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Applications of Fuzzy Differential Subordination for a New Subclass of Analytic Functions

Shahid Khan [1], Jong-Suk Ro [2,3,*], Fairouz Tchier [4] and Nazar Khan [1]

[1] Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22500, Pakistan; shahidmath761@gmail.com or shahidmath761@aust.edu.pk (S.K.); nazarmaths@gmail.com or nazarmaths@aust.edu.pk (N.K.)
[2] School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea
[3] Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea
[4] Mathematics Department, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia; ftchier@ksu.edu.sa
* Correspondence: jongsukro@gmail.com or jsro@cau.ac.kr

Abstract: This work is concerned with the branch of complex analysis known as geometric function theory, which has been modified for use in the study of fuzzy sets. We develop a novel operator $L^m_{\alpha,\lambda}$:$A_n \rightarrow A_n$ in the open unit disc Δ using the Noor integral operator and the generalized Sălăgean differential operator. First, we develop fuzzy differential subordination for the operator $L^m_{\alpha,\lambda}$ and then, taking into account this operator, we define a particular fuzzy class of analytic functions in the open unit disc Δ, represented by $R^\lambda_F(m,\alpha,\delta)$. Using the idea of fuzzy differential subordination, several new results are discovered that are relevant to this class. The fundamental theorems and corollaries are presented, and then examples are provided to illustrate their practical use.

Keywords: convex function; fuzzy best dominant; noor integral operator; fuzzy differential subordination; generalized Sălăgean differential operator; fuzzy operators; fuzzy sets

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1. Introduction and Definitions

Mathematician and computer scientist Zadeh [1,2] is remembered for his contributions to the fields of fuzzy logic and fuzzy set theory. The mathematical theory of fuzzy sets is useful for representing and working with data that are incomplete, imprecise, or otherwise difficult to put down with precision. It has found applications in fields such as artificial intelligence, control systems, decision making, and pattern recognition [3,4]. It was in 1965 that Lotfi A. Zadeh made his seminal contribution to the development of fuzzy set theory with the publication of his paper "Fuzzy Sets" [5]. This paper introduced the concept of a fuzzy set, which is a set that allows for degrees of membership between 0 and 1, rather than the traditional binary distinction between belonging and not belonging to a set. Initially, Zadeh's ideas were met with skepticism and resistance but, over time, the ideas gained widespread acceptance and have had a significant impact on a range of fields, including artificial intelligence, control systems, decision making, and pattern recognition. In their 2017 review article [6], the authors honor Lotfi A. Zadeh's scholarly impact by discussing the development of the concept of a fuzzy set and its applications to a wide range of disciplines. Mathematicians and researchers like Professor Dzitac [7,8] have since been exploring and adapting fuzzy set theory to various branches of mathematics, as well as applying it to other areas, such as engineering, economics, and social sciences. This has led to many interesting and useful connections being made, further expanding the range of applications of fuzzy set theory.

Fuzzy set theory has found applications in many areas of mathematics, including complex analysis, the study of analytic functions of complex variables. In 2011 [9], the concept of fuzzy subordination was introduced as a way to generalize the classical notion of subordination in complex analysis of the fuzzy setting. Fuzzy subordination is a notion that allows for a more flexible and nuanced understanding of the relationships between analytic functions. In particular, it allows for degrees of subordination to be expressed in a fuzzy manner, rather than the traditional binary notion of subordination. In 2012, the notion of fuzzy differential subordination was introduced [10], which significantly advanced the research of fuzzy subordination. Fuzzy differential subordination extends the notion of fuzzy subordination to include the use of derivatives of functions, providing a more refined way to express the relationships between functions in the fuzzy setting.

These developments have opened up new avenues for research in geometric function theory and other areas of mathematics, and have demonstrated the power and versatility of fuzzy set theory as a mathematical tool. The classical theory of differential subordination, which was first introduced by the same authors in 1978 [11] and 1981 [12], was due to its similarity to fuzzy set theory. This likely involved extending the classical theory to include the concept of fuzzy differential subordination, as mentioned earlier.

Starting in 2013 [13–16], the study of fuzzy differential subordination in connection with different operators began to be explored in more detail, as seen in the papers referenced in this sentence. This likely involved investigating how the concept of fuzzy differential subordination could be applied to specific classes of operators, such as those that arise in geometric function theory. These developments may have led to new insights and techniques for studying complex functions and their properties. The fuzzy differential subordination continues to be of active research interest, and numerous articles [17–22] published in recent years provide more evidence. These papers likely explore different aspects of fuzzy differential subordination and its applications to various mathematical problems. The papers in question focus on obtaining fuzzy differential subordination using a particular differential operator that has been defined and studied in previous work [23–25]. It is possible that the authors of the papers aim to extend the existing theory of fuzzy differential subordination by considering its relationship with a newly defined differential operator. This type of work is important for advancing our understanding of the connections between different mathematical concepts and for developing new tools and techniques for solving mathematical problems.

We next review the background studies that developed the notion of fuzzy differential subordination and their respective definitions.

Definition 1 ([5])**.** *Let X be a nonempty set. The pair $(\mathrm{B}, F_{\mathrm{B}})$ is the fuzzy subset of X, where $F_{\mathrm{B}} : X \to [0,1]$ is the membership function of the fuzzy set $(\mathrm{B}, F_{\mathrm{B}})$ and the set*

$$\mathrm{B} = \{x \in X : 0 < F_{\mathrm{B}}(x) \leq 1\}$$

is the support of the fuzzy set $(\mathrm{B}, F_{\mathrm{B}})$, denoted by

$$\mathrm{B} = \sup(\mathrm{B}, F_{\mathrm{B}}). \tag{1}$$

Remark 1. *If $\mathrm{B} \subset X$, then*

$$F_{\mathrm{B}}(x) = \begin{cases} 1, & \text{if } x \in \mathrm{B}, \\ 0, & \text{if } x \notin \mathrm{B}. \end{cases}$$

For the fuzzy subset, the real number 0 indicates the smallest membership of degree of $x \in X$ to B, while the real number 1 indicates the highest membership of degree of $x \in X$ to B.

Remark 2. *The empty set $\varnothing \subset X$ is characterized by*

$$F_{\varnothing}(x) = 0, \quad x \in X \tag{2}$$

and the total set X is characterized by

$$F_X(x) = 1, \ \ x \in X. \tag{3}$$

The collection of all analytic functions on the open unit disc $\Delta = \{\tau \in \mathbb{C} : |\tau| < 1\}$ denoted by $H(\Delta)$. Let the formula

$$\mathrm{A}_n = \left\{ f \in H(\Delta) : f(\tau) = \tau + \mathrm{b}_{n+1}\tau^{n+1} + \ldots, \ \tau \in \Delta \right\}$$

be the subclass of normalized analytic functions with the identity $\mathrm{A}_1 = \mathrm{A}$. When $\mathrm{b} \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; then, we write

$$H[\mathrm{b}, n] = \left\{ f \in H(\Delta) : f(\tau) = \mathrm{b} + \mathrm{b}_n\tau^n + \mathrm{b}_{n+1}\tau^{n+1} + \ldots \right\}$$

and $H[0,1] = H_0$.

The class of convex functions of order α, $(0 \le \alpha < 1)$, is denoted by $K(\alpha)$ and defined as follows:

$$K(\alpha) = \left\{ f \in \mathrm{A} : Re\left(1 + \frac{\tau f''(\tau)}{f'(\tau)}\right) > \alpha \right\},$$

and, when $\alpha = 0$, the class K of convex functions is obtained.

Definition 2 ([9]). *Let $D \subset \mathbb{C}$, and a fixed point $\tau_0 \in D$, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be a fuzzy subordinate to g (written as $\prec$)*

$$f \prec_{\mathcal{F}} g \ \ or \ \ f(\tau) \prec_{\mathcal{F}} g(\tau) \tag{4}$$

if

$$f(\tau_0) = g(\tau_0)$$

and

$$F_{f(D)} f(\tau) \le F_{g(D)} g(\tau), \ \ \tau \in D.$$

Definition 3 ([10]). *Let $\psi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ with $\psi(\mathrm{b}, 0; 0) = \mathrm{b}$ and let h be univalent in Δ with $h(0) = \mathrm{b}$. If φ is analytic in Δ with $\varphi(0) = \mathrm{b}$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3 \times \Delta)} \psi\left(\varphi(\tau), \tau\varphi'(\tau), \tau^2\varphi''(\tau); \tau\right) \le F_{h(\Delta)} h(\tau), \ \ \tau \in \Delta, \tag{5}$$

Then φ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is said to be a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or, more simply, a fuzzy dominant, if

$$F_{\varphi(\Delta)} \varphi(\tau) \le F_{q(\Delta)} q(\tau), \ \ \tau \in \Delta,$$

for all φ satisfying (5). A fuzzy dominant $\tilde{q}$ satisfying

$$F_{\tilde{q}(\Delta)} \tilde{q}(\tau) \le F_{q(\Delta)} q(\tau), ..\tau \in \Delta,$$

for all fuzzy dominants q of (5) is said to be the fuzzy best dominant of (5).

Definition 4 ([14]). *Let*

$$\begin{aligned} f(D) &= \sup\left(f(D), F_{f(D)}\right) \\ &= \left\{ \tau \in D : 0 < F_{f(D)} f(\tau) \le 1 \right\}, \end{aligned}$$

where $f(D)$ is the fuzzy set $F_{f(D)}$ membership function that is connected to the function f.

There is a one-to-one correspondence between the membership functions of the fuzzy sets $(\mu f)(D)$ associated with the functions μf coinciding with the membership function of the fuzzy set $f(D)$ associated to the function, i.e,

$$F_{(\mu f)(D)}((\mu f)(\tau)) = F_{f(D)} f(\tau), \ \ \tau \in D.$$

Remark 3. *As $0 < F_{f(D)} f(\tau) \leq 1$, and $0 < F_{g(D)} g(\tau) \leq 1$, it is obvious that*

$$0 < F_{(f+g)(D)}((f+g)(\tau)) \leq 1, \ \ \tau \in D.$$

Using the Sălgăean–Ruscheweyh differential operators and the Noor integral operator, some very useful results were obtained in the literature. These operators are well known in geometric function theory. In the following definitions and remark, we will cover their essential features and characteristics.

Ruscheweyh (see [26]) defined the $R^m f(\tau)$ of the m^{th} order as follows:

Definition 5 ([26])**.** *Let $f \in \mathcal{A}_n$. The Ruscheweyh derivative operator $R^m : \mathcal{A}_n \to \mathcal{A}_n$ is defined by*

$$R^m f(\tau) = \frac{\tau}{(1-\tau)^{m+1}} * f(\tau), \quad m > -1,$$

which implies that

$$R^m f(\tau) = \frac{\tau\left(\tau^{m-1} f(\tau)\right)^m}{m!}.$$

We note that

$$R^0 f(\tau) = f(\tau) \ \ \text{and} \ \ R^1 f(\tau) = \tau f'(\tau).$$

Many investigations—for example, in [24]—have produced fuzzy differential subordination by using Ruscheweyh and Sălăgean differential operators. In [27], the Riemann–Liouville fractional integral is used to calculate the derivative of a Gaussian hypergeometric function and, in [28], the fractional integral is used to calculate the derivative of a confluent hypergeometric function. Both of them are new operators introduced by the application of the fractional integral to fuzzy differential subordination theory. In this paper, we define a new differential operator involving the Noor integral operator and the Sălăgean differential operator, and then define a fuzzy differential subordination.

Noor [25] investigated the Noor integral operator I^m, which is defined as follows:

Definition 6 ([25])**.** *Let $f \in \mathcal{A}$; the operator $I^m : \mathcal{A}_n \to \mathcal{A}_n$ is defined as follows:*

Let

$$f_m(\tau) = \frac{\tau}{(1-\tau)^{m+1}}, \ \ m > -1$$

and let $f_m^{-1}(\tau)$ be defined such that

$$f_m(\tau) * f_m^{-1}(\tau) = \frac{\tau}{(1-\tau)^2}.$$

Then,

$$\begin{aligned} I^m f(\tau) &= f_m^{-1}(\tau) * f(\tau) \\ &= \left[\frac{\tau}{(1-\tau)^{m+1}}\right]^{-1} * f(\tau). \end{aligned} \quad (6)$$

It follows from (6) that

$$\begin{aligned} I^m f(\tau) &= \tau + \sum_{j=n+1}^{\infty} \frac{n!\Gamma(m+1)}{\Gamma(m+n)} \mathrm{b}_n \tau^n \\ &= \tau + \sum_{j=n+1}^{\infty} \Psi^m(n) \mathrm{b}_n \tau^n, \end{aligned}$$

where

$$\Psi^m(n) = \frac{n!\Gamma(m+1)}{\Gamma(m+n)}.$$

Remark 4. *The following identity holds for $I^m f(\tau)$:*

$$\tau(I^m f(\tau))' = (m+1)I^m f(\tau) - mI^{m+1} f(\tau), \quad \tau \in \Delta \tag{7}$$

and

$$I^0 f(\tau) = \tau f'(\tau), \text{ and } I^1 f(\tau) = f(\tau).$$

Al-Oboudi [29] investigated the generalization of the Sălăgean differential operator by introducing the parameter λ as follows:

Definition 7 ([29])**.** *For $\lambda \geq 0$, $m \in \mathbb{N}_0$, and $f \in A_n$, the operator $S_\lambda^m : A_n \to A_n$ is defined by*

$$\begin{aligned} S_\lambda^0 f(\tau) &= f(\tau), \\ S_\lambda^1 f(\tau) &= (1-\lambda) f(\tau) + \lambda \tau f'(\tau) = S_\lambda f(\tau) \\ &\cdots \\ S_\lambda^m f(\tau) &= (1-\lambda) S^{m-1} f(\tau) + \lambda \tau \Big(S_q^{m-1} f(\tau)\Big)' = S_\lambda (S_\lambda^{m-1} f(\tau). \end{aligned}$$

Using

$$f(\tau) = \tau + \sum_{j=n+1}^{\infty} \mathrm{b}_j \tau^j$$

and after some simple calculation, we have

$$S_\lambda^m f(\tau) = \tau + \sum_{j=n+1}^{\infty} \{\lambda(j-1)+1\}^m \mathrm{b}_n \tau^n. \tag{8}$$

Remark 5. *The following holds for $S_\lambda^m f(\tau)$:*

$$\begin{aligned} S_0^0 f(\tau) &= f(\tau),\ S_1^1 f(\tau) = \tau f'(\tau), \ldots \\ S_\lambda^{m+1} f(\tau) &= \Big((1-\lambda) S^m f(\tau) + \lambda \tau \Big(S_q^m f(\tau)\Big)\Big)', \quad \tau \in \Delta. \end{aligned}$$

Here, we define a new operator with the help of the generalized Sălăgean differential operator given in Definition 7 and the Noor integral operator given in Definition 6.

Definition 8. *Let $\alpha \geq 0$, $m > -1$, and $n \in \mathbb{N}$. The operator $L_{\alpha,\lambda}^m : A_n \to A_n$ is defined by*

$$L_{\alpha,\lambda}^m f(\tau) = (1-\alpha) S_\lambda^m f(\tau) + \alpha I^m f(\tau).$$

Remark 6. *$L_{\alpha,\lambda}^m$ is a linear operator and, if $f \in A_n$ and*

$$f(\tau) = \tau + \sum_{j=n+1}^{\infty} \mathrm{b}_j \tau^j$$

then, after some simple calculations, we have

$$L^m_{\alpha,\lambda} f(\tau) = \tau + \sum_{j=n+1}^{\infty} \left[(1-\alpha)\{\lambda(j-1)+1\}^m + \alpha \Psi^m(n)\right] \mathrm{b}_j \tau^j, \;\; \tau \in \Delta. \tag{9}$$

Remark 7. *If* $\alpha = 0$*, then* $L^m_{0,\lambda} f(\tau) = S^m_\lambda f(\tau)$ *and, for* $\alpha = 1$*,* $L^m_{1,\lambda} f(\tau) = I^m f(\tau)$*.*

Remark 8. *For* $\lambda = m = 0$*,* $L^0_{\alpha,0} f(\tau) = (1-\alpha) S^0_0 f(\tau) + \alpha I^0 f(\tau) = (1-\alpha) f(\tau) + \alpha \tau f^{'}(\tau)$*.*

Remark 9. *For* $m = 1$*, and* $\lambda = 1$*, in (9),*

$$\begin{aligned} L^1_{\alpha,1} f(\tau) &= (1-\alpha) S^1_1 f(\tau) + \alpha I^1 f(\tau) \\ &= (1-\alpha)\tau f^{'}(\tau) + \alpha f(\tau), \;\; \tau \in \Delta. \end{aligned}$$

By using the operator defined in Definition 8, we define a new class of fuzzy analytic functions as follows:

Definition 9. *Let* $R^\lambda_F(m, \alpha, \delta)$ *be a fuzzy class and contain all functions* $f \in A_n$*, including all* $f \in A_n$ *functions that fulfill the fuzzy inequality*

$$F_{\left(L^m_{\alpha,\lambda} f\right)'(\Delta)} \left(L^m_{\alpha,\lambda} f(\tau)\right)' > \delta, \; \tau \in \Delta, \tag{10}$$

where $\delta \in (0,1]$*, and also* $m > -1$*, and* $n \in \mathbb{N}$*.*

2. Set of Lemmas

To prove our main results, the following lemmas will be used:

Lemma 1 ([11])**.** *Let* $h \in \mathrm{A}_n$ *and*

$$\begin{aligned} L[f](\tau) &= F(\tau) \\ &= \frac{1}{n\tau^{\frac{1}{n}}} \int_0^\tau h(t) t^{\frac{1}{n}-1} dt, \;\; \tau \in \Delta. \end{aligned}$$

If

$$Re\left(\frac{\tau h''(\tau)}{h'(\tau)} + 1\right) > -\frac{1}{2}, \;\; \tau \in \Delta,$$

then $L(f) = F \in \mathcal{K}$*.*

Lemma 2 ([18])**.** *Let* $\gamma \in \mathbb{C}^*$ *be a complex number and* $Re\gamma \geq 0$ *and* h *be a convex function with* $h(0) = b$*. If* $\varphi \in \mathcal{H}[\mathrm{b}, n]$ *with* $\varphi(0) = \mathrm{b}$*,* $\psi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$*,*

$$\psi(\varphi(\tau), \tau\varphi'(\tau); \tau) = \varphi(\tau) + \frac{1}{\gamma}\tau\varphi'(\tau)$$

an analytic function in Δ *and*

$$F_{\psi(\mathbb{C}^2 \times \Delta)}\left(\varphi(\tau) + \frac{1}{\gamma}\tau\varphi'(\tau)\right) \leq F_{h(\Delta)} h(\tau), \tag{11}$$

i.e.,

$$\varphi(\tau) + \frac{1}{\gamma}\tau\varphi'(\tau) \prec_{\mathcal{F}} h(\tau), \;\; \tau \in \Delta,$$

then

$$\begin{aligned} F_{\varphi(\Delta)}\varphi(\tau) &\leq F_{g(\Delta)}g(\tau) \leq F_{h(\Delta)}h(\tau), \\ i.e,\ \varphi(\tau) &\prec {}_{\mathcal{F}}g(\tau) \prec_{\mathcal{F}} h(\tau),\ \ \tau \in \Delta, \end{aligned}$$

where

$$g(\tau) = \frac{\gamma}{n\tau^{\gamma/n}} \int_0^\tau h(t)t^{\gamma/n-1}dt,\ \ \tau \in \Delta.$$

The q is convex and is the fuzzy best dominant.

Lemma 3 ([18])**.** *Suppose that g represents a convex function in* Δ *and also suppose that*

$$h(\tau) = g(\tau) + n\alpha\tau g'(\tau),\ \ \tau \in \Delta$$

where $\alpha > 0$ *and n is a positive integer.*

If

$$\varphi(\tau) = g(0) + \varphi_n\tau^n + \varphi_{n+1}\tau^{n+1} + \ldots,\quad \tau \in \Delta,$$

is analytic in Δ *and*

$$F_{\varphi(\Delta)}\big(\varphi(\tau) + \alpha\tau\varphi'(\tau)\big) \leq F_{h(\Delta)}h(\tau),$$

i.e.,

$$\varphi(\tau) + \alpha\tau\varphi'(\tau) \prec \mathcal{F}h(\tau),\quad \tau \in \Delta,$$

then

$$F_{\varphi(\Delta)}\varphi(\tau) \leq F_{g(\Delta)}g(\tau),$$

i.e.,

$$\varphi(\tau) \prec_{\mathcal{F}} g(\tau),\ \tau \in \Delta.$$

This result is sharp.

This study follows a current trend in the study of fuzzy differential subordination, which focuses on the development and study of new fuzzy classes of functions using the introduction of innovative operators. New fuzzy differential subordination was obtained in Section 1 by using the newly established linear differential operator $L^m_{\alpha,\lambda}$ described in Definition 8. To produce fuzzy differential subordination, we next establish and investigate a new fuzzy class $R^\lambda_F(m,\alpha,\delta)$ in light of the operator $L^m_{\alpha,\lambda}$. Our primary findings will be proven by using the known lemmas as presented in Section 2. Section 3 presents the main part of the paper findings. In this section, we establish that our newly defined class is convex, and we also obtain some fuzzy differential subordination by using the operator $L^m_{\alpha,\lambda}$. The fuzzy best dominants are given for the considered fuzzy differential subordination in the main results, which generate interesting corollaries. Examples are provided to demonstrate the usefulness of the new findings. The last section provides an illustrative summary and future direction.

3. Main Results

Theorem 1. *The set* $R^\lambda_F(m,\alpha,\delta)$ *is convex.*

Proof. Let

$$f_j(\tau) = \tau + \sum_{j=n+1}^{\infty} \mathrm{b}_{jk}\tau^j \in R^\lambda_F(m,\alpha,\delta).$$

For obtaining the required conclusion, the function

$$h(\tau) = \mu_1 f_1(\tau) + \mu_1 f_2(\tau)$$

must belong to the class $R_F^{\lambda}(m,\alpha,\delta)$, with μ_1 and μ_1 non-negative such that $\mu_1+\mu_1=1$. We next show that $h\in R_F^{\lambda}(m,\alpha,\delta)$:

$$h^{'}(\tau)=(\mu_1 f_1(\tau)+\mu_2 f_2(\tau))^{'}(\tau)=\mu_1 f_1^{'}(\tau)+\mu_2 f_2^{'}(\tau)$$

and

$$\left(L^m_{\alpha,\lambda}h(\tau)\right)^{'}=\left(L^m_{\alpha,\lambda}(\mu_1 f_1(\tau)+\mu_2 f_2(\tau))\right)^{'}(\tau)=\mu_1\left(L^m_{\alpha,\lambda}f_1(\tau)\right)^{'}+\mu_2\left(L^m_{\alpha,\lambda}f_2(\tau)\right)^{'}.$$

From Definition 4, we obtain that

$$\begin{aligned}
& F_{(L^m_{\alpha,\lambda}h)'(\Delta)}\left(L^m_{\alpha,\lambda}h(\tau)\right)^{'} \\
= \; & F_{\left(\left(L^m_{\alpha,\lambda}(\mu_1 f_1+\mu_2 f_2)\right)^{'}(\Delta)\right.}\left(L^m_{\alpha,\lambda}(\mu_1 f_1+\mu_2 f_2)(\tau)\right)^{'} \\
= \; & F_{\left(\left(L^m_{\alpha,\lambda}(\mu_1 f_1+\mu_2 f_2)\right)^{'}(\Delta)\right.}\left(\mu_1\left(L^m_{\alpha,\lambda}f_1(\tau)\right)^{'}+\mu_2\left(L^m_{\alpha,\lambda}f_2(\tau)\right)^{'}\right) \\
= \; & \frac{F_{\left(\mu_1\left(L^m_{\alpha,\lambda}f_1(\tau)\right)^{'}(\Delta)\right.}\left(\mu_1\left(L^m_{\alpha,\lambda}f_1(\tau)\right)^{'}\right)+F_{\left(\mu_2\left(L^m_{\alpha,\lambda}f_2(\tau)\right)^{'}(\Delta)\right.}\left(\mu_2\left(L^m_{\alpha,\lambda}f_2(\tau)\right)^{'}\right)}{2} \\
= \; & \frac{F_{\left(L^m_{\alpha,\lambda}f_1(\tau)\right)^{'}(\Delta)}\left(\left(L^m_{\alpha,\lambda}f_1(\tau)\right)^{'}\right)+F_{\left(L^m_{\alpha,\lambda}f_2(\tau)\right)^{'}(\Delta)}\left(L^m_{\alpha,\lambda}f_2(\tau)\right)^{'}}{2}.
\end{aligned}$$

If $f_1,f_2\in R_F^{\lambda}(m,\alpha,\delta)$, then

$$\delta<F_{(L^m_{\alpha,\lambda}f_1)'(\Delta)}\left(L^m_{\alpha,\lambda}f_1(\tau)\right)'\leq 1$$

as well as

$$\delta<F_{(L^m_{\alpha,\lambda}f_2)'(\Delta)}\left(L^m_{\alpha,\lambda}f_2(\tau)\right)'\leq 1,\;\;\tau\in\Delta.$$

Therefore,

$$\delta<\frac{F_{(L^m_{\alpha,\lambda}f_1)'(\Delta)}\left(L^m_{\alpha,\lambda}f_1(\tau)\right)'+F_{(\Upsilon L^m_{\alpha,\lambda}f_2)'(\Delta)}\left(L^m_{\alpha,\lambda}f_2(\tau)\right)'}{2}\leq 1.$$

We obtain

$$\delta<F_{(L^m_{\alpha,\lambda}h)'(\Delta)}\left(L^m_{\alpha,\lambda}h(\tau)\right)'\leq 1,$$

which implies that $h\in R_F^{\lambda}(m,\alpha,\delta)$ and $R_F^{\lambda}(m,\alpha,\delta)$ is convex. □

The following theorem provides a result for fuzzy subordination, and an associated example is provided thereafter.

Theorem 2. *Suppose that g is a convex function in Δ and is defined as*

$$h(\tau)=g(\tau)+\frac{1}{c+2}\tau g'(\tau)$$

and, also $c>0$, $\tau\in\Delta$. Let $f\in R_F^{\lambda}(m,\alpha,\delta)$ as well as

$$G(\tau)=I_c(f)(\tau)=\frac{c+2}{\tau^{c+1}}\int_0^{\tau}t^c f(t)dt,\;\tau\in\Delta;$$

then, the fuzzy differential subordination

$$\begin{aligned} F_{(L^m_{\alpha,\lambda}f)'(\Delta)}\left(L^m_{\alpha,\lambda}f(\tau)\right)' &\leq F_{h(\Delta)}h(\tau), \\ i.e, \left(L^m_{\alpha,\lambda}f(\tau)\right)' &\prec {}_{\mathcal{F}}h(\tau),\ \tau\in\Delta, \end{aligned} \tag{12}$$

obtaining

$$\begin{aligned} F_{(L^m_{\alpha,\lambda}G)'(\Delta)}\left(L^m_{\alpha,\lambda}G(\tau)\right)' &\leq F_{g(\Delta)}g(\tau), \\ i.e,\ \left(L^m_{\alpha,\lambda}G(\tau)\right)' &\prec \mathcal{F}g(\tau),\ \tau\in\Delta. \end{aligned} \tag{13}$$

Proof. By using the formula for the function $G(\tau)$, we obtain

$$\tau^{c+1}G(\tau) = (c+2)\int_0^\tau t^c f(t)dt. \tag{14}$$

Differentiating Equation (14) with respect to τ, we have

$$(c+1)G(\tau) + \tau G'(\tau) = (c+2)f(\tau)$$

and also

$$\begin{aligned} & (c+1)L^m_{\alpha,\lambda}G(\tau) + \tau\left(L^m_{\alpha,\lambda}G(\tau)\right)' \\ = \ & (c+2)L^m_{\alpha,\lambda}f(\tau),\ \tau\in\Delta. \end{aligned} \tag{15}$$

Differentiating (15), we have

$$\left(L^m_{\alpha,\lambda}G(\tau)\right)' + \frac{1}{c+2}\tau\left(L^m_{\alpha,\lambda}G(\tau)\right)'' = \left(L^m_{\alpha,\lambda}f(\tau)\right)',\ \tau\in\Delta. \tag{16}$$

By Equation (16), the fuzzy differential subordination is

$$\begin{aligned} & F_{L^m_{\alpha,\lambda}G(\Delta)}\left(\left(L^m_{\alpha,\lambda}G(\tau)\right)' + \frac{1}{c+2}\,\tau\left(L^m_{\alpha,\lambda}G(\tau)\right)''\right) \\ \leq \ & F_{g(\Delta)}\left(g(\tau) + \frac{1}{c+2}\tau g'(\tau)\right). \end{aligned} \tag{17}$$

Let

$$\varphi(\tau) = \left(L^m_{\alpha,\lambda}G(\tau)\right)',\ \tau\in\Delta \tag{18}$$

and $\varphi\in\mathcal{H}[1,n]$. By substituting (18) in (17), we obtain

$$F_{\varphi(\Delta)}\left(\varphi(\tau) + \frac{1}{c+2}\tau\varphi'(\tau)\right) \leq F_{g(\Delta)}\left(g(\tau) + \frac{1}{c+2}\tau g'(\tau)\right),\ \tau\in\Delta.$$

Lemma 3 allows us to have

$$\begin{aligned} F_{\varphi(\Delta)}\varphi(\tau) &\leq F_{g(\Delta)}g(\tau), \\ i.e, F_{(L^m_{\alpha,\lambda}G)'(\Delta)}\left(L^m_{\alpha,\lambda}G(\tau)\right)' &\leq F_{g(\Delta)}g(\tau),\ \tau\in\Delta, \end{aligned}$$

the best dominant is g. We have obtained

$$\left(L^m_{\alpha,\lambda}G(\tau)\right)' \prec_{\mathcal{F}} g(\tau),\ \tau\in\Delta.$$

□

The following are several theorems and corollaries that prove various conclusions using fuzzy subordination. Some are elaborated on using examples.

Theorem 3. *Suppose that*

$$h(\tau) = \frac{1+(2\beta-1)\tau}{1+\tau},\ \beta \in [0,1)$$

as $c > 0$. *Let* $m > -1$ *and*

$$I_c(f)(\tau) = \frac{c+2}{\tau^{c+1}} \int_0^\tau t^c f(t)dt,\ \tau \in \Delta;$$

then,

$$I_c\left[R_F^{\lambda}(m,\alpha,\beta)\right] \subset R_F^{\lambda,t}(m,\alpha,\beta^*), \tag{19}$$

where

$$\beta^* = 2\delta - 1 + \frac{2(c+2)(1-\delta)}{n} \int_0^1 \frac{t^{\frac{c+2}{n}} - 1}{1+t} dt.$$

Proof. Since the provided function h is convex, we may apply the same logic that we used to prove Theorem 2 here. The Theorem 3 hypothesis might be interpreted as follows:

$$F_{\varphi(\Delta)}\left(\varphi(\tau) + \frac{1}{c+2}\tau\varphi'(\tau)\right) \leq f_{h(\Delta)}h(\tau),$$

where $\varphi(\tau)$ is given by (18). Using the Lemma 2, we obtain

$$F_{\varphi(\Delta)}\varphi(\tau) \leq F_{g(\Delta)}g(\tau) \leq F_{h(\Delta)}h(\tau),$$

i.e.,

$$F_{\left(L^m_{\alpha,\lambda}G\right)'(\Delta)}\left(L^m_{\alpha,\lambda}G(\tau)\right)' \leq F_{g(\Delta)}g(\tau) \leq F_{h(\Delta)}h(\tau),$$

and

$$\begin{aligned} g(\tau) &= \frac{c+2}{n\tau^{\frac{c+2}{n}}} \int_0^\tau t^{\frac{c+2}{n}-1} \frac{1+(2\delta-1)t}{1+t} dt \\ &= 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n\tau^{\frac{c+2}{n}}} \int_0^1 \frac{t^{\frac{c+2}{n}} - 1}{1+t} dt. \end{aligned}$$

Since $g(\Delta)$ is real axis symmetric using the convexity hypothesis for g, we may write

$$\begin{aligned} & F_{L^m_{\alpha,\lambda}G(\Delta)}\left(L^m_{\alpha,\lambda}G(\tau)\right)' \\ \geq\ & \min_{|\tau|=1} F_{g(\Delta)}g(\tau) = F_{g(\Delta)}g(1) \end{aligned} \tag{20}$$

and

$$\beta^* = g(1) = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n} \int_0^1 \frac{t^{\frac{c+2}{n}} - 1}{1+t} dt.$$

Inclusion (19) follows obviously from (20). □

Theorem 4. *Let the function* g *be a convex function with* $g(0) = 1$.

$$h(\tau) = g(\tau) + \tau g'(\tau),\ \tau \in \Delta.$$

Let $f \in \mathcal{A}_n$ satisfy

$$F_{(L^m_{\alpha,\lambda}f)'(\Delta)}\left(L^m_{\alpha,\lambda}f(\tau)\right)' \leq F_{h(\Delta)}h(\tau),$$
$$i.e, \left(L^m_{\alpha,\lambda}f(\tau)\right)' \prec {}_{\mathcal{F}}h(\tau),\ \tau \in \Delta. \tag{21}$$

Let $m > -1$; then, we obtain the fuzzy differential subordination

$$F_{L^m_{\alpha,\lambda}f(\Delta)}\frac{L^m_{\alpha,\lambda}f(\tau)}{\tau} \leq F_{g(\Delta)}g(\tau),$$
$$i.e, \frac{L^m_{\alpha,\lambda}f(\tau)}{\tau} \prec {}_{\mathcal{F}}g(\tau),\ \tau \in \Delta.$$

Proof. Using (9), we can write

$$L^m_{\alpha,\lambda}f(\tau) = \tau + \sum_{j=n+1}^{\infty}\left[\alpha\{\lambda(j-1)+1\}^m + (1-\alpha)\Psi^m(n)\right]\mathrm{b}_j\tau^j,\ \ \tau \in \Delta.$$

Consider

$$\begin{aligned}\varphi(\tau) &= \frac{L^m_{\alpha,\lambda}f(\tau)}{\tau} \\ &= \frac{\tau + \sum_{j=n+1}^{\infty}\left[\alpha\{\lambda(j-1)+1\}^m + (1-\alpha)\Psi^m(n)\right]\mathrm{b}_n\tau^n}{\tau} \\ &= 1 + \varphi_n\tau + \varphi_{n+1}\tau^{n+1} + \ldots\end{aligned}$$

From this, we conclude that $\varphi \in H[1,n]$.
Let $\tau\varphi(\tau) = L^m_{\alpha,\lambda}f(\tau)$, for $\tau \in \Delta$. The obtained expression is differentiated by

$$\left(L^m_{\alpha,\lambda}f(\tau)\right)' = \varphi(\tau) + \tau\varphi'(\tau). \tag{22}$$

Using (22) in (21), we can write

$$\begin{aligned}F_{\varphi(\Delta)}\Big(\varphi(\tau) + \tau\varphi'(\tau)\Big)' &\leq F_{h(\Delta)}h(\tau) \\ &= F_{g(\Delta)}\Big(g(\tau) + \tau g'(\tau)\Big).\end{aligned}$$

We use Lemma 3 and obtain

$$F_{\varphi(\Delta)}\varphi(\tau) \leq F_{g(\Delta)}g(\tau),$$

i.e.,

$$F_{(L^m_{\alpha,\lambda}f)'(\Delta)}\frac{L^m_{\alpha,\lambda}f(\tau)}{\tau} \leq F_{g(\Delta)}g(\tau),\ \tau \in \Delta.$$

Therefore,

$$\frac{L^m_{\alpha,\lambda}f(\tau)}{\tau} \prec_{\mathcal{F}} g(\tau),\ \ \tau \in \Delta.$$

□

Theorem 5. *Suppose that h denotes the convex function of order $-\frac{1}{2}$ with $h(0) = 1$. Let $f \in A_n$ satisfy*

$$F_{(L^m_{\alpha,\lambda})f)'(\Delta)}\left(L^m_{\alpha,\lambda}f(\tau)\right)' \leq F_{h(\Delta)}h(\tau),$$

i.e.,

$$\left(L_{\alpha,\lambda}^{m} f(\tau)\right)' \prec \mathcal{F}_{h(\tau)},\ \tau \in \Delta.$$

For $m > -1$,

$$\begin{aligned} F_{L_{\alpha,\lambda}^{m} f(\Delta)} \frac{L_{\alpha,\lambda}^{m} f(\tau)}{\tau} &\leq F_{q(\Delta)} q(\tau), \\ i.e,\ \frac{L_{\alpha,\lambda}^{m} f(\tau)}{\tau} &\prec {}_{\mathcal{F}} q(\tau),\ \tau \in \Delta, \end{aligned} \tag{23}$$

where

$$q(\tau) = \frac{1}{n\tau^{\frac{1}{n}}} \int_0^{\tau} h(t) t^{\frac{1}{n}-1} dt$$

is the best fuzzy dominant and a convex set.

Proof. Let

$$\begin{aligned} \varphi(\tau) &= \frac{L_{\alpha,\lambda}^{m} f(\tau)}{\tau} = \frac{\tau + \sum_{j=n+1}^{\infty} \left[(1-\alpha)\{\lambda(j-1)+1\}^{m} + \alpha\Psi^{m}(n)\right] \mathrm{b}_j \tau^{j}}{\tau} \\ &= 1 + \sum_{j=n+1}^{\infty} \left[(1-\alpha)\{\lambda(j-1)+1\}^{m} + \alpha\Psi^{m}(n)\right] \mathrm{b}_j \tau^{j-1} \\ &= 1 + \sum_{j=n+1}^{\infty} \varphi_j \mathrm{b}_j \tau^{j-1},\ \ \tau \in \Delta,\ \varphi \in [1, n], \end{aligned}$$

where

$$\varphi_j = \left[(1-\alpha)\{\lambda(j-1)+1\}^{m} + \alpha\Psi^{m}(n)\right].$$

as

$$Re\left(1 + \frac{\tau h''(\tau)}{h'(\tau)}\right) > \frac{-1}{2},\ \ \tau \in \Delta.$$

Using Lemma 1, we obtain

$$q(\tau) = \frac{1}{n\tau^{\frac{1}{n}}} \int_0^{\tau} h(t) t^{\frac{1}{n}-1} dt$$

which is a convex function and verifies the differential equation associated to the fuzzy differential subordination (23)

$$q(\tau) + \tau q'(\tau) = h(\tau).$$

Therefore, it is the fuzzy best dominant. Differentiating, we obtain

$$\left(L_{\alpha,\lambda}^{m} f(\tau)\right)' = \varphi(\tau) + \tau\varphi'(\tau), \tau \in \Delta.$$

Then, we obtain

$$F_{\varphi(\Delta)}\left(\varphi(\tau) + \tau\varphi'(\tau)\right) \leq F_{h(\Delta)} h(\tau),\ \tau \in \Delta.$$

Using Lemma 3, we have

$$\begin{aligned} F_{\varphi(\Delta)} \varphi(\tau) &\leq F_{q(\Delta)} q(\tau),\ \tau \in \Delta \\ i.e,\ F_{L_{\alpha,\lambda}^{m} f(\Delta)} \frac{L_{\alpha,\lambda}^{m} f(\tau)}{\tau} &\leq F_{q(\Delta)} q(\tau),\ \tau \in \Delta. \end{aligned}$$

We obtain

$$\frac{L^m_{\alpha,\lambda} f(\tau)}{\tau} \prec_{\mathcal{F}} q(\tau),\ \tau \in \Delta.$$

□

Corollary 1. *Suppose that*

$$h(\tau) = \frac{1 + (2\beta - 1)\tau}{1 + \tau}$$

is a convex function in Δ, $0 \le \beta < 1$. *Let* $m > -1$, $\lambda \ge 0$, $\alpha \ge 0$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in A_n$ *and verify the fuzzy differential subordination*

$$F_{\left(L^m_{\alpha,\lambda} f\right)'(\Delta)}\left(L^m_{\alpha,\lambda} f(\tau)\right)' \le F_{h(\Delta)} h(\tau), \tag{24}$$

i.e.,

$$\left(L^m_{\alpha,\lambda} f(\tau)\right)' \prec_{\mathcal{F}} h(\tau), \tau \in \Delta.$$

Then,

$$\begin{aligned} F_{L^m_{\alpha,\lambda} f(\Delta)} \frac{L^m_{\alpha,\lambda} f(\tau)}{\tau} &\le F_{q(\Delta)} q(\tau), \\ i.e,\ \frac{L^m_{\alpha,\lambda} f(\tau)}{\tau} &\prec \mathcal{F} q(\tau),\ \tau \in \Delta \end{aligned}$$

and

$$q(\tau) = 2\beta - 1 + \frac{2(1-\beta)}{n\tau^{\frac{1}{n}}} \int_0^\tau \frac{t^{\frac{1}{n}-1}}{1+t} dt, \tau \in \Delta.$$

The function $q(\tau)$ *is convex and fuzzy best dominant.*

Proof.

$$h(\tau) = \frac{1 + (2\beta - 1)\tau}{1 + \tau}.$$

Also,

$$h(0) = 1 \text{ and } h'(\tau) = \frac{-2(1-\beta)}{(1+\tau)^2},$$

as well as

$$h''(\tau) = \frac{4(1-\beta)}{(1+\tau)^3},$$

and, also,

$$\begin{aligned} & Re\left(\frac{\tau h''(\tau)}{h'(\tau)} + 1\right) \\ = & Re\left(\frac{1-\tau}{1+\tau}\right) \\ = & Re\left(\frac{1 - \phi\cos\theta - i\phi\sin\theta}{1 + \phi\cos\theta + i\phi\sin\theta}\right) \\ = & \frac{1-\phi^2}{1 + 2\phi\cos\theta + \phi^2} > 0 > -\frac{1}{2}. \end{aligned}$$

Using the same steps of proof of Theorem 5 and considering

$$\varphi(\tau) = \frac{L^m_{\alpha,\lambda} f(\tau)}{\tau},$$

the fuzzy differential subordination (24) becomes

$$F_{L^m_{\alpha,\lambda}f(\Delta)}\left(\varphi(\tau)+\tau\varphi'(\tau)\right)\le F_{h(\Delta)}h(\tau),\quad \tau\in\Delta.$$

According to Lemma 2, for $\gamma=1$,

$$\begin{aligned} F_{\varphi(\Delta)}\varphi(\tau) &\le F_{q(\Delta)}q(\tau), \\ F_{L^m_{\alpha,\lambda}f(\Delta)}\frac{L^m_{\alpha,\lambda}f(\tau)}{\tau} &\le F_{q(\Delta)}q(\tau), \end{aligned}$$

where

$$q(\tau)=\frac{1}{n\tau^{\frac{1}{n}}}\int_0^\tau h(t)t^{\frac{1}{n}-1}dt,\tau\in\Delta$$

and

$$\begin{aligned} q(\tau) &= \frac{1}{n\tau^{\frac{1}{n}}}\left(\int_0^\tau t^{\frac{1}{n}-1}\frac{1+(2\beta-1)t}{1+t}dt,\tau\in\Delta\right) \\ &= 2\beta-1+\frac{2(1-\beta)}{n\tau^{\frac{1}{n}}}\int_0^\tau \frac{t^{\frac{1}{n}-1}}{1+t}dt,\tau\in\Delta. \end{aligned}$$

□

Example 1. *Suppose that*

$$h(\tau)=\frac{1-\tau}{1+\tau}$$

with

$$h(0)=1,\ \ h'(\tau)=\frac{-2}{(1+\tau)^2},$$

and also

$$h''(\tau)=\frac{4}{(1+\tau)^3}.$$

In addition,

$$\begin{aligned} & Re\left(\frac{\tau h''(\tau)}{h'(\tau)}+1\right) \\ =\ & Re\left(\frac{1-\tau}{1+\tau}\right) \\ =\ & Re\left(\frac{1-\phi\cos\theta-i\phi\sin\theta}{1+\phi\cos\theta+i\phi\sin\theta}\right) \\ =\ & \frac{1-\phi^2}{1+2\phi\cos\theta+\phi^2}>0>-\frac{1}{2}, \end{aligned}$$

and thus the function h is convex in Δ.

Suppose that

$$f(\tau)=\tau+\tau^2,\ \ \tau\in\Delta.$$

For $n=1$, $\lambda=1$, $\alpha=2$, *and* $m=0$, *we obtain*

$$L^m_{\alpha,\lambda}f(\tau)=(1-\alpha)S^m_\lambda f(\tau)+\alpha I^m f(\tau).$$

$$\begin{aligned} L^0_{2,1}f(\tau) &= -S^0_1 f(\tau) + 2I^0 f(\tau) \\ &= -f(\tau) + 2\tau f'(\tau) \\ &= -\tau - \tau^2 + 2\tau + 4\tau^2 \\ &= \tau f'(\tau) \\ &= \tau + 3\tau^2. \end{aligned}$$

Then,

$$\left(L^1_{2,1}f(\tau)\right)' = 1 + 6\tau$$

and

$$\frac{L^1_{2,1}f(\tau)}{\tau} = 1 + 3\tau.$$

$$q(\tau) = \frac{1}{\tau}\int_0^\tau \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+\tau)}{\tau}.$$

Using Theorem 5, we obtain

$$1 + 6\tau \prec \mathcal{F}\frac{1-\tau}{1+\tau},\ \tau \in \Delta.$$

This induces

$$1 + 3\tau \prec \mathcal{F} - 1 + \frac{2\ln(1+\tau)}{\tau},\ \tau \in \Delta.$$

Theorem 6. *Let function $h(\tau) = g(\tau) + \tau g'(\tau)$, $\tau \in \Delta$, and g be a convex function in Δ with $g(0) = 1$. Let $f \in A_n$ satisfy*

$$F_{L^m_{\alpha,\lambda}f(\Delta)}\left(\frac{\tau L^m_{\alpha,\lambda}f(\tau)}{L^m_{\alpha,\lambda}f(\tau)}\right)' \leq F_{h(\Delta)}h(\tau),\ i.e,\ \left(\frac{\tau L^m_{\alpha,\lambda}f(\tau)}{L^m_{\alpha,\lambda}f(\tau)}\right)' \prec \mathcal{F}h(\tau),\ \ \tau \in \Delta. \tag{25}$$

As $\alpha \geq 0$, $m > -1$, and $n \in \mathbb{N}$, we obtain sharp fuzzy differential subordination

$$\begin{aligned} F_{L^m_{\alpha,\lambda}f(\Delta)}\frac{L^{m+1}_{\alpha,\lambda}f(\tau)}{L^m_{\alpha,\lambda}f(\tau)} &\leq F_{g(\Delta)}g(\tau), \\ i.e,\ \frac{L^{m+1}_{\alpha,\lambda}f(\tau)}{L^m_{\alpha,\lambda}f(\tau)} &\prec {}_{\mathcal{F}}g(\tau),\ \ \tau \in \Delta. \end{aligned}$$

Proof. As

$$f \in \mathrm{A}_n \text{ and } f(\tau) = \tau + \sum_{j=n+1}^{\infty} \mathrm{b}_j\tau^j,$$

we have

$$L^m_{\alpha,\lambda}f(\tau) = \tau + \sum_{j=n+1}^{\infty}\left[(1-\alpha)\{\lambda(j-1)+1\}^m + \alpha\Psi^m(n)\right]\mathrm{b}_j\tau^j,\ \ \tau \in \Delta.$$

Consider

$$\varphi(\tau) = \frac{L^{m+1}_{\alpha,\lambda}f(\tau)}{L^m_{\alpha,\lambda}f(\tau)} = \frac{\tau + \sum\limits_{j=n+1}^{\infty}\left[(1-\alpha)\{\lambda(j-1)+1\}^{m+1} + \alpha\Psi^{m+1}(n)\right]\mathrm{b}_j\tau^j}{\tau + \sum\limits_{j=n+1}^{\infty}\left[(1-\alpha)\{\lambda(j-1)+1\}^m + \alpha\Psi^m(n)\right]\mathrm{b}_j\tau^j}.$$

We have

$$\varphi'(\tau) = \frac{\left(L^{m+1}_{\alpha,\lambda}f(\tau)\right)'}{L^m_{\alpha,\lambda}f(\tau)} - \varphi(\tau)\cdot\frac{\left(L^m_{\alpha,\lambda}f(\tau)\right)'}{L^m_{\alpha,\lambda}f(\tau)}$$

and we obtain

$$\varphi(\tau)+\tau\varphi'(\tau)=\left(\frac{\tau L_{\alpha,\lambda}^{m+1}f(\tau)}{L_{\alpha,\lambda}^{m}f(\tau)}\right)'.$$

Relation (25) becomes

$$F_{\varphi(\Delta)}\big(\varphi(\tau)+\tau\varphi'(\tau)\big)\leq F_{h(\Delta)}h(\tau)=F_{g(\Delta)}\big(g(\tau)+\tau g'(\tau)\big),\quad \tau\in\Delta.$$

According to Lemma 3, we have

$$F_{\varphi(\Delta)}\varphi(\tau)\leq F_{g(\Delta)}g(\tau),\quad \tau\in\Delta,$$

i.e.,

$$F_{L_{\alpha,\lambda}^{m}f(\Delta)}\frac{L_{\alpha,\lambda}^{m+1}f(\tau)}{L_{\alpha,\lambda}^{m}f(\tau))}\leq F_{g(\Delta)}g(\tau),\quad \tau\in\Delta.$$

We obtain

$$\frac{L_{\alpha,\lambda}^{m+1}f(\tau)}{L_{\alpha,\lambda}^{m}f(\tau))}\prec_{\mathcal{F}} g(\tau),\quad \tau\in\Delta.$$

□

Theorem 7. *Given a convex function g with $g(0)=1$, define function*

$$h(\tau)=g(\tau)+\tau g'(\tau),\ \tau\in\Delta.$$

Let $\alpha\geq 0$, with $n\in\mathbb{N}$, $m>-1$, and $f\in A_n$ satisfying

$$F_{L_{\alpha,\lambda}^{m}f(\Delta)}\left(L_{\alpha,\lambda}^{m+1}f(\tau)\right)'+\frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{m+2}\leq F_{h(\Delta)}h(\tau),$$

i.e.,

$$\left(L_{\alpha,\lambda}^{m+1}f(\tau)\right)'+\frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{m+2}\prec_{\mathcal{F}} h(\tau);\tag{26}$$

then, the sharp fuzzy differential subordination obtains

$$\begin{aligned}F_{(L_{\alpha,\lambda}^{m}f)'(\Delta)}\left[L_{\alpha,\lambda}^{m}f(\tau)\right]' &\leq F_{g(\Delta)}g(\tau),\\ i.e,\ \left[L_{\alpha,\lambda}^{m}f(\tau)\right]' &\prec_{\mathcal{F}} g(\tau),\quad \tau\in\Delta.\end{aligned}$$

Proof. According to the definition of operator $L_{\alpha,\lambda}^{m}$, we obtain

$$L_{\alpha,\lambda}^{m+1}f(\tau)=(1-\alpha)S^{m+1}f(\tau)+\alpha I_{\lambda}^{m+1}f(\tau).\tag{27}$$

Using Equation (26), we obtain

$$F_{L_{\alpha,\lambda}^{m}f(\Delta)}\left(\left((1-\alpha)S^{m+1}f(\tau)+\alpha I_{\lambda}^{m+1}f(\tau)\right)'+\frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{m+2}\right)\leq F_{h(\Delta)}h(\tau),$$

which may be rearranged as follows:

$$F_{L_{\alpha,\lambda}^{m}f(\Delta)}\left(\begin{array}{c}\left((1-\alpha)(S^m f(\tau))'+\alpha\left(I_{\lambda}^{m}f(\tau)\right)'\right)\\ +\tau\left((1-\alpha)(S^m f(\tau))''+\alpha\left(I_{\lambda}^{m}f(\tau)\right)''\right)\end{array}\right)\leq F_{h(\Delta)}h(\tau),\ \tau\in\Delta.$$

Let

$$\begin{aligned}\varphi(\tau) &= \left((1-\alpha)(S^m f(\tau))' + \alpha(I_\lambda^m f(\tau))'\right) \\ &= \left(L_{\alpha,\lambda}^m f(\tau)\right)' \\ &= 1 + \sum_{j=n+1}^{\infty} j\left[\alpha\{\lambda(j-1)+1\}^m + (1-\alpha)jC_{m+j-1,t}^m\right]\mathrm{b}_j\tau^j \\ &= 1 + \varphi_n\tau^n + \varphi_{n+1}\tau^{n+1} + \ldots\end{aligned} \quad (28)$$

We deduce that $\varphi \in \mathcal{H}[1,n]$. Using the notation in (28), the fuzzy differential subordination becomes

$$\begin{aligned}F_{\varphi(\Delta)}\left(\varphi(\tau) + \tau\varphi'(\tau)\right) &\leq F_{h(\Delta)}h(\tau) \\ &= F_{g(\Delta)}\left(g(\tau) + \tau g'(\tau)\right).\end{aligned}$$

By using Lemma 3, we have

$$F_{\varphi(\Delta)}\varphi(\tau) \leq F_{g(\Delta)}g(\tau), \quad \tau \in \Delta,$$

i.e.,

$$F_{L_{\alpha,\lambda}^m f(\Delta)}\left(L_{\alpha,\lambda}^m f(\tau)\right)' \leq F_{g(\Delta)}g(\tau), \quad \tau \in \Delta.$$

□

Theorem 8. *Suppose that h is a convex function with order $-\frac{1}{2}$, satisfying the condition $h(0) = 1$. Let $f \in \mathcal{A}_n$ satisfy*

$$F_{L_{\alpha,\lambda}^m f(\Delta)}\left(L_{\alpha,\lambda}^{m+1} f(\tau)\right)' + \frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{m+2} \leq F_{h(\Delta)}h(\tau)$$

i.e.,

$$\left(L_{\alpha,\lambda}^{m+1} f(\tau)\right)' + \frac{(1-\alpha)m\tau(I^m f(\tau))''}{m+1} \prec_{\mathcal{F}} h(\tau), \quad (29)$$

for $\alpha \geq 0$, $m > -1$, $m > -1$, and $n \in N$. Then, the sharp fuzzy differential subordination obtains

$$F_{(L_{\alpha,\lambda}^m f)(\Delta)}\left(L_{\alpha,\lambda}^m f(\tau)\right)' \leq F_{q(\Delta)}q(\tau),$$

i.e.,

$$\left[L_{\alpha,\lambda}^m f(\tau)\right]' \prec_{\mathcal{F}} q(\tau), \quad \tau \in \Delta.$$

where

$$q(\tau) = \frac{1}{n\tau^{\frac{1}{n}}}\int_0^\tau h(t)t^{\frac{1}{n}-1}dt$$

is convex and the best fuzzy dominant.

Proof. As h is a convex of order $-\frac{1}{2}$, and using Lemma 1, we have that

$$q(\tau) = \frac{1}{n\tau^{\frac{1}{n}}}\int_0^\tau h(t)t^{\frac{1}{n}-1}dt$$

is a convex function and verifies the differential equation associated to the fuzzy differential subordination (29)

$$q(\tau) + \tau q'(\tau) = h(\tau).$$

So, it is the fuzzy best dominant. By using the characteristics of operator $L^m_{\alpha,\lambda}$ and taking into account $\varphi(\tau) = \left(L^m_{\alpha,\lambda}\right)'$, we are able to obtain

$$\begin{aligned} &\left(L^{m+1}_{\alpha,\lambda} f(\tau)\right)' + \frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{m+2} \\ = \quad & \varphi(\tau) + \tau\varphi'(\tau), \quad \tau \in \Delta. \end{aligned}$$

Then, (29) becomes

$$F_{\varphi(\Delta)}\big(\varphi(\tau) + \tau\varphi'(\tau)\big) \le F_{h(\Delta)} h(\tau),\ \tau \in \Delta.$$

As $\varphi \in \mathcal{H}[1, n]$, by Lemma 3, we obtain

$$F_{\varphi(\Delta)}\varphi(\tau) \le F_{q(\Delta)} q(\tau),\ \tau \in \Delta,$$

i.e.,

$$F_{L^m_{\alpha,\lambda} f(\Delta)}\big(L^m_{\alpha,\lambda} f(\tau)\big)' \le F_{q(\Delta)} q(\tau),\ \ \tau\ \in \Delta;$$

then, we obtain

$$\big(L^m_{\alpha,\lambda} f(\tau)\big)' \prec_{\mathcal{F}} q(\tau),\ \tau \in \Delta.$$

□

Corollary 2. *Consider* $h(\tau) = \frac{1+(2\beta-1)\tau}{1+\tau}$*, where* $0 \le \beta < 1$*. If* $\alpha \ge 0$*,* $n \in \mathbb{N}$*,* $m > -1$ *and* $f \in \mathrm{A}_n$ *and satisfies the differential subordination*

$$F_{L^m_{\alpha,\lambda} f(\Delta)}\left(\left(L^{m+1}_{\alpha,\lambda} f(\tau)\right)' + \frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{(m+2)}\right) \le F_{h(\Delta)} h(\tau)$$

i.e.,

$$\left(L^{m+1}_{\alpha,\lambda} f(\tau)\right)' + \frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{m+2} \prec_{\mathcal{F}} h(\tau),\ \tau \in \Delta, \tag{30}$$

then

$$F_{L^m_{\alpha,\lambda} f(\Delta)}\big(L^m_{\alpha,\lambda} f(\tau)\big)' \le F_{q(\Delta)} q(\tau),$$

i.e.,

$$i.e,\ \big(L^m_{\alpha,\lambda} f(\tau)\big)' \prec \mathcal{F} q(\tau),\ \tau \in \Delta,$$

where q is given by

$$q(\tau) = 2\beta - 1 + \frac{2(1-\beta)}{n\tau^{\frac{1}{n}}} \int_0^\tau \frac{t^{\frac{1}{n}-1}}{1+t} dt, \quad \tau \in \Delta.$$

The q is convex and fuzzy best dominant.

Proof. Following the same steps of proof of Theorem 7 and taking

$$\varphi(\tau) = \big(L^m_{\alpha,\lambda} f(\tau)\big)',$$

the fuzzy differential subordination (30) becomes

$$F_{\varphi(\Delta)}\big(\varphi(\tau) + \tau\varphi'(\tau)\big) \le F_{h(\Delta)} h(\tau),\ \tau \in \Delta.$$

According to Lemma 2 and $\gamma = 1$, we obtain

$$F_{\varphi(\Delta)}\varphi(\tau) \leq F_{q(\Delta)}q(\tau),$$

$$F_{\left(L^m_{\alpha,\lambda}f\right)'(\Delta)}\left(L^m_{\alpha,\lambda}f(\tau)\right)' \leq F_{q(\Delta)}q(\tau),$$

i.e.,

$$\left(L^m_{\alpha,\lambda}f(\tau)\right)' \prec \mathcal{F}q(\tau), \ \ \tau \in \Delta.$$

Also,

$$\begin{aligned} q(\tau) &= \frac{1}{n\tau^{\frac{1}{n}}}\int_0^\tau h(t)t^{\frac{1}{n}-1}dt \\ &= \frac{1}{n\tau^{\frac{1}{n}}}\int_0^\tau t^{\frac{1}{n}-1}\frac{1+(2\beta-1)t}{1+t}dt \\ &= 2\beta - 1 + \frac{2(1-\beta)}{n\tau^{\frac{1}{n}}}\int_0^\tau \frac{t^{\frac{1}{n}-1}}{1+t}dt, \quad \tau \in \Delta. \end{aligned}$$

□

Example 2. *Suppose that* $h(\tau) = \frac{1-\tau}{1+\tau}$ *is a convex function in* Δ *and* $h(0) = 1$*; then,*

$$Re\left(\frac{\tau h''(\tau)}{h'(\tau)} + 1\right) > -\frac{1}{2}.$$

Suppose that $f(\tau) = \tau + \tau^2$, $\tau \in \Delta$*. For* $n = 1$, $m = 0$*, and* $\lambda = 1$*, as well as* $\alpha = 2$*, we obtain*

$$L^0_{2,1} = 2\tau f^{'}(\tau) - f(\tau) = \tau + 3\tau^2.$$

Also,

$$\left(L^0_{2,1}f(\tau)\right)' = 1 + 6\tau.$$

We obtain

$$\begin{aligned} &\left(L^{m+1}_{\alpha,\lambda}f(\tau)\right)' + \frac{(1-\alpha)(m+1)\tau(I^m f(\tau))''}{(m+2)} \\ =& \left(L^1_{2,1}f(\tau)\right)' - \frac{\tau\left(I^0 f(\tau)\right)''}{2} \\ =& \left(\tau + 3\tau^2\right)' - \frac{\tau\left(\tau + 4\tau^2\right)''}{2} \\ =& 1 - 2\tau. \end{aligned}$$

We have

$$q(\tau) = \frac{1}{\tau}\int_0^\tau \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+\tau)}{\tau}.$$

By Theorem 8, we obtain

$$1 - 2\tau \prec_{\mathcal{F}} \frac{1-\tau}{1+\tau'}, \ \tau \in \Delta,$$

inducing

$$1 + 6\tau \prec_{\mathcal{F}} -1 + \frac{2\ln(1+\tau)}{\tau}, \ \tau \in \Delta.$$

4. Conclusions

Geometric function theory, a subfield of complex analysis, was applied in this work to the study of fuzzy sets. Using the Noor integral operator and the generalized Sălăgean differential operator, we introduced a definition of linear differential operator $L^m_{\alpha,\lambda} : \mathrm{A}_n, \to \mathrm{A}_n$ in an open unit disc Δ. Applying the theory of fuzzy differential subordination, we defined and studied a subclass $R^\lambda_F(m, \alpha, \delta)$ of analytic function considering the operator $L^m_{\alpha,\lambda}$ and, for this operator, new fuzzy differential subordinations were obtained. Several new results were found that are pertinent to this class by using the concept of fuzzy differential subordination. After introducing the necessary theorems and corollaries, we showed how they may be used in application with the help of certain examples.

Other subclasses of analytic functions can be introduced regarding this operator and some properties for these subclasses can be investigated, such as estimates for coefficients, distortion theorems, and closure theorems. Moreover, the method used to propose this class might motivate studies of other intriguing fuzzy classes. More research into the bound of $0 < \delta \leq 1$ might lead to the discovery of alternative values of δ that provide appropriate definitions of fuzzy classes.

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