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; methodology, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; software, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; validation, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; formal analysis, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; investigation, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; writing—original draft preparation, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; writing—review and editing, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; visualization, N.A., A.S.H., M.E., C.C. and A.R.E.-S.; funding acquisition, N.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data supporting the reported results are given in the text, and the associated references are given.

Acknowledgments: Researchers Supporting Project number (RSPD2023R548), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Marshall, A.; Olkin, I. A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika* **1997**, *84*, 641–652.
2. Gupta, R.D.; Kundu, D. Exponentiated exponential family: An alternative to Gamma and Weibull distributions. *Biom. J.* **2001**, *43*, 117–130.
3. Eugene, N.; Lee, C.; Famoye, F. Beta-normal distribution and its applications. *Commun. Stat. Theory Methods* **2002**, *31*, 497–512.
4. Cordeiro, G.M.; de Castro, M. A new family of generalized distributions. *J. Stat. Comput. Simul.* **2011**, *81*, 883–898.
5. Alzaatreh, A.; Lee, C.; Famoye, F. A new method for generating families of continuous distributions. *Metron* **2013**, *71*, 63–79.
6. Chesneau, C.; Djibrila, S. The generalized odd inverted exponential-G family of distributions: Properties and applications. *Eurasian Bull. Math.* **2019**, *2*, 86–110.
7. Alizadeh, M.; Afify, A.Z.; Eliwa, M.S.; Ali, S. The odd loglogistic Lindley-G family of distributions: Properties, Bayesian and non-Bayesian estimation with applications. *Comput. Stat.* **2020**, *35*, 281–308.
8. Shaw, W.T.; Buckley, I.R. The Alchemy of probability distributions: Beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv* **2009**, arXiv:0901.0434.
9. Reyes, J.; Iriarte, Y.A. A New Family of Modified Slash Distributions with Applications. *Mathematics* **2023**, *11*, 3018.
10. Gillariose, J.; Balogun, O.S.; Almetwally, E.M.; Sherwani, R.A.K.; Jamal, F.; Joseph, J. On the Discrete Weibull Marshall-Olkin Family of Distributions: Properties, Characterizations, and Applications. *Axioms* **2021**, *10*, 287.
11. Liu, B.; Ananda, M.M.A. A Generalized Family of Exponentiated Composite Distributions. *Mathematics* **2022**, *10*, 1895.

12. Kharazmi, O.; Alizadeh, M.; Contreras-Reyes, J.E.; Haghbin, H. Arctan-Based Family of Distributions: Properties, Survival Regression, Bayesian Analysis and Applications. *Axioms* **2022**, *11*, 399.
13. Kumar, D.; Singh, U.; Singh, S.K. A method of proposing new distribution and its application to bladder cancer patient data. *J. Stat. Probab. Lett.* **2015**, *2*, 235–245.
14. Kumar, D.; Singh, U.; Singh, S.K. the new distribution using sine function- its application to bladder cancer patients data. *J. Stat. Appl. Probab.* **2015**, *4*, 417–427.
15. Maurya, S.K.; Kaushik, A.; Singh, R.K.; Singh, U. A new class of distribution having decreasing, increasing, and bathtub-shaped failure rate. *Commun. Stat. Theory Methods* **2017**, *46*, 10359–10372.
16. Kavya, P.; Manoharan, M. Some parsimonious models for lifetimes and applications. *J. Stat. Comput. Simul.* **2021**, *91*, 3693–3708.
17. Alotaibi, N.; Hashem, A.F.; Elbatal, I.; Alyami, S.A.; Al-Moisheer, A.S.; Elgarhy, M. Inference for a Kavya–Manoharan inverse length biased exponential distribution under progressive-stress model based on progressive type-II censoring. *Entropy* **2022**, *24*, 1033. [CrossRef]
18. Hassan, O.H.M.; Elbatal, I.; Al-Nefaie, A.H.; Elgarhy, M. On the Kavya-Manoharan-Burr X Model: Estimations under ranked set sampling and applications. *Risk Financ. Manag.* **2023**, *16*, 19.
19. Al-Nefaie, A.H. Applications to bio-medical data and statistical inference for a Kavya-Manoharan log-logistic model. *J. Radiat. Res. Appl. Sci.* **2023**, *16*, 100523. [CrossRef]
20. Alotaibi, N.; Elbatal, I.; Shrahili, M.; Al-Moisheer, A.S.; Elgarhy, M.; Almetwally, E.M. Statistical Inference for the Kavya–Manoharan Kumaraswamy model under ranked set sampling with applications. *Symmetry* **2023**, *15*, 587. [CrossRef]
21. Alotaibi, N.; Elbatal, I.; Almetwally, E.M.; Alyami, S.A.; Al-Moisheer, A.S.; Elgarhy, M. Bivariate step-stress accelerated life tests for the Kavya–Manoharan exponentiated Weibull model under progressive censoring with applications. *Symmetry* **2022**, *14*, 1791. [CrossRef]
22. Dubey, S.D. Compound Gamma, Beta and F Distributions. *Metrika* **1970**, *16*, 27–31. [CrossRef]
23. Voda, V.G. On the inverse Rayleigh distributed random variable. *Rep. Stat. Appl. Res.* **1972**, *19*, 13–21.
24. Folks, J.L.; Chhikara, R.S. The inverse Gaussian distribution and its statistical application—A review. *J. R. Stat. Soc. Ser. B (Methodol.)* **1978**, *40*, 263–289.
25. Calabria, R.; Pulcini, G. On the maximum likelihood and least-squares estimation in the inverse Weibull distribution. *Stat. Appl.* **1990**, *2*, 53–66.
26. Sharma, V.K.; Singh, S.K.; Singh, U.; Agiwal, V. The inverse Lindley distribution: A stress-strength reliability model with application to head and neck cancer data. *J. Ind. Eng. Int.* **2015**, *32*, 162–173.
27. Barco, K.V.P.; Mazucheli, J.; Janeiro, V. The inverse power Lindley distribution. *Commun. Stat. Simul. Comput.* **2017**, *46*, 6308–6323. [CrossRef]
28. Tahir, M.H.; Cordeiro, G.M.; Ali, S.; Dey, S.; Manzoor, A. The inverted Nadarajah–Haghighi distribution: Estimation methods and applications. *J. Stat. Comput. Simul.* **2018**, *88*, 2775–2798.
29. Abd AL-Fattah, A.M.; El-Helbawy, A.A.; Al-Dayian, G.R. Inverted Kumaraswamy distribution: Properties and estimation. *Pak. J. Stat.* **2017**, *33*, 37–61.
30. Iqbal, Z.; Tahir, M.M.; Riaz, N.; Ali, S.A.; Ahmad, M. Generalized inverted Kumaraswamy distribution: Properties and application. *Open J. Stat.* **2017**, *7*, 645–662.
31. Abdelwahab, M.M.; Ghorbal, A.B.; Hassan, A.S.; Elgarhy, M.; Almetwally, E.M.; Hashem, A.F. Classical and Bayesian inference for the Kavya–Manoharan generalized exponential distribution under gneralized progressively hybrid censored data. *Symmetry* **2023**, *15*, 1193. [CrossRef]
32. Rényi, A. On measures of entropy and information. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, CA, USA, 20–30 June 1960; pp. 547–561.
33. Cover, T.M.; Thomas, J.A. *Elements of Information Theory*; John Wiley & Sons: Hoboken, NJ, USA, 2006.
34. Nassr, S.G.; Hassan, A.S.; Alsultan, R.; El-Saeed, A.R. Acceptance sampling plans for the three-parameter inverted Topp–Leone model. *Math. Biosci. Eng.* **2022**, *19*, 13628–13659. [CrossRef]
35. Abushal, T.A.; Hassan, A.S.; El-Saeed, A.R.; Nassr, S.G. Power inverted Topp-Leone in acceptance sampling plans. *Comput. Mater. Contin.* **2021**, *67*, 991–1011.
36. Singh, S.; Tripathi, Y.M. Acceptance sampling plans for inverse Weibull distribution based on truncated life test. *Life Cycle Reliab. Saf. Eng.* **2017**, *6*, 169–178. [CrossRef]
37. Varian, H.R. A Bayesian approach to real estate assessment. In *Variants in Economic Theory: Selected Works of H. R. Varian*; Varian, H.R., Ed.; Edward Elgar Publishing: Cheltenham, UK, 2000; pp. 144–155.
38. Doostparast, M.; Akbari, M.G.; Balakrishnan, N. Bayesian analysis for the two-parameter Pareto distribution based on record values and times. *J. Stat. Comput. Simul.* **2011**, *81*, 1393–1403. [CrossRef]
39. Ravenzwaaij, D.V.; Cassey, P.; Brown, S.D. A simple introduction to Markov Chain Monte-Carlo sampling. *Psychon. Bull. Rev.* **2018**, *25*, 143–154. [CrossRef]

40. Linhart, H.; Zucchini, W. *Model Selection*; Wiley: New York, NY, USA, 1986.
41. Gross, A.J.; Clark, V.A. *Survival Distributions: Reliability Applications in the Biomedical Sciences*; John Wiley and Sons: New York, NY, USA, 1975.